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Sequences of singularly perturbed functionals generating free-discontinuity problems

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Abstract

We prove that a wide class of singurlarly perturbed functionals generates as Γ-limit a functional related to a free-discontinuity problem. Several applications of the result are shown.

Key words: free-discontinuity problems, singular perturbations, Γ-convergence

1 Introduction

Many problems arising from Fracture Mechanics and Computer Vision lead to free-discontinuity problems, that is to the minimization of functionals defined in spaces of discontinuous functions (namely BV and SBV) involving energies with a bulk part and a surface part concentrated along the (free) discontinuity zone. This paper is concerned with the variational approximation in the sense of De Giorgi's Γ-convergence of such energies by smooth functionals defined on Sobolev spaces. In the last years a lot of work has been addressed in this direction in view of numerical applications and in view of defining an evolution model (as limit of the gradient flows of the approximating functionals): for a general survey on free-discontinuity problems and their approximation we refer to [10] and [6].

When the volume part of the energy is given by $\int_\Omega |\nabla u|^2\,dx$, heuristic considerations suggest to use, as approximating functionals, energies of the form

$$\frac{1}{\varepsilon}\int_\Omega f(\sqrt{\varepsilon}|\nabla u|)\,dx,$$

where $f:[0,+\infty)\to[0,+\infty)$ is quadratic near the origin and with finite limit at infinity. However, an easy convexity argument shows that energies of this kind Γ-converge to the zero functional. Various methods have been developed to bypass this convexity constraint, most of them exploiting the De Giorgi's suggestion of replacing the functionals above with suitable non-local versions (see [13], [21], [17]). The approach we consider here is based on singular perturbations and consists in adding a "small" term depending on higher derivatives: the idea is to impose a bound on the oscillations of minimizing sequences by penalizing abrupt changes of the gradient. So we are led to consider energies of the form

$$\frac{1}{\varepsilon}\int_\Omega f(\sqrt{\varepsilon}|\nabla u|)\,dx + r(\varepsilon)\int_\Omega \|\nabla^2 u\|^2\,dx, \tag{1.1}$$

where $r(\varepsilon)$ is a function which vanishes as $\varepsilon\to 0^+$.

The first step in this direction was taken by Alicandro, Braides & Gelli in [2]: they showed that the one dimensional functionals

$$\frac{1}{\varepsilon}\int_0^1 f(\sqrt{\varepsilon}|u'|)\,dx + \varepsilon^3\int_0^1 |u''|^2\,dx,$$

with $f(t) = \alpha t^2 \wedge \beta$, Γ-converge with respect to the L^1-norm to the functional

$$\alpha \int_0^1 |u'|^2 \, dx + c(\beta) \sum_{S_u} \sqrt{u^+ - u^-},$$

where $c(\beta)$ is a constant depending on β; later Alicandro & Gelli treated the N-dimensional case (see [3]). We aim to extend the results above to general functionals of the form (1.1), where f is still quadratic near the origin, but possibly unbounded. In fact we face the problem in a more general framework, by investigating the asymptotic behaviour of

$$F_\varepsilon(u) := \int_\Omega f_\varepsilon(|\nabla u|) \, dx + (r(\varepsilon))^3 \int_\Omega \|\nabla^2 u\|^2 \, dx, \tag{1.2}$$

where (f_ε) is any family of positive non-decreasing functions with a convex or convex-concave shape (i.e. there exists $x_\varepsilon > 0$ such that f_ε is convex in $[0, x_\varepsilon]$ and concave in $[x_\varepsilon, +\infty)$); let us remark that such a structure assumption is quite natural for this kind of problems (see, for example, [14], [15], [22]). In the main theorem of the paper (Theorem 3.2) we prove that the Γ-limits of (1.2) are related to the pointwise limits of $f_\varepsilon(t)$ and of $r(\varepsilon) f_\varepsilon(t/r(\varepsilon))$: if for an infinitesimal subsequence (ε_n) we have

a) $f_{\varepsilon_n} \to g$ pointwise,

b) $r(\varepsilon_n) f_{\varepsilon_n}(\cdot/r(\varepsilon_n)) \to b$ pointwise,

then (F_{ε_n}) Γ-converges to a functional F defined on $BV(\Omega)$ and taking the form

$$F(u) = \int_\Omega f(|\nabla u|) \, dx + \int_{S_u} \varphi(u^+ - u^-) + C|D^c u|, \tag{1.3}$$

where C (possibly equal to $+\infty$, meaning that F is finite only on SBV), f, and φ can be characterized in terms of g and b.

The "regularizing" effect due to the presence of the second derivatives in the approximating functionals, determines a restriction on the regularity and on the growth of the jump-function φ, which turns out to satisfy the growth condition

$$C_1(\sqrt{z} - 1) \le \varphi(z) \le C_2(z + 1) \qquad \forall z \ge 0,$$

for suitable C_1, $C_2 > 0$, whenever $\lim_{t \to 0^+} b(t)/t \neq 0$; moreover, we always have $\varphi(0) = 0$. In particular, the Mumford-Shah functional is not reachable by our procedure. However, since for any positive, convex, and superlinear function g and for any positive and concave function b with $\lim_{t \to 0^+} b(t)/t = +\infty$, it is possible to construct a family (f_ε) and a rescaling function $r(\varepsilon)$ such that conditions a) and b) above are fulfilled, we see that a wide class of free-discontinuity functionals with φ satisfying (3.57) can be approximated. Letting b vary among the possible choices, we may conjecture to recover most of the admissible asymptotic behaviours as the following fact seems to suggest: all the functions of the form $\varphi(t) = ct^\gamma$, with $c > 0$ and γ varying in $[1/2, 1]$ are reachable, and for every $\gamma \in [1/2, 1)$ a function φ can be generated such that

$$\lim_{z \to +\infty} \frac{\varphi(z)}{z^\gamma} = +\infty \qquad \text{and} \qquad \lim_{z \to +\infty} \frac{\varphi(z)}{z^{\gamma+\varepsilon}} = 0 \qquad \forall \varepsilon > 0.$$

As announced, in Section 4 we apply our theorem to prove that if f is quadratic near the origin, sublinear, and concave at infinity, there exists a rescaling function $r(\varepsilon)$ (explicitly given in terms of f) such that the family (1.1) Γ-converges, up to passing to a subsequence, to a free-discontinuity functional like (1.3). All the possible Γ-limits of that family are classified. The rescaling $r(\varepsilon)$ is unique up to asymptotic equivalence, in the sense that when we use functions with a different behaviour near the origin, we obtain in the limit either $F \equiv 0$ or the functional $\alpha \int |\nabla u|^2 \, dx$ defined only on $H^1(\Omega)$.

In a recent paper ([9]) Bouchitté, Dubs & Seppecher considered the one-dimensional functionals

$$F_\varepsilon(u) := \int_I \frac{|u'|^2}{1 + (\varepsilon |u'|)^p} \, dx + \varepsilon^{\frac{3p}{p-1} \vee 4} \int_I |u''|^2 \, dx$$

defined in $W^{2,2}(I)$ and proved that they Γ-converge to the functional F (defined in $SBV(I)$) given by

$$F(u) := \int_I |u'|^2\,dx + k_p \sum_{x\in S_u} (u^+ - u^-)^{\frac{4-p}{2+p}\vee 0}.$$

When $p \le 2$ their result is a particular case of ours (but it is proved by the use of different techniques); on the contrary, the case $p > 2$ is not included in our treatment since the potential $f(t)$ becomes decreasing and degenerates at infinity; note that the use of a degenerate potential allows the approximation of the Mumford-Shah functional (the case $p > 4$).

Let us also point out that our theorem applies to the study of the singular perturbations of the rescaled Perona-Malik energy

$$\frac{1}{\varepsilon}\int_\Omega \log(1+\varepsilon|\nabla u|^2)\,dx :$$

we will show that the right rescaling function is given by $r(\varepsilon) = \frac{\varepsilon}{\log\frac{1}{\varepsilon}}$ and that the family

$$\frac{1}{\varepsilon}\int_\Omega \log(1+\varepsilon|\nabla u|^2)\,dx + \left(\frac{\varepsilon}{\log\frac{1}{\varepsilon}}\right)^3 \int_\Omega \|\nabla^2 u\|^2\,dx$$

Γ-converges to

$$\int_\Omega |\nabla u|^2\,dx + c\int_{S_u} \sqrt{u^+ - u^-}\,d\mathcal{H}^{N-1},$$

with $c > 0$ explicitly computable (see Example 4.8). The Perona-Malik functional was introduced in the context of Image Processing. Let us briefly recall the problem: if g is the input grey level function representing the original image, the simplest way to smooth and denoise it is to apply a gaussian convolution kernel; this procedure turns out to be equivalent to letting g evolve according to the heat equation, i.e. to taking as processed image the solution $u(x,t)$ of the heat diffusion equation

$$\frac{\partial}{\partial t}u = \Delta u \qquad u(x,0) = g(x), \tag{1.4}$$

computed at time t ("t" can be seen as a scale parameter: the greater it is, the smaller is the scale at which the smoothing occurs).

The main drawback of this approach is that it produces an inconditional smoothing which cannot distinguish between objects and contours, since also edges begin soon to diffuse! To overcome these difficulties Perona and Malik proposed in [26] a model of selective smoothing where the contours are preserved as much as possible: it consists in replacing (1.4) by the nonlinear equation

$$\frac{\partial}{\partial t}u = \operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \qquad u(x,0) = g(x), \tag{1.5}$$

which is the gradient flow of the (Perona-Malik) functional $\int_\Omega \log(1+|\nabla u|^2)\,dx$. The underlying idea is the following: where $|\nabla u|$ is large, in particular, near the edges, the diffusion is low and the contour is "kept", while far from the edges, where the gradient is small, u diffuses as in the heat equation. Note that the simultaneous smoothing and edge detection effects of the equation strongly depend on the particular structure of the function $\log(1+t^2)$: the quadratic behaviour near the origin is responsible of the denoising process while the concave and sublinear behaviour at infinity is responsible of the edge detection. Our Γ-convergence result says that there is an alternative procedure, based on minimizing the (rescaled) energy instead of considering its gradient flow, which exploits the structure of $\log(1+t^2)$ generating again a smoothing and edge detection effect.

Actually, the same considerations apply to all functions f satisfying our structure assumptions and we can think the functionals $\int f(|\nabla u|)\,dx$ as "generalized Perona-Malik energies" giving rise to "generalized Perona-Malik equations" of the form

$$\frac{\partial}{\partial t}u = \operatorname{div}\,(g(|\nabla u|)\nabla u) \qquad u(x,0) = g(x),$$

with g bounded and decreasing to 0 when $|\nabla u|$ is large.

We want to mention, as a further application of our main result, the study of the asymptotic behaviour of the family

$$\frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|)\,dx+\varepsilon^3\int_\Omega \|\nabla^2 u\|^2\,dx,$$

where f is non-decreasing, differentiable at the origin, with non-zero derivative, and concave at infinity: the Γ-limit turns out to be a functional defined in $BV(\Omega)$ and taking the form

$$f'(0)\int_\Omega |\nabla u|\,dx+\int_{S_u}\varphi(u^+-u^-)\,d\mathcal{H}^{N-1}+f'(0)|D^c u|, \tag{1.6}$$

with φ explicitly characterized in terms of f. Again, as f varies among all the admissible potentials, a wide class of jump-functions (satisfying (3.57)) can be generated (see Theorem 4.12 and Example 4.13).

Some final remarks are in order. All the convergence results we mentioned above are completely proved in the one-dimensional case; in N dimensions one can prove the following. Let (F_n) be a sequence of one-dimensional functionals converging to F and denote by (F_n^N) and F^N their respective N-dimensional versions; then we show that Γ-$\lim_n F_n^N(u)=F^N(u)$ if u satisfies

$$\exists\, u_k\to u \quad \text{s.t} \quad \mathcal{H}^{N-1}(S_{u_k})<+\infty \quad \text{and} \quad F^N(u_k)\to F^N(u).$$

The class of such functions coincides with the whole space if F^N is finite in BV so that in this case the Γ-convergence is completely proved; we believe that the same occurs when F^N is defined in SBV but, at the moment, such a technical result is not available, and, in fact, the representation of the Γ-limit is performed for functions with discontinuity set of finite $\mathcal{H}^{N-1}$-measure. Let us finally remark that these difficulties arise in the proof of the Γ-lim sup inequality; on the other hand, the Γ-lim inf inequality is completely proved as well as the equicoerciveness of the approximating functionals which guarantees the convergence of minimizers.

2 Preliminary results

2.1 Definitions and general properties of BV functions

In this subsection we fix notations and we briefly recall basic definitions and properties from the theory of BV functions: for a general treatment we refer to [6]. The Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure of a set $B\subset\mathbb{R}^N$ are denoted by $\mathcal{L}^N(B)$ and $\mathcal{H}^{N-1}(B)$ respectively. We will often write $|B|$ instead of $\mathcal{L}^N(B)$. Given a measure μ we denote its total variation by $|\mu|$; moreover $\mu\lfloor B$ denotes the restriction of the measure μ to the set B given by $(\mu\lfloor B)(A)=\mu(B\cap A)$.

Let $\Omega\subset\mathbb{R}^N$ be an open set, let $u:\Omega\to\mathbb{R}$ be a measurable function, and let $x\in\Omega$. We denote by $u^+(x)$ and $u^-(x)$, respectively, the upper and lower limit of u at x, defined by

$$\begin{aligned} u^+(x) &:= \inf\left\{t\in\mathbb{R}:\ \lim_{\rho\to0^+}\frac{|\{y\in\Omega:\ |x-y|<\rho,\ u(y)>t\}|}{\rho^N}=0\right\},\\ u^-(x) &:= \sup\left\{t\in\mathbb{R}:\ \lim_{\rho\to0^+}\frac{|\{y\in\Omega:\ |x-y|<\rho,\ u(y)<t\}|}{\rho^N}=0\right\}. \end{aligned}$$

If $u^+(x)=u^-(x)\in\mathbb{R}$, then the common value of $u^+(x)$ and $u^-(x)$ is called the *approximate limit* of u at the point x, and is denoted by $\text{ap-}\lim_{y\to x}u(y)$.

We say that u is a *function of bounded variation* in Ω, and we write $u\in BV(\Omega)$, if $u\in L^1(\Omega)$ and its distributional derivative is a vector-valued measure Du with finite total variation $|Du|(\Omega)$. Given $u\in BV(\Omega)$, we denote by by J_u the set where $u^+>u^-$ and by S_u the *essential discontinuity set* of u made up of those points x which are not Lebesgue points. It turns out that $J_u\subseteq S_u$ and $\mathcal{H}^{N-1}(S_u\setminus J_u)=0$. For every $x\notin S_u$ we denote by $\tilde u(x)$ the approximate limit of u at x.

The *complete graph* of a function $u\in BV(\Omega)$ is the set

$$\Gamma_u:=\{(x,z)\in\Omega\times\mathbb{R}:\ u^-(x)\le z\le u^+(x)\}.$$

If $u \in BV(\Omega)$, then it can be proved that S_u is countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable, i.e.

$$S_u = N \cup \bigcup_{i\in\mathbb{N}} K_i,$$

where $\mathcal{H}^{N-1}(N) = 0$, and each K_i is a compact set contained in a C^1 hypersurface; as a consequence we have that for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ it is possible to define an approximate tangent plane $T_x(S_u)$ and therefore an approximate normal unit vector $\nu_u(x)$ which can be chosen in such a way that

$$\lim_{\rho\to 0^+} \int_{B_\rho^{\nu_u(x)}(x)} |u(y) - u^+(x)|\, dy = 0,$$

where $B_\rho^{\nu_u(x)}(x) := \{y \in B_\rho(x) : (y-x)\cdot \nu_u(x) > 0\}$ (here and in the sequel, given x and y in $\mathbb{R}^N$, we denote the scalar product of x and y by $x \cdot y$). For every $u \in BV(\Omega)$, by the Radon-Nykodim Theorem we can write $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular with respect to the Lebesgue measure. We denote the density of $D^a u$ with respect to the Lebesgue measure by ∇u. Moreover, we denote the restriction of $D^s u$ to S_u by $D^j u$, and the restriction of $D^s u$ to $\Omega\backslash S_u$ by $D^c u$. It turns out that $D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1}\lfloor S_u$ so that in particular

$$|D^s u| = |D^c u| + (u^+ - u^-)\mathcal{H}^{N-1}\lfloor S_u.$$

We will say that a set E is of *finite perimeter* in Ω if χ_E (i.e. the characteristic function of E) is of bounded variation in Ω. We define $\partial^* E \cap \Omega := S_{\chi_E} \cap \Omega$ the *reduced boundary* of E in Ω. Let us recall now the Fleming-Rishel *coarea formula.* Let u be a Lipschitz function and let v belong to $BV(\Omega)$. Then for almost every $t \in \mathbb{R}$ we have that $\{x \in \Omega : u > t\}$ is a set of finite perimeter in Ω and

$$\int_\Omega v|\nabla u|\, dx = \int_{-\infty}^{+\infty} \left(\int_{\partial^*\{u>t\}\cap\Omega} \tilde{v}\, d\mathcal{H}^{N-1} \right) dt \tag{2.1}$$

We say that u is a *special function of bounded variation*, and we write $u \in SBV(\Omega)$, if $u \in BV(\Omega)$ and $D^c u = 0$. For each $p \geq 1$ the space of all functions $u \in SBV(\Omega)$ such that

$$\nabla u \in L^p(\Omega) \qquad \text{and} \qquad \mathcal{H}^{N-1}(S_u) < +\infty$$

is denoted by $SBV^p(\Omega)$. We consider also the larger space $GBV(\Omega)$, which is composed by all measurable functions $u : \Omega \to \mathbb{R}$ whose truncations $u_k = (u \wedge k) \vee (-k)$ belong to $BV(\Omega)$ for every $k > 0$; finally we set

$$GSBV := \{u \in GBV(\Omega) : |D^c u_k| = 0\ \forall k > 0\} = \{u \in L^1(\Omega) : u_k \in SBV(\Omega)\ \forall k > 0\},$$

and

$$GSBV^p(\Omega) := \{u \in L^1(\Omega) : u_k \in SBV^p(\Omega)\ \forall k > 0\}.$$

Every $u \in GBV(\Omega) \cap L^1_{loc}(\Omega)$ has a countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable discontinuity set S_u.

We conclude this subsection by recalling a "slicing" result due to Ambrosio (see [5]) and a L^1-precompactness criterion by slicing proved in [1]. We introduce first some notation. Let $\xi \in S^{N-1}$ and let $\Pi_\xi := \{y \in \mathbb{R}^N : y \cdot \xi = 0\}$ be the linear hyperplane orthogonal to ξ. Given $E \subset \mathbb{R}^N$ we denote by $E_\xi \subseteq \Pi_\xi$ the orthogonal projection of E on Π_ξ and for $y \in \Pi_\xi$ we set $E_\xi^y := \{t \in \mathbb{R} : y + t\xi \in E\}$. Finally for $u : E \to \mathbb{R}$ we define $u_\xi^y : E_\xi^y \to \mathbb{R}$ by $u_\xi^y(t) := u(y + t\xi)$.

Theorem 2.1 *a) Let $u \in BV(\Omega)$. Then, for all $\xi \in S^{N-1}$ the function u_ξ^y belongs to $BV(\Omega_\xi^y)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi_\xi$. For such y we have*

$$\begin{aligned} (u_\xi^y)'(t) &= \nabla(y+t\xi)\cdot\xi \qquad \text{for a.e. } t \in \Omega_\xi^y, \\ S_{u_\xi^y} &= (S_u)_\xi^y, \\ u_\xi^y(t\pm) &= u^\pm(y+t\xi) \qquad \text{or} \qquad u_\xi^y(t\pm) = u^\mp(y+t\xi), \end{aligned}$$

according to the case $\nu_u \cdot \xi > 0$ or $\nu_u \cdot \xi\rangle < 0$ (the case $\nu_u \cdot \xi\rangle = 0$ being negligible). Moreover we have

$$\int_{\Pi_\xi} |D^c u^y_\xi|(A^y_\xi)\, d\mathcal{H}^{N-1}(y) = |D^c u \cdot \xi|(A),$$

for all open subset $A \subseteq \Omega$, and for all Borel functions g

$$\int_{\Pi_\xi} \sum_{t \in S_{u^y_\xi}} g(t)\, d\mathcal{H}^{N-1}(y) = \int_{S_u} g(x)|\nu_u \cdot \xi|\, d\mathcal{H}^{N-1}.$$

b) Conversely, if $u \in L^1(\Omega)$ and for all $\xi \in \{e_1, \dots, e_N\}$ and for a.e. $y \in \Pi_\xi$ $u^y_\xi \in BV(\Omega^y_\xi)$ $(SBV(\Omega^y_\xi))$ and

$$\int_{\Pi_\xi} |Du^y_\xi|\, d\mathcal{H}^{N-1}(y) < +\infty,$$

then $u \in BV(\Omega)$ $(SBV(\Omega))$.

Given a family $\mathcal{F}$ of functions, for every $\xi \in S^{N-1}$ and $y \in \Pi_\xi$ we set $\mathcal{F}^y_\xi := \{u^y_\xi : u \in \mathcal{F}\}$; moreover we say that a family $\mathcal{F}'$ is δ-close to $\mathcal{F}$ if $\mathcal{F}'$ is contained in a δ-neighbourhood of $\mathcal{F}$.

Lemma 2.2 *Let $\mathcal{F}$ be a family of equiintegrable functions belonging to $L^1(A)$ and assume that there exists a basis of unit vectors $\{\xi_1, \dots, \xi_N\}$ with the property that for every $i = 1, ..., N$, for every $\delta > 0$, there exists a family $\mathcal{F}_\delta$ δ-close to $\mathcal{F}$ such that $(\mathcal{F}_\delta)^y_{\xi_i}$ is precompact in $L^1(A^y_{\xi_i})$ for $\mathcal{H}^{N-1}$-a.e $y \in A_{\xi_i}$. Then $\mathcal{F}$ is precompact in $L^1(A)$.*

2.2 Semicontinuity and relaxation in BV and SBV

Let $f : \mathbb{R} \to [0, +\infty]$ be convex. Then we define the *recession function* f^∞ of f by

$$f^\infty(z) = \lim_{t \to +\infty} \frac{f(tz)}{t}.$$

Let $\theta : \mathbb{R} \to [0, +\infty]$ be lower semicontinuous and such that there exists $\lim_{t \to 0^+} \theta(t)/t$. Then we can define the *recession function* θ^0 of θ by

$$\theta^0(z) = \lim_{t \to 0^+} \frac{\theta(tz)}{t}.$$

The functions f^∞ and θ^0 turn out to be 1-homogeneous. For every g, $h : \mathbb{R} \to [0, +\infty]$, we define the *inf-convolution of g and h* as the function $g \triangle h$ given by

$$(g \triangle h)(z) = \inf\{g(x) + h(z - x) : x \in \mathbb{R}\}.$$

Finally we recall that given a function $F : X \to \mathbb{R} \cup +\infty$, where X is a topological space, we denote by $\overline{F}$ the *relaxed functional* of F, i.e. the greatest lower semicontinuous (with respect to the X-topology) functional which is less than F.

The following relaxation result is proved in [8].

Theorem 2.3 (Relaxation in BV) *Let $f : [0, +\infty) \to [0, +\infty)$ be a non-decreasing convex function and let $\varphi : [0, +\infty) \to [0, +\infty)$ be a concave function. Let $F : BV(\Omega) \to [0, +\infty]$ be defined by*

$$F(u) := \begin{cases} \displaystyle\int_\Omega f(|\nabla u|)\, dx + \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1} & \text{if } u \in SBV^2(\Omega) \cap L^\infty(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

Then the relaxed functional of F with respect to the L^1-metric is given on BV by

$$\overline{F}(u) := \int_\Omega f_1(|\nabla u|)\, dx + \int_{S_u} \varphi_1(u^+ - u^-)\, d\mathcal{H}^{N-1} + \big(f^\infty(1) \wedge \varphi^0(1)\big)\, |D^c u|,$$

where $f_1 := f \triangle \varphi^0$ and $\varphi_1 := \varphi \triangle f^\infty$.

It is possible to prove that $f\triangle\varphi^0 = [f \wedge (\varphi^0 + f(0))]^{**}$, where h^{**} denotes the convexification of h, i.e. the greatest convex and lower semicontinuous function which is smaller than h and, analogously, $\varphi\triangle f^\infty = \mathrm{sub}\,[\varphi \wedge (f^\infty + \varphi(0))]$ where $\mathrm{sub}\, h$ denotes the subadditive envelope, i.e the greatest lower semicontinuous and subadditive function which is smaller than h. Given two Borel functions $\varphi :]0,+\infty[\to [0,+\infty)$ and $f : [0,+\infty] \to [0,+\infty)$, we consider the functional F defined by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(|\nabla u|)\, dx + \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

In [5] the following semicontinuity result is proved.

Theorem 2.4 *(Ambrosio's Semicontinuity Theorem) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $f : [0,+\infty) \to [0,+\infty)$ be a non-decreasing convex function such that $f^\infty(1) = +\infty$ and let $\varphi :]0,+\infty] \to [0,+\infty)$ be a non-decreasing subadditive function such that $b^0(1) = \infty$. Then the functional F defined in (2.3) is lower semicontinuous with respect to the L^1 convergence.* □

2.3 A density result in SBV

In analogy with the strong density results of smooth functions in $W^{1,p}(\Omega)$, functions in $SBV^p(\Omega)$ can be approximated in a "strong sense" by functions which have a "regular" jump set and are smooth outside. This can be formally expressed as follows.

Let Ω be an open bounded subset in $\mathbb{R}^N$ with Lipschitz boundary and denote by $\mathcal{W}(\Omega)$ the space of all function $w \in SBV(\Omega)$ enjoying the following properties:

i) $\mathcal{H}^{N-1}(\overline{S}_w \setminus S_w) = 0$;

ii) $\overline{S}_w$ is the intersection of Ω with the union of a finite number of pairwise disjoint $(N-1)$-simplexes;

iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$ for every $k \in \mathbb{N}$.

Cortesani and Toader have proved in [18] the following density result.

Theorem 2.5 *Let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence $(w_j)_j$ in $\mathcal{W}(\Omega)$ such that $w_j \to u$ strongly in $L^1(\Omega)$, $\nabla w_j \to \nabla u$ strongly in $L^p(\Omega, \mathbb{R}^N)$, $\lim_j \|w_j\|_\infty = \|u\|_\infty$ and*

$$\limsup_{j\to\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j})\, d\mathcal{H}^{N-1} \le \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1},$$

for every upper semicontinuous function $\phi : \mathbb{R} \times \mathbb{R} \times S^{N-1} \to [0,+\infty)$ such that $\phi(a,b,\nu) = \phi(b,a,-\nu)$, for every a, $b \in \mathbb{R}$ and for every $\nu \in S^{N-1}$.

2.4 Γ-convergence

We recall here the definition and the main properties of Γ-convergence: for the general theory we refer to [19] (see also the forthcoming [11]).

Definition 2.6 *Let (X,d) be a metric space and let $F_h : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functions. We set*

$$\Gamma\text{-}\liminf_{h\to\infty} F_h(x) := \inf\left\{ \liminf_{h\to\infty} F_h(x_h) :\ x_h \to x \right\}$$

and

$$\Gamma\text{-}\limsup_{h\to\infty} F_h(x) := \inf\left\{ \limsup_{h\to\infty} F_h(x_h) :\ x_h \to x \right\}.$$

We say that the sequence $(F_h)_{h\in\mathbb{N}}$ Γ*-converges if*

$$\Gamma\text{-}\liminf_{h\to\infty} F_h(x) = \Gamma\text{-}\limsup_{h\to\infty} F_h(x) \qquad \forall x \in X.$$

The common value is called Γ-limit *and is denoted by* $\Gamma\text{-}\lim_{h\to\infty} F_h$.

Definition 2.7 *We say that the maps* $F_h : X \to \mathbb{R}\cup\{+\infty\}$ *are equicoercive if for every* $t\in\mathbb{R}$ *there exists a compact subset* $K_t \subseteq X$ *such that*

$$\{x\in X : F_h(x) \le t\} \subseteq K_t \qquad \forall h \in \mathbb{N}.$$

The following theorem explains the variational meaning of this kind of convergence.

Theorem 2.8 *Let* $(F_h)_h$ *be a sequence of equicoercive maps which* Γ*-converges to* F*. Then, if* $(x_h)_h$ *is a sequence such that*

$$\lim_{h\to\infty} F_h(x_h) = \lim_{h\to\infty} \inf_X F_h,$$

then x_h *is precompact and any cluster point is a minimizer of* F*.*

We finally recall that given $F : X \to \mathbb{R}\cup\{+\infty\}$, the relaxed functional $\overline{F}$ can be characterized as the Γ-limit of the constant sequence $F_n = F$ for every $n\in\mathbb{N}$.

3 The main convergence result in the one-dimensional case

Let $f_n : [0,+\infty) \to [0,+\infty)$ be a family of continuous non-decreasing functions and let r_n be an infinitesimal sequence of positive real numbers. For any open bounded subset $I\subset\mathbb{R}$, we define

$$F_n(u) := \begin{cases} \displaystyle\int_I f_n(|u'|)\,dx + (r_n)^3 \int_I |u''|^2\,dx & \text{if } u\in W^{2,2}(I),\\ +\infty & \text{otherwise in } L^1(I). \end{cases} \tag{3.1}$$

Moreover, given two functions $b,\ g : [0,+\infty)\to[0,+\infty)$, we set

$$\mathcal{F}_{b,g}(u) := \begin{cases} \displaystyle\int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-) & \text{if } u\in SBV(I),\\ +\infty & \text{otherwise in } L^1(I), \end{cases} \tag{3.2}$$

where

$$\varphi(z) := \inf_{\eta>0}\inf\left\{\int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx : u\in W^{2,2}(0,\eta),\right.$$
$$\left. u(0)=0,\ u(\eta)=z, u'(0)=u'(\eta)=0\right\}. \tag{3.3}$$

If g is convex and b is convex or concave or convex-concave then we can finally define

$$F_{b,g}(u) := \begin{cases} \displaystyle\int_I g_1(|u'|)\,dx + \sum_{S_u} \varphi_1(u^+ - u^-) + (g^\infty(1)\wedge b^0(1))|D^c u| & \text{if } u\in BV(I),\\ +\infty & \text{otherwise in } L^1(I), \end{cases} \tag{3.4}$$

where $g_1 := g\triangle b^0 = [g\wedge(b^0+g(0))]^{**}$ and $\varphi_1 := \varphi\triangle g^\infty = \mathrm{sub}\,(\varphi\wedge g^\infty)$ (g^∞ and b^0 are the recession functions of g and b respectively defined in Subsection 2.2).

Remark 3.1 Note that if $g^\infty(1) = b^0(1) = +\infty$ then $F_{b,g} = \mathcal{F}_{b,g}(u)$.

Our main result is stated in the following theorem.

Theorem 3.2 *Let f_n and r_n be as above and satisfying in addition the following hypotheses:*

i) there exists a non-decreasing function $g:[0,+\infty)\to[0,+\infty)$ such that

$$f_n(t)\to g(t) \qquad \forall t\in[0,+\infty); \tag{3.5}$$

ii) there exists a non-decreasing and continuous function $b:(0,+\infty)\to(0,+\infty)$ such that

$$r_n f_n\left(\frac{t}{r_n}\right)\to b(t) \qquad \forall t>0. \tag{3.6}$$

Then

$$\Gamma\text{-}\limsup_{n\to\infty} F_n \le \overline{\mathcal{F}_{b,g}}(u),$$

with respect to the $L^1(I)$-convergence, where F_n are the functionals defined in (3.1) while $\overline{\mathcal{F}_{b,g}}$ denotes the L^1-relaxation of the functional $\mathcal{F}_{b,g}$ introduced in (3.2). If in addition we assume

iii) one of the two following structure conditions holds true:

- **st1)** *f_n is convex for every $n\in\mathbb{N}$;*
- **st2)** *there exists a sequence $(x_n)\subset(0,+\infty)$ such that $x_n\to+\infty$ and f_n is convex in $[0,x_n]$ and concave in $[x_n,+\infty)$,*

then

$$\Gamma\text{-}\lim_{n\to\infty} F_n=\overline{\mathcal{F}_{b,g}}=F_{b,g},$$

where $F_{b,g}$ is the functional defined in (3.4). Finally, every sequence u_n such that $sup_n(F_n(u_n)+\|u_n\|_1)<+\infty$ is strongly precompact in L^p for every $p\ge 1$.

Remark 3.3 If *iii)* holds then g is convex; concerning b, assumption **st1)** implies that it is in turn convex while **st2)** implies that it is either concave or convex-concave. In all these cases the recession function b^0 is well defined. We finally point out that the equality $\overline{\mathcal{F}_{g,b}}=F_{b,g}$ stated in the last part of the theorem is a consequence of Theorem 2.3 and of the equality $\varphi^0=b^0$ which will be proved in the sequel (see Lemma 3.6).

Remark 3.4 If **st2)** holds with $\limsup_{n\to\infty} x_n r_n = c>0$, then

$$b(t)\ge g^\infty(1)t=g^\infty(t) \qquad \forall t\in[0,c], \tag{3.7}$$

so that, in particular, $b^0(1)\ge g^\infty(1)$. Passing to a subsequence, if needed, we can suppose that $\lim_n x_n r_n=c$; since the functions f_n pointwise converge to g and becomes convex in larger and larger intervals, we have

$$g'(t-)\le\liminf_{n\to\infty} f_n'(t-)\le\limsup_{n\to\infty} f_n'(t+)\le g'(t+), \tag{3.8}$$

for every $t>0$. Suppose that $g^\infty(1)\neq 0$, otherwise the statement is trivial and let $y_k\to+\infty$ such that $g'(y_k)\to g^\infty(1)$. For a fixed k and $\delta\in(0,1)$ there exists $n_{k,\delta}$ such that $x_n\ge y_k$, $f_n(t)\ge g(y_k)/2$, and $f_n'(t)\ge(1-\delta)g'(y_k)$ for every $n\ge n_{k,\delta}$, so that, by convexity,

$$f_n(t)\ge\frac{g(y_k)}{2}+(1-\delta)g'(y_k)(t-y_k) \qquad \forall t\in[y_k,x_n]\ \forall n\ge n_{k,\delta}. \tag{3.9}$$

Fix $t<c$; then, by (3.9),

$$r_n f_n\left(\frac{t}{r_n}\right)\ge r_n\frac{g(y_k)}{2}+r_n(1-\delta)g'(y_k)\left(\frac{t}{r_n}-y_k\right),$$

for every $n\ge\overline{n}$, where $\overline{n}\ge n_{k,\delta}$ is such that $y_k\le t/r_n\le x_n$ for every $n\ge\overline{n}$. Passing to the limit in n in the above inequality and taking into account (3.6) we obtain $b(t)\ge(1-\delta)g'(y_k)t$, from which (3.7) follows letting k tend to infinity and then δ tend to zero. With the same proof we see that **st1)** implies that $b(t)\ge g^\infty(t)$ for every $t\ge 0$.

Before giving the proof of the theorem we need to state and prove some preparatory lemmas.

Lemma 3.5 *Suppose that* $b(t) = Mt$ *for some* $M > 0$ *and let* φ *be the function defined in* (3.3). *Then* $\varphi(z) = Mz$ *for every* $z > 0$.

PROOF. Fix $z > 0$ and let (v, η) be an admissible pair for problem (3.3), then

$$\int_0^\eta M|v'|\,dt + \int_0^\eta |v''|^2\,dt \geq \int_0^\eta M|v'|\,dt \geq Mz,$$

and therefore $\varphi(z) \geq Mz$. Let us prove now the reverse inequality. To this aim we construct a sequence of admissible pairs (v_n, η_n) by setting $\eta_n := nz$ and

$$v_n(t) := \begin{cases} \frac{\phi(t)}{n} & \text{if } t \in [0,1) \\ \frac{1}{n} + \frac{1}{n}(t-1) & \text{if } t \in [1, nz-1) \\ z - \frac{\phi(nz-t)}{n} & \text{if } t \in [nz-1, nz], \end{cases}$$

where ϕ is a function belonging to $C^2([0,1])$ and satisfying $\phi(0) = \phi'(0) = 0$, $\phi(1) = \phi'(1) = 1$. We can now estimate

$$\begin{aligned} \varphi(z) &\leq \int_0^{\eta_n} M|v_n'|\,dt + \int_0^{\eta_n} |v_n''|^2\,dt = 2\frac{M}{n}\int_0^1 |\phi'|\,dt + M\frac{nz-2}{n} + 2\frac{1}{n}\int_0^1 |\phi''|^2\,dt \\ &= Mz + O\left(\frac{1}{n}\right) \end{aligned}$$

and therefore, letting $n \to \infty$, we obtain $\varphi(z) \leq Mz$. □

Lemma 3.6 *Let* b *as in Remark 3.3. Then the function* $\varphi : [0, +\infty) \to [0, +\infty)$ *defined in* (3.3) *is continuous, non-decreasing, subadditive, and* $\varphi^0(1) = b^0(1)$.

PROOF. The first three properties are easy; let us prove only the last one. We begin with the case

$$b^0(1) = +\infty. \tag{3.10}$$

Claim. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $z < \delta$ and if (η, u) is an *admissible pair* for problem (3.3) satisfying

$$\int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx < (1+\varepsilon)\varphi(z), \tag{3.11}$$

then $|u'| \leq \varepsilon$ in $(0, \eta)$.

Suppose by contradiction the existence of $\varepsilon > 0$ and of a sequence $\delta_n \downarrow 0$ such that, for every $n \in \mathbb{N}$, there exist $z_n < \delta_n$ and (η_n, u_n) which satisfies

$$\int_0^{\eta_n} b(|u_n'|)\,dx + \int_0^{\eta_n} |u_n''|^2\,dx < (1+\varepsilon)\varphi(z_n), \tag{3.12}$$

and

$$\|u_n'\|_{L^\infty(0,\eta_n)} > \varepsilon. \tag{3.13}$$

Note that we can suppose $\eta_n > 1$ for every n (if needed u_n can be extended outside the original interval as the constant function z_n); using Hölder's Inequality we can estimate, for every $x, y \in (0, \eta_n)$

$$|u_n'(x) - u_n'(y)| \leq \int_x^y |u_n''|\,dt \leq \sqrt{|x-y|}\left(\int_0^{\eta_n} |u_n''|^2\,dt\right)^{\frac{1}{2}} \leq C\sqrt{|x-y|},$$

where $C>0$ is independent of n; by the above estimate and by (3.13) we can state the existence of an interval $I_n \subseteq (0,\eta_n)$ such that $|I_n| \geq C'$, with C' independent of n, and $|u_n'| \geq \varepsilon/2$ in I_n. As a consequence we deduce

$$\int_0^\eta b(|u_n'|)\,dx + \int_0^\eta |u_n''|^2\,dx \geq \int_{I_n} b(|u_n'|)\,dx \geq b\left(\frac{\varepsilon}{2}\right) C'$$

which is in contradiction with (3.12) since $\varphi(z_n)$, by continuity, tends to 0. The claim is proved. Given $M>0$, thanks to (3.10), we can choose ε such that $b(t)/t \geq M$ for every $t \in (0,\varepsilon]$; if $\delta > 0$ is as in the above Claim, for $0 \leq z < \delta$ we can estimate

$$\begin{aligned}(1+\varepsilon)\varphi(z) &\geq \int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx \\ &\geq \int_0^\eta \frac{b(|u'|)}{|u'|}|u'|\,dx \geq M\int_0^\eta |u'|\,dx \geq Mz,\end{aligned}$$

where (η,u) is an admissible pair satisfying (3.11); this concludes the proof when (3.10) holds. Let us suppose now that

$$b^0(1) = C < +\infty. \tag{3.14}$$

Fix $\sigma>0$ and choose $\varepsilon_\sigma > 0$ such that $b(t) < (C+\sigma)t$ for any $t \in (0,\varepsilon_\sigma)$. Consider the sequence of admissible pairs (η_n, v_n) constructed in the previous lemma; for n large we have $\|v_n'\|_\infty \leq \varepsilon_\sigma$ and therefore

$$\begin{aligned}\varphi(z) &\leq \int_0^{\eta_n} b(|v_n'|)\,dt + \int_0^{\eta_n} |v_n''|^2\,dt \leq (C+\sigma)\int_0^{\eta_n} |v_n'|\,dt + \int_0^{\eta_n} |v_n''|^2\,dt \\ &= (C+\sigma)z + O\left(\frac{1}{n}\right).\end{aligned}$$

Letting $n\to\infty$ and $\sigma \to 0$ we obtain

$$\varphi(z) \leq Cz \qquad \forall z > 0. \tag{3.15}$$

Finally, arguing exactly as for the other case, we easily obtain $\liminf_{z\to 0^+}\varphi(z)/z \geq C$, which concludes the proof of the lemma. □

Lemma 3.7 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions such that $\sup_n F_n(u_n) < +\infty$ and, for a fixed $c>0$, consider the sets $D_n := \{x \in I : |u_n'(x)| > c/r_n\}$. Then there exists $\overline{n} \in \mathbb{N}$, depending on c, such that*

$$|D_n| \leq \left(\frac{2\sup_n F_n(u_n)}{b(c)}\right) r_n,$$

for every integer $n \geq \overline{n}$.

Proof. We can estimate

$$F_n(u_n) \geq \int_{D_n} f_n(|u_n'|)\,dx \geq \frac{r_n}{r_n} f_n\left(\frac{c}{r_n}\right)|D_n| \geq \frac{1}{2r_n} b(c)|D_n|,$$

if n is large enough, thanks to (3.6). □

Lemma 3.8 *Suppose that also iii) of Theorem 3.2 holds true and let (u_n) be such that*

$$\sup_n F_n(u_n) < +\infty.$$

Then

$$r_n\|u_n'\|_\infty \leq 2\sup_n F_n(u_n)+1, \tag{3.16}$$

for n large enough. Moreover, if $g \not\equiv 0$, there exists a positive constant C depending only on $|I|$, g, and b such that

$$\operatorname{Var} u_n \leq C\left(\sup_n F_n(u_n)+1\right)^2, \tag{3.17}$$

for n large enough.

Proof. Take $c=1$ and consider the sets D_n defined in the previous lemma; since they are open, we can write $D_n=\bigcup_{k=1}^\infty(a_n^k,b_n^k)$. Let y be a point of D_n; therefore there exists $k\in\mathbb{N}$ such that $y\in(a_n^k,b_n^k)$. By Lemma (3.7) and using Hölder's Inequality, we have

$$\begin{aligned}
|u_n'(y)| &\leq |u_n'(a_n^k)|+\int_{a_n^k}^y |u_n''(t)|\,dt\\
&\leq \frac{1}{r_n}+\frac{|D_n|^{\frac12}}{(r_n)^{\frac32}}\left((r_n)^3\int_{a_n^k}^{b_n^k}|u_n''|^2\,dt\right)^{\frac12}\\
&\leq \frac{1}{r_n}+2\frac{(r_n)^{\frac12}}{(r_n)^{\frac32}}\sup_n F_n(u_n)=\left(2\sup_n F_n(u_n)+1\right)\frac{1}{r_n},
\end{aligned}$$

so that (3.16) is proved. Concerning the second part of the Lemma, we first observe that, by a translation argument, we can suppose that $f_n(0)=g(0)=0$ for every $n\in\mathbb{N}$. Let x_0 be the last point such that $g(x_0)=0$ and define

$$\tilde g(x):=\begin{cases}0 & \text{if } x\in[0,x_0],\\ g(x-x_0) & \text{if } x\geq x_0;\end{cases}$$

it is easy to see that $\tilde g$ is still convex, $\tilde g^\infty=g^\infty$, and, taking into account the fact that $f_n\to g$ uniformly on compact subsets of $[0,+\infty)$ (the uniformity follows from the pointwise convergence and from the monotonicity of f_n),

$$\forall\delta\in(0,1),\ \forall K>0,\ \exists\overline{n}\ \text{s.t.}\ f_n\geq(1-\delta)\tilde g \text{ in } [0,K],\ \forall n\geq\overline{n}. \tag{3.18}$$

Fix $\overline{y}>0$ such that $\tilde g'(\overline{y})>g^\infty(1)/2$; set $k:=2\sup_n F_n(u_n)+1$ and let $\overline{x}$ be the first point such that $\tilde g'(\overline{x}+)/2\geq\min\{g^\infty(1)/3,b(1)/(3k)\}$. Since either $\overline{x}=0$ or

$$\frac{\tilde g'(\overline{x}-)}{2}\leq\min\left\{\frac{g^\infty(1)}{3},\frac{b(1)}{3k}\right\}\leq\frac{\tilde g'(\overline{x}+)}{2}, \tag{3.19}$$

it is clear that $\overline{x}<\overline{y}$. So, by virtue of (3.18), (3.19), (3.8), and (3.6), we can find $\overline{n}$ such that

a) $f_n\geq\tilde g(x)/2$ in $[0,\overline{x}+1]$,

b) $f_n'((\overline{x}+1)+)\geq\min\{g^\infty(1)/3,b(1)/(3k)\}$,

c) $f_n(k/r_n)/(k/r_n)\geq b(k)/(3k)$,

for every $n\geq\overline{n}$; we define $a(t):=\tilde g(\overline{x})/2+\min\{g^\infty(1)/3,b(1)/(3k)\}(t-\overline{x})$. Exploiting the convexity of $\tilde g$, we observe that, by (3.19) and a),

$$a(t)\leq\frac{\tilde g(\overline{x})}{2}\leq f_n(t)\qquad\text{in }[0,\overline{x}+1]. \tag{3.20}$$

If **st1)** holds, that is if f_n is convex, then, taking into account b) and (3.20), we also have

$$a(t) \leq f_n(\overline{x}+1) + f_n'\big((\overline{x}+1)+\big)(t-\overline{x}-1) \leq f_n(t) \qquad \forall t \geq \overline{x}+1.$$

Suppose now that **st2)** holds; by replacing f_n with

$$\tilde{f}_n(t) := \begin{cases} f_n(t) + (r_n t)^2 & \text{if } t \leq x_n, \\ f_n(t) + (r_n x_n)^2 & \text{if } t > x_n, \end{cases}$$

if needed, we can assume that f_n is strictly convex in $[0, x_n]$ (recall that x_n is the point appearing in condition **st2)**). Arguing as above, we obtain

$$a(t) \leq f_n(t) \qquad \forall t \in [\overline{x}+1, x_n \wedge k/r_n]. \tag{3.21}$$

Let us denote by y_n the first strictly positive point such that $f_n(y_n) = [f_n(k/r_n)/(k/r_n)]y_n$; by the strict convexity assumption we have that $0 < y_n \leq k/r_n$. If $x_n < y_n < k/r_n$, we can first observe that, by concavity,

$$f_n'(t\pm) \geq f_n'(y_n-) \geq \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) \qquad \forall t \in (x_n, y_n), \tag{3.22}$$

where the last inequality is a consequence of the following one

$$f_n(t) \geq f_n(y_n) + \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right)(t-y_n) = \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) t \qquad \text{in } [y_n, k/r_n],$$

where we used again the concavity of f_n in (x_n, y_n). Using (3.22) and c) we then have

$$a(t) \leq f_n(x_n) + \min\left\{\frac{g^\infty(1)}{3}, \frac{b(1)}{3k}\right\}(t-x_n) \leq f_n(t) \qquad \text{in } [x_n, y_n] \tag{3.23}$$

and therefore

$$a(t) \leq f_n(y_n) + \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right)(t-y_n) \leq f_n(t) \qquad \text{in } [y_n, k/r_n].$$

If $x_n < y_n = k/r_n$, then either

$$f_n(t) > \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) t \geq \frac{b(k)}{3k} t \qquad \text{in } [0, k/r_n],$$

or

$$f_n(t) < \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) t.$$

In the first case, (3.17) follows immediately; in the second case, we observe that (3.22) and therefore (3.23) are still true. Summarizing, we have proved that

$$a(t) \leq f_n(t) \qquad \text{in } [0, k/r_n], \tag{3.24}$$

if $x_n < y_n$; arguing in a similar way, we obtain the same estimate also if $y_n \leq x_n$. Using the definition of $a(t)$ and the fact that $\overline{x} < \overline{y}$, from (3.24) we easily obtain

$$\min\left\{\frac{g^\infty(1)}{3}, \frac{b(1)}{3}\right\} t \leq k f_n(t) + \frac{k g^\infty(1)}{3}\overline{y} \qquad \forall t \in [0, k/r_n],$$

from which, recalling the definition of k and (3.16), the inequality (3.17) follows with

$$C := (6 + 2g^\infty(1)\overline{y}|I|)(\min\{g^\infty(1)/3, b(1)/3\})^{-1}.$$

□

Remark 3.9 Let us remark that if $u_n \to u$ in L^1 and $\sup_n F_n(u_n) < +\infty$ then $u_n \to u$ in L^p for every $p \geq 1$: indeed from (3.17) it easily follows that u_n is equibounded in L^∞. As a consequence we have that in one dimension the functionals F_n Γ-converge with respect to the L^1-norm if and only if they Γ-converge with respect to the L^p-norm, for every $p \geq 1$.

Lemma 3.10 *Assume that also condition iii) of Theorem 3.2 holds and let $(u_n)_{n\in\mathbb{N}} \subset SBV(I)$ be such that $r_n\|u_n'\|_\infty \to 0$ as $n \to \infty$. Then there exists an increasing sequence $(\psi_i)_{i\in\mathbb{N}}$ of positive convex functions enjoying the following properties:*

i) $\psi_i(t) \uparrow g_1$ for every $t > 0$ as $i \to \infty$ (we recall that g_1 is the function appearing in (3.4));

ii) $\psi_i^\infty(1) = g_1^\infty(1) = b^0(1) \wedge g^\infty(1)$ for every i;

iii) passing to a subsequence, still denoted by $(u_n)_n$, we have that for every i there exists n_i such that

$$f_n(|u_n'|) \geq \psi_i(|u_n'|),$$

for every $n \geq n_i$.

PROOF. We can assume that $\min\{g^\infty(1), b^0(1)\} \neq 0$, otherwise the statement is trivial. We will distinguish two cases.
Case 1: $g^\infty(1) > b^0(1)$.

Note that in this case, by Remark 3.4, we have that necessarily **st2)** holds true, with $\lim_n x_n r_n = 0$. We can suppose without loss of generality that $f_n(0) = g(0) = 0$ for every $n \in \mathbb{N}$ (otherwise translate). We begin by assuming

$$g'(0+) < b^0(1), \tag{3.25}$$

so that, letting $\overline{x}$ be the last point such that $g'(\overline{x}-) \leq b^0(1) \leq g'(\overline{x}+)$ and setting $\overline{y} := \sup\{y \geq 0 : g(t) \leq b^0(1)t, \forall t \in [0, y]\}$, we have $\overline{x} < \overline{y}$. We make also the following assumption:

$$\forall \delta \in (0,1),\ \forall K > 0,\ \exists\, n_{\delta,K} \text{ s.t } f_n(t) \geq (1-\delta)g(t) \quad \forall t \in [0, K] \text{ and } \forall n \geq n_{\delta,K}. \tag{3.26}$$

It is clear that we can find $\delta_0 \in (0,1)$ such that for every $0 < \delta \leq \delta_0$ there holds $(1-\delta)g^\infty(1) > b^0(1)$ and $\overline{x}_\delta < \overline{y}$, where $\overline{x}_\delta$ is the last point such that $(1-\delta)g'(\overline{x}_\delta-) \leq b^0(1) \leq (1-\delta)g'(\overline{x}_\delta+)$. In particular, for every $\delta \leq \delta_0$, there exists $x_\delta \in [\overline{x}_\delta, \overline{y})$ satisfying

$$(1-\delta)g'(x_\delta) \geq b^0(1). \tag{3.27}$$

Let us choose now a sequence d_n increasing to $+\infty$ with the following properties:

a) $d_n > \|u_n'\|_\infty$ and $d_n > x_n$ for every $n \in \mathbb{N}$, where x_n is the point appearing in assumption **st2)**;

b) $d_n r_n \to 0$ so slowly that $r_n f_n(d_n)/b(d_n r_n) \to 1$ (this is possible thanks to (3.6)).

Setting $\tilde{s}_n := f_n(d_n)/d_n$, from b) it easily follows that $\lim_{n\to\infty} \tilde{s}_n = b^0(1)$, so that, passing to a subsequence if needed and denoting $s_n := \tilde{s}_n \wedge b^0(1)$, we have that s_n is a non-decreasing sequence converging to $b^0(1)$. Finally, denoting by y_n the first strictly positive point such that $f_n(y_n) = s_n y_n$, the convergence of s_n to $b^0(1)$ and of f_n to g implies $y_n \to \overline{y}$. Taking into account all this facts and recalling (3.8), it is now evident that we can find $\overline{n}_\delta > 0$ such that

*) $f_n(t) \geq (1-\delta)g(t)$, for every $t \in [0, x_\delta]$,

**) $f_n'(x_\delta-) > b^0(1) \geq s_n$ and $x_\delta < y_n$, for every $n \geq \overline{n}_\delta$. At this point, for $k > \overline{n}_\delta$ we define the function ψ^k_δ by induction in the following way:

$$\psi^k_\delta = [(1-\delta)g \wedge s_k t]^{**} \text{ in } [0, d_k] \quad \text{and} \quad \psi^k_\delta = \psi^k_\delta(d_j) + s_{j+1}(t-d_j) \text{ in } [d_j, d_{j+1}], \text{ for } j \geq k.$$

Recalling that s_n increases to $b^0(1)$ it is easily seen that ψ^k_δ is convex with $(\psi^k_\delta)^\infty(1) = b^0(1)$ and $\psi^k_\delta \uparrow [(1-\delta)g \wedge b^0(1)t]^{**}$ as k tends to infinity. Defining

$$\tilde{f}_n(t) := \begin{cases} f_n(t) & \text{if } t \in [0, x_\delta], \\ f_n(x_\delta) + s_n(t - x_\delta) & \text{otherwise,} \end{cases}$$

by *) and **), we have

$$\psi^k_\delta(t) \leq \tilde{f}_n(t) \qquad \text{in } [0, d_n]. \tag{3.28}$$

Moreover it turns out

$$\tilde{f}_n(t) \leq f_n(t) \qquad \text{in } [0, y_n]: \tag{3.29}$$

actually, this is true in $[0, x_n]$ by **) and by convexity (since $x_n \to +\infty$ we have that $x_n > y_n$ provided n is large enough). Exploiting the concave or convex-concave structure of f_n in $[y_n, d_n]$ it is also easy to prove (see the Figure 1 and the proof of the previous Lemma for the details of the argument) that

$$\tilde{f}_n(t) \leq s_n t \leq f_n(t) \qquad \text{in } [y_n, d_n]. \tag{3.30}$$

Combining (3.28), (3.29), and (3.30), we obtain that $\psi^k_\delta \leq f_n$ in $[0, d_n]$ for every $n > k$ and so, in particular,

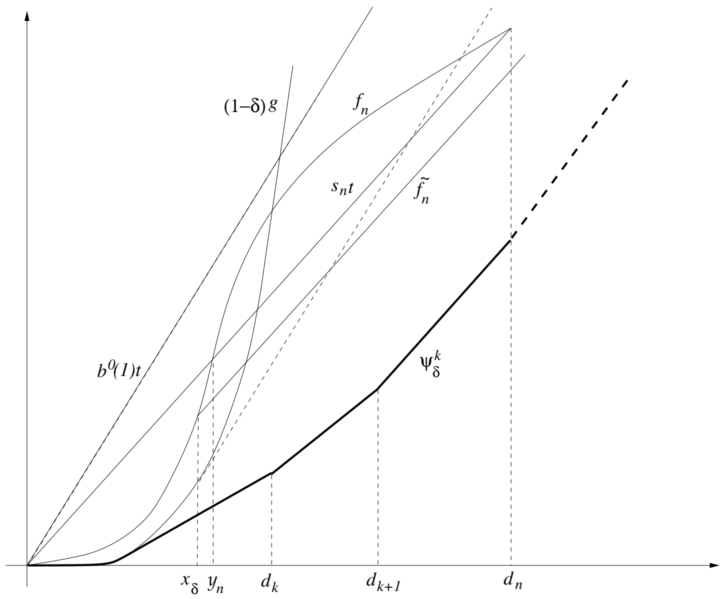

Figure 1: The construction of ψ^k_δ in the case $g^\infty(1) > b^0(1)$.

$f_n(|u'_n|) \geq \psi^k_\delta(|u'_n|)$ almost everywhere for $n > k$. Finally, choosing a sequence $\delta_n \downarrow 0$, by diagonalization, from the family $(\psi^k_{\delta_n})_{k,n}$ we can extract a subfamily $(\psi_i)_i$ having all the required properties. If g does not satisfy (3.26), we can proceed in the following way: let x_0 be the last point where g vanishes and define

$$g_k(x) := \begin{cases} 0 & \text{if } x \in [0, x_0], \\ g\left(x - \frac{1}{k}\right) & \text{if } x \geq x_0; \end{cases}$$

It turns out that $g_k^\infty(1) = g^\infty(1)$, $g_k \uparrow g$ as $k \to \infty$, and g_k satisfies (3.26). Hence we can repeat the construction above for every g_k and conclude by diagonalization. If g does not satisfy (3.25), then in particular $g(t) \geq b^0(1)t$ for every $t > 0$ and therefore $g_1 = b^0(1)t$; moreover there exists $\overline{n} \in \mathbb{N}$ such that $f_n'(0+) > (1-\delta)b^0(1)$ for every $n \geq \overline{n}$. If (d_n) and (s_n) are as above, for $k > \overline{n}$ we define ψ_δ^k by induction in the following way:

$$\psi_\delta^k(t) = (1-\delta)s_k t \text{ in } [0, d_k]$$

and

$$\psi_\delta^k(t) = \psi_\delta^k(d_{k+j}) + \left(1 - \frac{\delta}{j+1}\right) s_{k+j+1}(t - d_{k+j}) \text{ in } [d_{k+j}, d_{k+j+1}] \text{ for } j \geq 0.$$

Arguing as above, it is easy to see that from the family $(\psi_\delta^k)_{k,\delta}$ we can extract by diagonalization a subfamily satisfying all the requirements.

Case: $b^0(1) \geq g^\infty(1)$. Note that in this case $g_1 = g$. As above it is not restrictive to suppose that $g(0) = f_n(0) = 0$ for every $n \in \mathbb{N}$ and that g satisfies (3.26). At first we choose a sequence $\delta_n \downarrow 0$ and, as above, a diverging sequence d_n satisfying $\|u_n'\|_\infty \leq d_n$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{f_n(d_n)}{d_n} = b^0(1) \geq g^\infty(1).$$

Recalling (3.8) we can define for every $i \in \mathbb{N}$

$$n_{0,i} := \inf \Big\{ j \in \mathbb{N} : j > i,\ f_n(t) \geq (1-\delta_i)g(t) \text{ in } [0, d_i],\ f_n'(d_i-) > (1-\delta_i)g'(d_i-),$$
$$\text{and } \frac{f_n(d_n)}{d_n} \geq (1-\delta_i)g'(d_i-),\ \forall n \geq j \Big\},$$

and, for $h \geq 1$,

$$n_{h,i} := \inf \Big\{ j > n_{h-1,i} : f_n(t) \geq (1-\delta_{i+h})g(t) \text{ in } [0, d_{i+h}],\ f_n'(d_{i+h}-) > (1-\delta_{i+h})g'(d_{i+h}-),$$
$$\text{and } \frac{f_n(d_n)}{d_n} \geq (1-\delta_{i+h})g'(d_{i+h}-),\ \forall n \geq j \Big\}.$$

We define the function ψ_i by induction on h in the following way:

$$\psi_i(t) := \begin{cases} (1-\delta_i)g(t) & \text{if } t \in [0, d_i], \\ (1-\delta_i)[g(d_i) + g'(d_i-)(t - d_i)] & \text{if } t \in (d_i, d_{n_{1,i}}], \end{cases}$$

and, for $h \geq 1$,

$$\psi_i(t) := \psi_k(d_{n_{h,i}}) + (1-\delta_{i+h})g'(d_{i+h}-)(t - d_{n_{h,i}}) \quad \text{in} \quad (d_{n_{h,i}}, d_{n_{h+1,i}}];$$

clearly $\psi_i^\infty = g^\infty(1)$ for every i and $\psi_i \uparrow g$ as $i \to \infty$. Set, for every $h \geq 0$,

$$\phi_{i+h}(t) := \begin{cases} (1-\delta_{i+h})g(t) & \text{if } t \in [0, d_{i+h}], \\ (1-\delta_{i+h})[g(d_{i+h}) + g'(d_{i+h}-)(t - d_{i+h})] & \text{if } t > d_{i+h}. \end{cases}$$

First of all, taking into account the definition of $n_{h,i}$ and exploiting the structure assumption on f_n exactly as we did before, one can prove that

$$\phi_{i+h} \leq f_n \qquad \forall\, n \geq n_{h,i}; \tag{3.31}$$

moreover we have

$$\psi_i \leq \phi_{i+h} \qquad \text{in } [0, d_{n_{h+1,i}}]. \tag{3.32}$$

The last inequality is an immediate consequence of the following one

$$\psi_i \leq (1-\delta_{i+h})g \qquad \text{in } [0, d_{i+h}],$$

which can be proved easily by induction on h.

Take $n \geq n_{0,i}$ and let h be such that $n_{h,i} \leq n \leq n_{h+1,i}$: combining (3.31) and (3.32) we finally obtain that $\psi_i \leq f_n$ in $[0, d_n]$. □

Lemma 3.11 *Suppose that also* (3.10) *and condition iii) of Theorem 3.2 hold and let* $(u_n)_{n\in\mathbb{N}} \subset W^{2,2}(I)$ *be such that* $\sup_n F_n(u_n) < +\infty$. *Then, for every* $\delta > 0$, *there exists a sequence* $(v_n)_{n\in\mathbb{N}} \subseteq SBV(I)$ *such that* $\|u_n - v_n\|_1 \to 0$, $r_n\|v_n'\|_\infty \to 0$ *as* $n \to \infty$, $|v_n'| \leq |u_n'|$ *everywhere, and*

$$F_n(u_n) \geq (1-\delta) \sum_{x\in S_{v_n}} \varphi(v_n^+(x) - v_n^-(x)),$$

for n *sufficiently large.*

Proof. By Lemma 3.8 there exists $K > 0$ such that

$$r_n\|u_n'\|_\infty \leq K; \tag{3.33}$$

for every $0 < s \leq K$ we define

$$\omega_n(s) := \sup_{t\in[s,K]} \left| r_n f_n\left(\frac{t}{r_n}\right) - b(t) \right|. \tag{3.34}$$

Recalling that if a family of monotone functions pointwise converges to a continuous function, then the convergence is actually uniform on compact subsets, by (3.6) we have that $\omega_n \to 0$ pointwise. As a first step we choose a sequence $(c_n)_n$ of positive real numbers converging to 0 so slowly that:

a) $\dfrac{r_n}{(c_n)^{\frac{5}{2}}} \to 0$ as $n \to \infty$;

b) $\displaystyle\lim_{n\to\infty} \frac{\omega_n(c_n)}{b(c_n)} = 0$.

We set $D_n := \{x \in I : |u_n'| > c_n/r_n\} = \bigcup_{k=1}^{\infty} I_n^k$, where (I_n^k) is the collection of the connected components of D_n; we also denote $I_n^k = (a_n^k, b_n^k)$. Arguing as in Lemma 3.8 and taking into account condition b), we obtain

$$|D_n| \leq \left(\frac{2\sup_n F_n(u_n)}{b(c_n)}\right) r_n, \tag{3.35}$$

for n large enough. For every $n \in \mathbb{N}$ we define

$$\tilde{v}_n(x) := \begin{cases} u_n(x) & \text{if } x \in I \setminus D_n, \\ u_n(a_n^k) & \text{if } x \in (a_n^k, b_n^k); \end{cases}$$

clearly $\tilde{v}_n \in L^1(I) \cap SBV(I)$. Moreover we set $w_n := u_n - \tilde{v}_n$; since $w_n' = u_n'$ and $w_n'' = u_n''$ on D_n, we have

$$F_n(u_n, I_n^k) = \int_{I_n^k} f_n(|w_n'|)\, dx + (r_n)^3 \int_{I_n^k} |w_n''|^2\, dx;$$

summing over k and setting $\tilde{z}_n(x) := w_n(r_n x)$ we therefore obtain

$$\begin{aligned} F_n(u_n, D_n) &= \sum_k \left(\int_{I_n^k} f_n(|w_n'|)\, dx + (r_n)^3 \int_{I_n^k} |w_n''|^2\, dx \right) \\ &= \sum_k \left(\int_{I_n^k} f_n\left(\frac{1}{r_n}\left|\tilde{z}_n'\left(\frac{x}{r_n}\right)\right|\right) dx + \frac{1}{r_n}\int_{I_n^k} \left|\tilde{z}_n''\left(\frac{x}{r_n}\right)\right|^2 dx \right) \\ &= \sum_k \left(r_n \int_{\frac{I_n^k}{r_n}} f_n\left(\frac{1}{r_n}|\tilde{z}_n'|\right) dy + \int_{\frac{I_n^k}{r_n}} |\tilde{z}_n''|^2\, dy \right). \end{aligned} \tag{3.36}$$

By (3.33) we have

$$c_n \le |\tilde{z}_n'| \le K \qquad \text{in } D_n/r_n; \tag{3.37}$$

moreover, we can prove that for every $\delta > 0$, there exists $\overline{n}$ such that $r_n f_n\left(\frac{t}{r_n}\right) \ge (1-\delta)b(t)$ for every $t \in [c_n, K]$ and for every $n \ge \overline{n}$ and thus, by (3.37),

$$r_n f_n\left(\frac{|\tilde{z}_n'|}{r_n}\right) \ge (1-\delta) b\left(|\tilde{z}_n'|\right) \qquad \text{in } D_n/r_n. \tag{3.38}$$

Indeed, by condition b), for every $\delta > 0$ we can find $\overline{n}$ such that $\omega_n(c_n) \le \delta b(c_n)$ for every $n \ge \overline{n}$, so that, recalling (3.34),

$$r_n f_n\left(\frac{t}{r_n}\right) \ge b(t) - \omega_n(c_n) \ge b(t) - \delta b(c_n) \ge (1-\delta)b(t) \qquad \forall t \in [c_n, K],$$

where we used the monotonicity of b. Let us define the functions z_n as

$$z_n(x) := \begin{cases} \tilde{z}_n(x) & \text{if } x \in (I \setminus D_n)/r_n, \\ \tilde{z}_n(x) - \tilde{z}_n'\left(\frac{a_k^n}{r_n}\right)\left(x - \frac{a_k^n}{r_n}\right) & \text{in } I_n^k/r_n. \end{cases}$$

By (3.36) and (3.38), by using the fact that $|z_n'| \le |\tilde{z}_n'|$ everywhere and the monotonicity of b, we have

$$\begin{aligned} F_n(u_n, D_n) &\ge (1-\delta)\sum_k \left(\int_{\frac{I_n^k}{r_n}} b\left(|\tilde{z}_n'|\right)\, dy + \int_{\frac{I_n^k}{r_n}} |\tilde{z}_n''|^2\, dy\right) \\ &\ge (1-\delta)\sum_k \left(\int_{\frac{I_n^k}{r_n}} b\left(|z_n'|\right)\, dy + \int_{\frac{I_n^k}{r_n}} |z_n''|^2\, dy\right) \\ &\ge (1-\delta)\sum_k \varphi\left(\left|z_n\left(\frac{b_n^k}{r_n}\right)\right|\right) = (1-\delta)\sum_k \varphi(v_n^+(b_n^k) - v_n^-(b_n^k)) = (**), \end{aligned} \tag{3.39}$$

where $v_n(x) := u_n(x) - z_n(x/r_n)$. Using the definition of z_n, it is easy to check that

$$v_n(x) = \begin{cases} u_n(x) & \text{if } x \in I \setminus D_n \\ u_n(a_n^k) + u_n'(a_n^k)(x - a_n^k) & \text{if } x \in (a_n^k, b_n^k) \end{cases}$$

and to see that $(**) = (1-\delta)\sum_{x\in S_{v_n}} \varphi(v_n^+(x) - v_n^-(x))$, which, combined with (3.39), gives the thesis of the lemma, once we have shown that

$$\|v_n - u_n\|_1 \to 0 \qquad \text{as } n \to \infty. \tag{3.40}$$

If $t \in I_n^k$, by Hölder's Inequality, we have

$$\begin{aligned} |v_n(t) - u_n(t)| &\le \int_{a_n^k}^t |v_n'(s) - u_n'(s)|\, ds \le \int_{a_n^k}^t \int_{a_n^k}^s |u_n''(z)|\, dz \\ &\le \left(\int_{I_n^k} |u_n''|^2\, dz\right)^{\frac{1}{2}} \int_{a_n^k}^t (s - a_n^k)^{\frac{1}{2}}\, ds \\ &= \frac{2}{3}\left(\int_{I_n^k} |u_n''|^2\, dz\right)^{\frac{1}{2}} (t - a_n^k)^{\frac{3}{2}}, \end{aligned}$$

therefore, integrating on I_n^k,

$$\int_{I_n^k} |v_n(t) - u_n(t)|\, dt \leq \frac{4}{15} \left(\int_{I_n^k} |u_n''|^2\, dz \right)^{\frac{1}{2}} |I_n^k|^{\frac{5}{2}};$$

using (3.35), we can conclude

$$\begin{aligned} \|u_n - v_n\|_1 &\leq \frac{4}{15} \sum_{k\in\mathbb{N}} \left(\int_{I_n^k} |u_n''|^2\, dz \right)^{\frac{1}{2}} |I_n^k|^{\frac{5}{2}} \\ &\leq \frac{4}{15 (r_n)^{\frac{3}{2}}} \left((r_n)^3 \int_{D_n} |u_n''|^2\, dz \right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} |I_n^k|^5 \right)^{\frac{1}{2}} \\ &\leq \frac{4 (\sup_n F_n(u_n))^{\frac{1}{2}}}{15 (r_n)^{\frac{3}{2}}} |D_n|^{\frac{5}{2}} \leq \frac{4 (\sqrt{2})^5 (\sup_n F_n(u_n))^3}{15} \frac{r_n}{(b(c_n))^{\frac{5}{2}}}; \end{aligned}$$

recalling condition a) and (3.10), we finally get (3.40). □

Lemma 3.12 *Let $g : [0,\infty) \to [0,\infty)$ be a convex superlinear function and let $u \in SBV(I)$ be such that $\int_I g(|u'|)\, dx + \mathcal{H}^0(S_u) < +\infty$. Then there exists a sequence $(u_n) \in SBV(I)$ such that $S_{u_n} \subseteq S_u$, $u_n \in W^{2,2}(I \setminus S_u)$, $u_n'(t\pm) = 0$ on S_u, $u_n \to u$ in $L^\infty(I)$, $u_n^\pm(t) \to u^\pm(t)$ on S_u, and $\int_I g(|u_n'|)\, dx \to \int_I g(|u'|)\, dx$.*

Proof. Let $I = (a,b)$ and $S_u = \{x_1, \dots, x_N\}$ with $x_i < x_{i+1}$ and set $x_0 = a$ and $x_{N+1} = b$. We can construct a family (g_k) of strictly convex and superlinear functions belonging to $C^2([0,+\infty))$ and satisfying

$$g_k'(0) = 0, \qquad g_k \downarrow g, \qquad \text{and} \qquad \lim_{t\to+\infty} \frac{g_k(t)}{g(t)} = 1.$$

For every $k \in \mathbb{N}$ and for every $i \in \{0, \dots, N\}$ let $u_{i,j}^k$ be the solution of the minimum problem

$$\min \left\{ \int_I g_k(|v'|)\, dx + j \int_I |v - u|^2\, dx : \, v \in W^{1,1}(x_i, x_{i+1}) \right\};$$

note that the existence of such a solution is guaranteed by the convexity and the superlinearity of g_k; moreover $u_{i,j}^k$ is a classical solution to the Euler equation $h_k''(w')w'' = j(w-u)$ with the Neumann conditions $w'(x_i) = w'(x_{i+1}) = 0$, where h_k is the function in $C^2(\mathbb{R})$ obtained by reflection of g_k. Therefore, taking into account the regularity and strict convexity assumptions on g_k and the fact that $u \in C([x_i, x_{i+1}])$, we get $u_{i,j}^k \in C^2([x_i, x_{i+1}])$, so that, denoting by u_j^k the function in $SBV(I)$ which coincides with $u_{i,j}^k$ on (x_i, x_{i+1}), we clearly have that the family $(u_j^k)_j$ satisfies all the required conditions except for the last one. Indeed, by construction, we know only $\int_I g_k(|(u_j^k)'|)\, dx \xrightarrow{j} \int_I g_k(|u'|)\, dx$, but recalling that $\int_I g_k(|u'|)\, dx \xrightarrow{k} \int_I g(|u'|)\, dx$, the desired approximating sequence can be obtained by diagonalization. □

We finally state a Lemma which will be useful in the sequel.

Lemma 3.13 *Denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω and let $\nu : \mathcal{A}(\Omega) \to [0,+\infty)$ be a superadditive set-function. Let λ be a positive measure on Ω and let $(\psi_i)_i$ a family of positive Borel functions such that $\nu(A) \geq \int_A \psi_i\, d\lambda$ for all $A \in \mathcal{A}(\Omega)$ and for all $i \in \mathbb{N}$. Then $\nu(A) \geq \int_A \sup_i \psi_i\, d\lambda$, for all $A \in \mathcal{A}(\Omega)$.*

Proof. See Proposition 1.16 of [10]. □

Proof of Theorem 3.2: the case $b^0(1) = +\infty$.

- Γ-$\limsup$ Inequality

Let us set for notational convenience $F'' := \Gamma\text{-}\limsup_n F_n$. We first remark that it is enough to show that $F''(u) \le \int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-)$ for every $u \in SBV(I)$ with $\mathcal{H}^0(S_u) < +\infty$, indeed, once we have this inequality, the thesis follows from the semicontinuity of F'' and the fact that $\overline{\mathcal{F}}_{b,g}$ coincides with the relaxed functional of

$$H(u) := \begin{cases} \displaystyle\int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-) & \text{if } u \in SBV(I) \text{ and } \mathcal{H}^0(S_u) < +\infty. \\ +\infty & \text{otherwise} \end{cases}$$

Claim 1. Let $u \in SBV(I)$ such that $\mathcal{H}^0(S_u) < +\infty$, $u \in W^{2,2}(I \setminus S_u)$, $F(u) < +\infty$, and $u'(t\pm) = 0$ for every $t \in S_u$. Then

$$F''(u) \le \int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-).$$

Since the construction is local, we may assume that $S_u = \{\bar t\}$ and $u(t\pm) = u^\pm(\bar t)$.

Fix $\delta > 0$ and choose an admissible pair (η, v) for problem (3.3) (with $z = u^+(\bar t) - u^-(\bar t)$) satisfying:

$$\int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx < \varphi(u^+(\bar t) - u^-(\bar t)) + \delta.$$

We define the recovery sequence in the following way:

$$u_n(x) := \begin{cases} u(x) & \text{if } x \le \bar t \\ v\left(\dfrac{x - \bar t}{r_n}\right) + u^-(\bar t) & \text{if } x \in (\bar t, \bar t + r_n\eta) \\ u(x - r_n\eta) + u^+(\bar t) & \text{if } x \ge \bar t + r_n\eta. \end{cases}$$

Clearly $u_n \to u$ in L^1. We can now compute

$$\begin{aligned} F_n(u_n) &= F_n(u_n, I \setminus (\bar t, \bar t + r_n\eta)) + \int_{\bar t}^{\bar t + r_n\eta} f_n\left(\frac{1}{r_n}\left|v'\left(\frac{x - \bar t}{r_n}\right)\right|\right) dx \\ &\quad + (r_n)^3 \int_{\bar t}^{\bar t + r_n\eta} \frac{1}{(r_n)^4}\left|v''\left(\frac{x - \bar t}{r_n}\right)\right|^2 dx \\ &= F_n(u_n, I \setminus (\bar t, \bar t + r_n\eta)) + \underbrace{\int_0^\eta r_n f_n\left(\frac{|v'|}{r_n}\right) dy + \int_0^\eta |v''|^2\,dy}_{=(*)_n}. \end{aligned} \tag{3.41}$$

Since

$$r_n f_n\left(\frac{|v'|}{r_n}\right) \to b(|v'|) \text{ in } \{x \in I : |v'(x)| \ne 0\}, \qquad r_n f_n\left(\frac{|v'|}{r_n}\right) \le r_n f_n\left(\frac{\|v'\|_\infty}{r_n}\right) \to b(\|v'\|_\infty),$$

by the Dominated Convergence Theorem and the fact that

$$\lim_{n\to\infty} \int_{\{x \in I:\, |v'(x)| \ne 0\}} r_n f_n\left(\frac{|v'|}{r_n}\right) dx = \lim_{n\to\infty} |\{x \in I : |v'(x)| \ne 0\}| r_n f_n(0) = 0,$$

we have

$$\limsup_{n\to\infty} (*)_n \le \int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \quad \text{as } n \to \infty;$$

moreover, again by the Dominated Convergence Theorem, we easily see that

$$\lim_{n\to\infty} F_n(u_n, I \setminus (\bar t, \bar t + r_n\eta)) = \int_I g(|u'|)\,dx.$$

From (3.41), we therefore obtain

$$\begin{aligned}\limsup_{n\to\infty} F_n(u_n) &\leq \int_I g(|u'|)\,dx + \int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \\ &\leq \int_I g(|u'|)\,dx + \varphi(u^+(\bar t) - u^-(\bar t)) + \delta.\end{aligned}$$

By the arbitrariness of δ, Claim 1 is proved. By a standard density argument based on the use of Lemma 3.12 we recover the same inequality for every $u \in SBV(I)$ with $\mathcal{H}^0(S_u) < +\infty$ and this concludes the proof of the Γ-$\limsup$ inequality, as we remarked above.

- Γ-$\liminf$ Inequality

We are supposing that the structure condition *iii)* holds so that g is convex and b is convex or concave or convex-concave; the functional $F_{b,g}$ is then well defined and, by Theorem 2.3, coincides with $\overline{\mathcal{F}_{b,g}}$. We distinguish two cases.

Case 1: $g^\infty(1) = +\infty$ i.e. g is superlinear.

Note that in this case $F_{b,g}(u)$ is finite only if $u \in SBV(I)$ and for such u we have

$$F_{b,g}(u) = \int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-).$$

Let $u_n \to u$ in L^1 and such that $\sup_n F_n(u_n) < +\infty$, let $(v_n)_n$ be the sequence constructed in Lemma 3.11 and $(\psi_i)_i$ the associated family of convex superlinear functions according to Lemma 3.10. For δ and $\mu \in (0,1)$, and for every open subset $J \subseteq I$, by Lemmas 3.11 and 3.10, we have, for n sufficiently large,

$$\begin{aligned}F_n(u_n, J) &= (1-\delta)\int_J f_n(|u_n'|)\,dx + \delta\left[\int_J f_n(|u_n'|)\,dx + ((1/\sqrt[3]{\delta})r_n)^3 \int_J |u_n''|^2\,dx\right] \\ &\geq (1-\delta)\int_J \psi_i(|v_n'|)\,dx + \delta(1-\mu)\sum_{x\in S_u} \varphi(v_n^+ - v_n^-). \end{aligned} \tag{3.42}$$

Therefore, by the Ambrosio Semicontinuity Theorem (recall also Lemma 3.6), we obtain that $u \in SBV(I)$ and

$$\liminf_{n\to\infty} F_n(u_n, J) \geq (1-\delta)\int_J \psi_i(|u'|)\,dx + \delta(1-\mu)\sum_{x\in S_u}\varphi(u^+ - u^-) \qquad \forall i;$$

letting $i \uparrow \infty$ and $\mu \downarrow 0$, we obtain

$$\begin{aligned}(\Gamma\text{-}\liminf_{n\to\infty} F_n)(u, J) &\geq (1-\delta)\int_J g(|u'|)\,dx + \delta\sum_{x\in S_u}\varphi(u^+ - u^-) \\ &= \int_J h^\delta(x)\,d\lambda \quad \forall \text{ open } J \subseteq I, \forall \delta \in (0,1), \end{aligned} \tag{3.43}$$

where we have set $\lambda := g(|u'|)\mathcal{L}^1 + \varphi(u^+ - u^-)\mathcal{H}^0$ and $h^\delta := (1-\delta)(1-\chi_{S_u}) + \delta\chi_{S_u}$. Let δ_n be a dense sequence in $(0,1)$; since $\sup_i h^{\delta_i} = 1$, from (3.43) we finally deduce

$$(\Gamma\text{-}\liminf_{n\to\infty} F_n)(u) \geq \int_I \sup_i h^{\delta_i}\,d\lambda = \int_I g(|u'|)\,dx + \sum_{x\in S_u}\varphi(u^+ - u^-),$$

where we applied Lemma 3.13 (with $\nu := (\Gamma\text{-}\liminf_{n\to\infty} F_n)(u,\cdot)$).

Case 2: $g^\infty(1) < +\infty$. Let v_n be as above: according to Lemma 3.10, construct a family $(\psi_i)_i$ of convex functions such that $\psi_i^\infty(1) = g^\infty(1)$ for every $i \in \mathbb{N}$, $\psi_i \uparrow g$ as $i \to \infty$, and $\psi_i(|v_n'|) \leq f_n(|v_n'|)$ for every i and for n sufficiently large. Therefore, by using Lemma 3.11, we can write

$$\begin{aligned}F_n(u_n, I) &= F_n(u_n, D_n) + F_n(u_n, I\setminus D_n) \\ &\geq (1-\delta)\left(\int_I \psi_i(|v_n'|)\,dx + \sum_{x\in S_{v_n}}\varphi(v_n^+ - v_n^-)\right) - \int_{D_n}\psi_i(|v_n'|)\,dx; \end{aligned} \tag{3.44}$$

using the inequality $\psi_i(t) \leq g(0) + g^\infty(1)t$, true for every $t > 0$, and recalling (3.35) and the fact that $\lim_n b(c_n)/c_n = +\infty$, we can estimate

$$\int_{D_n} \psi_i(|v_n'|)\,dx = \psi_i\left(\frac{c_n}{r_n}\right)|D_n| \leq \left(g(0) + g^\infty(1)\frac{c_n}{r_n}\right)\left(\frac{2\sup_n F_n(u_n)}{b(c_n)}\right) r_n = O(1)$$

hence, invoking the Relaxation Theorem 2.3, from (3.44), we get

$$\liminf_{n\to\infty} F_n(u_n) \geq (1-\delta)\left(\int_I \psi_i(|u'|)\,dx + \sum_{S_u}(\varphi \triangle g^\infty)(u^+ - u^-) + g^\infty(1)|D^c u|\right).$$

Letting $i \uparrow +\infty$ and $\delta \downarrow 0$ we complete the proof of the Γ-$\liminf$ inequality.

Concerning the last part of the theorem, we first observe that, thanks to (3.17) the approximating functionals are equicoercive: the conclusion then follows from Remark 3.9. □

To treat the case

$$b^0(1) = \lim_{t\to 0^+} \frac{b(t)}{t} < +\infty,$$

we need first the following lemma.

Lemma 3.14 *Let b be satisfy (3.14), and, for every $\delta > 0$, let $\varphi^\delta : (0,\infty) \to (0,\infty)$ be the function defined by*

$$\varphi^\delta(z) := \inf_{\eta>0} \inf\left\{\int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx \,:\, u \in W^{2,2}(0,\eta),\right.$$
$$\left. u(0) = 0,\; u(\eta) = z, u'(0) = u'(\eta) = \delta\right\}.$$

Then the following properties hold true:

i) $\lim_{\delta\to 0^+} \varphi^\delta(z) = \varphi(z)$, uniformly in $[k, +\infty)$, for every $k > 0$;

ii) for every $\varepsilon \in (0,1)$, there exists $\overline{\delta}$ such that $\varphi^\delta(z) \geq (1-\varepsilon)\varphi(z)$ for every $\delta \leq \overline{\delta}$ and for every $z > 0$.

Proof. Fix $k > 0$ and let $\phi \in C^2([0,1])$ be such that $\phi(0) = \phi'(0) = 0$ and $\phi(1) = \phi'(1) = 1$. Moreover choose $0 < \overline{\delta} < k$ such that

$$\int_0^1 b(\delta|\phi'|)\,dx + \delta^2 \int_0^1 |\phi''|^2\,dx \leq \frac{\varepsilon}{4}, \tag{3.45}$$

for every $\delta \leq \overline{\delta}$. Fix now $\delta \in (0, \overline{\delta})$ and for a given $z \geq k$ set $z' := z - 2\delta$ and take (v,η), admissible pair for the minimum problem defining $\varphi(z')$ such that

$$\int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \leq \varphi(z') + \frac{\varepsilon}{2} \leq \varphi(z) + \frac{\varepsilon}{2}. \tag{3.46}$$

We now define $\tilde{\eta} := \eta + 2$ and $\tilde{v} \in W^{2,2}(0,\tilde{\eta})$ by

$$\tilde{v}(t) := \begin{cases} \delta - \delta\phi(1-t) & \text{if } t \in [0,1), \\ v(t-1) + \delta & \text{if } t \in [1, \eta+1), \\ z - \delta + \delta\phi(t - \eta - 1) & \text{if } t \in [\eta+1, \tilde{\eta}]. \end{cases}$$

It is clear that $(\tilde{v}, \tilde{\eta})$ is an admissible pair for the minimum problem defining $\varphi^\delta(z)$, so that we have

$$\begin{aligned} \varphi^\delta(z) &\leq \int_0^{\tilde{\eta}} b(|\tilde{v}'|)\,dx + \int_0^{\tilde{\eta}} |\tilde{v}''|^2\,dx \\ &= 2\left(\int_0^1 b(\delta|\phi'|)\,dx + \delta^2 \int_0^1 |\phi''|^2\,dx\right) + \int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \\ &\leq \varphi(z) + \varepsilon, \end{aligned} \tag{3.47}$$

where the last inequality follows from (3.45) and (3.46). Let now (v,η) be an admissible pair for $\varphi^\delta(z)$ satisfying

$$\int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \le \varphi^\delta(z) + \frac{\varepsilon}{2}. \tag{3.48}$$

We now define $\tilde\eta := \eta + 2$ and $\tilde v$ by

$$\tilde v(t) := \begin{cases} \delta\phi(t) & \text{if } t \in [0,1), \\ v(t-1) + \delta & \text{if } t \in [1, \eta+1), \\ z + 2\delta - \delta\phi(\tilde\eta - t) & \text{if } t \in [\eta+1, \tilde\eta]. \end{cases}$$

As above, we have

$$\begin{aligned} \varphi(z) &\le \varphi(z+2\delta) \le \int_0^{\tilde\eta} b(|\tilde v'|)\,dx + \int_0^{\tilde\eta} |\tilde v''|^2\,dx \\ &= 2\left(\int_0^1 b(\delta|\phi'|)\,dx + \delta^2 \int_0^1 |\phi''|^2\,dx\right) + \int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \\ &\le \varphi^\delta(z) + \varepsilon, \end{aligned}$$

thanks to (3.45) and (3.48); recalling (3.47), *i)* is proved.

For the last part we suppose by contradiction that there exist $\varepsilon \in (0,1)$, a sequence $\delta_n \downarrow 0$, and sequence x_n such that

$$\varphi^{\delta_n}(x_n) < (1-\varepsilon)\varphi(x_n), \tag{3.49}$$

for every $n \in \mathbb{N}$. Testing with the pair $(v(t) := \delta t, z/\delta)$, we easily obtain

$$\varphi^\delta(z) \le \frac{b(\delta)}{\delta} z \le C' z, \qquad \forall \delta < 1. \tag{3.50}$$

Taking into account *i)* we see that (3.49) and (3.50) imply

$$x_n \to 0 \qquad \text{and} \qquad \varphi^{\delta_n}(x_n) \to 0. \tag{3.51}$$

Let (v_n, η_n) be an admissible pair for the minimum problem defining $\varphi^{\delta_n}(x_n)$ such that

$$\int_0^{\eta_n} b(|v_n'|)\,dx + \int_0^{\eta_n} |v_n''|^2\,dx \le \varphi^{\delta_n}(x_n) + (\varphi^{\delta_n}(x_n))^2; \tag{3.52}$$

arguing as in the proof of Lemma 3.6 we deduce that $\|v_n'\|_\infty \to 0$. Choose $\sigma > 0$ such that

$$b(t) \ge \left(1 - \frac{\varepsilon}{2}\right) Ct \qquad \forall t \le \sigma \tag{3.53}$$

and let $\overline{n}$ be such that $\|v_n'\|_\infty \le \sigma$ for every $n \ge \overline{n}$. Then, using (3.52), (3.53), (3.50), (3.51) and recalling that $\varphi(z) \le Cz$ for every $z > 0$ (see (3.15)), we estimate

$$\begin{aligned} \varphi^{\delta_n}(x_n) &\ge \int_0^{\eta_n} b(|v_n'|)\,dx - (\varphi^{\delta_n}(x_n))^2 \ge \left(1 - \frac{\varepsilon}{2}\right) Cx_n - (C'x_n)^2 \\ &\ge \left(1 - \frac{3}{4}\varepsilon\right) Cx_n \ge \left(1 - \frac{3}{4}\varepsilon\right)\varphi(x_n), \end{aligned}$$

for n large enough, a contradiction with (3.49). □

We are now in a position to conclude the proof of Theorem 3.2.

Proof of Theorem 3.2: the case $b^0(1) < +\infty$.

The Γ-lim sup inequality can be proved as in the other case. So suppose that the structure condition *iii)* appearing in the statement of the theorem holds so that $F_{b,g}$ is well defined and coincides with $\overline{\mathcal{F}_{b,g}}$. We may also suppose that $g \not\equiv 0$, otherwise the Γ-liminf inequality is trivial. Let $\varepsilon_n \to 0$ and $u_{\varepsilon_n} \to u$ in L^1 and such that $\exists \lim_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}) < +\infty$. Choose now an infinitesimal sequence c_n with the same properties as in the proof of Lemma 3.11; set

$$D_n := \left\{ x \in I : \ |u'_{\varepsilon_n}| > \frac{c_n}{r(\varepsilon_n)} \right\} = \bigcup_{k=1}^{\infty} (a_n^k, b_n^k) = \bigcup_{k=1}^{\infty} I_n^k$$

and define

$$v_{\varepsilon_n}(x) := \begin{cases} u_{\varepsilon_n}(x) & \text{if } x \in I \setminus D_n \\ u_{\varepsilon_n}(a_n^k) & \text{if } x \in (a_n^k, b_n^k) \end{cases}.$$

Finally set $w_{\varepsilon_n} := u_{\varepsilon_n} - v_{\varepsilon_n}$ and $z_{\varepsilon_n}(x) := w_{\varepsilon_n}(r(\varepsilon_n)x)$. For fixed $\delta \in (0,1)$, with exactly the same arguments of Lemma 3.11, we obtain

$$\begin{aligned} F_{\varepsilon_n}(u_{\varepsilon_n}, D_n) &\geq (1-\delta) \sum_k \left(\int_{I_n^k/(r(\varepsilon_n))} b(|z'_{\varepsilon_n}|)\,dx + \int_{I_n^k/(r(\varepsilon_n))} |z''_{\varepsilon_n}|^2\,dx \right) \\ &\geq (1-\delta) \sum_k \inf_{\eta>0} \inf \left\{ \int_0^\eta b(|z'|)\,dx + \int_0^\eta |z''|^2\,dx : \ z \in W^{2,2}(0,\eta), \right. \\ &\qquad\qquad \left. z(0) = 0,\ z(\eta) = |w_{\varepsilon_n}(b_n^k)|,\ z'(0) = z'(\eta) = c_n \right\} \\ &= (1-\delta) \sum_k \varphi^{c_n}(|w_{\varepsilon_n}(b_n^k)|) = (1-\delta) \sum_{S_{v_{\varepsilon_n}}} \varphi^{c_n}(v^+_{\varepsilon_n} - v^-_{\varepsilon_n}), \end{aligned} \tag{3.54}$$

for n large enough, where φ^{c_n} is the function defined in Lemma 3.14 (with $\delta = c_n$). Using *ii)* of Lemma 3.14, from (3.54) we deduce

$$F_{\varepsilon_n}(u_{\varepsilon_n}, D_n) \geq (1-\delta)^2 \sum_{S_{v_{\varepsilon_n}}} \varphi(v^+_{\varepsilon_n} - v^-_{\varepsilon_n}),$$

for n large enough. Combining the estimate above with Lemma 3.10, we therefore obtain (passing to a subsequence, if needed),

$$\begin{aligned} F_{\varepsilon_n}(u_{\varepsilon_n}) &\geq (1-\delta)^2 \left(\int_{I\setminus D_n} \psi_i(|v'_{\varepsilon_n}|)\,dx + \sum_{S_{v_{\varepsilon_n}}} \varphi(v^+_{\varepsilon_n} - v^-_{\varepsilon_n}) \right) \\ &= (1-\delta)^2 \left(\int_I \psi_i(|v'_{\varepsilon_n}|)\,dx + \sum_{S_{v_{\varepsilon_n}}} \varphi(v^+_{\varepsilon_n} - v^-_{\varepsilon_n}) \right), \end{aligned} \tag{3.55}$$

where, according to Lemma 3.10, ψ_i is convex, $\psi_i^\infty(1) = g^\infty(1) \wedge b^0(1)$ and $\psi_i \uparrow g_1$ as $i \to \infty$. Since, by Lemma 3.8, we have $\sup_n \operatorname{Var} v_{\varepsilon_n} \leq \sup_n \operatorname{Var} u_{\varepsilon_n} < +\infty$, Rellich's Theorem implies that v_{ε_n} is precompact in L^1 and since $v_{\varepsilon_n} \to u$ in measure (recall that $|D_n| \to 0$), we get $v_{\varepsilon_n} \to u$ in L^1. Applying Theorem 2.3 (recall that $b^0(1) = \varphi^0(1)$, by virtue of Lemma 3.6), from (3.55) we deduce

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}) \geq (1-\delta)^2 \left(\int_I \psi_i(|u'|)\,dx + \sum_{S_u} \varphi_1(u^+ - u^-) + (g^\infty(1) \wedge b^0(1))|D^c u| \right);$$

letting $i \uparrow +\infty$ and $\delta \downarrow 0$ we finally obtain the desired Γ-lim inf inequality. □

Remark 3.15 Looking carefully at the proof we see that the structure assumption *iii)* of Theorem 3.2 can be slightly weakened without changing the result; more precisely it is sufficient to suppose that there exists a family $(g_n^k)_{n,k}$ of positive continuous non-decreasing functions enjoying the following properties:

i) $f_n \geq g_n^k$ for very $n,\ k \in \mathbb{N}$;

ii) for every $k \in \mathbb{N}$ the family $(g_n^k)_n$ satisfies either **st1**) or **st2**);

iii) $g_n^k(t) \to g^k(t)$ for every $t \geq 0$ and $r_n g_n^k(t/r_n) \to b^k(t)$ for every $t > 0$, as $n \to \infty$, with g^k and b^k satisfying
$$g^k \uparrow g \qquad \text{and} \qquad (b^k)^0(1) \uparrow b^0(1) \qquad \text{as } k \to \infty .$$

Indeed call G_n^k the functional associated with g_n^k; then, for every $k \in \mathbb{N}$, by Theorem 3.2 we have
$$\Gamma\text{-}\liminf_{n\to\infty} F_n \geq \Gamma\text{-}\lim_{n\to\infty} G_n^k = F_{b^k,g^k},$$
where $F_{b^k,g^k} \uparrow F_{b,g}$ as $k \to \infty$.

We want now to show that if $g : [0,+\infty) \to [0,+\infty)$ is any superlinear non-decreasing convex function and $b : [0,+\infty) \to [0,+\infty)$ is an arbitrary concave function with $b^0(1) = +\infty$, then $F_{b,g}$ can be reached by functionals of the form (3.1).

Theorem 3.16 *Let $g : [0,+\infty) \to [0,+\infty)$ be non-decreasing, convex, and superlinear ($g^\infty(1) = +\infty$) and let $b : [0,+\infty) \to [0,+\infty)$ be non-decreasing and concave with $b(0) = 0$ and $b^0(1) = +\infty$. Then there exists a family (f_ε) of positive, continuous, and non-decreasing functions such that the functionals*
$$F_\varepsilon := \begin{cases} \displaystyle\int_I f_\varepsilon(|u'|)\,dx + \varepsilon^3 \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I); \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$
Γ-converge with respect to the L^1-metric to $F_{b,g}$, as $\varepsilon \to 0^+$.

The theorem is an immediate consequence of Theorem 3.2 and of the following proposition which is proved in [22] (see Lemmas 6.6 and 6.7).

Proposition 3.17 *Let g and b be as in the previous theorem. Then the functions f_ε defined by*
$$f_\varepsilon(t) := \min\left\{ g(s) + \frac{1}{\varepsilon} b(\varepsilon(t-s)) :\ s \in [0,t] \right\},$$
are continuous, non-decreasing and satisfy the following properties:

i) $f_\varepsilon(t) \to g(t)$ for every $t \geq 0$;

ii) $\varepsilon f(t/\varepsilon) \to b(t)$ for every $t > 0$;

iii) setting $x_\varepsilon := \sup\{x \geq 0 :\ f_\varepsilon(x) = g(x)\}$, there holds that $f_\varepsilon = g$ in $[0, x_\varepsilon]$ and f_ε is concave in $[x_\varepsilon, +\infty)$; moreover $x_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$.

We conclude this subsection with some considerations on the asymptotic behaviour of the function φ defined in (3.3).

Proposition 3.18 *i) Let $b(t) = ct^p$ with $c > 0$ and $p \in [0,1)$. Then $\varphi(z) = m(p) c^{\frac{3}{4-p}} z^{\frac{2+p}{4-p}}$, where*
$$m(p) := \min\Bigg\{ \left[\left(\frac{3}{1-p}\right)^{\frac{1-p}{4-p}} + \left(\frac{1-p}{3}\right)^{\frac{3}{4-p}} \right] \left(\int_0^1 |v''|^2\,dt\right)^{\frac{1-p}{4-p}} \left(\int_0^1 |v'|^p\,dt\right)^{\frac{3}{4-p}} :$$
$$v \in W^{2,2}(0,1),\ v(0) = 0,\ v(1) = 1,\ v'(0) = v'(1) = 0 \Bigg\}. \quad (3.56)$$

ii) Let $b : [0,+\infty) \to [0,+\infty)$ be concave with $b^0(1) \neq 0$. Then the function φ defined in (3.3) satisfies the growth condition

$$C_1(\sqrt{z}-1) \le \varphi(z) \le C_2(z+1) \qquad \forall z \ge 0, \tag{3.57}$$

for suitable C_1, $C_2 > 0$.

iii) For every $\gamma \in [1/2,1)$ there exists a concave function b satisfying the hypotheses of Theorem 3.16 such that the associated φ satisfies

$$\lim_{z\to+\infty} \frac{\varphi(z)}{z^\gamma} = +\infty \qquad \textit{and} \qquad \lim_{z\to+\infty} \frac{\varphi(z)}{z^{\gamma+\varepsilon}} = 0 \qquad \forall \varepsilon > 0. \tag{3.58}$$

PROOF. *i):* For notational convenience we set

$$S_{\eta,z} := \{u \in W^{2,2}(0,\eta) : \ u(0)=0, \ u(\eta)=z, \ u'(0)=u'(\eta)=0\};$$

then, noting that for every $v \in S_{\eta,z}$ we can write $v(\cdot) = w(\cdot/\eta)$ with $w \in S_{1,z}$, we can use the definition of φ to compute

$$\begin{aligned}
\varphi(z) &= \inf_\eta \inf_{v\in S_{\eta,z}} \left(c\int_0^\eta |v'|^p\,dt + \int_0^\eta |v''|^2\,dt \right) \\
&= \inf_{w\in S_{1,z}} \inf_\eta \left(c\int_0^\eta \frac{1}{\eta^p}\left|w'\left(\frac{t}{\eta}\right)\right|^p dt + \int_0^\eta \frac{1}{\eta^4}\left|w''\left(\frac{t}{\eta}\right)\right|^2 dt \right) \\
&= \inf_{w\in S_{1,z}} \inf_\eta \left(c\eta^{1-p}\int_0^1 |w'|^p\,ds + \frac{1}{\eta^3}\int_0^1 |w''|^2\,ds \right) \\
&= \inf_{w\in S_{1,z}} \left\{ \left[\left(\frac{3}{1-p}\right)^{\frac{1-p}{4-p}} + \left(\frac{1-p}{3}\right)^{\frac{3}{4-p}} \right] c^{\frac{3}{4-p}} \left(\int_0^1 |w''|^2\,ds\right)^{\frac{1-p}{4-p}} \left(\int_0^1 |w'|^p\,ds\right)^{\frac{3}{4-p}} \right\} \\
&= \inf_{w\in S_{1,1}} \left\{ \left[\left(\frac{3}{1-p}\right)^{\frac{1-p}{4-p}} + \left(\frac{1-p}{3}\right)^{\frac{3}{4-p}} \right] \left(\int_0^1 |w''|^2\,ds\right)^{\frac{1-p}{4-p}} \left(\int_0^1 |w'|^p\,ds\right)^{\frac{3}{4-p}} \right\} c^{\frac{3}{4-p}} z^{\frac{2+p}{4-p}} \\
&= m(p) c^{\frac{3}{4-p}} z^{\frac{2+p}{4-p}}.
\end{aligned}$$

It is clear also from the computations above that there exists an optimal pair (η, v) for the problem defining φ: let v be a solution of problem (3.56), then, setting

$$\eta := \left[\frac{3}{c^p(1-p)}\right]^{\frac{1}{4-p}} \left(\frac{\int_0^1 |v''|^2\,ds}{\int_0^1 |v'|^p\,ds}\right)^{\frac{1}{4-p}} z^{\frac{2+p}{4-p}} \qquad \text{and} \qquad w(t) := zv\left(\frac{t}{\eta}\right), \tag{3.59}$$

we have that (η, w) is an optimal pair.

ii): Under our assumptions there exists $C>0$ such that $b(t) \le C(1+t)$ for every $t \ge 0$. Take (η, v) such that $v \in S_{\eta,z}$, v is non-decreasing and

$$C\eta + \int_0^\eta |v''|^2\,dx = m(0)C^{3/4}\sqrt{z} :$$

this is possible thanks to the previous point. Then

$$\varphi(z) \le C\int_0^\eta |v'|\,dx + C\eta + \int_0^\eta |v''|^2\,dx = Cz + m(0)C^{3/4}\sqrt{z} \le C'(1+z).$$

Concerning the other inequality, since, under our hypotheses, there exist α, $\beta > 0$ such that $b(t) \ge \alpha t \wedge \beta$, it will be enough to prove the following claim.

Claim. Let $b(t) = \alpha t \wedge \beta$ with $\alpha,\ \beta > 0$. Then

$$\lim_{z\to+\infty} \frac{\varphi(z)}{m(0)\beta^{3/4}\sqrt{z}} = 1.$$

First of all, since $b(t) \le \beta$, by comparison and by the previous point we immediately obtain

$$\varphi(z) \le m(0)\beta^{3/4}\sqrt{z}. \tag{3.60}$$

Let $z_n \uparrow +\infty$ and let (η_n, z_n) be an admissible pair for $\varphi(z_n)$ such that v_n is non-decreasing and

$$\int_0^{\eta_n} (\alpha|v_n'| \wedge \beta)\,dx + \int_0^{\eta_n} |v_n''|^2\,dx < \varphi(z_n) + 1. \tag{3.61}$$

Let $\sigma_n \in (0,1)$ be such that $\int_{\{x\in I:\,|v_n'|\le\beta/\alpha\}} |v_n'|\,dx = \sigma_n z_n$; since, by (3.60) and (3.61),

$$m(0))\beta^{3/4}\sqrt{z_n} + 1 \ge \int_0^{\eta_n} (\alpha|v_n'| \wedge \beta)\,dx \ge \int_{\{x\in I:\,|v_n'|\le\beta/\alpha\}} \alpha|v_n'|\,dx = \alpha\sigma_n z_n,$$

it follows that $\sigma_n \to 0$. Consider the sets $D_n := \{x \in I : |v_n'| > \beta/\alpha\} = \cup_{k=1}^{\infty} I_n^k$, where $(I_n^k)_k$ is the collection of the connected components of D_n. We denote also $I_n^k := (a_n^k, b_n^k)$. Let $\Phi \in C^2([0,1])$ be such that $\Phi(0) = \Phi'(0) = 0$, $\Phi(1) = 1$, and $\Phi'(1) = \beta/\alpha$ and, for every $t \in [1, 1 + |D_n|]$ set

$$i_n(t) := \min\left\{ k : \sum_{j=1}^{k} |I_n^j| \ge t - 1 \right\} \qquad \tau_n(t) := t - 1 - \sum_{j=1}^{i_n(t)-1} |I_n^j|.$$

We can now define the new sequence of admissible pairs $(\tilde\eta_n, \tilde v_n)$ by $\tilde\eta_n := |D_n| + 2$ and $\tilde v_n(t) = \int_0^t \tilde v_n'(s)\,ds$, where

$$\tilde v_n' := \begin{cases} \Phi'(s) & \text{if } s \in [0,1], \\ v_n'(a_n^{i_n(s)} + \tau_n(s)) & \text{if } s \in [1, \tilde\eta_n - 1], \\ \Phi'(\tilde\eta_n - s) & \text{if } s \in [\tilde\eta_n - 1, \tilde\eta_n]. \end{cases}$$

Note that $\tilde v_n$ is constructed by gluing together the pieces of v_n defined on the sets I_n^k; since $v_n'(a_n^k) = v'(b_n^k) = \beta/\alpha$ for every k, we have $\tilde v_n \in W^{2,2}(0, \tilde\eta_n)$. Therefore $\tilde v_n \in S_{\tilde\eta_n, \tilde z_n}$ with $\tilde z_n := (1-\sigma_n)z_n + 2$ and, since by construction

$$\int_{D_n} (\alpha|v_n'| \wedge \beta)\,dx + \int_{D_n} |v_n''|^2\,dx = \beta\tilde\eta_n + \int_0^{\tilde\eta_n} |\tilde v_n''|^2\,dx - 2\left(\int_0^1 |\Phi'|\,dx + \int_0^1 |\Phi''|^2\,dx\right),$$

recalling (3.61) and *i)* we can estimate

$$\begin{aligned}
\varphi(z_n) + 1 &\ge \beta\tilde\eta_n + \int_0^{\tilde\eta_n} |\tilde v_n''|^2\,dx - 2\left(\int_0^1 |\Phi'|\,dx + \int_0^1 |\Phi''|^2\,dx\right) \\
&\ge \inf_{\eta>0} \inf_{S_{\eta,\tilde z_n}} \left(\beta\eta + \int_0^{\eta} |\tilde v''|^2\,dx\right) - 2\left(\int_0^1 |\Phi'|\,dx + \int_0^1 |\Phi''|^2\,dx\right) \\
&= m(0)\beta^{3/4}\sqrt{(1-\sigma_n)z_n + 2} - 2\left(\int_0^1 |\Phi'|\,dx + \int_0^1 |\Phi''|^2\,dx\right),
\end{aligned}$$

whence, taking into account that $\sigma_n \to 0$,

$$\liminf_{z\to+\infty} \frac{\varphi(z)}{m(0)\beta^{3/4}\sqrt{z}} \ge 1,$$

which combined with (3.60), gives the thesis of the claim.

iii): For simplicity we treat in details only the case $\gamma = 1/2$. We take $b(t) := 1 + \log(1+t)$ for $t > 0$ and $b(0) = 0$. Fix $p \in (0,1)$ and take (η, w) with $w \in S_{\eta,z}$ and satisfying

$$\int_0^\eta |w'|^p \, dx + \int_0^\eta |w''|^2 \, dx = m(p) z^{\frac{2-p}{4-p}} \quad \text{and} \quad \eta \leq c(p) z^{\frac{2-p}{4-p}} :$$

this is possible by virtue of *i)* (see (3.59)). Then, since $b(t) \leq 1 + t^p$ we have

$$\varphi(z) \leq (m(p) + c(p)) z^{\frac{2-p}{4-p}}; \tag{3.62}$$

since as p varies in $(0,1)$ the exponent $(2-p)/(4-p)$ varies in $(1/2, 1)$, from (3.62) we deduce that

$$\lim_{z \to +\infty} \frac{\varphi(z)}{z^{(1/2)+\varepsilon}} = 0 \qquad \forall \varepsilon > 0.$$

Now take two positive sequences (α_n) and (β_n) with $\beta_n \to +\infty$ such that $b(t) \geq b_n(t) := \alpha_n t \wedge \beta_n$, for every $t \geq 0$ and for every $n \in \mathbb{N}$. Calling φ_n the function associated with b_n, by the claim proved above, we have

$$\liminf_{z \to +\infty} \frac{\varphi(z)}{\sqrt{z}} \geq \lim_{z \to +\infty} \frac{\varphi_n(z)}{\sqrt{z}} = m(0) \beta_n^{3/4},$$

for every $n \in \mathbb{N}$; letting $n \to \infty$ we eventually complete the proof of (3.58). If γ is any number in $(1/2, 1)$, take $b(t) = t^p \log(1+t)$, where p is such that $\gamma = (2-p)/(4-p)$, and argue as above. □

4 Some applications

In this subsection we are going to apply the results of the previous one to study the singular perturbations of the one-dimensional functionals of the form

$$G_\varepsilon(u) = \frac{1}{\varepsilon} \int_I f(\varepsilon^{1/q} |u'|) \, dx,$$

where $q \geq 1$. More precisely, given a positive function $p(\varepsilon)$ such that $\lim_{\varepsilon \to 0^+} p(\varepsilon) = 0$, we set

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_I f(\varepsilon^{1/q} |u'|) \, dx + (p(\varepsilon))^3 \int_I |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases} \tag{4.1}$$

and we aim to classify all the possible Γ-limits generated by the family (F_ε) depending on the asymptotic behaviour of the "rescaling" function p. Let us begin with the case $q > 1$. Let $f : [0, +\infty) \to [0, +\infty)$ be non-decreasing, continuous, and satisfying the following properties:

H1) f is concave in $(x_1, +\infty)$ for some $x_1 > 0$;

H2) $\displaystyle\lim_{x \to 0^+} \frac{f(x)}{x^q} = \alpha > 0$, with $q > 1$;

H3) $\displaystyle\lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

We will show that there exists a unique (up to asymptotic equivalence) rescaling function $r(\varepsilon)$ which generates non-trivial (i.e. non-zero) "free-discontinuity" functionals. Setting $h(x) := f(x)/x$, such a rescaling function is defined as:

$$r(\varepsilon) := \frac{\varepsilon^{1/q}}{h^{-1}\left(\varepsilon^{1/q'}\right)}, \tag{4.2}$$

where q is the exponent appearing in *H2)* while q' denotes its Lebesgue conjugate exponent satisfying $1/q + 1/q' = 1$.

Remark 4.1 Note that, for ε small enough, r is well defined, indeed, being f concave at infinity and sublinear, h becomes decreasing for x large enough. Moreover

$$\lim_{n\to\infty}\frac{\sqrt{\varepsilon}_n}{r(\varepsilon_n)}=\lim_{n\to\infty}h^{-1}(\sqrt{\varepsilon_n})=+\infty$$

since $h\downarrow 0$ as $x\to+\infty$.

Our main result is the following theorem.

Theorem 4.2 *Let $I\subset\mathbb{R}$ be a bounded interval and let $f:[0,+\infty)\to[0,+\infty)$ be a non-decreasing continuous function satisfying hypotheses H1), H2), and H3) and $p(\varepsilon)$ be a positive function such that $\lim_{\varepsilon\to0^+}p(\varepsilon)=0$. Finally let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an infinitesimal sequence such that*

$$\lim_{n\to\infty}\frac{p(\varepsilon_n)}{r(\varepsilon_n)}=a>0\qquad\textit{and}\qquad\exists\lim_{n\to+\infty}\frac{f\left(t\frac{{\varepsilon_n}^{1/q}}{r(\varepsilon_n)}\right)}{f\left(\frac{{\varepsilon_n}^{1/q}}{r(\varepsilon_n)}\right)}=:b(t)\qquad\forall t>0. \tag{4.3}$$

If $b^0(1)=+\infty$ then the functionals F_{ε_n} (defined in (4.1)) Γ-converge with respect to the L^1-metric to

$$F(u):=\begin{cases}\alpha\displaystyle\int_I|u'|^q\,dx+\sum_{x\in S_u}\varphi^{(a)}(u^+(x)-u^-(x)) & \textit{if } u\in SBV(I),\\ +\infty & \textit{otherwise in } L^1(I);\end{cases} \tag{4.4}$$

where $\varphi^{(a)}$ is defined by (3.3) with $b^{(a)}(t):=ab(t/a)$ instead of $b(t)$.

Conversely, if $b^0(1)=C<+\infty$, then $\Gamma\text{-}\lim_{n\to\infty}F_{\varepsilon_n}=F$ with F given by

$$F(u):=\begin{cases}\displaystyle\int_I g(|u'|)\,dx+\sum_{x\in S_u}\varphi^{(a)}(u^+(x)-u^-(x))+C|D^cu| & \textit{if } u\in BV(I),\\ \\ +\infty & \textit{if } x\in L^1(I)\setminus BV(I),\end{cases} \tag{4.5}$$

where $g:=(\alpha x^q\wedge Cx)^{**}$ while $\varphi^{(a)}$ is as above. Moreover, in both cases, every sequence u_n such that $\sup_n(F_n(u_n)+\|u_n\|_1)<+\infty$ is strongly precompact in L^p for every $p\geq 1$.

An easy consequence of the theorem is the fact that, up to asymptotic equivalence, the function r defined in (4.2) is the unique nontrivial rescaling function; this is made precise by the following Corollary whose easy proof is left to the reader (see [2]).

Corollary 4.3 *Let I, f, and r be as in Theorem 3.2. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ be two sequences converging to 0 and, for every n, set*

$$F_n(u)=\begin{cases}\displaystyle\frac{1}{\varepsilon_n}\int_I f({\varepsilon_n}^{1/q}|u'|)\,dx+(a_n)^3\int_I|u''|^2\,dx & \textit{if } u\in W^{2,2}(I),\\ +\infty & \textit{otherwise in } L^1(I).\end{cases}$$

If $\lim_{n\to\infty}a_n/r(\varepsilon_n)=0$, then $\Gamma\text{-}\lim_{n\to\infty}F_n=0$ with respect to the L^1-metric; if $\lim_{n\to\infty}a_n/r(\varepsilon_n)=+\infty$, then the functionals F_n Γ-converge to

$$F(u):=\begin{cases}\alpha\displaystyle\int_I|u'|^q\,dx & \textit{if } u\in W^{1,q}(I),\\ +\infty & \textit{otherwise in } L^1(I).\end{cases}$$

Remark 4.4 Since f is concave for x large, from the previous remark and from the definition (4.3), it follows that $b(\cdot)$ is in turn concave. Moreover, again by the sublinearity and the concavity assumption we get the existence of $x_2 \geq x_1$ such that

$$f(a+b) \leq f(a) + f(b) \qquad \forall a, b > x_2; \tag{4.6}$$

if f is unbounded, we deduce

$$\begin{aligned} b(t) &\leq \limsup_{x\to+\infty} \frac{f(tx)}{f(x)} \leq \limsup_{x\to+\infty} \frac{f(([t]+1)x)}{f(x)} \\ &\leq \limsup_{x\to+\infty} \frac{([t]+1)f(x)}{f(x)} = [t]+1, \end{aligned} \tag{4.7}$$

where $[t]$ denotes the integer part of t; if f is bounded we get trivially $b(t) \equiv 1$. Finally, since $b(1) = 1$, taking into account the concavity it turns out that $b(t) > 0$ for any $t > 0$.

Proof of Theorem 4.2. Setting $r_n := p(\varepsilon_n)$ and $f_n(t) := \frac{1}{\varepsilon_n} f\left(\varepsilon_n^{1/q} t\right)$, by *H2)* we get immediately that $f_n(t) \to \alpha t^q$ for every $t \geq 0$; moreover, using the identity $f\left(\frac{\varepsilon_n^{1/q}}{r(\varepsilon_n)}\right) = \frac{r(\varepsilon_n)}{\varepsilon_n}$, which follows easily from the definition of r (see (4.2)), for $t > 0$ it turns out

$$r_n f_n\left(\frac{t}{r_n}\right) = \frac{p(\varepsilon_n)}{r(\varepsilon_n)} \frac{r(\varepsilon_n)}{\varepsilon_n} f\left(\frac{\varepsilon_n^{1/q}}{r(\varepsilon_n)} \frac{r(\varepsilon_n)}{p(\varepsilon_n)} t\right) = \frac{p(\varepsilon_n)}{r(\varepsilon_n)} \frac{f\left(\frac{\varepsilon_n{}^{1/q}}{r(\varepsilon_n)} \frac{r(\varepsilon_n)}{p(\varepsilon_n)} t\right)}{f\left(\frac{\varepsilon_n{}^{1/q}}{r(\varepsilon_n)}\right)} \overset{n\to\infty}{\longrightarrow} ab\left(\frac{t}{a}\right) = b^{(a)}(t), \tag{4.8}$$

where we used (4.3). By the first part of Theorem 3.2 we therefore obtain the Γ-lim sup inequality. Concerning the other inequality, by Theorem 3.2 and Remark 3.15, it will be proved if for every $\delta > 0$ we are able to construct a family of functions (f_n^δ) such that $f_n \geq f_n^\delta$, f_n^δ satisfies the structure condition **st2)** and finally

$$f_n^\delta(t) \to (1-\delta)\alpha t^q \ \forall t \geq 0 \qquad \text{and} \qquad r_n f_n^\delta\left(\frac{t}{r_n}\right) \to b^{(a)}(t) \ \forall t > 0. \tag{4.9}$$

It is also clear that if we exhibit a function f^δ verifying

a) $f \geq f^\delta$,

b) $\displaystyle\lim_{t\to0^+} \frac{f^\delta(t)}{t^q} = (1-\delta)\alpha$,

c) $\displaystyle\lim_{t\to+\infty} \frac{f^\delta(t)}{f(t)} = 1$,

d) there exists $\overline{x}$ such that f^δ is convex in $[0, \overline{x}]$ and concave in $[\overline{x}, +\infty)$,

then the family $f_n^\delta(t) := \frac{1}{\varepsilon_n} f^\delta(\varepsilon_n^{1/q} t)$ enjoys all the required conditions. Therefore it remains only to construct such a f^δ. By assumption we know that there exist $x' < x''$ such that $f(t) \geq (1-\delta)\alpha t^q$ for every $t \in [0, x']$ and f is concave in $[x'', +\infty)$. Define $a(t) := (1-\delta)\alpha \frac{(x')^q}{x''} t$, $g := [\min\{(1-\delta)\alpha t^q, a(t)\}]^{**}$, and finally

$$f^\delta(t) := \begin{cases} g(t) & \text{if } t \leq x'', \\ f(t) + g(x'') - f(x'') & \text{if } t \geq x''; \end{cases}$$

it is easy to see that f^δ satisfies all conditions a),..., d) (see Figure 2). Finally the equicoerciveness of the family (F_{ε_n}) follows again from Theorem 3.2. $\square$

An easy consequence of Theorem 4.2 is the following compactness result.

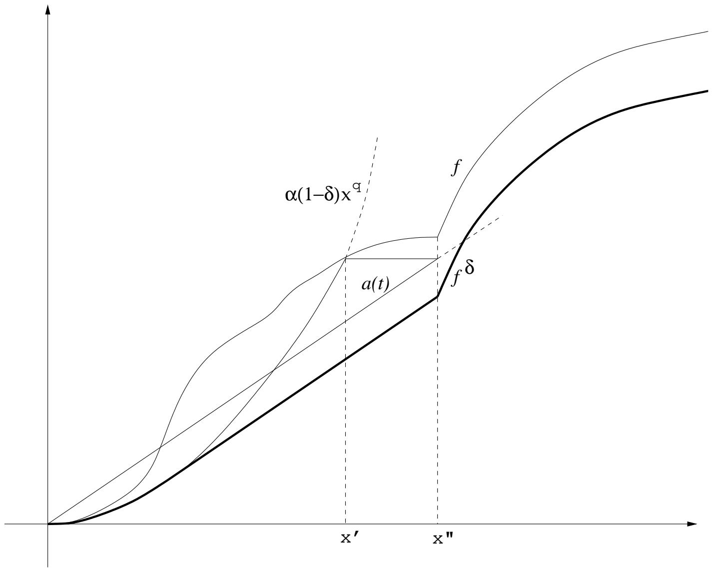

Figure 2: The construction of f^δ.

Theorem 4.5 *Let I, f, and r as above and consider the family of functionals F_ε defined in* (4.1) *with $p(\varepsilon)$ satisfying $0 < \liminf_{\varepsilon\to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} \leq \limsup_{\varepsilon\to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} < +\infty$. Then, for every infinitesimal sequence $(\varepsilon_n)_n$, there exist a subsequence, still denoted by $(\varepsilon_n)_n$ and a concave, non-decreasing function b such that $\exists\Gamma\text{-}\lim_n F_{\varepsilon_n} = F$ with respect to the L^1-convergence, where F is either as in* (4.4) *or as in* (4.5).

Proof. It is sufficient to extract a subsequence such that (4.3) holds and then to apply Theorem 4.2. The existence of such a subsequence is an easy consequence of Helly's Theorem. □

Proposition 4.6 *Let f be a function satisfying the hypotheses H1), H2), and H3) of Theorem 4.2 and let us suppose in addition that*

$$\exists \lim_{x\to+\infty} \frac{f(tx)}{f(x)} := b(t) \qquad \forall t > 0. \tag{4.10}$$

Let F_ε the functional defined in (4.1), *with p satisfying $\lim_{\varepsilon\to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} = a > 0$, where r is the rescaling function defined in* (4.2). *If $b^0(1) = +\infty$, then $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = F$ with respect to the L^1-metric, with F given by*

$$F(u) := \begin{cases} \displaystyle\int_I |u'|^q\,dx + m(\gamma)a^{\frac{3(1-\gamma)}{4-\gamma}} \sum_{x\in S_u} (u^+ - u^-)^{\frac{2+\gamma}{4-\gamma}} & \text{if } u \in SBV(I) \\ +\infty & \text{in } L^1(I)\setminus SBV(I), \end{cases}$$

where $\gamma = \log b(\mathrm{e})$ and $m(\gamma)$ is the constant defined in (3.56). *If $b^0(1) < +\infty$ the family F_ε Γ-converges to the functional F given by*

$$F(u) := \begin{cases} \displaystyle\int_I g_\alpha(|u'|)\,dx + |D^s u| & \text{if } u \in BV(I) \\ +\infty & \text{in } L^1(I)\setminus BV(I), \end{cases}$$

where $g_\alpha = (\alpha x^q \wedge x)^{**}$.

PROOF. From Theorem 4.2, it is clear that $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = F$ where F is the functional defined either in (4.4) or in (4.5). It remains only to prove that

$$\varphi^{(a)}(z) = m(\gamma) a^{\frac{3(1-\gamma)}{4-\gamma}} z^{\frac{2+\gamma}{4-\gamma}} \qquad \forall z > 0, \tag{4.11}$$

if (4.3) hold true, or

$$\varphi^{(a)}(z) = z \qquad \forall z > 0, \tag{4.12}$$

otherwise. First of all note that from (4.10) it follows immediately that $b(st) = b(s)b(t)$ for $t,\ s > 0$ and therefore $b(t) = t^\gamma$ for $t > 0$, with $\gamma = \log b(\mathrm{e})$; by remark 4.4 (and in particular by (4.7)), we have that $\gamma \in [0,1]$. If $\gamma < 1$, (4.11) follows from Proposition 3.18 since $b^{(a)}(t)ab(t/a) = a^{1-\gamma}t^\gamma$. If $\gamma = 1$, then by Lemma 3.5, we get (4.12). □

Let us see now some examples. We will use the following notation: given two functions r_1 and r_2 we will write $r_1 \simeq r_2$ if they are asymptotically equivalent, that is if $\lim_{\varepsilon\to 0^+} \frac{r_1(\varepsilon)}{r_2(\varepsilon)} = 1$.

Example 4.7 Let γ belong to $[0,1)$ and set $f(x) := \frac{\alpha x^2}{1+x^{2-\gamma}}$; using the definitions (see (4.2) and (4.3)), it is easy to see that $r(\varepsilon) \simeq \varepsilon^{\frac{2-\gamma}{2-2\gamma}}$ and $b(t) = t^\gamma$, and therefore, setting

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_I \frac{\alpha |u'|^2}{1+\varepsilon^{\frac{2-\gamma}{2}}|u'|^{2-\gamma}}\,dx + a^3 \varepsilon^{\frac{6-3\gamma}{2-2\gamma}} \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

by Proposition 4.6, we have that the functionals F_ε Γ-converge to

$$F^\gamma(u) := \begin{cases} \displaystyle\alpha \int_I |u'|^2\,dx + m(\gamma) a^{\frac{3(1-\gamma)}{4-\gamma}} \sum_{x\in S_u} (u^+ - u^-)^{\frac{2+\gamma}{4-\gamma}} & \text{if } u \in SBV(I) \\ +\infty & \text{in } L^1(I)\setminus SBV(I), \end{cases} \tag{4.13}$$

as $\varepsilon \to 0^+$, with respect to the L^1-metric, so that we recover the result of Dubs, Bouchitté & Seppecher (see [9]). Note that as γ varies in $[0,1)$, the exponent $\frac{2+\gamma}{4-\gamma}$ varies in $\left[\frac{1}{2},1\right)$. Moreover, note that $m(0)$ can be easily computed and it is equal to $2\sqrt{3/2}+\sqrt{2/3}$ (see [2]).

Example 4.8 Let $f(x) := (1+x^\gamma)\log(1+\alpha x^2)$ with $\gamma \in [0,1)$. We show now that

$$r(\varepsilon) \simeq (1-\gamma)^{\frac{1}{1-\gamma}} \frac{\varepsilon^{\frac{2-\gamma}{2-2\gamma}}}{\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{1-\gamma}}}. \tag{4.14}$$

Indeed, with the same notations of Theorem 3.2, we have

$$\lim_{\varepsilon\to 0^+} r(\varepsilon)\frac{\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{2-\gamma}{2-2\gamma}}} = \lim_{\varepsilon\to 0^+} \frac{\sqrt{\varepsilon}}{h^{-1}(\sqrt{\varepsilon})} \frac{\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{2-\gamma}{2-2\gamma}}} = \lim_{y\to+\infty} \frac{\left(\log\frac{1}{h^2}\right)^{\frac{1}{1-\gamma}}}{y h^{\frac{1}{1-\gamma}}} = (1-\gamma)^{\frac{1}{1-\gamma}},$$

where we performed the change of variable $y = h^{-1}(\sqrt{\varepsilon})$.

We finally observe that $b(t) = \lim_{x\to+\infty} \frac{f(tx)}{f(x)} = t^\gamma$ for all $t > 0$; therefore, setting

$$F_\varepsilon^\gamma(u) := \begin{cases} \displaystyle\frac{1}{\varepsilon}\int_I (1+\varepsilon^{\frac{\gamma}{2}}|u'|^\gamma)\log(1+\varepsilon\alpha|u'|^2)\,dx & \\ \displaystyle\qquad + \left(a^{1-\gamma}(1-\gamma)\frac{\varepsilon^{\frac{2-\gamma}{2}}}{\log\frac{1}{\varepsilon}}\right)^{\frac{3}{1-\gamma}} \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

by (4.14) and by Proposition 4.6 we obtain that the sequence F_ε^γ Γ-converges, as $\varepsilon \to 0^+$, to the functional F^γ defined in (4.13). In particular, taking $\gamma = 0$, we prove that the singular perturbations of the rescaled Perona-Malik functionals

$$F_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I \log(1+\varepsilon\alpha|u'|^2)\,dx + \left(\dfrac{a\varepsilon}{\log\frac{1}{\varepsilon}}\right)^3 \displaystyle\int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

Γ-converge to F^0, as announced in the Introduction.

Remark 4.9 Let f_1 and f_2 be two functions satisfying the hypotheses of Theorem 3.2 and let r_1 and r_2 be the rescaling functions associated with f_1 and f_2 respectively according to (4.2) and, for $\varepsilon > 0$ and $i = 1, 2$ denote by $F_{i,\varepsilon}$ the functional

$$F_{i,\varepsilon}(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f_i(\sqrt{\varepsilon}|u'|)\,dx + (r_i(\varepsilon))^3 \displaystyle\int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

Suppose in addition that

$$\lim_{x\to+\infty} \frac{f_1(x)\log^\gamma(x)}{f_2(x)} = 1; \tag{4.15}$$

then, for every infinitesimal sequence $(\varepsilon_n)_n$, $\Gamma\text{-}\lim_{n\to\infty} F_{1,\varepsilon_n} = F \iff \Gamma\text{-}\lim_{n\to\infty} F_{2,\varepsilon_n} = F$; in other words, functions asymptotically differing by a logarithmic factor generate the same Γ-limits. To prove this fact we pass to a subsequence such that

$$\exists \lim_{n\to+\infty} \frac{f_1\left(t\frac{\sqrt{\varepsilon_n}}{r_1(\varepsilon_n)}\right)}{f_1\left(\frac{\sqrt{\varepsilon_n}}{r_1(\varepsilon_n)}\right)} =: b_1(t) \quad \forall t > 0 \qquad \text{and} \qquad \exists \lim_{n\to+\infty} \frac{f_2\left(t\frac{\sqrt{\varepsilon_n}}{r_2(\varepsilon_n)}\right)}{f_2\left(\frac{\sqrt{\varepsilon_n}}{r_2(\varepsilon_n)}\right)} =: b_2(t) \qquad \forall t > 0,$$

and we observe that, by virtue of (4.15), we have $b_1 \equiv b_2$; we conclude by applying Theorem 4.2. Note that the results of Example 4.8 can be derived from Examples 4.7, using the present remark.

Example 4.10 Let $f_\alpha(x) := \frac{\alpha x^2}{(1+x)\log(e+x)}$; then, by easy computations, the rescaling function r defined in (4.2) satisfies $r(\varepsilon) \simeq \frac{\sqrt{\varepsilon}}{e^{\frac{1}{\sqrt{\varepsilon}}}}$. Moreover $b(t) = \lim_{x\to+\infty} \frac{f(tx)}{f(x)} = t$ for every $t > 0$; therefore, setting

$$F_{\alpha,\varepsilon}(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f_\alpha(\sqrt{\varepsilon}|u'|)\,dx + \left(\dfrac{\sqrt{\varepsilon}}{e^{\frac{1}{\sqrt{\varepsilon}}}}\right)^3 \displaystyle\int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

by Proposition 4.6 we obtain that the family F_ε Γ-converges, with respect to L^1-metric, to

$$F_\alpha(u) := \begin{cases} \displaystyle\int_I g_\alpha(|u'|)\,dx + |D^s u| & \text{if } u \in BV(I) \\ +\infty & \text{in } L^1(I) \setminus BV(I), \end{cases}$$

where $g_\alpha = (\alpha x^2 \wedge x)^{**}$. Note that $g_\alpha(x) \uparrow x$ as $\alpha \to +\infty$, and therefore

$$\Gamma\text{-}\lim_{\alpha\to+\infty}(\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_{\alpha,\varepsilon}) = \Gamma\text{-}\lim_{\alpha\to+\infty} F_\alpha = G,$$

with G given by

$$G(u) := \begin{cases} |Du| & \text{if } u \in BV(I), \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

Remark 4.11 The hypothesis *H3)* is in some sense necessary; indeed suppose that f is an increasing function satisfying *H1)*, *H2)*, and $\lim_{x\to+\infty}\frac{f(x)}{x}=C>0$. Then it is easy to see that the functionals

$$G_\varepsilon(u):=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_I f(\sqrt{\varepsilon}|u'|)\,dx, & \text{if } u\in C^1(I),\\ +\infty & \text{otherwise in } L^1(I),\end{cases}$$

Γ-converge in the L^1-topology, to the functional

$$G(u):=\begin{cases}\alpha\displaystyle\int_I |u'|^q dx, & \text{if } u\in W^{1,q}(I),\\ +\infty & \text{otherwise in } L^1(I),\end{cases}$$

as $\varepsilon\to0^+$. We leave the details to the reader.

Let us see what happens when in (4.1) the function f has a finite strictly positive derivative at the origin so that $q=1$.

Theorem 4.12 *Let $f:[0,+\infty)\to[0,+\infty)$ be continuous, non-decreasing , differentiable in 0 with $f'(0)>0$, and concave in $(x_1,+\infty)$ for a suitable $x_1>0$. Then the family*

$$F_\varepsilon:=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|u'|)\,dx+\varepsilon^3\int_I|u''|^2\,dx & \text{if } u\in W^{2,2}(I),\\ +\infty & \text{otherwise in } L^1(I),\end{cases}$$

Γ-converges with respect to the L^1-norm to the functional

$$F(u):=\begin{cases}f'(0)\displaystyle\int_I|u'|\,dx+\sum_{S_u}\varphi(u^+-u^-)+f'(0)|D^cu| & \text{if } u\in BV(I),\\ +\infty & \text{otherwise in } L^1(I),\end{cases}$$

where φ is the function defined in (3.3) with $b=f$. Moreover every sequence u_ε such that $\sup_\varepsilon(F_\varepsilon(u_\varepsilon)+\|u_\varepsilon\|_1)<+\infty$ is strongly precompact in L^p for every $p\geq1$.

PROOF. Take an infinitesimal sequence (ε_n) and consider the family of functions $f_n:=(1/\varepsilon_n)f(\varepsilon_n\cdot)$: we clearly have that $f_n(t)\to f'(0)t$ for every $t>0$ and $\varepsilon_nf_n(t/\varepsilon_n)=f(t)$ for every $t\geq0$ and every $n\in\mathbb{N}$, so that (f_n) verifies (3.5) and (3.6) with $g=f'(0)t$ and $b=f$. Now construct a sequence of functions (f^k) such that $f\geq f^k$ for every k, $(f^k)'(0)\uparrow f'(0)$ as $k\to\infty$, and f^k is linear in $[0,y_k]$ and concave in $[y_k,+\infty)$ for a suitable $y_k>0$ (it is clear that under our assumptions such a construction is possible); then, setting $f^k_n(t):=(1/\varepsilon_n)f^k(\varepsilon_nt)$, we have that the family $(f^k_n)_{n,k}$ satisfies the weaker structure assumption introduced in Remark 3.15. At this point we can conclude by applying Theorem 3.2. □

The following example is in the spirit of Theorem 3.17.

Example 4.13 Given a convex non-decreasing positive function g and a concave positive function b satisfying $b^0(1)=g^\infty(1)=C\in(0,+\infty)$, we have that the family

$$F_\varepsilon:=\begin{cases}\displaystyle\int_I\left[g(|u'|)\wedge\left(\frac{1}{\varepsilon}b(\varepsilon|u'|)+g(0)\right)\right]\,dx+\varepsilon^3\int_I|u''|^2\,dx & \text{if } u\in W^{2,2}(I),\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converges to the functional $F_{b,g}$ defined in (3.4) which, under our assumptions, takes the form

$$\int_I g(|u'|)\,dx+\sum_{S_u}\varphi(u^+-u^-)+C|D^cu| \qquad u\in BV(I),$$

where φ is the function associated with b according to (3.3). It is enough to apply Theorem 3.2 to the family $f_\varepsilon := g(t) \wedge \left(\frac{1}{\varepsilon}b(\varepsilon t) + g(0)\right)$, after noting that

$$f_\varepsilon(t) \to g(t) \wedge (b^0(t) + g(0)) = g(t) \qquad \text{and} \qquad \varepsilon f_\varepsilon\left(\frac{t}{\varepsilon}\right) \to b(t) \wedge g^\infty(t) = b(t).$$

5 The N-dimensional case

In the section we aim to extend the results of the previous ones to the N-dimensional case. Let us fix first some notations: for $u \in W^{2,2}(\Omega)$, we denote its hessian matrix by $\nabla^2 u$ and, given a square matrix A we consider the norm defined by

$$\|A\| := \sup_{|\xi|=1} A\,\xi \cdot \xi\,.$$

It is convenient to introduce the following definition.

Definition 5.1 *Given $X \subseteq L^1(\Omega)$ we say that the sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$* steadily Γ-converges in X *to $F : X \to \mathbb{R} \cup \{+\infty\}$ (and we will write $\Gamma^s\text{-}\lim_{n\to\infty} F_n = F$ or, shortly, $F_n \stackrel{\Gamma^s}{\to} F$) if, for every $p \geq 1$, $F_n|_{X\cap L^p(\Omega)}$ Γ-converges to $F|_{X\cap L^p(\Omega)}$ with respect to the L^p-convergence. Equivalently we have that $\Gamma^s\text{-}\lim_{n\to\infty} F_n = F$ if and only if the two following conditions are satisfied:*

i) for every $(u_n)_n \subset X$ such that $u_n \to u \in X$ in L^1, we have

$$\liminf_{n\to\infty} F_n(u_n) \geq F(u);$$

ii) for every $u \in X \cap L^p(\Omega)$, there exists a sequence $(u_n)_n \subset X \cap L^p(\Omega)$ such that

$$u_n \to u \text{ in } L^p \qquad \text{and} \qquad \limsup_{n\to\infty} F_n(u_n) \leq F(u).$$

We will also say that G is the steady relaxed functional *of F if G is the Γ^s-limit of the constant sequence $F_n = F$.*

We underline that, thanks to Remark 3.9, in the one dimensional case we have in fact proved the steady Γ-convergence in the whole $L^1(I)$ of the functionals F_n. The main result of the section is the following theorem.

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let f_n, r_n satisfy hypotheses i), ii), and iii) of Theorem 3.2. For every $n \in \mathbb{N}$, consider the following N-dimensional version of the functional F_n defined in (3.1):*

$$F_n^N(u) = \begin{cases} \displaystyle\int_\Omega f_n(|\nabla u|)\,dx + (r_n)^3 \int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{5.1}$$

Then

$$\Gamma\text{-}\liminf_{n\to\infty} F_n^N \geq F_{b,g}^N, \tag{5.2}$$

with respect to the L^1-convergence, where $F_{b,g}^N$ is the N-dimensional version of $F_{b,g}$ given by

$$F_{b,g}^N(u) := \begin{cases} \displaystyle\int_\Omega g_1(|\nabla u|)\,dx + \int_{S_u} \varphi_1(u^+(x) - u^-(x))\,d\mathcal{H}^{N-1} + (g^\infty(1) \wedge b^0(1))|D^c u| & \text{if } u \in GBV(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Suppose now that, for every n, f_n satisfies the following additional growth conditions:

gr1) *there exists* C, $C_0 > 0$ *and* $q \geq 1$ *such that* $f_n(t) \leq C(1+t^q)$ *and* $C_0 t^q \leq g(t) \leq C(1+t^q)$ *for every* $t \geq 0$,

gr2) *for every* $\alpha > 0$ *there exists* $c(\alpha) > 0$ *such that* $f_n(\alpha t) \leq c(\alpha) f_n(t)$ *for every* $t \geq 0$,

where C, C_0, q, *and* $c(\alpha)$ *are independent of* n. *Then the sequence* F_n^N Γ^s*-converges in* $GSBV^q$ *to* $F_{b,g}^N$. *Moreover if* $g^\infty(1) \wedge b^0(1) < +\infty$, *actually we have that* $\Gamma^s\text{-}\lim_{n\to\infty} F_n^N = F_{b,g}^N$ *in the whole* $L^1(\Omega)$.

Remark 5.3 Note that, for technical reasons, when $g^\infty(1) \wedge b^0(1) = +\infty$ in the N-dimensional case we are able to represent the Γ-limit only on $GSBV^q(\Omega)$; to complete the result one would have to show that every $u \in GSBV(\Omega)$ can be approximated in L^1 by a sequence $u_j \in GSBV^q(\Omega)$ such that $F_{b,g}^N(u_j) \to F_{b,g}^N(u)$ (see the introduction).

Remark 5.4 Note that if g satisfies both **gr1)** and **gr2)**, also the family f_ε constructed in Proposition 3.17 verifies the same growth conditions.

PROOF. Let us prove (5.2). The inequality will be proved by means of the so called slicing method, which relies on the use of Theorem 2.1.

Let us suppose for simplicity that $g^\infty(1) \wedge b^0(1) = +\infty$; the other case can be treated in an analogous way. First of all we observe that, for $\xi \in S^{n-1}$, for $u \in W^{2,2}(\Omega)$, and for $A \in \mathcal{A}(\Omega)$ we have, by Fubini's Theorem and by the monotonicity of f_n,

$$\begin{aligned} F_n^N(u,A) &= \int_{\Pi_\xi}\int_{A_\xi^y} \left(f_n(|\nabla u(y+t\xi)|) + (r_n)^3\|\nabla^2 u(y+t\xi)\|^2\right) dt\, d\mathcal{H}^{N-1}(y) \\ &\geq \int_{\Pi_\xi}\int_{A_\xi^y} \left(f_n(|(u_\xi^y)'|) + (r_n)^3|(u_\xi^y)''|^2\right) dt\, d\mathcal{H}^{N-1}(y) \\ &= \int_{\Pi_\xi} F_n(u_\xi^y, A_\xi^y)\, d\mathcal{H}^{N-1}(y), \end{aligned}$$

where, Π_ξ is the hyperplane orthogonal to ξ while A_ξ^y and u_ξ^y are the one dimensional sections defined in Subsection 2.1. Let $u_n \to u$ in $L^1(A)$ be such that $\sup_n F_n^N < +\infty$ and note that, for every $\xi \in S^{n-1}$ and for almost every $y \in A_\xi^y$, $(u_n)_\xi^y \in W^{2,2}(A_\xi^y)$ and $(u_n)_\xi^y \to (u)_\xi^y$ in $L^1(A_\xi^y)$, hence, recalling that, by our assumptions, $F_n(\cdot, A_\xi^y) \xrightarrow{\Gamma} F_{b,g}(\cdot, A_\xi^y)$ and using Fatou's Lemma,

$$\begin{aligned} \liminf_{n\to\infty} F_n^N(u_n, A) &\geq \int_{\Pi_\xi} \liminf_{n\to\infty} F_n((u_n)_\xi^y, A_\xi^y)\, d\mathcal{H}^{N-1}(y) \\ &\geq \int_{\Pi_\xi}\left(\int_{A_\xi^y} g(|(u_\xi^y)'|) + \sum_{(S_u)_\xi^y \cap A_\xi^y} \varphi((u_\xi^y)^+ - (u_\xi^y)^-)\right) \mathcal{H}^{N-1}(y). \end{aligned} \tag{5.3}$$

From (5.3), by virtue of Theorem 2.1, we get $u \in GSBV(\Omega)$ and

$$\Gamma\text{-}\liminf_{n\to\infty} F_n^N(u,A) \geq \alpha \int_A g(|\nabla u\cdot\xi|)\, dx + \int_{S_u\cap A} \varphi(u^+ - u^-)|\nu_u\cdot\xi|\, d\mathcal{H}^{N-1} = \int_A \psi_\xi(x)\, \lambda, \tag{5.4}$$

where we have set

$$\lambda := \mathcal{L}^N + \varphi(u^+ - u^-)\mathcal{H}^{N-1}\lfloor S_u,$$

and

$$\psi_\xi := g(|\nabla u\cdot\xi|)(1-\chi_{S_u}) + |\nu_u\cdot\xi|\chi_{S_u};$$

since (5.4) holds true for every ξ and for every $A \in \mathcal{A}(\Omega)$, choosing a sequence $(\xi_i)_{i\in\mathbb{N}}$ dense in S^{n-1}, and by applying Lemma 3.13 (with $\nu(\cdot) := \Gamma\text{-}\liminf_{n\to\infty} F_n^N(u,\cdot)$), we finally obtain

$$\Gamma\text{-}\liminf_{n\to\infty} F_n^N(u) \geq \int_\Omega \sup_i \psi_{\xi_i}\, d\lambda = \int_\Omega g(|\nabla u|)\, dx + \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1},$$

as desired.

Concerning the Γ-lim sup inequality, we adapt the proof given in [3]. In the sequel we will assume that **gr1)** and **gr2)** hold true. For every $p \geq 1$ we denote by $G_p : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ the following functional

$$G_p(u, A) := \inf\{\limsup_{n\to\infty} F_n^N(u_n, A) : \ u_n \to u \text{ in } L^p(A)\};$$

our thesis is then equivalent to prove that $G_p(u, \Omega) \leq F_{b,g}^N(u, \Omega)$ for every $u \in GSBV^q(\Omega) \cap L^p(\Omega)$. It is clear that

$$G_{p_1}(u, A) \leq G_{p_2}(u, A), \tag{5.5}$$

for every $1 \leq p_1 < p_2$, for every $u \in L^{p_2}(\Omega)$, and for every $A \in \mathcal{A}(\Omega)$.

STEP 1. Let Π be an affine hyperplane, and denote by Π^+ and Π^- the two open half-spaces whose union gives $\mathbb{R}^N \setminus \Pi$ and by ν the unit normal vector to Π which points towards Π^+. Then, for every $A \in \mathcal{A}(\Omega)$ and for every $z \in \mathbb{R}$,

$$G_p(z\chi_{\Pi^+}, A) \leq \varphi(|z|)\mathcal{H}^{N-1}(\Pi \cap A) = F_{b,g}^N(z\chi_{\Pi^+}, A) \qquad \forall p \geq 1.$$

First of all, since

$$\lim_{t\to 0} \mathcal{H}^{N-1}(\{x \in A : \ d(x) = t\}) = \mathcal{H}^{N-1}(\Pi \cap A),$$

for $\delta \in (0, 1)$, we can choose $\eta > 0$ such that

$$\sup_{t\in(-\eta,\eta)} \ \mathcal{H}^{N-1}(\{x \in A : \ d(x) = t\}) \leq (1 + \delta)\mathcal{H}^{N-1}(\Pi \cap A). \tag{5.6}$$

Let $u_n \to z\chi_{(0,+\infty)}$ be the one dimensional recovery sequence constructed in the previous section which satisfies $\|u_n\|_\infty \leq |z|$, $u_n \equiv z\chi_{(0,+\infty)}$ in $\mathbb{R} \setminus (-\eta, \eta)$, and

$$\lim_{n\to\infty} F_n(u_n, (-\eta, \eta)) = F_{b,g}(z\chi_{(0,+\infty)}, (-\eta, \eta)) = \varphi(|z|); \tag{5.7}$$

we recall also that, for a suitable $K > 0$,

$$r_n\|u_n'\|_\infty \leq K. \tag{5.8}$$

We define, for every $x \in \Omega$, $v_n(x) := u_n(d(x))$, where d is the signed distance function from Π, positive in Π^+ and negative in Π^-. Clearly $v_n \in W^{2,2}(\Omega)$ and $v_n \to z\chi_{\Pi^+}$ in $L^p(\Omega)$ for every $p \geq 1$, moreover, using co-area formula (see (2.1)), (5.6), and (5.8) , we can estimate

$$\begin{aligned}
F_n^N(v_n, A) &= \int_{A\cap(\Pi)_\eta} f_n(|u'(d)|)\,dx + (r_n)^3 \int_{A\cap(\Pi)_\eta} \|u_n''(d)\nabla d \otimes \nabla d + u_n'(d)\nabla^2 d\|^2\,dx \\
&\leq \int_{-\eta}^{\eta}\int_{\{x\in A:\, d(x)=t\}} f_n(|u_n'(t)|)\,d\mathcal{H}^{N-1}\,dt \\
&\quad + \int_{-\eta}^{\eta} (r_n)^3 \int_{\{x\in A:\, d(x)=t\}} ((1+\varepsilon)|u_n''(t)|^2 + c_\varepsilon|u_n'(t)|^2\|\nabla^2 d\|)\,d\mathcal{H}^{N-1}\,dt \\
&\leq \int_{-\eta}^{\eta} F_n(u_n, (-\eta, \eta))\mathcal{H}^{N-1}(\{x \in A : \ d(x) = t\})\,dt \\
&\quad + c_\varepsilon K r_n\|\nabla^2 d\|_\infty \int_{-\eta}^{\eta} \mathcal{H}^{N-1}(\{x \in A : \ d(x) = t\})\,dt \\
&\leq (1+\varepsilon)(1+\delta)\mathcal{H}^{N-1}(\Pi \cap A)F_n(u_n, (-\eta, \eta)) + c_\varepsilon(1+\delta)\mathcal{H}^{N-1}(\Pi \cap A)2K\eta r_n\|\nabla^2 d\|_\infty,
\end{aligned}$$

where we denoted by $(\Pi)_\eta$ the η-neighbourhood of Π. From the last inequality, taking into account (5.7), we deduce

$$\limsup_{n\to\infty} F_n^N(v_n, A) \leq (1+\varepsilon)(1+\delta)\mathcal{H}^{N-1}(\Pi \cap A)\varphi(|z|);$$

since δ and ε are arbitrary, STEP 1 is proved.
STEP 2. Let $u=\sum_{i=1}^k z_i\chi_{E_i}$ with E_i closed polyhedra such that $\overset{\circ}{E}_i\cap\overset{\circ}{E}_j=\emptyset$ for $i\neq j$. Then

$$G_p(u,A)\leq F^N_{b,g}(u,A),$$

for all $A\in\mathcal{A}(\Omega)$ and for every $p\geq 1$.

The proof is based on a standard partition of unity argument, we refer to Proposition 2.6 of [3] for the details.
STEP 3. Let A', A, $B\in\mathcal{A}(I)$ such that $A'\subset\subset A$ and let ϕ be a cut-off function between A' and A. Then there exists a positive constant $C>0$ such that, for every u, $v\in W^{2,2}(\Omega)\cap L^q(\Omega)$, we have

$$\begin{aligned}F^N_n(\phi u+(1-\phi)v,A'\cup B)&\leq F^N_n(u,A)+F^N_n(v,B)\\&\quad+C(F^N_n(u,S)+F^N_n(v,S))+C\|\nabla\phi\|^q_\infty\|u-v\|^q_{L^q(S)}+C\mathcal{L}^N(S)\\&\quad+C(r_n)^3(\|\nabla\phi\|^2_\infty\|\nabla u-\nabla v\|^2_{L^2(S)}+\|\nabla^2\phi\|^2_\infty\|u-v\|^2_{L^2(S)}),\end{aligned}\tag{5.9}$$

where $S:=(A\setminus\overline{A'})\cap B$.

Using the monotonicity of f, we can estimate

$$\begin{aligned}&F^N_n(\phi u+(1-\phi)v,A'\cup B)\\ \leq\;&F^N_n(u,A)+F^N_n(v,B)+\int_S f_n(|(u-v)\nabla\phi|+\phi|\nabla u|+(1-\phi)|\nabla v|)\,dx\\&+C_1(r_n)^3\int_S(|\nabla\phi|^2|\nabla u-\nabla v|^2+|u-v|^2\|\nabla^2\phi\|^2+\|\nabla^2u-\nabla^2v\|^2)\,dx\\ \leq\;&F^N_n(u,A)+F^N_n(v,B)+\int_S(f_n(3|(u-v)\nabla\phi|)+f_n(3\phi|\nabla u|)+f_n(3(1-\phi)|\nabla v|))\,dx\\&+C_1(r_n)^3\int_S(|\nabla\phi|^2|\nabla u-\nabla v|^2+|u-v|^2\|\nabla^2\phi\|^2+\|\nabla^2u\|^2+\|\nabla^2v\|^2)\,dx=:(*);\end{aligned}$$

using **gr2)** we can continue our estimate

$$\begin{aligned}(*)\;\leq\;&F^N_n(u,A)+F^N_n(v,B)+(C_1+c(3))(F^N_n(u,S)+F^N_n(v,S))+\int_S f_n(3|(u-v)\nabla\phi|)\,dx\\&+C_1(r_n)^3\int_S(|\nabla\phi|^2|\nabla u-\nabla v|^2+|u-v|^2\|\nabla^2\phi\|^2)\,dx;\end{aligned}$$

recalling **gr1)**, from the last inequality we easily get (5.9).
STEP 4. Let A', A, B, and S be as in STEP 3 and $p\geq q$. Then for every u, $v\in L^p(\Omega)$ and for every $K\in\mathbb{N}$ there exists a cut-off function ϕ_K between A and A', such that

$$G_p(\phi_Ku+(1-\phi_K)v,A'\cup B)\leq\left(1+\frac{C}{K}\right)(G_p(u,A)+G_p(v,B))+C\frac{K^{q-1}}{d^q}\|u-v\|^q_{L^q(S)}+\frac{C}{K}\mathcal{L}^N(S),\tag{5.10}$$

where $d:=\mathrm{dist}(A',\Omega\setminus A)$.

First of all choose u_n, $v_n\in W^{2,2}(\Omega)$ such that $u_n\to u$, $v_n\to v$ in $L^p(\Omega)$ and

$$G_p(u,A)=\lim_{n\to\infty}F^N_n(u_n,A)\qquad\text{and}\qquad G_p(u,B)=\lim_{n\to\infty}F^N_n(v_n,B).$$

For $j\in\{0,1,\dots,K\}$ we consider the set

$$A^K_j=\left\{x\in A:\ \mathrm{dist}(x,A')<j\frac{d}{K}\right\};$$

for any $j \in \{0, 1, \ldots, K-1\}$ we choose a cut-off function ϕ_j^K between A_j^K and A_{j+1}^K such that

$$\|\nabla \phi_j^K\|_\infty \leq 2\frac{K}{d}; \tag{5.11}$$

finally we set $S_j^K := \left(A_{j+1}^K \setminus \overline{A}_j^K\right) \cap B$. By using (5.9) and (5.11), we get

$$\begin{aligned} F_n^N(\phi_j^K u_n \ &+ \ (1-\phi_j^K)v_n, A' \cup B) \\ &\leq \ F_n^N(u_n, A) + F_n^N(v_n, B) + C(F_n^N(u_n, S_j^K) + F_n^N(v_n, S_j^K)) + C\frac{K^q}{d^q}\|u - v\|_{L^q(S_j^K)}^q \\ &\quad + C(r_n)^3(\|\nabla\phi_j^K\|_\infty^2 \|\nabla u_n - \nabla v_n\|_{L^2(S_j^K)}^2 + \|\nabla^2\phi_j^K\|_\infty^2 \|u_n - v_n\|_{L^2(S_j^K)}^2). \end{aligned}$$

Passing to a subsequence, if needed, it follows that there exists $j_K \in \{0, 1, \ldots, K-1\}$ such that

$$\begin{aligned} F_n^N(\phi_{j_K}^K u_n + (1-\phi_{j_K}^K)v_n, A' \cup B) \ &\leq \ \frac{1}{K}\sum_{j=0}^{K-1} F_n^N(\phi_j^K u_n + (1-\phi_j^K)v_n, A' \cup B) \\ \leq \ \left(1 + \frac{C}{K}\right)(F_n^N(u_n, A) &+ F_n^N(v_n, B)) + C\frac{K^{q-1}}{d^q}\|u_n - v_n\|_{L^q(S)}^q \\ &+ \frac{C}{K}\mathcal{L}^N(S) + C(K)(r_n)^3(\|\nabla u_n - \nabla v_n\|_{L^2(S)}^2 + \|u_n - v_n\|_{L^2(S)}^2), \end{aligned} \tag{5.12}$$

for every $n \in \mathbb{N}$. Recall that by Nirenberg inequality (see [25]), there exists $M > 0$ such that for all $u \in W^{2,2}(S)$

$$\|\nabla u\|_{L^2(S)} \leq M(\|\nabla^2 u\|_{L^2(S)}^{1/2}\|u\|_{L^2(S)}^{1/2} + \|u\|_{L^2(S)}),$$

therefore from the equiboundedness of

$$(\|u_n - v_n\|_{L^2(S)} + (r_n)^3\|\nabla^2 u_n - \nabla^2 v_n\|_{L^2(S)}^2)_n$$

we get

$$(r_n)^3\|\nabla u_n - \nabla v_n\|_{L^2(S)}^2 \to 0 \qquad \text{as } n \to \infty.$$

Thus (5.10) follows letting n tend to $+\infty$ in (5.12).
STEP 5. For every $u \in GSBV^q(\Omega) \cap L^\infty(\Omega)$ we have

$$G_p(u, \Omega) \leq \int_\Omega g(|\nabla u|)\, dx + \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1} \qquad \forall p \geq 1. \tag{5.13}$$

We start with $u \in \mathcal{W}(\Omega)$ (see Subsection 2.3) and, for every $h \in \mathbb{N}$, we consider the sets

$$B_h := (S_u)_{1/h} \cap \Omega = \left\{x \in \Omega : \operatorname{dist}(x, S_u) < \frac{1}{h}\right\};$$

by the regularity assumptions on S_u we have that $\mathcal{L}^N(B_h) = O(1/h)$ and therefore, setting

$$\rho_h := h^{-\frac{1}{2}}\left(\int_{B_h} |\nabla u|^2\right)^{\frac{1}{4}},$$

we have

$$\lim_{h\to\infty} \frac{1}{\rho_h}\int_{B_h} |\nabla u|\, dx = 0. \tag{5.14}$$

By a standard argument based on the use of coarea formula (2.1) (see for example [12]) it is possible to find a sequence u_h satisfying the hypotheses of STEP 2 such that

$$\|u-u_h\|_{L^\infty(B_h)} \le \rho_h, \qquad \mathcal{H}^{N-1}((S_{u_h}\cap B_h)\setminus S_u) \le \frac{1}{\rho_h}\int_{B_h}|\nabla u|\,dx + O(1). \tag{5.15}$$

We are going to apply STEP 4 with $A=B_h$, $A'=B_{2h}$, $B=\Omega\setminus\overline{B}_{3h}$, obtaining for every $K\in\mathbb{N}$ the existence of a cut-off function ϕ_K^h such that

$$\begin{aligned} G_p(\phi_K^h u_h+(1-\phi_K^h)u,\Omega) &\le \left(1+\frac{C}{K}\right)(G_p(u,\Omega\setminus\overline{S}_u)+G_p(u_h,B_h)) + C\frac{K^{q-1}}{d^q}\|u-u_h\|^q_{L^q(S)} \\ &\quad +\frac{C}{K}\mathcal{L}^N(B_h) \\ &\le \left(1+\frac{C}{K}\right)(G_p(u,\Omega\setminus\overline{S}_u)+G_p(u_h,B_h)) + \left(CK^{q-1}h^q\rho_h^q+\frac{C}{K}\right)\mathcal{L}^N(B_h), \end{aligned} \tag{5.16}$$

where $p\ge q$. By STEP 2 we have

$$\begin{aligned} G_p(u_h,B_h) &\le \int_{S_{u_h}\cap B_h}\varphi(u_h^+-u_h^-)\,d\mathcal{H}^{N-1} \\ &\le \int_{S_u}\varphi(u^+-u^-)\,d\mathcal{H}^{N-1}+\varphi(2\|u\|_\infty)\mathcal{H}^{N-1}((S_{u_h}\cap B_h)\setminus S_u) \\ &\quad +\int_{S_u}(\varphi(u_h^+-u_h^-)-\varphi(u^+-u^-))\,d\mathcal{H}^{N-1}; \end{aligned}$$

using (5.15) and (5.14) and noting that, by the Dominated Convergence Theorem

$$\lim_{h\to\infty}\int_{S_u}(\varphi(u_h^+-u_h^-)-\varphi(u^+-u^-)\,d\mathcal{H}^{N-1}=0,$$

we therefore obtain

$$\limsup_{h\to\infty} G_p(u_h,B_h) \le \int_{S_u}\varphi(u^+-u^-)\,d\mathcal{H}^{N-1}. \tag{5.17}$$

Moreover, taking as approximating sequence $u_n=u$ for every $n\in\mathbb{N}$, we discover that

$$G_p(u,\Omega\setminus\overline{S}_u) \le \int_\Omega g(|\nabla u|)\,dx; \tag{5.18}$$

combining (5.17) and (5.18), letting $h\to+\infty$ in (5.16), and taking into account the lower semicontinuity of G_p we finally get (5.13), for $p\ge q$ and therefore for every $p\ge 1$, by virtue of (5.5). For a general $u\in GSBV^q(\Omega)\cap L^\infty(\Omega)$ we conclude by a standard density argument based on Theorem 2.5.

We are now in a position to conclude the proof of the Theorem. Take $u\in GSBV^q(\Omega)\cap L^p(\Omega)$ and set $u_k:=(-k\vee u)\wedge k$; then by (5.13), and the Monotone Convergence Theorem we have

$$\begin{aligned} G_p(u,\Omega) &\le \liminf_{k\to\infty} G_p(u_k,\Omega) \\ &\le \lim_{k\to\infty}\int_\Omega g(|\nabla u_k|)\,dx+\int_{S_u}\varphi(u_k^+-u_k^-)\,d\mathcal{H}^{N-1} \\ &= \int_\Omega g(|\nabla u|)\,dx+\int_{S_u}\varphi(u^+-u^-)\,d\mathcal{H}^{N-1}. \end{aligned}$$

So if $g^\infty(1)\wedge b^0(1)=+\infty$ then we are done; if it is not the case, then the conclusion follows by the fact that, thanks to Theorem 2.3 and an easy truncation argument, $F^N_{b,g}$ coincides with the steady relaxed functional (see

Definition 5.1) of

$$H(u) := \begin{cases} \displaystyle\int_\Omega g(|\nabla u|)\,dx + \int_{S_u} \varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} & \text{if } u \in GSBV^q(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus GSBV^q(\Omega). \end{cases}$$

□

The two following corollaries are an immediate consequence of Theorems 5.2, 4.2, 4.12.

Corollary 5.5 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let f, r, p as in Theorem 4.2. Let $(\varepsilon_n)_n$ be an infinitesimal sequence such that (4.3) holds. If $b^0(1) = +\infty$ then the functionals*

$$F_n^N(u) = \begin{cases} \displaystyle\frac{1}{\varepsilon_n}\int_\Omega f(\sqrt{\varepsilon_n}|\nabla u|)\,dx + (p(\varepsilon_n))^3 \int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Γ^s-converge in $GSBV^q(\Omega)$ to

$$F^N(u) := \begin{cases} \alpha \displaystyle\int_\Omega |\nabla u|^q\,dx + \int_{S_u} \varphi^{(a)}(u^+(x) - u^-(x))\,d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where $\varphi^{(a)}$ is as in Theorem 4.2; conversely, if $b^0(1) = C$ then the sequence F_n^N Γ^s-converges in $L^1(\Omega)$ to

$$F^N(u) := \begin{cases} \displaystyle\int_\Omega g(|\nabla u|)\,dx + \int_{S_u} \varphi^{(a)}(u^+(x) - u^-(x))\,d\mathcal{H}^{N-1} + C|D^c u| & \text{if } u \in GBV(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus GBV(\Omega), \end{cases}$$

with $\varphi^{(a)}$ still given by (3.3) and $g = (\alpha x^q \wedge Cx)^{**}$.

Corollary 5.6 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let f be as in Theorem 4.12. Then the family*

$$F_\varepsilon^N := \begin{cases} \displaystyle\frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|)\,dx + \varepsilon^3 \int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

Γ^s-converges in $L^1(\Omega)$ to the functional

$$F^N(u) := \begin{cases} f'(0) \displaystyle\int_\Omega |\nabla u|\,dx + \int_{S_u} \varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} + f'(0)|D^c u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where φ is the function defined in (3.3) with $b = f$.

To conclude the N-dimensional analysis it remains to prove the equicoerciveness of the approximating functionals: this is done in the following proposition.

Proposition 5.7 *Under the same hypotheses of Theorem 5.2, let $(u_n)_n \subset L^1(\Omega)$ be equiintegrable and such that*

$$\sup_n F_n^N(u_n) < M < +\infty;$$

then $(u_n)_n$ is strongly precompact in $L^1(\Omega)$. Suppose in addition that $F_n^N \overset{\Gamma^s}{\to} G$ in $L^1(\Omega)$; then, for every $g \in L^p(\Omega)$ $(p > 1)$ and $\beta > 0$, the solutions u_n of

$$\min\left\{F_n^N(v) + \beta \int_\Omega |v - g|^p\,dx \,:\, v \in W^{2,2}(\Omega)\right\}$$

converge, up to subsequences, in the $L^p(\Omega)$-norm to a solution of

$$\min\left\{G(v) + \beta \int_\Omega |v - g|^p\,dx \,:\, v \in L^1(\Omega)\right\}.$$

Proof. As at the beginning of the proof of Theorem 5.2, we fix $\xi \in S^{n-1}$ and we get

$$M \geq F_n^N(u_n) \geq \int_{\Omega_\xi} g_n(y)\,d\mathcal{H}^{N-1}(y), \tag{5.19}$$

where $g_n(y) := F_n((u_n)^y_\xi, \Omega^y_\xi)$. Using the equiintegrability assumption, for fixed $\delta > 0$, we find $\sigma_\delta > 0$ such that

$$\mathcal{L}^N(B) \leq \sigma_\delta \Rightarrow \int_B |u_n|\,dx < \delta \qquad \forall n \in \mathbb{N}. \tag{5.20}$$

Choose $k > 0$ such that

$$\frac{M\,\mathrm{diam}(\Omega)}{k} \leq \sigma_\delta; \tag{5.21}$$

set $A_{n,k} := \{y \in \Omega_\xi \,:\, g_n(y) > k\}$ and denote by P_ξ the orthogonal projection on Π_ξ. We now define the new sequence v_n in the following way

$$v_n(x) := \begin{cases} u_n(x) & \text{if } P_\xi(x) \in \Omega_\xi \setminus A_{n,k} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\|u_n - v_n\|_{L^1(\Omega)} = \int_{\{x\in\Omega:\, P_\xi(x)\in A_{n,k}\}} |u_n|\,dx$; since by Chebicev Inequality, (5.19), and (5.21)

$$\begin{aligned} \mathcal{L}^N(\{x \in \Omega \,:\, P_\xi(x) \in A_{n,k}\}) &\leq \mathcal{H}^{N-1}(A_{n,k})\mathrm{diam}(\Omega) \\ &\leq \frac{\|g_n\|_{L^1(\Omega_\xi)}}{k}\mathrm{diam}(\Omega) \leq \frac{M}{k}\mathrm{diam}(\Omega) \leq \sigma_\delta, \end{aligned}$$

recalling (5.20) we finally obtain $\|u_n - v_n\|_{L^1(\Omega)} \leq \delta$.

Moreover $F_n((v_n)^y_\xi, \Omega^y_\xi) \leq g_n(y)(1 - \chi_{A_{n,k}}(y)) \leq k$ and therefore, by the one-dimensional results, $(v_n)^y_\xi$ is precompact in $L^1(\Omega^y_\xi)$ for every $y \in \Omega_\xi$. Since the construction can be repeated for every $\delta > 0$ and for every $\xi \in S^{n-1}$, the thesis follows by applying Lemma 2.2.

Concerning the second part, we first observe that

$$\sup_n \left(F_n^N(u_n) + \beta \int_\Omega |u_n - g|^p\,dx\right) \leq \sup_n f_n(0)|\Omega| + \beta \int_\Omega |g|^p\,dx < +\infty \tag{5.22}$$

and therefore, by the first part of the theorem, there exist $u \in L^1(\Omega)$ and a subsequence, still denoted by u_n, such that $u_n \to u$ in L^1. Note that by (5.22) $\sup_n \|u_n\|_{L^p} < +\infty$ which implies that $u_n \rightharpoonup u$ weakly in L^p. Since $F_n^N \overset{\Gamma^s}{\to} G$, there exists $v_n \to v$ in L^p such that $F_n^N(v_n) \to G(u)$ and therefore, by exploiting the minimality of u_n,

$$\begin{aligned} G(u) + \beta \int_\Omega |u - g|^p\,dx &\leq \liminf_{n\to\infty} \left(F_n^N(u_n) + \beta \int_\Omega |u_n - g|^p\,dx\right) \\ &\leq \limsup_{n\to\infty} \left(F_n^N(u_n) + \beta \int_\Omega |u_n - g|^p\,dx\right) \\ &\leq \lim_{n\to\infty} \left(F_n^N(v_n) + \beta \int_\Omega |v_n - g|^p\,dx\right) = G(u) + \beta \int_\Omega |u - g|^p\,dx \end{aligned}$$

whence,

$$\begin{aligned}G(u)+\beta\int_\Omega |u-g|^p\,dx &= \lim_{n\to\infty}\left(F_n^N(u_n)+\beta\int_\Omega |u_n-g|^p\,dx\right)\\ &\geq G(u)+\limsup_{n\to\infty}\beta\int_\Omega |u_n-g|^p\,dx\\ &\geq G(u)+\liminf_{n\to\infty}\beta\int_\Omega |u_n-g|^p\,dx \geq G(u)+\beta\int_\Omega |u-g|^p\,dx.\end{aligned}$$

We deduce $\int_\Omega |u_n-g|^p\,dx \to \int_\Omega |u-g|^p\,dx$ and since $u_n-g \rightharpoonup u-g$ weakly in L^p, we conclude that $u_n \to u$ in L^p. The minimality of u follows now from the properties of Γ-convergence. □

We conclude the section by remarking that all the examples contained in Section 4 can be generalized to the N-dimensional case by means of Theorem 5.2. In particular let us underline the following ones.

Example 5.8 *(Perona-Malik Energy)* By Example 4.8 and by Theorem 5.2 we obtain that the functionals

$$F_\varepsilon^N(u) := \begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_\Omega \log(1+\varepsilon\alpha|\nabla u|^2)\,dx+\left(\frac{a\varepsilon}{\log\frac{1}{\varepsilon}}\right)^3\int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u\in W^{2,2}(\Omega),\\ +\infty & \text{otherwise in } L^1(\Omega),\end{cases}$$

Γ^s-converge, as $\varepsilon\to 0^+$, to

$$F^N(u) := \begin{cases}\alpha\displaystyle\int_\Omega |\nabla u|^2\,dx+m(0)a^{\frac{3}{4}}\int_{S_u}\sqrt{u^+-u^-}\,d\mathcal{H}^{N-1} & \text{if } u\in GSBV(\Omega)\\ +\infty & \text{in } L^1(\Omega)\setminus GSBV(\Omega),\end{cases}$$

in $GSBV^2(\Omega)$.

Example 5.9 Let b and g be as in Example 4.13 (and suppose for simplicity $g(0)=0$); then the family

$$F_\varepsilon^N := \begin{cases}\displaystyle\int_\Omega\left(g(|\nabla u|)\wedge\frac{1}{\varepsilon}b(\varepsilon|\nabla u|)\right)dx+\varepsilon^3\int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u\in W^{2,2}(\Omega),\\ +\infty & \text{otherwise}\end{cases}$$

Γ^s-converges in $L^1(\Omega)$ to the functional

$$\begin{cases}\displaystyle\int_\Omega g(|\nabla u|)\,dx+\int_{S_u}\varphi(u^+-u^-)\,d\mathcal{H}^{N-1}+C|D^c u| & \text{if } u\in GBV(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

where φ is the function associated with b according to (3.3).

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