Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability

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January 9, 2010

Abstract

We discuss some recent developments of the theory of BV functions and sets of finite perimeter in infinite-dimensional Gaussian spaces. In this context the concepts of Hausdorff measure, approximate continuity, rectifiability have to be properly understood. After recalling the known facts, we prove a Sobolev-rectifiability result and we list some open problems.

1 Introduction

This paper is devoted to the theory of sets of finite perimeter in infinite-dimensional Gaussian spaces. We illustrate some recent results, we provide some new ones and eventually we discuss some open problems.

We start first with a discussion of the finite-dimensional theory, referring to [13] and [3] for much more on this subject. Recall that a Borel set $E \subset \mathbb{R}^m$ is said to be of finite perimeter if there exists a vector valued measure $D\chi_E = (D_1\chi_E, \ldots, D_m\chi_E)$ with finite total variation in $\mathbb{R}^m$ satisfying the integration by parts formula:

$$\int_E \frac{\partial \phi}{\partial x_i}\, dx = -\int_{\mathbb{R}^m} \phi \, dD_i\chi_E \qquad \forall i = 1, \ldots, m, \ \forall \phi \in C^1_c(\mathbb{R}^m). \tag{1}$$

De Giorgi proved in [10] a deep result on the structure of $D\chi_E$, which could be considered as the starting point of modern Geometric Measure Theory. First of all he identified a

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set $\mathscr{F}E$, called by him reduced boundary, on which $|D\chi_E|$ is concentrated, and defined a pointwise inner normal $\nu_E(x) = (\nu_{E,1}(x), \dots, \nu_{E,m}(x))$ on it (see (26) for the precise definition); then, through a suitable blow-up procedure, he proved that $\mathscr{F}E$ is countably C^1 rectifiable, i.e., it is contained in the union of countably many graphs of C^1 functions defined on hyperplanes of $\mathbb{R}^m$; finally, he proved the representation formula $D\chi_E = \nu_E \mathscr{H}^{m-1} \llcorner \mathscr{F}E$, where $\mathscr{H}^{m-1}$ is the $(m-1)$-dimensional Hausdorff measure in $\mathbb{R}^m$. Actually, because of the rectifiability property, many other measures could be used in place of $\mathscr{H}^{m-1}$, for instance the spherical Hausdorff measure $\mathscr{S}^{m-1}$. In light of these results, the integration by parts formula reads

$$\int_E \frac{\partial \phi}{\partial x_i} \, dx = -\int_{\mathscr{F}E} \phi \nu_{E,i} \, d\mathscr{H}^{m-1} \qquad \forall i = 1, \dots, m, \ \forall \phi \in C^1_c(\mathbb{R}^m).$$

This is much closer to the classical Gauss–Green formula, the only difference being that boundary and inner normal are understoood in a measure-theoretic sense. A few years later Federer proved in [12] that the same representation result of $D\chi_E$ holds for another concept of boundary, the so-called essential boundary:

$$\partial^* E := \left\{ x \in \mathbb{R}^m : \ \limsup_{r \downarrow 0} \frac{\mathscr{L}^m(B_r(x) \cap E)}{\mathscr{L}^m(B_r(x))} > 0, \ \limsup_{r \downarrow 0} \frac{\mathscr{L}^m(B_r(x) \setminus E)}{\mathscr{L}^m(B_r(x))} > 0 \right\},$$

where $\mathscr{L}^m$ is the m-dimensional Lebesgue measure. Indeed, a consequence of De Giorgi's blow-up procedure is that $\mathscr{F}E \subset \partial^* E$ (because tangent sets to E at all points in the reduced boundary are halfspaces), and in [12] it is shown that $\mathscr{H}^{m-1}(\partial^* E \setminus \mathscr{F}E) = 0$.

If we now move to Gaussian spaces we have to change the reference measure from $\mathscr{L}^m$ to $\gamma = G_m \mathscr{L}^m$, where G_m is the standard Gaussian kernel. Then, since $\partial_{x_i} G_m = -x_i G_m$, we have the Gaussian integration by parts formula

$$\int_{\mathbb{R}^m} f \frac{\partial \phi}{\partial x_i} \, d\gamma = -\int_{\mathbb{R}^m} \phi \frac{\partial f}{\partial x_i} \, d\gamma + \int_{\mathbb{R}^m} x_i \phi f \, d\gamma. \tag{2}$$

This leads to the definition of Gaussian set of finite perimeter: we require the existence of a measure $D_\gamma \chi_E = (D_{\gamma,1}\chi_E, \dots, D_{\gamma,m}\chi_E)$ satisfying

$$\int_E \frac{\partial \phi}{\partial x_i} \, d\gamma = -\int_{\mathbb{R}^m} \phi \, dD_{\gamma,i}\chi_E + \int_E x_i \phi \, d\gamma \quad \forall i = 1, \dots, m, \ \forall \phi \in C^1_b(\mathbb{R}^m). \tag{3}$$

Both (2) and (3) can be extended to infinite-dimensional Gaussian spaces (X, γ), the so-called Wiener spaces (see [7], [20]) by looking at directions in the Cameron-Martin space H of (X, γ), see Section 4 for more details. Along these lines the theory of sets of finite perimeter and BV functions has been initiated by Fukushima and Hino in [15, 16, 17]. More recently, we revisited the theory in [4, 5, 6], where we provided also compactness criteria in BV. Also, in these papers it is shown that BV functions can be characterized in terms of the Ornstein-Uhlenbeck semigroup

$$T_t f(x) = \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\, d\gamma(y). \tag{4}$$

This last point is particularly relevant, also because the original definition of De Giorgi [9] was not based on the integration by parts formula (1), but precisely on the finiteness of the limit (where $t \mapsto R_t f$ is the heat flow starting from f)

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^m} |\nabla R_t \chi_E|\, dx. \tag{5}$$

When looking for the counterpart of De Giorgi's and Federer's results in infinite-dimensional spaces, several difficulties arise:

(i) The classical concept of Lebesgue approximate continuity, underlying also the definition of essential boundary, seems to fail or seems to be not reproducible in Gaussian spaces (X,γ). For instance, in [21] it is shown that in general the balls of X can't be used, and in any case the norm of X is not natural from the point of view of the calculus in Wiener spaces, where no intrinsic metric structure exists and the differentiable structure is induced by H.

(ii) Suitable notions of codimension-1 Hausdorff measure, of rectifiability and of essential/reduced boundary have to be devised.

Nevertheless, some relevant progresses have been obtained by Feyel-De la Pradelle in [14] and by Hino in [19]. In [14] a family of Hausdorff pre-measures $\mathscr{S}^{\infty-1}_F$ (of spherical type) have been introduced by looking at the factorization $X = \mathrm{Ker}(\pi_F) \otimes F$, with F m-dimensional subspace of H, and considering the measures $\mathscr{S}^{m-1}$ on the m-dimensional fibers of the decomposition. A crucial monotonicity property of these pre-measures with respect to F allows to define $\mathscr{S}^{\infty-1}$ as $\lim_F \mathscr{S}^{\infty-1}_F$ (the limit being taken in the sense of directed sets). In [19] this approach has been used to build a Borel set $\partial^*_{\mathcal{F}} E$, called cylindrical essential boundary, for which the representation formula

$$|D_\gamma \chi_E| = \mathscr{S}^{\infty-1}_{\mathcal{F}} \llcorner \partial^*_{\mathcal{F}} E \tag{6}$$

holds. Here $\mathcal{F} = \{F_n\}_{n\geq 1}$ is an increasing family of finite-dimensional subspaces of QX^* (here X^* is the dual of X and Q is the covariance operator from X^* to $H \subset X$) whose union is dense in H and $\mathscr{S}^{\infty-1}_{\mathcal{F}} = \lim_n \mathscr{S}^{\infty-1}_{F_n}$. Notice that, while the left hand side in the representation formula is independent of the choice of $\mathcal{F}$, both the cylindrical essential boundary and $\mathscr{S}^{\infty-1}_{\mathcal{F}}$ depend on $\mathcal{F}$. The problem of getting a representation formula in terms of the coordinate-free measure $\mathscr{S}^{\infty-1}$ of [14] is still open. This seems to be strongly related to the problem of finding coordinate-free definitions of reduced/essential boundary.

In connection with (ii), adapting some ideas from [1, 2] and answering in part to one of the questions raised in [19], we are able to show that $|D_\gamma \chi_E|$ is concentrated on the union of countably many graphs of entire $W^{1,1}$ functions defined on hyperplanes of X orthogonal to vectors in H (because of this, γ factors as a product of Gaussian measures and the notion of Sobolev function defined on the hyperplane makes sense).

The paper is organized as follows: in Section 2 we fix some basic notation, in Section 3 we study the codimension 1 spherical Hausdorff measures of [14]. In Section 4 we recall some preliminary results on BV functions and sets of finite perimeter needed in the proofs. In Section 5 we prove, following with minor variants the original proof in [19], Hino's representation formula (6) of the perimeter measure. In Section 6 we prove that $|D\chi_E|$ is concentrated on countably many graphs of entire Sobolev functions. Finally, in Section 7 we discuss open problems and research perspectives: the improvement from Sobolev to Lipschitz in our rectifiability result and potential coordinate-free definitions of essential/reduced boundary. Our proposals are in the same spirit as De Giorgi's pioneering intuition (5), and based on the Ornstein-Uhlenbeck semigroup.

It is a pleasure and an honour for us to dedicate this paper to Louis Nirenberg, on the occasion of his 85th birthday.

2 Basic Notation

First, we recall some measure-theoretic notation. We denote by $\mathscr{B}(X)$ the σ-algebra of Borel sets in a metric space X and by $\mathscr{M}_+(X)$ the space of Borel nonnegative and finite measures in X; we often use the fact that if I is a directed set and $\mu_i \in \mathscr{M}_+(X)$ satisfy $\mu_i \leq \mu_j$ for $i \leq j$, $\sup_i \mu_i(X) < \infty$, then $\lim_i \mu_i$ belongs to $\mathscr{M}_+(X)$. We also denote by $\mathscr{P}(X)$ the subspace of probability measures and by $\mathscr{M}(X)$ the space of Borel signed measures with finite total variation in X. We use the notation $f_\sharp$ for the push-forward operator from measures in X to measures in Y, induced by a Borel map $f : X \to Y$, and the notation $\mu \llcorner E$ for the restriction operator, defined by $\mu \llcorner E(B) = \mu(E \cap B)$.

If E is a subset of a finite-dimensional Hilbert space F, we denote by $\partial^* E$ its essential boundary, namely the set of points $x \in F$ where the volume density does not exist or it is different from 0, 1. Here "volume" refers to the unique Lebesgue measure on F, but any Gaussian measure in F would lead to an equivalent definition of $\partial^* E$.

We mostly use the notation X for a separable Banach space endowed with a centered and non-degenerate Gaussian measure γ (i.e., the support of γ is the whole of X). We denote by X^* the dual of X and by $Q : X^* \to X$ the covariance operator of γ: it is a continuous linear operator uniquely determined by

$$\langle y^*, Qx^*\rangle = \int_X \langle x^*, x\rangle\langle y^*, x\rangle \, d\gamma(x) \qquad \forall x^*,\ y^* \in X^*.$$

It can be easily proved that $Q = RR^*$, where $R^* : X^* \to L^2(X,\gamma)$ is given by $R^*x^*(x) = \langle x^*, x\rangle$, and $R : L^2(X,\gamma) \to X$ is given by

$$Rf := \int_X xf(x)\, d\gamma(x).$$

The Cameron-Martin space H of (X,γ) is given by $R\mathscr{H}$, where $\mathscr{H}$ is the closure in $L^2(X,\gamma)$ of R^*X^*. Since R is injective on $\mathscr{H}$ we can define a Hilbert norm on H in such a way that $R : \mathscr{H} \to H$ is an isometry, and with this choice QX^* is dense in H, while H embeds continuously and densely in X. We use the notation $\langle\cdot,\cdot\rangle$ also for the inner product in H and the notation $|\cdot|$ for the induced norm; since the typical element of X^* is denoted by x^*, y^*, etc., this should not create a real ambiguity.

The symbol $\mathcal{F}C^1_b(X)$ denotes the space of continuously differentiable cylindrical functions with bounded derivatives, that is, $u \in \mathcal{F}C^1_b(X)$ if

$$u(x) = v(\langle x_1^*, x\rangle, \ldots, \langle x_m^*, x\rangle)$$

for some $v \in C^1_b(\mathbb{R}^m)$ and $x_1^*, \ldots, x_m^* \in X^*$.

3 Spherical Hausdorff measures in X

If $F \subset X$ is a m-dimensional subspace of H, $B \subset F$ and $\delta > 0$, we denote by $\mathscr{S}^k_\delta(B)$ the spherical k-dimensional Hausdorff pre-measure of B, namely

$$\inf\left\{\sum_i \omega_k r_i^k : \ B \subset \bigcup_i B_{r_i}(x_i),\ r_i < \delta\right\}$$

(ω_k being the Lebesgue volume of the unit ball in $\mathbb{R}^k$) and by $\mathscr{S}^k(B)$ their monotone limit as $\delta \downarrow 0$. When $k = m$ this measure coincides with the (outer) Lebesgue measure in F, and we shall mostly consider the case $k = m-1$. We stress that the balls used in the minimization above are understood with respect to the H distance and we do not emphasize the dependence on F. Occasionally we canonically identify F with $\mathbb{R}^m$, choosing a suitable orthonormal basis.

Let $F \subset QX^*$ be an m–dimensional subspace of H. We denote by $z = \pi_F(x)$ the canonical projection induced by an orthonormal basis $e_i = Q(e_i^*)$ of F, namely

$$\pi_F(x) = \sum_{i=1}^m \langle e_i^*, x\rangle e_i$$

and set $x = y + z$, so that $y = x - \pi_F(x)$ belongs to $\mathrm{Ker}(\pi_F)$, the kernel of π_F. This decomposition induces the factorization $\gamma = \gamma^\perp \otimes \gamma_F$ with γ_F standard Gaussian in F and $\gamma^\perp$ Gaussian in $\mathrm{Ker}(\pi_F)$ (whose Cameron-Martin space is $F^\perp$).

Following [14], we can now define spherical $(\infty - 1)$-dimensional Hausdorff measures in X by

$$\mathscr{S}^{\infty-1}_F(B) := \int^*_{\mathrm{Ker}(\pi_F)} \int_{B_y} G_m(z)\, d\mathscr{S}^{m-1}(z)\, d\gamma^\perp(y) \qquad \forall B \subset X. \tag{7}$$

Here and in the sequel

$$G_m(z) := (2\pi)^{-m/2} \exp(-|z|^2/2)$$

is the m-dimensional Gaussian kernel in F and, for $y \in \mathrm{Ker}(\pi_F)$,

$$B_y := \{z \in F :\ y + z \in B\}. \tag{8}$$

The internal integral in (7) is understood in the Choquet sense, namely

$$\int_{B_y} G_m(z)\, d\mathscr{S}^{m-1}(z) = \int_0^\infty \mathscr{S}^{m-1}\big(\{z :\ G_m(z) > \tau\}\big)\, d\tau.$$

Of course if $B_y \in \mathscr{B}(X)$, as it happens in the case $B \in \mathscr{B}(X)$, the integral reduces to a standard one. Furthermore, the external integral in (7) is understood as outer integral, in order to avoid at least at the level of the definition the issue of the measurability of the map $y \mapsto \int_{B_y} G_m\, d\mathscr{S}^{m-1}$.

The next basic additivity result has been proved in [14] (the result therein is slightly more general, since general finite-dimensional subspaces F of H, not only of QX^*, and Suslin sets are considered).

Proposition 3.1. *$\mathscr{S}^{\infty-1}_F$ is a σ-additive Borel measure on $\mathscr{B}(X)$. In addition, for all Borel sets B the map $y \mapsto \int_{B_y} G_m\, d\mathscr{S}^{m-1}$ is $\gamma^\perp$-measurable in* $\mathrm{Ker}(\pi_F)$.

A remarkable fact is the monotonicity of $\mathscr{S}^{\infty-1}_F$ with respect to F, which crucially depends on the fact that we are considering spherical Hausdorff measures.

Lemma 3.2. *$\mathscr{S}^{\infty-1}_F \le \mathscr{S}^{\infty-1}_G$ on $\mathscr{B}(X)$ whenever $F \subset G$.*

Proof. We write $G = F \oplus L$ and denote by m and $k - m$ the dimensions of F and L, respectively, so that G is k-dimensional. We consider the orthogonal decomposition $H = G^\perp \oplus L \oplus F$, so that γ can be written as the product $\gamma^\perp \otimes \gamma_L \otimes \gamma_F$, γ_F, γ_L being standard Gaussians in F and L, respectively. Since for all $B \in \mathscr{B}(X)$ we have

$$\mathscr{S}^{\infty-1}_F(B) = \int_{\mathrm{Ker}(\pi_F)} \int_{B_{w,y}} G_m(z)\, d\mathscr{S}^{m-1}(z)\, d(\gamma_L \otimes \gamma^\perp)(w, y)$$

and

$$\mathscr{S}_G^{\infty-1}(B) = \int_{\mathrm{Ker}(\pi_G)} \int_{B_y} G_m(w,z)\, d\mathscr{S}^{k-1}(w,z) d\gamma^{\perp}(y),$$

the statement follows by applying to all sections $C = B_y \subset G$ the following finite-dimensional inequality:

$$\int_L \int_{C_w} G_m(z)\, d\mathscr{S}^{m-1}(z) d\gamma_L(w) \le \int_C G_k(w,z)\, d\mathscr{S}^{k-1}(w,z). \tag{9}$$

In turn, since $G_k(w,z) = G_m(z)G_{k-m}(w)$, inequality (9) follows from

$$\int_L \mathscr{S}^{m-1}(A_w)\, d\mathscr{S}^{k-m}(w) \le \mathscr{S}^{k-1}(A) \qquad A \in \mathscr{B}(G). \tag{10}$$

For completeness we provide a proof of (10). Fix $(u,v) \in F \oplus L$ and the projection map $\pi : G \to L$; we then have that

$$\begin{aligned}(B_r(u,v))_w =& B_r(u,v) \cap \pi^{-1}(w) = \{z \in F :\ (w,z) \in B_r(u,v)\}\\ =& \{z \in F :\ |w-u|^2 + |z-v|^2 < r^2\} = B_{r(u,w)}(v),\end{aligned}$$

where $r(u,w) = \sqrt{r^2 - |w-u|^2}$ and the ball is understood in F. Then, with the same argument as Federer [13, 2.10.27], we obtain that

$$\begin{aligned}\omega_{m-1} \int_L \left(\frac{\mathrm{diam}(B_r(u,v))_w}{2}\right)^{m-1} d\mathscr{S}^{k-m}(w) =& \omega_{m-1} \int_L (r(u,w))^{m-1}\, d\mathscr{S}^{k-m}(w)\\ =& \omega_{k-1} r^{k-1} = \omega_{k-1}\left(\frac{\mathrm{diam} B_r(u,v)}{2}\right)^{k-1}.\end{aligned}$$

We assume with no loss of generality that $\mathscr{S}^{k-1}(A)$ is finite; by definition, for any $\delta > 0$ there exists a covering of balls $B_j = B_{r_j}(x_j)$ with $r_j < \delta$ and $\mathscr{S}^{k-1}(A) + \delta \ge \omega_{k-1} \sum_j r_j^{k-1}$; the balls $(B_j)_w$ cover A_w and have radii less than δ for any $w \in L$, whence

$$\begin{aligned}\mathscr{S}^{k-1}(A) + \delta \ge \omega_{k-1} \sum_j r_j^{k-1} =& \int_L \omega_{m-1} \sum_j \left(\frac{\mathrm{diam}(B_j)_w}{2}\right)^{m-1} d\mathscr{S}^{k-m}(w)\\ \ge& \int_L^* \mathscr{S}_\delta^{m-1}(A_w) d\mathscr{S}^{k-m}(w).\end{aligned}$$

By letting $\delta \downarrow 0$, (10) follows. □

Thanks to Lemma 3.2 we can define the spherical $(\infty-1)$-Hausdorff measure $\mathscr{S}^{\infty-1}$ in $\mathscr{B}(X)$ by

$$\mathscr{S}^{\infty-1}(B) := \sup_F \mathscr{S}_F^{\infty-1}(B) = \lim_F \mathscr{S}_F^{\infty-1}(B), \tag{11}$$

the limits being understood in the directed set of finite-dimensional subspaces of QX^*. Notice that this measure does not coincide directly with the one of [14], since we consider subspaces of QX^* only. A direct consequence of Proposition 3.1 is that $\mathscr{S}^{\infty-1}$ is σ-additive on $\mathscr{B}(X)$.

Finally, we conclude this section with the following elementary proposition.

Proposition 3.3. *Let $\mathcal{F}$ be a countable family of finite-dimensional subspaces of QX^* stable under finite unions. For $F \in \mathcal{F}$, let $A_F \in \mathscr{B}(X)$ be such that*

(i) $\mathscr{S}^{\infty-1}_F(A_F \setminus A_G) = 0$ *whenever* $F \subset G$*;*

(ii) $\sup_F \mathscr{S}^{\infty-1}_F(A_F) < \infty$.

Then $\lim_F(\mathscr{S}^{\infty-1}_F \llcorner A_F)$ *exists, and it is representable as* $(\lim_F \mathscr{S}^{\infty-1}_F) \llcorner A$ *with*

$$A := \bigcup_{F \in \mathcal{F}} \bigcap_{G \in \mathcal{F},\, G \supset F} A_G \in \mathscr{B}(X).$$

Proof. First of all, we notice that assumptions (i) implies that $F \mapsto \mathscr{S}^{\infty-1}_F \llcorner A_F$ is monotone w.r.t. F, hence the limit exists; it is obviously additive and, because of assumption (i), finite and σ-additive. If we define $A'_F := \bigcap_{G \supset F} A_G$ then, because of assumption (i), $\mathscr{S}^{\infty-1}_F \llcorner A_F = \mathscr{S}^{\infty-1}_F \llcorner A'_F$. The monotonicity of $F \mapsto A'_F$ now yields

$$\mathscr{S}^{\infty-1}_F \llcorner A'_F \leq \mathscr{S}^{\infty-1}_F \llcorner A \leq (\lim_F \mathscr{S}^{\infty-1}_F) \llcorner A$$

with $A := \cup_F A'_F$; on the other hand, for all G we have

$$\lim_F (\mathscr{S}^{\infty-1}_F \llcorner A'_F) \geq (\lim_F \mathscr{S}^{\infty-1}_F) \llcorner A'_G$$

and since G is arbitrary we conclude. $\square$

4 Preliminary results on BV functions

Before defining the class $BV(X,\gamma) \cap L^2(X,\gamma)$ we recall the notation for the partial derivative and its adjoint:

$$\partial_h f(x) := \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}, \qquad \partial^*_h f(x) := \partial_h f(x) - f(x)\hat{h}(x)$$

where $h = R\hat{h} \in H$ with $\hat{h} \in \mathscr{H}$.

Definition 4.1 (BV space)**.** *Let $u \in L^2(X,\gamma)$. We say that $u \in BV(X,\gamma)$ if there exists a H-valued measure $\mu \in \mathscr{M}(X,H)$ with finite total variation such that*

$$\int_X u(x)\partial_h^*\phi(x)d\gamma(x) = -\int_X \phi(x)d\mu_h(x) \qquad \forall h \in H,\ \phi \in \mathcal{F}C_b^1(X) \tag{12}$$

where $\mu_h = \langle \mu, h\rangle$. The measure μ is uniquely determined by (12) *and will be denoted by $D_\gamma u$. Finally, we denote by $\nu_u : X \to H$ the Borel vector field with $|\nu_u| = 1$ providing the polar decomposition*

$$D_\gamma u = \nu_u |D_\gamma u|. \tag{13}$$

Notice that the L^2 assumption is not natural in the context of BV functions (the natural space is the Orlicz space $L\log^{1/2}L(X,\gamma)$, see [5]) but it allows for a simpler definition, with an integration by parts formula along all directions in H. This is possible because $u\partial_h^*\phi$ is integrable.

When $u \in BV(X,\gamma)$ and $D_\gamma u \ll \gamma$ we say that $u \in W^{1,1}(X,\gamma)$ and denote by ∇u the density of $D_\gamma u$, so that $D_\gamma u = \nabla u\, \gamma$.

We say that E has γ-finite perimeter if $u = \chi_E \in BV(X,\gamma)$ and, accordingly, we denote by $|D_\gamma \chi_E|$ the perimeter measure. The unit vector ν_{χ_E} in (13) is simply denoted by ν_E.

In the next theorem we provide a representation of the measures

$$\int_{\mathrm{Ker}(\pi_F)} |D_{\gamma_F}\chi_{E_y}|(B_y)\, d\gamma^\perp(y) \qquad B \in \mathscr{B}(X)$$

in terms of the global derivative (we state the result in terms of BV functions, since in this context the proof and the statement are more natural). We denote by $\Pi_F : H \to F$ the orthogonal projection on F, to keep a notation distinct from the projections $\pi_F : X \to F$ of Section 3.

In the proof of the next theorem we use the Ornstein-Uhlenbeck semigroup T_t in (4) and the following inequality:

$$\limsup_{t\downarrow 0} \int_X |\Pi_F(\nabla T_t u)|\, d\gamma \le |\Pi_F(D_\gamma u)|(X) = \int_X |\Pi_F(\nu_u)|\, d|D_\gamma u| \ \ \forall u \in BV(X,\gamma). \tag{14}$$

The first inequality is proved in [5, Remark 4.2], while the second equality follows from (13).

Theorem 4.2. *Assume that F is a finite dimensional subspace of H and $u \in BV(X,\gamma)$. Then, with the notation of Section 3, $u_y(z) = u(y,z) \in BV(F,\gamma_F)$ for $\gamma^\perp$-a.e. $y \in$* $\mathrm{Ker}(\pi_F)$ *and the following identity of Borel measures holds:*

$$\int_B |\Pi_F(\nu_u)|\, d|D_\gamma u| = \int_{\mathrm{Ker}(\pi_F)} |D_{\gamma_F} u_y|(B_y)\, d\gamma^\perp(y) \qquad \forall B \in \mathscr{B}(X). \tag{15}$$

PROOF. Let $u_n = T_{t_n} u$, with $t_n \to 0$, and assume with no loss of generality that $(u_n)_y(z) = u_n(y,z)$ converge to u_y in $L^1(F,\gamma_F)$ for $\gamma^\perp$-a.e. y. We have

$$\int_{\mathrm{Ker}(\pi_F)} \int_F |\nabla (u_n)_y| \, d\gamma_F d\gamma^\perp = \int_X |\Pi_F(\nabla u_n)| \, d\gamma$$

and, passing to the limit as $n \to \infty$, Fatou's lemma and (14) give

$$\int_{\mathrm{Ker}(\pi_F)} \liminf_{n\to\infty} \int_F |\nabla (u_n)_y|(x) \, d\gamma_F(x) d\gamma^\perp(y) \leq \int_X |\Pi_F(\nu_u)| \, d|D_\gamma u|.$$

From [5, Theorem 4.1] we deduce $u_y \in BV(F,\gamma_F)$ for $\gamma^\perp$-a.e. $y \in \mathrm{Ker}(\pi_F)$ and the inequality

$$\int_X |\Pi_F(\nu_u)| \, d|D_\gamma u| \geq \int_{\mathrm{Ker}(\pi_F)} |D_{\gamma_F} u_y|(X) \, d\gamma^\perp(y). \tag{16}$$

Now, the factorization $\gamma = \gamma^\perp \otimes \gamma_F$ yields

$$\langle h, D_\gamma u \rangle = \int_{\mathrm{Ker}(\pi_F)} \langle h, D_{\gamma_F} u_y \rangle d\gamma^\perp(y)$$

for all $h \in F$ (indeed, both measures satisfy the integration by parts formula in the direction h), hence

$$\Pi_F(\nu_u)|D_\gamma u| = \int_{\mathrm{Ker}(\pi_F)} D_{\gamma_F} u_y \, d\gamma^\perp(y).$$

This immediately gives the inequality of measures

$$|\Pi_F(\nu_u)||D_\gamma u| \leq \int_{\mathrm{Ker}(\pi_F)} |D_{\gamma_F} u_y| \, d\gamma^\perp(y)$$

that, combined with (16), yields (15). □

If we apply Theorem 4.2 to $u = \chi_E$ and use the finite-dimensional representation of $|D_\gamma \chi_L|$ as $G_m |D\chi_L| = G_m \mathscr{S}^{m-1} \llcorner \partial^* L$ we get

$$\int_B |\Pi_F(\nu_E)| \, d|D_\gamma \chi_E| = \int_{\mathrm{Ker}(\pi_F)} \int_{B_y \cap \partial^* E_y} G_m \, d\mathscr{S}^{m-1} d\gamma^\perp(y) \qquad \forall B \in \mathscr{B}(X), \tag{17}$$

where $m = \dim(F)$. Now, by applying the result to a finite-dimensional space $X = G$ we obtain the following lemma, providing a kind of inclusion between essential boundaries of different dimensions.

Lemma 4.3. *Let G be a k-dimensional Hilbert space, let $F \subset G$ be a m-dimensional subspace and let E be a set with finite perimeter in G. Then, with the orthogonal decomposition $G = F \oplus L$ and the notation*

$$E_w := \{z \in F : \ w + z \in E\} \qquad\qquad w \in L$$

we have that $\mathscr{S}^{m-1}(\{z \in F : \ z \in \partial^* E_w, \, w + z \notin \partial^* E\}) = 0$ for $\mathscr{S}^{k-m}$-a.e. $w \in L$.

Proof. Take $B = G \setminus \partial^* E$ in (17), so that the left hand side vanishes. It follows that $\int_{\partial^* E_w \setminus (\partial^* E)_w} G_m \, d\mathscr{S}^{m-1} = 0$ for $\gamma^\perp$-a.e. w, and therefore for $\mathscr{S}^{k-m}$-a.e. w, and, since $G_m > 0$, $\mathscr{S}^{m-1}(\partial^* E_w \setminus (\partial^* E)_w) = 0$. $\square$

5 Representation of the perimeter measure

In this section we reproduce with minor variants Hino's representation result of the perimeter measure, recently obtained in [19].

Definition 5.1 (Cylindrical essential boundary)**.** *Let $\mathcal{F}$ be a countable set of finite-dimensional subspaces of H stable under finite union, with $\cup_{F \in \mathcal{F}} F$ dense in H. For $F \in \mathcal{F}$, with the notation* (8)*, we define*

$$\partial^*_F E := \{y + z : \ y \in \mathrm{Ker}(\pi_F), \ z \in \partial^* E_y\}$$

where $\partial^* E_y$ is the essential boundary of E_y in F. It is not difficult to show that $\partial^*_F E$ is a Borel set. Then, we define cylindrical essential boundary $\partial^*_{\mathcal{F}} E$ along $\mathcal{F}$ the set

$$\partial^*_{\mathcal{F}} E := \bigcup_{F \in \mathcal{F}} \bigcap_{G \in \mathcal{F}, \, G \supset F} \partial^*_G E. \tag{18}$$

Accordingly, it is also be useful the notation $\mathscr{S}^{\infty-1}_{\mathcal{F}} = \lim_{F \in \mathcal{F}} \mathscr{S}^{\infty-1}_F$.

By applying the finite-dimensional De Giorgi theorem and the obvious relation $D_\gamma \chi_E = G_m D\chi_E = G_m \mathscr{S}^{m-1} \llcorner \partial^* E$ in m-dimensional spaces F, we obtain a useful representation formula of the Hausdorff pre-measures $\mathscr{S}^{\infty-1}_F \llcorner \partial^*_F E$:

$$\mathscr{S}^{\infty-1}_F(B \cap \partial^*_F E) = \int_{\mathrm{Ker}(\pi_F)} |D_{\gamma_F} \chi_{E_y}|(B_y) \, d\gamma^\perp(y) \qquad \forall B \in \mathscr{B}(X). \tag{19}$$

Theorem 5.2. *Let $E \in \mathscr{B}(X)$ be a set with finite γ-perimeter in X, let $\mathcal{F}$ be as in Definition 5.1 and let $\partial^*_{\mathcal{F}} E$ be the corresponding cylindrical essential boundary. Then*

$$|D_\gamma \chi_E|(B) = \mathscr{S}^{\infty-1}_{\mathcal{F}}(B \cap \partial^*_{\mathcal{F}} E) \qquad \forall B \in \mathscr{B}(X). \tag{20}$$

In particular, $\partial^*_{\mathcal{F}} E$ is uniquely determined by (20) *up to $\mathscr{S}^{\infty-1}_{\mathcal{F}}$-negligible sets.*

Proof. The basic property we claim is that $\partial^*_F E \setminus \partial^*_G E$ is contained in a $\mathscr{S}^{\infty-1}_F$-negligible set whenever $F \subset G$. Indeed, if this property holds we can apply Proposition 3.3 with $A_F = \partial^*_F E$ to obtain the existence of the limit of the measures $\mathscr{S}^{\infty-1}_F \llcorner \partial^*_F E$ and its coincidence with $\mathscr{S}^{\infty-1}_{\mathcal{F}} \llcorner \partial^*_{\mathcal{F}} E$.
The proof of the claim follows from the purely finite-dimensional result proved in Lemma 4.3. Indeed, let us write $G = L \oplus F$ and let $m = \dim(F)$, $k = \dim(G)$, so that L is $(k-m)$-dimensional. We consider the orthogonal decomposition $H = G^\perp \oplus L \oplus F$, so that γ can

be written as the product $\gamma^\perp \otimes \gamma_L \otimes \gamma_F$, γ_F, γ_L being standard Gaussians in F and L, respectively. Then, denoting the variable in $F^\perp$ by (y, w) with $y \in \mathrm{Ker}(\pi_G)$ and $w \in L$, we have

$$\mathscr{S}_F^{\infty-1}(B) = \int_{\mathrm{Ker}(\pi_G)} \int_L \mathscr{S}^{m-1}(B_{y,w})\, d\gamma_L(w) d\gamma^\perp(y).$$

Hence, B is $\mathscr{S}_F^{\infty-1}$-negligible if, for $\gamma^\perp$-a.e. $y \in \mathrm{Ker}(\pi_G)$, the set $B_{y,w} \subset F$ is $\mathscr{S}^{m-1}$-negligible for γ_L-a.e. w.

Now, let us check that $B := \partial_F^* E \setminus \partial_G^* E$ has this property; indeed, the slicing theory of sets of finite perimeter illustrated in Section 4 shows that

$$E_y := \{w + z \in L \oplus F :\ y + w + z \in E\} \subset G$$

has finite γ-perimeter, and hence locally finite Euclidean perimeter, for $\gamma^\perp$-a.e. $y \in \mathrm{Ker}(\pi_G)$. For any such y, by applying Lemma 4.3 to E_y, and taking into account that

$$B_{y,w} = \{z \in F :\ z \in \partial^*(E_y)_w,\ (w, z) \notin \partial^* E_y\}$$

we have that $B_{y,w}$ is $\mathscr{S}^{m-1}$-negligible for $\mathscr{S}^{k-m}$-a.e. w, and then for γ_L-a.e. w. This proves the claim.

Now we show that $\mathscr{S}_{\mathcal{F}}^{\infty-1} \llcorner \partial_{\mathcal{F}}^* E = |D_\gamma \chi_E|$. Indeed, we can use (19) and Theorem 4.2 with $u = \chi_E$ to get

$$\mathscr{S}_F^{\infty-1} \llcorner \partial_F^* E = |\Pi_F(\nu_E)||D_\gamma \chi_E|.$$

This proves that all measures in the left hand side are less than $|D_\gamma \chi_E|$, and considering an increasing family $(F_n) \subset \mathcal{F}$ whose union is dense one obtains that the limit of these measures is $|D_\gamma \chi_E|$. □

6 Sobolev-rectifiability of the essential boundary

In this Section we prove that the perimeter measure $|D_\gamma \chi_E|$ is concentrated on a countable union of (entire) graphs of Sobolev functions. Since the perimeter measure is representable as in (20), this yields as a byproduct that the same property holds for the cylindrical essential boundary $\partial_{\mathcal{F}}^* E$ built from a countable family $\mathcal{F}$ of finite-dimensional subspaces of H, namely this set is contained, up to $\mathscr{S}_{\mathcal{F}}^{\infty-1}$-negligible sets, in a countable union of graphs of Sobolev functions.

We fix a set E with finite perimeter and a unit vector in the Cameron-Martin space $k = Qx^*$, $x^* \in X^*$, and uniquely write any element $x \in X$ as $y + tk$ with

$$t = \langle x^*, x \rangle, \qquad y \in \mathrm{Ker}(\pi_F),$$

$\pi_F(x) = \langle x^*, x\rangle k$ being the canonical projection map from X to $F := \mathbb{R}k$. The measure γ_k in the resulting product decomposition $\gamma = \gamma_k \otimes \gamma^\perp$ is the law under $t \mapsto tk$ of the standard 1-dimensional Gaussian measure $\gamma_1 = G_1\mathscr{L}^1$ in $\mathbb{R}$. If $E \subset X$ is a set with finite perimeter, with the notation

$$E_y := \{t \in \mathbb{R} :\ y + tk \in E\}$$

(unlike the previous sections, here we directly view E_y as subsets of $\mathbb{R}$), Theorem 4.2 gives

$$\int_B |\langle \nu_E, k\rangle|\, d|D_\gamma \chi_E| = \int_{\mathrm{Ker}(\pi_F)} |D_{\gamma_1}\chi_{E_y}|(B_y)\, d\gamma^\perp(y) \qquad \forall B \in \mathscr{B}(X). \tag{21}$$

In particular this gives

$$(Id - \pi_F)_\sharp |\langle \nu_E, k\rangle||D_\gamma\chi_E| \ll \gamma^\perp. \tag{22}$$

In the sequel we use the notation ψ^* for the function $\psi'(t) - t\psi(t)$; notice that $\psi^* = \partial^*\psi$ in the 1-dimensional Gaussian space $(\mathbb{R}, G_1\mathscr{L}^1)$.

Lemma 6.1. *For all $\psi \in C^1_c(\mathbb{R})$ the map $\hat{\psi} : \mathrm{Ker}(\pi_F) \to \mathbb{R}$ defined by*

$$\hat{\psi}(y) := \int_{\mathbb{R}} \psi\, dD_{\gamma_1}\chi_{E_y}$$

belongs to $BV(\mathrm{Ker}(\pi_F), \gamma^\perp)$ and

$$|D_{\gamma^\perp}\hat{\psi}| \le \sup|\psi^*|(Id - \pi_F)_\sharp |\Pi_F^\perp \nu_E||D_\gamma\chi_E|, \tag{23}$$

where $\Pi_F^\perp : H \to H$ denotes the orthogonal projection on $k^\perp$. In addition, it belongs to $W^{1,1}(\mathrm{Ker}(\pi_F), \gamma^\perp)$ if

$$|D_\gamma\chi_E|\big(\{x \in X :\ \langle \nu_E(x), k\rangle = 0\}\big) = 0. \tag{24}$$

Proof. First of all we notice that an integration by parts gives that $\|\hat{\psi}\|_\infty \le \sup|\psi^*|$. We fix a unit vector $h \in k^\perp$ and $g \in \mathcal{F}C^1_b(X)$. Then, we get

$$\begin{aligned}
\int_{\mathrm{Ker}(\pi_F)} \partial_h^* g(y)\hat{\psi}(y)\, d\gamma^\perp(y) &= \int_{\mathrm{Ker}(\pi_F)} \partial_h^* g(y) \int_{\mathbb{R}} \psi(t)\, dD_{\gamma_1}\chi_{E_y}(t)\, d\gamma^\perp(y) \\
&= -\int_{\mathrm{Ker}(\pi_F)} \partial_h^* g(y) \int_{E_y} \psi^*(t) G_1(t)\, dt d\gamma^\perp(y) \\
&= -\int_E \partial_h^* g(x - \langle x^*, x\rangle k)\psi^*(\langle x^*, x\rangle)\, d\gamma(x) \\
&= -\int_E \partial_h^* \left[g(x - \langle x^*, x\rangle k)\psi^*(\langle x^*, x\rangle)\right]\, d\gamma(x) \\
&= \int_X g(x - \langle x^*, x\rangle k)\psi^*(\langle x^*, x\rangle)\langle \nu_E(x), h\rangle |D_\gamma\chi_E|(x),
\end{aligned}$$

where we used the product rule $\partial_h^*(fg) = f\partial_h^* g$ if $\partial_h f = 0$.

Hence, estimating $|\langle \nu_E(x), h\rangle|$ with $|\Pi_F^\perp(\nu_E)|$ gives

$$\left|\int_{\mathrm{Ker}(\pi_F)} \partial_h^* g(y)\hat{\psi}(y)\, d\gamma^\perp(y)\right| \leq \sup|\psi^*| \int_{\mathrm{Ker}(\pi_F)} |g|\, d(Id - \pi_F)_\sharp |\Pi_F^\perp(\nu_E)||D_\gamma \chi_E|.$$

Since g is arbitrary, Daniell's theorem (see e.g. [8, Theorem 7.8.1]) gives that $\hat{\psi}$ has a weak derivative along the direction h, given by a measure ν_h satisfying $|\nu_h| \leq \sup|\psi^*|(Id - \pi_F)_\sharp |\Pi_F^\perp(\nu_E)||D_\gamma\chi_E|$. Since h is arbitrary and the upper bound does not depend on h, it is easy to derive from this fact (see Proposition 3.4 of [5] for details) that $\hat{\psi} \in BV(\mathrm{Ker}(\pi_F), \gamma^\perp)$ and (23) holds. If (24) holds, then

$$|D_{\gamma^\perp}\hat{\psi}| \leq \sup|\psi^*|(Id - \pi_F)_\sharp |D_\gamma\chi_E| \ll \sup|\psi^*|(Id - \pi_F)_\sharp |\langle \nu_E, k\rangle||D_\gamma\chi_E|$$

because $\langle \nu_E, k\rangle \neq 0$ $|D_\gamma\chi_E|$-a.e. in X. We conclude from (22) that $|D_{\gamma^\perp}\hat{\psi}| \ll \gamma^\perp$, i.e., the Sobolev regularity of $\hat{\psi}$. □

Remark 6.2. The previous result can be also interpreted in a more abstract form by saying that the map $y \mapsto D_{\gamma_1}\chi_{E_y}$ is a BV function with values in the space of bounded measures in $\mathbb{R}$ endowed with a suitable (weak) distance. Even though this interpretation does not play any role in this paper, we think it is worthwhile to present this point of view. Let (Z, θ) be a Gaussian space. Following [1], a function $u : Z \to Y$ with values in a metric space Y is said to have bounded variation if:

(i) for any 1–Lipschitz map $\varphi : Y \to \mathbb{R}$ the map $\varphi \circ u : Z \to \mathbb{R}$ belongs to $BV(Z, \theta)$;

(ii) the supremum of the family of measures $|D_\theta(\varphi \circ u)|$, among all $\varphi : Y \to \mathbb{R}$ 1–Lipschitz, is a positive finite measure in Z.

The supremum in (ii) is again called total variation measure of u, and denoted by $|D_\theta u|$.

In our case $Z = \mathrm{Ker}(\pi_F)$, $\theta = \gamma^\perp$, $Y = \mathscr{M}(\mathbb{R})$ endowed with the distance

$$d(\mu, \nu) := \sup\left\{\int_{\mathbb{R}} \psi\, d\mu - \int_{\mathbb{R}} \psi\, d\nu : \ \psi \in C_c^1(\mathbb{R}),\ \max\{\sup|\psi|, \sup|\psi^*|\} \leq 1\right\}.$$

If $\psi \in C_c^1(\mathbb{R})$ with $|\psi| \leq 1$ and $|\psi^*| \leq 1$, then $\hat{\psi}(\mu) := \int_{\mathbb{R}} \psi\, d\mu$ is clearly 1–Lipschitz in (Y, d); on the other hand, we have also

$$|D_{\gamma^\perp}\hat{\psi}| \leq (Id - \pi_F)_\sharp |\Pi_F^\perp \nu_E||D_\gamma\chi_E|.$$

Hence, it follows that $y \mapsto D_{\gamma_1}\chi_{E_y}$ is BV from $\mathrm{Ker}(\pi_F)$ to $(\mathscr{M}(\mathbb{R}), d)$ and that its total variation is less than $(Id - \pi_F)_\sharp |\Pi_F^\perp \nu_E||D_\gamma\chi_E|$.

The next lemma (applied with $\sigma = |D_\gamma \chi_E|$ and $f = \nu_E$) shows that the set of "good" directions $k \in QX^*$ satisfying (24) is dense.

Lemma 6.3. *Let $\sigma \in \mathscr{M}_+(X)$ and $f : X \to H$ with $f \neq 0$ σ-a.e. in X. Then the set of vectors $k \in QX^*$ satisfying*

$$\sigma\big(\{x \in X : \ \langle f(x), k\rangle = 0\}\big) = 0$$

is dense in H.

Proof. First we build $\theta \in \mathscr{P}(H)$ concentrated on QX^*, with (topological) support equal to the whole of H and satisfying $\theta(h^\perp) = 0$ for all $h \in H \setminus \{0\}$. To this aim, we consider in the space $\mathbb{R}^{\mathbb{N}_*}$, $\mathbb{N}_* = \mathbb{N} \setminus \{0\}$, a product $\tilde\theta$ of centered Gaussian measures γ_i with variance $c_i^2 > 0$; since

$$\int_{\mathbb{R}^{\mathbb{N}_*}} \sum_i |x_i| \, d\tilde\theta = \int_{\mathbb{R}} |t| G_1(t)\, dt \sum_i c_i,$$

if $(c_i) \in \ell_1$ we can consider $\tilde\theta$ as a Gaussian measure in ℓ_1. It is not difficult to check that its support is the whole of ℓ_1: indeed, if $(d_i) \in \ell_1$ and $d_i = 0$ for i sufficiently large, then $(d_i/c_i^2) \in (\ell_1)^*$ and $Q(d_i/c_i^2) = (d_i)$; this proves that the Cameron-Martin space of $\tilde\theta$ is dense in ℓ_1. Now, if B were a ball with $\tilde\theta(B) = 0$ we could shift B along Cameron-Martin directions to obtain a countable family of $\tilde\theta$-negligible balls covering the whole of ℓ_1, a contradiction.

Then, we consider the continuous map

$$\Phi : \ell_1 \mapsto H, \qquad \Phi((x_i)) := \sum_i x_i e_i,$$

where $e_i = Qe_i^*$ is an orthonormal basis of H made by vectors in QX^*, and define $\theta := \Phi_\sharp \tilde\theta$. The continuity of Φ ensures the inclusion $\mathrm{supp}(\theta) \supset \Phi(\mathrm{supp}(\tilde\theta))$, hence the support of θ contains $\Phi(\ell_1)$ and therefore coincides with the whole of H. On the other hand, since the image of Φ is contained in

$$\left\{ \sum_i x_i Q e_i^* : \ (x_i) \in \ell_1 \right\}$$

and this set is contained in QX^*, we obtain that θ is concentrated on QX^*. Finally, we check that $\theta(k^\perp) = 0$ for all $k = \sum_i v_i e_i \in H \setminus \{0\}$: by the definition of θ, we need to check that

$$\tilde\theta\big(\{x \in \ell_1 : \ \sum_i x_i v_i = 0\}\big) = 0. \tag{25}$$

Since (x_i) are independent, Gaussian with variance c_i^2, it turns out that $\sum_i x_i v_i$ is a Gaussian random variable with variance $\sum_i v_i^2 c_i^2 > 0$, hence (25) holds.

Since the support of θ is the whole of H, it suffices to show that the set of vectors $k \in QX^*$ satisfying

$$\sigma\big(\{x \in X : \ \langle f(x), k\rangle = 0\}\big) > 0$$

is θ-negligible. By applying Fubini's theorem, it suffices to show that the set of all $x \in X$ satisfying

$$\theta\big(\{k \in H : \ \langle f(x), k\rangle = 0\}\big) > 0$$

is σ-negligible. But, by our construction of θ, the latter set is empty. □

We make a particular choice of the functions ψ in Lemma 6.1; we start by considering cut-off functions $\eta_{t_0,r,\epsilon}$ of class C^1, having support contained in $[t_0 - r - \epsilon, t_0 + r + \epsilon]$ and identically equal to 1 on $[t_0 - r, t_0 + r]$. We then have that the family

$$\mathcal{D} := \left\{ \frac{t}{G_1(t)} \eta_{t_0,r,\varepsilon}(t) : t_0 \in \mathbb{Q}, \ r, \varepsilon \in \mathbb{Q}_+ \right\}$$

is countable and any function $\hat{\psi}$, $\psi \in \mathcal{D}$, belongs to $BV(\mathrm{Ker}(\pi_F), \gamma^\perp)$ and to the Sobolev space $W^{1,1}(\mathrm{Ker}(\pi_F), \gamma^\perp)$ if (24) holds.

Definition 6.4 (H-graph)**.** *A set $\Gamma \subset X$ is called a H–graph if there exist a unit vector $k \in QX^*$ and $u : D \subset \mathrm{Ker}(\pi_F) \to \mathbb{R}$ such that*

$$\Gamma = \{y + u(y)k : \ y \in D\}.$$

We say that Γ is an entire Sobolev H–graph if $D \in \mathscr{B}(\mathrm{Ker}(\pi_F))$, $\gamma^\perp\big(\mathrm{Ker}(\pi_F) \setminus D\big) = 0$ and $u \in W^{1,1}(\mathrm{Ker}(\pi_F), \gamma^\perp)$.

Theorem 6.5. *For any set $E \subset X$ with finite perimeter the measure $|D_\gamma \chi_E|$ is concentrated on a countable union of entire Sobolev H–graphs.*

Proof. We fix a good direction $k = Qx^*$ satisfying (24) and prove the property for the measure $|\langle \nu_E, k\rangle||D_\gamma \chi_E|$, using the fact that all functions $\hat{\psi}$ are Sobolev; then, Lemma 6.3 provides the density of good directions and the validity of the statement for the whole measure $|D_\gamma \chi_E|$.

We consider the set $D \subset \mathrm{Ker}(\pi_F)$ of all y such that E_y has finite perimeter in $(\mathbb{R}, \gamma_1)$; this set has full $\gamma^\perp$-measure in $\mathrm{Ker}(\pi_F)$. Then, we study the map

$$y \in D \mapsto D_{\gamma_1} \chi_{E_y}.$$

Such measures are atomic and, because of the identity $D_{\gamma_1}\chi_L = G_1 D\chi_L$, have the form

$$\sum_{i \in I} G_1(t_i)\delta_{t_i}$$

with I finite or countable. For $y \in D$ we denote by $\mathscr{A}_y$ the set of atoms of $D_{\gamma_1}\chi_{E_y}$; notice that the set is discrete, since G_1 is locally bounded away from 0 and $D_{\gamma_1}\chi_{E_y}$ has finite mass. We fix a point $\bar{t} \in \mathscr{A}_y$; then, there exists $\psi(t) = tG_1^{-1}(t)\eta_{t_0,r,\varepsilon}(t) \in \mathcal{D}$ such that

$$[t_0 - r - \varepsilon, t_0 + r + \varepsilon] \cap \mathscr{A}_y = \{\bar{t}\}, \qquad \bar{t} \in [t_0 - r, t_0 + r].$$

We then have

$$\hat{\psi}(y) = \int_{\mathbb{R}} \psi \, dD_{\gamma_1}\chi_{E_y} = \sum_{t \in \mathscr{A}_y} t\eta_{t_0,r,\varepsilon}(t) = \bar{t},$$

so that

$$\{y + tk : \ y \in D, \ t \in \mathscr{A}_y\} \subset \bigcup_{\psi \in \mathcal{D}} \mathrm{graph}(\hat{\psi}).$$

Since, by (21), $|\langle \nu_E, k\rangle||D_\gamma\chi_E|$ is concentrated on $\{y + tk : \ y \in D, \ t \in \mathscr{A}_y\}$ the proof is achieved. $\square$

7 Further extensions, open problems

In this section we discuss some open problems related to the rectifiability result and potential alternative definitions of essential and reduced boundary. In connection with the latter problem, the motivation we have in mind is to try to achieve coordinate-free definitions, i.e., independent of the family of finite-dimensional subspaces.

A first natural question is whether the Sobolev rectifiability result can be improved to a Lipschitz one, namely whether $|D_\gamma\chi_E|$ is concentrated on countably many graphs of $W^{1,\infty}$ functions (i.e., Lipschitz in the Cameron-Martin directions). In the Euclidean space $\mathbb{R}^m$ there is not a real difference between the two concepts, since Sobolev (and even BV) functions can be approximated in the Lusin sense by Lipschitz maps (and even by C^1 maps, using Whitney's extension theorem [13, 3.1.14]). By Lusin approximation we mean, here, that $\mathscr{L}^m$-almost all of $\mathbb{R}^m$ can be covered by a sequence of sets E_λ such that $u|_{E_\lambda}$ is Lipschitz. In order to obtain this approximation for $u \in W^{1,1}(\mathbb{R}^m)$ it suffices to consider the pointwise inequality

$$|\tilde{u}(x) - \tilde{u}(y)| \leq c(m)|x - y|\big[M|\nabla u|(x) + M|\nabla u|(y)\big]$$

(see for instance [3, Theorem 5.34]). Here $M|\nabla u|$ is the Hardy-Littlewood maximal function of $|\nabla u|$ and $\tilde{u}$ is the approximate limit of u at x, coinciding with u at $\mathscr{L}^m$-a.e. point. Considering the restriction of $\tilde{u}$ to $\{M|\nabla u| \leq \lambda\} \cap \{u = \tilde{u}\}$ we obtain that $\tilde{u}$ is a Lipschitz function; these Lipschitz functions, possibly extended to the whole of $\mathbb{R}^m$, provide the desired Lusin approximation of u.

Let us now discuss other potential definitions of essential and reduced boundary. We start by recalling the following maximal inequality for the Ornstein-Uhlenbeck semigroup T_t in (4), see [22, Page 73]:

$$\| \sup_{t>0} T_t f \|_p \le C_p \| f \|_p \qquad f \in L^p(X,\gamma),\ 1 < p < \infty.$$

Since $C_b(X)$ is dense in $L^p(X,\gamma)$ and $T_t f \to f$ pointwise as $t \downarrow 0$ on C_b, for all $p \in (1,\infty)$ we can use a classical decomposition argument to infer

$$\lim_{t\downarrow 0} T_t f(x) = f(x) \qquad \text{for } \gamma\text{-a.e. } x \in X, \text{ for all } f \in L^p(X,\gamma)$$

from the maximal inequality. Of course this is still valid for L^∞ functions and motivates the following definition:

Definition 7.1 (Gaussian Essential boundary)**.** *Let $E \in \mathscr{B}(X)$; we denote by $\partial^* E$ the set of points $x \in X$ where either $T_t \chi_E(x)$ does not have a limit as* t $\downarrow 0$*, or the limit exists and belongs to* $(0,1)$*.*

Obviously $\partial^* \chi_E$ is γ-negligible and the definition is easily seen to be consistent with the finite-dimensional case (heuristically, T_t amounts to an average of averages on balls with radius $r \sim \sqrt{t}$). Notice however that, in order to turn really Definition 7.1 into a pointwise definition, one should work with precise representatives of Sobolev functions, well defined out of sets with 0 capacity: indeed, even though $T_t \chi_E$ is not continuous, is known to be a function in $W^{1,1}(X,\gamma)$ (see [16] or [5]). Additional difficulties arise from the fact that a continuum of times t is considered, and that $t \mapsto T_t(x)$ is not necessarily continuous.

Let us now discuss the reduced boundary. De Giorgi's definition in Euclidean spaces requires $x \in \operatorname{supp} |D\chi_E|$, existence of the limit

$$\nu_E(x) := \lim_{r\downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))|} \tag{26}$$

and $|\nu_E(x)| = 1$, so that at points in $\mathscr{F}E$ the unit inner normal is also pointwise defined.

In our infinite-dimensional framework we start from the observation that T_t, $t > 0$, can be extended to an operator $T_t^* : \mathscr{M}(X) \to \mathscr{M}(X)$ by duality:

$$\langle T_t^* \mu, \phi \rangle = \langle \mu, T_t \phi \rangle \qquad \phi \text{ bounded Borel.}$$

Componentwise, the definition can be extended to vector-valued measures as well. Moreover, we know that

$$T_t^* D_\gamma u = e^t (\nabla T_t u) \gamma.$$

Now we have all the ingredients for a potential definition of Gaussian reduced boundary:

Definition 7.2 (Gaussian Reduced boundary)**.** *Let $E \in \mathscr{B}(X)$; we denote by $\mathscr{F}E$ the set of points $x \in X$ where the limit*

$$\nu_E(x) := \lim_{t\downarrow 0} T_t\left(\frac{T_t^* D_\gamma \chi_E}{T_t^*|D_\gamma \chi_E|}\right)(x)$$

exists and satisfies $|\nu_E(x)| = 1$.

Even this definition is fully consistent with the finite-dimensional case. But, in this framework no analog of Besicovitch differentiation theorem holds, hence it is not clear whether $|D\chi_E|$-a.e. point belongs to the Gaussian reduced boundary $\mathscr{F}E$. In addition, the same difficulties (precise representatives, continuity in time) as Definition 7.1 are present here.

More generally, the relations between cylindrical essential boundary, dependent on the family of subspaces, essential boundary and reduced boundary are still to be understood, and this seems to be a challenging open question. To conclude, we recall the relations, already mentioned in the introduction, between these concepts in the case of the Euclidean space $\mathbb{R}^m$:

$$\mathscr{F}E \subset \partial^* E, \qquad \mathscr{S}^{m-1}(\partial^* E \setminus \mathscr{F}E) = 0, \qquad D\chi_E = \nu_E \mathscr{S}^{m-1} \llcorner \mathscr{F}E.$$

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