CONVEX ISOPERIMETRIC SETS IN THE HEISENBERG GROUP

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Abstract. We characterize convex isoperimetric sets in the Heisenberg group. We first prove Sobolev regularity for a certain class of $\mathbb{R}^2$-valued vector fields of bounded variation in the plane related to the curvature equations. Then we show that the boundary of convex isoperimetric sets is foliated by geodesics of the Carnot-Carathéodory distance.

1. Introduction

We identify the Heisenberg group $\mathbb{H}^1$ with $\mathbb{C} \times \mathbb{R}$ endowed with the group law

$$(z,t)(z',t') = \big(z+z', t+t' + 2\mathrm{Im}(z\bar{z}')\big),$$

where $t, t' \in \mathbb{R}$ and $z = x+iy, z' = x'+iy' \in \mathbb{C}$. The Lie algebra of left-invariant vector fields is spanned by

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t} \quad \text{and} \quad T = \frac{\partial}{\partial t},$$

and the distribution of planes spanned by X and Y, called horizontal distribution, generates the Lie algebra by brackets.

The natural volume in $\mathbb{H}^1$ is the Haar measure, which, up to a positive factor, coincides with Lebesgue measure in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. Lebesgue measure is also the Riemannian volume of the left-invariant metric for which X, Y and T are orthonormal. We denote by $|E|$ the volume of a (Lebesgue) measurable set $E \subset \mathbb{H}^1$. The horizontal perimeter (or simply perimeter) of E is

$$P(E) = \sup\left\{ \int_E \big(X\varphi_1 + Y\varphi_2\big)\, dxdydt \;\Big|\; \varphi_1, \varphi_2 \in C_c^1(\mathbb{R}^3),\ \varphi_1^2 + \varphi_2^2 \le 1 \right\}. \tag{1.1}$$

If $P(E) < +\infty$, the set E is said to be of finite perimeter. Perimeter is left-invariant and 3-homogeneous with respect to the group of dilations $\delta_\lambda : \mathbb{H}^1 \to \mathbb{H}^1$, $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$, $\lambda > 0$, that is $P(\delta_\lambda(E)) = \lambda^3 P(E)$. Definition (1.1) of perimeter is modelled on De Giorgi's notion of perimeter in Euclidean spaces which was generalized in [GN] to Carnot-Carathéodory spaces. For smooth sets (e.g. of class C^2), perimeter coincides with the Minkowski content and with the 3-dimensional Hausdorff measure of ∂E constructed by means of the standard Carnot-Carathéodory metric in $\mathbb{H}^1$ (see [MSC]

Date: 26th March, 2008.

and [FSSC], respectively). Volume and perimeter are related via the isoperimetric inequality

$$|E| \leq C_{\rm isop} P(E)^{4/3}, \tag{1.2}$$

where $C_{\rm isop} > 0$ and $E \subset \mathbb{H}^1$ is any measurable set with finite perimeter and volume. This inequality is proved by P. Pansu in [P1] and [P2] for smooth domains with the 3-dimensional Hausdorff measure of ∂E in $\mathbb{H}^1$ replacing $P(E)$. In the general form (1.2), the inequality is proved in [GN] (see also [FGW]). The problem of computing the sharp constant $C_{\rm isop}$ leads to the notion of isoperimetric set. A set $E \subset \mathbb{H}^1$ with $0 < |E| < +\infty$ is an *isoperimetric set* if it minimizes the isoperimetric ratio

$$\mathrm{Isop}(E) = \frac{P(E)^{4/3}}{|E|}. \tag{1.3}$$

Isoperimetric sets do exist: this is proved in [LR] by a concentration-compactness argument. Pansu notes that the boundary of a smooth isoperimetric set has "constant mean curvature" and that a smooth surface has "constant mean curvature" if and only if it is foliated by horizontal lifts of plane circles with constant radius. Then he conjectures that an isoperimetric set is obtained by rotating around the center of the group a geodesic joining two points in the center. Recently, Pansu's conjecture reappeared in [LM].

The problem of determining isoperimetric sets is interesting for two reasons. On the one hand, the non commutative group law makes it difficult to prove by a rearrangement argument that the isoperimetric ratio (1.3) is minimized by rotationally symmetric sets, as it is natural to conjecture. On the other hand, there is no regularity theory for measurable sets in $\mathbb{H}^1$ minimizing perimeter with (or without) a volume constraint. So, new techniques and ideas are needed.

All known results assume either symmetry or regularity (or both). In fact, assuming rotational symmetry and regularity it is easy to determine the isoperimetric profile (see [Mo1], and [RR1] for the general case). Actually, it suffices to assume the rotational symmetry of a certain horizontal section (see [DGN] and especially the calibration argument in [R]). The solution of the rotationally symmetric case with no regularity assumption is in [Mo2]. On the other hand, isoperimetric sets can be also determined assuming only the C^2 regularity of the boundary and no symmetry (see [RR2]). Further evidence supporting Pansu's conjecture is provided by [MM], where a 2-dimensional version of the problem is solved. We refer to the monograph [CDST] for a more detailed introduction to the isoperimetric problem in the Heisenberg group.

In this article, we characterize isoperimetric sets which are convex. By convex set we mean a subset of $\mathbb{H}^1 = \mathbb{R}^3$ which is convex with respect to the standard vector space structure of $\mathbb{R}^3$. Convexity is a left invariant property in $\mathbb{H}^1$ because left translations are linear mappings.

Theorem 1.1 (Convex isoperimetric sets). *Up to a left translation and a dilation, any closed convex isoperimetric set in $\mathbb{H}^1$ coincides with*

$$E_{\rm isop} = \Big\{(z,t) \in \mathbb{H}^1 \ \Big|\ |t| \le \arccos|z| + |z|\sqrt{1-|z|^2},\ |z| \le 1\Big\}. \tag{1.4}$$

The boundary of the set in (1.4) is foliated by Heisenberg geodesics, it is globally of class C^2, but it fails to be of class C^3 at the north and south poles $(0, \pm\pi/2)$. At these points, the plane spanned by the vector fields X and Y, the horizontal plane, is tangent to the boundary.

We explain the main steps in the proof of Theorem 1.1. Let us consider a bounded convex set of the form

$$E = \big\{(z,t) \in \mathbb{H}^1 \mid z \in D,\ f(z) \le t \le g(z)\big\}, \tag{1.5}$$

where $D \subset \mathbb{R}^2$ is a compact convex set in the plane with nonempty interior, and $-g, f : D \to \mathbb{R}$ are convex functions. If E is isoperimetric, then the function $f : D \to \mathbb{R}$ satisfies the partial differential equation

$$\operatorname{div}\left(\frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|}\right) = \frac{3P(E)}{4|E|} \tag{1.6}$$

in $\operatorname{int}(D) \setminus \Sigma(f)$. Here and in the following we let $z = (x, y)$ and $z^\perp = (-y, x)$. We call the singular set $\Sigma(f) = \big\{z \in \operatorname{int}(D) \mid -2z^\perp \in \partial f(z)\big\}$ the characteristic set of f. This set is always contained in a line segment. The basic facts concerning $\Sigma(f)$ are discussed in the Appendix. Equation (1.6) is derived in Section 2.

The curvature operator in (1.6) has been studied by several authors under different regularity assumptions (besides the previous references, see also [Pa1], [Pa2], [CHY1], [CHMY], [CHY2]). In the smooth case, the number

$$H = \frac{3P(E)}{4|E|} \tag{1.7}$$

is known as the (horizontal) curvature of ∂E. In our case, equation (1.6) is to be interpreted in distributional sense. The distributional derivatives of $u(z) = \nabla f(z) + 2z^\perp$ are measures, because f is a convex function, and so the equation states that the distributional divergence of $u/|u|$ is constant.

Our goal is to give equation (1.6) a pointwise meaning along integral curves of the vector field orthogonal to $u/|u|$. The first step is to show that the equation holds in the Sobolev sense. Precisely, we prove that the distributional derivative of $u/|u|$ is a measure which is absolutely continuous w.r.t. Lebesgue measure. This result is a corollary of the following regularity theorem for BV vector fields, which is interesting in itself. We prove a slightly more general statement in the third section.

Theorem 1.2 (Improved regularity)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let $u \in BV(\Omega;\mathbb{R}^2)$ be a vector field. Suppose that:*

(i) *There exists $\delta > 0$ such that $|u(z)| \geq \delta$ for $\mathcal{L}^2$-a.e. $z \in \Omega$;*
(ii) $\operatorname{div} u^\perp \in L^1(\Omega)$*;*
(iii) $\operatorname{div}\Big(\dfrac{u}{|u|}\Big) \in L^1(\Omega)$.

Then we have $u/|u| \in W^{1,1}(\Omega;\mathbb{R}^2)$.

The improved regularity of the boundary is the starting point for the geometric characterization of convex isoperimetric sets. The vector field $v(z) = -u^\perp(z) = 2z - \nabla f^\perp(z)$ belongs to $BV_{\mathrm{loc}}(\mathrm{int}(D);\mathbb{R}^2)$. Moreover, its distributional divergence is in L^∞, in fact

$$\operatorname{div} v = 4 \quad \text{in } \mathrm{int}(D). \tag{1.8}$$

Thanks to the theory on the Cauchy Problem for BV vector fields recently developed by Ambrosio in [A], the bound on the divergence ensures the existence of a unique regular Lagrangian flow $\Phi : K\times[-\varrho,\varrho] \to D$ starting from a compact set $K \subset \mathrm{int}(D)$, where $\varrho > 0$ is small enough. For $\mathcal{L}^2$-a.e. $z \in K$, the curve $s \mapsto \Phi(z,s)$ is an integral curve of v passing through z at time $s = 0$. In Section 4, we show that $\mathcal{L}^2$-a.e. integral curve of v is an arc of circle.

Theorem 1.3 (Foliation by circles)**.** *Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set of the form* (1.5) *and let $K \subset \mathrm{int}(D) \setminus \Sigma(f)$ be a compact set. Then, for $\mathcal{L}^2$-a.e. $z \in K$, the curve $s \mapsto \Phi(z,s)$ is an arc of circle having radius $1/H$, with $H > 0$ as in* (1.7).

The proof of Theorem 1.3 relies upon a reparameterization argument. The vector field v has a regular flow Φ starting from $K \subset \mathrm{int}(D) \setminus \Sigma(f)$ but it is only in $BV_{\mathrm{loc}}(\mathrm{int}(D);\mathbb{R}^2)$. On the other hand, $v/|v|$ is in $W^{1,1}_{\mathrm{loc}}(\mathrm{int}(D);\mathbb{R}^2)$, but its divergence is only in $L^1_{\mathrm{loc}}(\mathrm{int}(D))$ and so we have no regular flow for $v/|v|$. In order to compute the second order derivative of a generic integral curve of v, we introduce a suitable reparameterization $\gamma(s) = \Phi(z,\tau(s))$. We show that for any vector field $w \in W^{1,1}(\Omega;\mathbb{R}^2)$ defined in some open neighborhood Ω of K, the curve $\kappa(s) = w(\gamma(s))$ is in $W^{1,1}$ for $\mathcal{L}^2$-a.e. $z \in K$ and moreover

$$\dot\kappa = (\nabla w \circ \gamma)\,\dot\gamma \quad \text{in the weak sense.} \tag{1.9}$$

Using the chain rule (1.9) with $w = v/|v|$, which is in $W^{1,1}$ by Theorem 1.2, equation (1.6) can be given a pointwise meaning along the flow, and the integral curves of v turn out to be arcs of circles.

Since the horizontal lift of the flow Φ foliates the graph of the function f, it follows that the bottom (and upper) part of the boundary of E is foliated by geodesics of the Heisenberg Carnot-Carathéodory metric. However, this is not yet enough to finish our argument. We still need to show that the bottom and upper parts of ∂E match together in the proper way. To do this, we write E in the form

$$E = \big\{(x, y, t) \in \mathbb{H}^1 \mid (y, t) \in F,\, h(y, t) \le x \le k(y, t)\big\}, \tag{1.10}$$

for some compact convex set $F \subset \mathbb{R}^2$ and convex functions $h, -k : F \to \mathbb{R}$. The analysis carried out for f can be also carried out for h even though in this case computations are, unfortunately, more complicated. This provides the last piece of information needed to get the set $E_{\rm isop}$ in (1.4).

A short overview is now in order. In Section 2, we derive the curvature equations for convex isoperimetric sets. In Section 3, we prove the generalized version of Theorem 1.2 which is needed in Section 4, where we establish the foliation by geodesics property for convex isoperimetric sets. Finally, in Section 5 we prove Theorem 1.1. The results concerning convex sets are collected in the Appendix at the end of the paper.

2. Curvature equations for convex isoperimetric sets

We derive partial differential equations for certain vector fields built from the functions which parameterize the boundary of convex isoperimetric sets. We study graphs of the form $t = f(x, y)$ and graphs of the form $x = h(y, t)$. We use the following representation formula for perimeter. If $E \subset \mathbb{H}^1$ is a bounded set such that ∂E is locally a Lipschitz surface, then

$$P(E) = \int_{\partial E} \sqrt{(X \cdot \nu)^2 + (Y \cdot \nu)^2}\, d\mathcal{H}^2, \tag{2.1}$$

where ν is a unit normal to ∂E. Here and in the following, $\cdot$ denotes the standard inner product in $\mathbb{R}^3$ or $\mathbb{R}^2$ (we think of X and Y as vectors in $\mathbb{R}^3$). $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure in $\mathbb{R}^3$ w.r.t. the usual Euclidean distance. For a proof of formula (2.1), see [FSSC].

Graphs of the form $t = f(x, y)$. We denote elements of $\mathbb{H}^1 = \mathbb{R}^2 \times \mathbb{R}$ by (z, t) with $t \in \mathbb{R}$ and $z = (x, y) \in \mathbb{R}^2$. We write $z^\perp = (-y, x)$. Let E be a convex set in $\mathbb{H}^1$ of the form (1.5). We define the *characteristic set* of the convex function $f : D \to \mathbb{R}$ as

$$\Sigma(f) = \big\{z \in \operatorname{int}(D) \mid -2z^\perp \in \partial f(z)\big\}, \tag{2.2}$$

where $\partial f(z)$ stands for the subdifferential of f at z.

Proposition 2.1 (Curvature equation I). *Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set with perimeter $P(E)$ and volume $|E|$. Then the function $f : D \to \mathbb{R}$ satisfies in distributional sense the partial differential equation*

$$\operatorname{div}\left(\frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|}\right) = H \tag{2.3}$$

in $\mathrm{int}(D)\setminus\Sigma(f)$*, with* $H = 3P(E)/4|E|$.

Proof. By Theorem 6.1 in the Appendix, $\Sigma(f)$ is contained in the union of at most two line segments.

Let $\varphi \in C_c^\infty(\mathrm{int}(D)\setminus\Sigma(f))$ and for $\varepsilon\in\mathbb{R}$, consider the set

$$E_\varepsilon = \big\{(z,t)\in\mathbb{H}^1 \mid z\in D,\ f(z)+\varepsilon\varphi(z)\le t\le g(z)\big\}.$$

We let $P(\varepsilon)=P(E_\varepsilon)$, $V(\varepsilon)=|E_\varepsilon|$ and $R(\varepsilon)=P(\varepsilon)^4/V(\varepsilon)^3$. If E is an isoperimetric set, then $R(\varepsilon)$ has a minimum at $\varepsilon=0$, and then

$$R'(0) = \frac{P^3}{V^4}\big(4P'V-3PV'\big)\Big|_{\varepsilon=0} = 0. \tag{2.4}$$

There exists $\varepsilon_0>0$ such that

$$V'(\varepsilon) = -\int_D \varphi(z)\,dz \quad \text{for } |\varepsilon|<\varepsilon_0. \tag{2.5}$$

Let ν_ε be the exterior unit normal to ∂E_ε and let $S_\varepsilon = \big\{(z, f(z)+\varepsilon\varphi(z))\in\mathbb{H}^1 \mid z\in D\big\}$ be the graph of $f+\varepsilon\varphi$. By the Heisenberg Area Formula (2.1) and from the standard Area Formula for graphs of functions in Euclidean spaces, we find

$$\begin{aligned} P'(\varepsilon) &= \frac{d}{d\varepsilon}\int_{S_\varepsilon}\sqrt{(X\cdot\nu_\varepsilon)^2+(Y\cdot\nu_\varepsilon)^2}\,d\mathcal{H}^2 \\ &= \frac{d}{d\varepsilon}\int_D |\nabla f(z)+\varepsilon\nabla\varphi(z)+2z^\perp|\,dz. \end{aligned} \tag{2.6}$$

By Proposition 6.2 in the Appendix, for any compact set $K\subset\mathrm{int}(D)\setminus\Sigma(f)$, there is $\delta>0$ such that $|\nabla f(z)+2z^\perp|\ge\delta$ for $\mathcal{L}^2$-a.e. $z\in K$. Then we can interchange derivative and integral for all small ε. At $\varepsilon=0$ we obtain

$$P'(0) = \int_D \frac{\nabla f(z)+2z^\perp}{|\nabla f(z)+2z^\perp|}\cdot\nabla\varphi(z)\,dz. \tag{2.7}$$

From (2.4), (2.5) and (2.7) we find

$$4|E|\int_D \frac{\nabla f(z)+2z^\perp}{|\nabla f(z)+2z^\perp|}\cdot\nabla\varphi(z)\,dz = -3P(E)\int_D\varphi(z)\,dz$$

for arbitrary $\varphi\in C_c^\infty(\mathrm{int}(D)\setminus\Sigma(f))$. This is (2.3). □

Graphs of the form $x=h(y,t)$**.** We denote points in $\mathbb{H}^1=\mathbb{R}\times\mathbb{R}^2$ by (x,ζ) with $x\in\mathbb{R}$ and $\zeta=(y,t)\in\mathbb{R}^2$. Let E be a convex set in $\mathbb{H}^1$ of the form (1.10). We define the characteristic set of the convex function $h:F\to\mathbb{R}$ as the set $\Sigma(h)$ of the points $\zeta\in F$ such that the horizontal plane spanned by the vector fields X and Y at the point $p=(h(\zeta),\zeta)\in\mathbb{R}\times\mathbb{R}^2$ is a supporting plane for E (i.e. the plane does not intersect the interior of E). By Theorem 6.1 in the Appendix, $\Sigma(h)$ is contained in the union of at most two line segments.

Proposition 2.2 (Curvature equation II). *Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set with perimeter $P(E)$ and volume $|E|$. Then the function $h : F \to \mathbb{R}$ satisfies in distributional sense the partial differential equation*

$$(\partial_y - 2h\partial_t)\Big(\frac{u_1}{|u|}\Big) - 2y\partial_t\Big(\frac{u_2}{|u|}\Big) = H \tag{2.8}$$

in $\mathrm{int}(F) \setminus \Sigma(h)$, *with* $H = 3P(E)/4|E|$ *and* $u = (u_1, u_2) = (h_y - 2hh_t, 1 - 2yh_t)$.

Proof. Notice that $u \in BV_{\mathrm{loc}}(\mathrm{int}(F);\mathbb{R}^2)$ and moreover, by Proposition 6.3 in the Appendix, for each compact set $K \subset (\mathrm{int}(F) \setminus \Sigma(h))$, there exists $\delta > 0$ such that $|u| \geq \delta$ $\mathcal{L}^2$-a.e. in K. Hence $u/|u| \in BV_{\mathrm{loc}}(\mathrm{int}(F) \setminus \Sigma(h);\mathbb{R}^2)$.

Let $\varphi \in C_c^\infty(\mathrm{int}(F) \setminus \Sigma(h))$ and for any $\varepsilon \in \mathbb{R}$ consider the set

$$E_\varepsilon = \big\{(x,\zeta) \in \mathbb{H}^1 \mid \zeta \in F,\, h(\zeta) + \varepsilon\varphi(\zeta) \leq x \leq k(\zeta)\big\}.$$

We let $P(\varepsilon) = P(E_\varepsilon)$, $V(\varepsilon) = |E_\varepsilon|$ and $R(\varepsilon) = P(\varepsilon)^4/V(\varepsilon)^3$. Denoting by $S_\varepsilon = \{(h(\zeta) + \varepsilon\varphi(\zeta), \zeta) \in \mathbb{H}^1 \mid \zeta \in F\}$ the graph of $h + \varepsilon\varphi$ and by ν_ε the exterior unit normal to ∂E_ε, from the Heisenberg Area Formula (2.1) and from the standard Area Formula we get

$$\begin{aligned} P'(\varepsilon) &= \frac{d}{d\varepsilon}\int_{S_\varepsilon} \sqrt{(X\cdot\nu_\varepsilon)^2 + (Y\cdot\nu_\varepsilon)^2}\, d\mathcal{H}^2 \\ &= \frac{d}{d\varepsilon}\int_F \Big(\big(1 - 2y(h_t + \varepsilon\varphi_t)\big)^2 + \big((h_y + \varepsilon\varphi_y) - 2(h+\varepsilon\varphi)(h_t + \varepsilon\varphi_t)\big)^2\Big)^{1/2} d\zeta. \end{aligned}$$

Because we have $(h_y - 2hh_t)^2 + (1 - 2yh_t)^2 \geq \delta^2 > 0$ $\mathcal{L}^2$-a.e. in a neighbourhood of $\mathrm{spt}(\varphi)$, we can differentiate under the integral sign and we obtain

$$P'(0) = \int_F \frac{(h_y - 2hh_t)(\varphi_y - 2(\varphi h)_t) - 2y\varphi_t(1 - 2yh_t)}{\sqrt{(h_y - 2hh_t)^2 + (1 - 2yh_t)^2}}\, d\zeta$$

at $\varepsilon = 0$. An integration by parts yields

$$P'(0) = -\int_F \varphi\, d\Big((\partial_y - 2h\partial_t)\Big(\frac{u_1}{|u|}\Big) - 2y\partial_t\Big(\frac{u_2}{|u|}\Big)\Big),$$

where the derivatives of $u/|u|$ are measures. If E minimizes the isoperimetric ratio then, as in the proof of Proposition 2.1, we get

$$4|E|\int_F \varphi\, d\Big((\partial_y - 2h\partial_t)\Big(\frac{u_1}{|u|}\Big) - 2y\partial_t\Big(\frac{u_2}{|u|}\Big)\Big) = 3P(E)\int_F \varphi\, d\zeta$$

for arbitrary $\varphi \in C_c^\infty(\mathrm{int}(F) \setminus \Sigma(h))$. This is (2.8). □

3. Improved regularity of the boundary

In this section, we prove a regularity result for vector fields with bounded variation in the plane arising from the parameterization of the boundary of convex isoperimetric sets.

Let $\Omega \subset \mathbb{R}^2$ be an open set and let $a, b \in C(\Omega)$ be continuous functions. We consider the differential operator $\mathcal{M}$ acting on $BV(\Omega;\mathbb{R}^2)$ defined by

$$\mathcal{M}u = (\partial_1 - a\partial_2)u_1 + b\partial_2 u_2. \tag{3.1}$$

In general, $\mathcal{M}u$ is a measure in Ω. We have $\mathcal{M} = \operatorname{div}$ when $a = 0$ and $b = 1$. When $a = 2h$ and $b = -2y$, $\mathcal{M}$ is the operator appearing in the left hand side of (2.8).

We recall that a vector field $u \in L^1(\Omega;\mathbb{R}^2)$ has approximate limit $\bar{u}(q) \in \mathbb{R}^2$ at the point $q \in \Omega$ if

$$\lim_{r\downarrow 0} \fint_{B(q,r)} |u - \bar{u}(q)|\, d\mathcal{L}^2 = 0. \tag{3.2}$$

Here, $B(q,r)$ denotes the open Euclidean ball of radius r centered at q. The approximate discontinuity set of u is the set S_u of points in Ω at which u has no approximate limit. The jump set of u is the set J_u of points $q \in \Omega$ for which there exist $u^+(q), u^-(q) \in \mathbb{R}^2$ and $\nu_u(q) \in S^1$ such that $u^+(q) \neq u^-(q)$ and

$$\lim_{r\downarrow 0} \fint_{B^\pm(q,r)} |u - u^\pm(q)|\, d\mathcal{L}^2 = 0, \tag{3.3}$$

where $B^\pm(q,r) = \big\{q' \in B(q,r) \mid \pm(q'-q)\cdot\nu_u(q) > 0\big\}$. Finally, the precise representative $u^* : \Omega \to \mathbb{R}^2$ of u is:

$$u^*(q) = \begin{cases} \bar{u}(q) & q \in \Omega \setminus S_u, \\ \dfrac{1}{2}\big(u^+(q) + u^-(q)\big) & q \in J_u, \\ 0 & q \in S_u \setminus J_u. \end{cases} \tag{3.4}$$

By the Lebesgue density theorem, it is $u = u^*$ $\mathcal{L}^2$-a.e. in Ω.

Theorem 3.1 (Improved regularity)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, let $u = (u_1, u_2) \in BV(\Omega;\mathbb{R}^2)$, and let $a, b \in C(\Omega)$ be continuous functions such that $b \neq 0$ in Ω. Assume that:*

(i) *There exists $\delta > 0$ such that $|u| \geq \delta$ $\mathcal{L}^2$-a.e. in Ω;*
(ii) *$\mathcal{M}u^\perp \in L^1(\Omega)$, where $u^\perp = (-u_2, u_1)$;*
(iii) *$\mathcal{M}\Big(\dfrac{u}{|u|}\Big) \in L^1(\Omega)$.*

Then $u/|u| \in W^{1,1}(\Omega;\mathbb{R}^2)$ and there exists a function $\mu : J_u \to (0,+\infty)$, such that $u^- = \mu u^+$ on J_u.

Proof. If $u \in BV(\Omega;\mathbb{R}^2)$ the measure Du has the decomposition $Du = D^a u + D^s u$ with $D^a u \ll \mathcal{L}^2$ and $D^s u \perp \mathcal{L}^2$. $D^a u = \nabla u\, \mathcal{L}^2$ is the absolutely continuous part of the measure, with $\nabla u \in L^1(\Omega; M^{2\times 2})$ and $M^{2\times 2}$ denotes the space of 2×2 matrices with real entries. Moreover, it is $D^s u = D^c u + D^j u$, where $D^c u = D^s u \llcorner \Omega \setminus S_u$ is the Cantor part and $D^j u = D^s u \llcorner S_u$ is the jump part. By the representation theorem of Federer–Vol'pert and by Alberti's rank one theorem, the measures $D^c u$ and $D^j u$ admit the representations

$$D^j u = (u^+ - u^-)\otimes \nu_u\, \mathcal{H}^1 \llcorner J_u \quad \text{and} \quad D^c u = \eta\otimes\xi\, |D^c u|, \tag{3.5}$$

where $|D^c u|$ is the total variation of $D^c u$ and η, $\xi : \Omega \to S^1$ are suitable Borel maps. The measure $|D^c u|$ is absolutely continuous with respect to $\mathcal{H}^1$. The set J_u is an $\mathcal{H}^1$-rectifiable Borel subset of S_u with $\mathcal{H}^1(S_u \setminus J_u) = 0$. For these and related results on BV vector fields we refer the reader to the monograph [AFP].

We claim that both the jump and the Cantor parts of $D(u/|u|)$ vanish. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth mapping such that

$$F(0) = 0, \quad F(q) = \frac{q}{|q|} \text{ for } |q| \ge \frac{\delta}{2} \quad \text{and} \quad |\nabla F| \in L^\infty(\mathbb{R}^2). \tag{3.6}$$

If $|q| > \delta/2$, the derivative of F is

$$\nabla F(q) = \frac{1}{|q|^3} q^\perp \otimes q^\perp. \tag{3.7}$$

By (i), the vector field $v = F\circ u$ is in $BV(\Omega;\mathbb{R}^2)$ (this fact is implicitly assumed in (iii)).

By the chain rule for BV vector fields, we have

$$D^a v = \nabla F(u^*)\nabla u\, \mathcal{L}^2 \quad \text{and} \quad D^c v = (\nabla F(u^*)\eta)\otimes\xi\, |D^c u|. \tag{3.8}$$

A point $q\in\Omega$ belongs to J_v if and only if $q \in J_u$ and $F(u^+(q)) \ne F(u^-(q))$. Moreover, it is $D^j v = \big(F(u^+) - F(u^-)\big)\otimes \nu_u\, \mathcal{H}^1 \llcorner J_u$. Notice that $|u^\pm(q)| \ge \delta$ for all $q \in J_u$, by (i). Then, the jump set of v is

$$J_v = \big\{ q \in J_u \mid u^+(q)/|u^+(q)| \ne u^-(q)/|u^-(q)| \big\} \tag{3.9}$$

and the jump part of Dv is

$$D^j v = \left(\frac{u^+}{|u^+|} - \frac{u^-}{|u^-|}\right)\otimes \nu_u\, \mathcal{H}^1 \llcorner J_u. \tag{3.10}$$

By assumption (ii), the part of the measure $-\partial_1 u_2 + b\partial_2 u_1 + a\partial_2 u_2$ concentrated on J_u vanishes. From the formula for $D^j u$ in (3.5), we can compute $D^j_k u_l$ for $k,l = 1,2$, and we obtain

$$\big((u_2^- - u_2^+), b(u_1^+ - u_1^-) + a(u_2^+ - u_2^-)\big)\cdot \nu_u = 0 \tag{3.11}$$

$\mathcal{H}^1$-a.e. on J_u.

By assumption (iii), the part of the measure $(\partial_1 - a\partial_2)(u_1/|u|) + b\partial_2(u_2/|u|)$ concentrated on J_v vanishes. Thus, using (3.10), we can compute $D^j_k(u_l/|u|)$ for $k, l = 1, 2$, and we get

$$\left(\frac{u_1^+}{|u^+|} - \frac{u_1^-}{|u^-|}, \frac{au_1^- - bu_2^-}{|u^-|} - \frac{au_1^+ - bu_2^+}{|u^+|}\right) \cdot \nu_u = 0 \tag{3.12}$$

$\mathcal{H}^1$-a.e. on J_v.

From (3.12) and (3.11), we deduce that there exists $\lambda \in \mathbb{R}$ such that the following system of equations is satisfied $\mathcal{H}^1$-a.e. on J_v:

$$\begin{cases} \dfrac{u_1^+}{|u^+|} - \dfrac{u_1^-}{|u^-|} = -\lambda(u_2^+ - u_2^-) \\ \dfrac{bu_2^+ - au_1^+}{|u^+|} - \dfrac{bu_2^- - au_1^-}{|u^-|} = \lambda\big(b(u_1^+ - u_1^-) + a(u_2^+ - u_2^-)\big). \end{cases}$$

By elementary linear algebra, using $b \neq 0$, we obtain the equivalent system

$$\begin{cases} \dfrac{u_1^+}{|u^+|} - \dfrac{u_1^-}{|u^-|} = -\lambda(u_2^+ - u_2^-) \\ \dfrac{u_2^+}{|u^+|} - \dfrac{u_2^-}{|u^-|} = \lambda(u_1^+ - u_1^-) \end{cases} \quad \Leftrightarrow \quad \left(\frac{u^+}{|u^+|} - \frac{u^-}{|u^-|}\right) \cdot \big(u^+ - u^-\big) = 0,$$

that is $u^- \cdot u^+ = |u^-||u^+|$, $\mathcal{H}^1$-a.e. on J_v. It follows that $u^- = \mu u^+$ $\mathcal{H}^1$-a.e. on J_v for some $\mu : J_v \to (0, +\infty)$, and then $u^+/|u^+| = u^-/|u^-|$ $\mathcal{H}^1$-a.e. on J_v. This proves that $\mathcal{H}^1(J_v) = 0$ and thus $D^j v = 0$.

By assumption (ii), the Cantor part of the measure $-\partial_1 u_2 + b\partial_2 u_1 + a\partial_2 u_2$ vanishes. From $D^c u = \eta \otimes \xi\, |D^c u|$, we can compute $D^c_k u_l$ for $k, l = 1, 2$, and we find

$$(-\eta_2, b\eta_1 + a\eta_2) \cdot \xi = 0 \tag{3.13}$$

$|D^c u|$-a.e. on Ω. By assumption (iii), the Cantor part of the measure $(\partial_1 - a\partial_2)(u_1/|u|) + b\partial_2(u_2/|u|)$ vanishes, too. Thus, letting $\vartheta = (\vartheta_1, \vartheta_2) = ((u^*)^\perp \otimes (u^*)^\perp)\eta$, we can use (3.7) and (3.8) to compute $D^c_k(u_l/|u|)$ for $k, l = 1, 2$, and we find

$$\big(\vartheta_1, b\vartheta_2 - a\vartheta_1\big) \cdot \xi = 0 \tag{3.14}$$

$|D^c u|$-a.e. on Ω. From (3.13) and (3.14), we deduce that there exists $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \vartheta_1 = -\lambda\eta_2 \\ b\vartheta_2 - a\vartheta_1 = \lambda\big(b\eta_1 + a\eta_2\big) \end{cases} \quad \Leftrightarrow \quad \begin{cases} \vartheta_1 = -\lambda\eta_2 \\ \vartheta_2 = \lambda\eta_1. \end{cases}$$

Here we used $b \neq 0$. This, in turn, is equivalent with

$$\vartheta \cdot \eta = \big(\big((u^*)^\perp \otimes (u^*)^\perp\big)\eta\big) \cdot \eta = 0$$

$|D^c u|$-a.e. on Ω. Using the identity $\big(\big((u^*)^\perp \otimes (u^*)^\perp\big)\eta\big) \cdot \eta = \big((u^*)^\perp \cdot \eta\big)^2$, we deduce that $(u^*)^\perp \cdot \eta = 0$, and thus $\big((u^*)^\perp \otimes (u^*)^\perp\big)\eta = 0$ as well. By (3.7), this proves that $(\nabla F(u^*)\eta) \otimes \xi = 0$ $|D^c u|$-a.e. on Ω, whence $D^c v = 0$ by (3.8).

□

Theorem 3.1 has the following corollaries.

Corollary 3.2. *Let* $E \subset \mathbb{H}^1$ *be a convex isoperimetric set and let* $f : D \to \mathbb{R}$ *be the function in* (1.5)*. Then we have*

$$\frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|} \in W^{1,1}_{\mathrm{loc}}(\mathrm{int}(D) \setminus \Sigma(f); \mathbb{R}^2). \tag{3.15}$$

Proof. The vector field $u(z) = (u_1(z), u_2(z)) = \nabla f(z) + 2z^\perp$ satisfies $\operatorname{div} u^\perp = -4$ in $\mathrm{int}(D)$. By Proposition 6.2 in the Appendix, $|u| \geq \delta > 0$ on a compact set in $\mathrm{int}(D) \setminus \Sigma(f)$. Moreover, by Proposition 2.1, we have $\operatorname{div}(u/|u|) = 3P(E)/4|E|$ in $\mathrm{int}(D) \setminus \Sigma(f)$. The claim follows from Theorem 3.1 with $a = 0$ and $b = 1$. □

Corollary 3.3. *Let* $E \subset \mathbb{H}^1$ *be a convex isoperimetric set and let* $h : F \to \mathbb{R}$ *be the function in* (1.10)*. Then we have*

$$\frac{u}{|u|} \in W^{1,1}_{\mathrm{loc}}(\mathrm{int}(F) \setminus (\{y = 0\} \cup \Sigma(h)); \mathbb{R}^2), \tag{3.16}$$

with $u = (u_1, u_2) = (h_y - 2hh_t, 1 - 2yh_t)$.

Proof. We use Theorem 3.1 with $a = 2h$ and $b = -2y$. The vector field u is in $BV_{\mathrm{loc}}(\mathrm{int}(F); \mathbb{R}^2)$ and satisfies

$$\mathcal{M}u^\perp = (\partial_y - 2h\partial_t)(2yh_t - 1) - 2y\partial_t(h_y - 2hh_t) = 2h_t(1 + 2yh_t) \in L^\infty_{\mathrm{loc}}(\mathrm{int}(F)).$$

From Proposition 6.3 in the Appendix, it follows that $u/|u| \in BV_{\mathrm{loc}}(\mathrm{int}(F) \setminus \Sigma(h))$, and moreover, by Proposition 2.2, $u/|u|$ satisfies

$$\mathcal{M}\Big(\frac{u}{|u|}\Big) = (\partial_y - 2h\partial_t)\Big(\frac{u_1}{|u|}\Big) - 2y\partial_t\Big(\frac{u_2}{|u|}\Big) = \frac{3P(E)}{4|E|}$$

in $\mathrm{int}(F) \setminus \Sigma(h)$. Now the claim follows from Theorem 3.1. □

4. Integration of the curvature equation

In this section we solve equations (2.3) and (2.8). These equations can be integrated along a regular Lagrangian flow and the solutions are suitable arcs of circle.

We recall the definition of regular flow in the plane. Let $\Omega \subset \mathbb{R}^2$ be an open set and let $u \in BV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$. For $q \in \Omega$ and $\varrho > 0$ small enough define

$$C_q([-\varrho, \varrho]; \Omega) = \left\{ \gamma \in C([-\varrho, \varrho]; \Omega) \;\middle|\; \gamma(s) = q + \int_0^s u(\gamma(\sigma))\, d\sigma,\ s \in [-\varrho, \varrho] \right\}.$$

Let $K \subset \Omega$ be a compact set. An $\mathcal{L}^2$-measurable map $\Phi : K \to C([-\varrho, \varrho]; \Omega)$ is a Lagrangian flow starting from K relative to u if $\Phi(q) \in C_q([-\varrho, \varrho]; \Omega)$ for $\mathcal{L}^2$-a.e. $q \in K$. With abuse of notation, Φ is identified with the map $\Phi : K \times [-\varrho, \varrho] \to \Omega$,

$\Phi(q, s) = \Phi(q)(s)$. The flow is said to be regular if there exists a constant $m \geq 1$ such that

$$\frac{1}{m}\mathcal{L}^2(A) \leq \mathcal{L}^2(\Phi(A, s)) \leq m\mathcal{L}^2(A) \tag{4.17}$$

for all $\mathcal{L}^2$-measurable sets $A \subset K$ and for all $s \in [-\varrho, \varrho]$.

Theorem 4.1 (Foliation by circles I). *Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set and let $f : D \to \mathbb{R}$ be the function in (1.5). Let $K \subset \mathrm{int}(D) \setminus \Sigma(f)$ be a compact set and $\Omega \subset \mathrm{int}(D) \setminus \Sigma(f)$ an open neighborhood of K.*

For a sufficiently small $\varrho > 0$, there exists a (unique) regular Lagrangian flow $\Phi : K \times [-\varrho, \varrho] \to \Omega$ of the vector field $v(z) = 2z - \nabla f^{\perp}(z)$. Moreover, for $\mathcal{L}^2$-a.e. $z \in K$, the integral curve $s \mapsto \Phi(z, s)$ is an arc of circle with radius $4|E|/3P(E)$ oriented clockwise.

For a convex isoperimetric set of the form (1.10) there is an analogous statement.

Theorem 4.2 (Foliation by circles II). *Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set and let $h : F \to \mathbb{R}$ be the function in (1.10). Let $K \subset \mathrm{int}(F) \setminus \big(\{y = 0\} \cup \Sigma(h)\big)$ be a compact set and $\Omega \subset \mathrm{int}(F) \setminus \big(\{y = 0\} \cup \Sigma(h)\big)$ be an open neighborhood of K.*

For a sufficiently small $\varrho > 0$, there exists a (unique) regular Lagrangian flow $\Phi : K \times [-\varrho, \varrho] \to \Omega$ of the vector field $v(\zeta) = \big(1 - 2yh_t, 2yh_y - 2h\big)$, $\zeta = (y, t) \in F$. Moreover, for $\mathcal{L}^2$-a.e. $\zeta \in K$, the projection onto the xy-plane of the curve

$$s \mapsto (h(\Phi(\zeta, s)), \Phi(\zeta, s)) \in \mathbb{H}^1, \quad s \in [-\varrho, \varrho], \tag{4.18}$$

is an arc of circle with radius $4|E|/3P(E)$ oriented clockwise.

The existence of a (unique) regular Lagrangian flow stated in Theorems 4.1 and 4.2 follows from Ambrosio's theory on the Cauchy Problem for BV vector fields.

Theorem 4.3 (Ambrosio). *Let $\Omega \subset \mathbb{R}^2$ be an open set and assume that:*

(i) $u \in BV(\Omega; \mathbb{R}^2) \cap L^{\infty}(\Omega; \mathbb{R}^2)$;

(ii) $\mathrm{div}\, u \in L^{\infty}(\Omega)$.

Then, for any compact set $K \subset \Omega$ and for a small enough $\varrho > 0$, there exists a (unique) regular Lagrangian flow $\Phi : K \times [-\varrho, \varrho] \to \Omega$ starting from K relative to u.

The existence statement is Theorem 6.2 and the uniqueness statement is Theorem 6.4 in [A]. We do not need the uniqueness of the flow in our argument. Ambrosio's theory holds more generally for non autonomous vector fields in any space dimension.

The characterization of the flow in Theorems 4.1 and 4.2 is obtained on proving that a suitable reparameterization γ of a generic integral curve of the flow has a second order derivative which satisfies the equation $\ddot{\gamma} = -H\dot{\gamma}^{\perp}$ in a weak sense with $H = 3P(E)/4|E|$.

In order to make this reparameterization argument precise, let us consider an open set $\Omega \subset \mathbb{R}^2$ and a vector field v in Ω satisfying the assumptions of Theorem 4.3, that is

$$v \in BV(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2) \quad \text{and} \quad \operatorname{div} v \in L^\infty(\Omega). \tag{4.19}$$

The function v is defined pointwise, i.e. we choose a representative in the equivalence class of v. Our results hold independently of this choice. However, there is an exceptional set of points which may a priori depend on the representative.

Given a compact set $K \subset \Omega$ and a sufficiently small $\varrho > 0$, there exists a unique regular Lagrangian flow $\Phi : K \times [-\varrho, \varrho] \to \Omega$ starting from K relative to v. Let $\lambda : \Omega \to \mathbb{R}$ be a measurable function such that

$$0 < c_1 \le \lambda \le c_2 \quad \mathcal{L}^2\text{-a.e. in } \Omega. \tag{4.20}$$

Then, for $\mathcal{L}^2$-a.e. $q \in K$, the curve $s \mapsto \lambda(\Phi(q,s))$ is measurable and

$$c_1 \le \lambda(\Phi(q,s)) \le c_2 \quad \text{for a.e. } s \in [-\varrho, \varrho].$$

This follows from (4.17) by a Fubini-type argument. In fact, if $N \subset \Omega$ is an $\mathcal{L}^2$-negligible set, then $\Phi^{-1}(N) \subset K \times [-\varrho, \varrho]$ is also negligible. Thus, for $\mathcal{L}^2$-a.e. $q \in K$, the change of parameter $\sigma_q : [-\varrho, \varrho] \to [\sigma_q(-\varrho), \sigma_q(\varrho)]$ defined by

$$\sigma_q(s) = \int_0^s \lambda(\Phi(q,\xi))\, d\xi$$

is bi-Lipschitz, strictly increasing and admits therefore a bi-Lipschitz, strictly increasing inverse $\tau_q : [\sigma_q(-\varrho), \sigma_q(\varrho)] \to [-\varrho, \varrho]$, which satisfies

$$\dot{\tau}_q(s) = \frac{1}{\dot{\sigma}_q(\tau_q(s))} = \frac{1}{\lambda(\Phi(q, \tau_q(s)))} \quad \text{for a.e. } s \in [\sigma_q(-\varrho), \sigma_q(\varrho)]. \tag{4.21}$$

Consequently, for $\mathcal{L}^2$-a.e. $q \in K$, the curve $\gamma_q : [\sigma_q(-\varrho), \sigma_q(\varrho)] \to \Omega$ defined by

$$\gamma_q(s) = \Phi(q, \tau_q(s)) \tag{4.22}$$

is absolutely continuous and satisfies

$$\dot{\gamma}_q(s) = \frac{v(\gamma_q(s))}{\lambda(\gamma_q(s))} \quad \text{for a.e. } s \in [\sigma_q(-\varrho), \sigma_q(\varrho)], \tag{4.23}$$

i.e. γ_q is an integral curve of the vector field v/λ.

Theorem 4.4 (Chain rule for integral curves)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open set, $K \subset \Omega$ be a compact subset, $w \in W^{1,1}(\Omega;\mathbb{R}^2)$ and $\varrho > 0$ small enough. For $\mathcal{L}^2$-a.e. $q \in K$, the curve $w \circ \gamma_q$ with γ_q defined in (4.22) belongs to $W^{1,1}((\sigma_q(-\varrho), \sigma_q(\varrho));\mathbb{R}^2)$ and its weak derivative is $(\nabla w \circ \gamma_q)\dot{\gamma}_q$.*

Proof. Let $\{w_k\}_{k\in\mathbb{N}}$ be a sequence of smooth vector fields $w_k \in W^{1,1}(\Omega;\mathbb{R}^2)$ converging to w in $W^{1,1}(\Omega;\mathbb{R}^2)$. The map $(q,s)\mapsto w(\Phi(q,s))$ is measurable and integrable on $K\times[-\varrho,\varrho]$. By Fubini's theorem and by the bounded volume distortion property (4.17) of the flow, we have

$$\int_K\int_{-\varrho}^{\varrho} |w_k(\Phi(q,s)) - w(\Phi(q,s))|\,ds\,dq \le m\int_{-\varrho}^{\varrho}\int_{\Phi(K,s)} |w_k(q)-w(q)|\,dq\,ds,$$

and

$$\begin{aligned}\int_K\int_{-\varrho}^{\varrho} \big|\big(\nabla w_k(\Phi(q,s)) - \nabla w(\Phi(q,s))\big)v(\Phi(q,s))\big|\,ds\,dq \le\\ \le \|v\|_\infty \int_{-\varrho}^{\varrho}\int_K |\nabla w_k(\Phi(q,s)) - \nabla w(\Phi(q,s))|\,dq\,ds\\ \le m\|v\|_\infty \int_{-\varrho}^{\varrho}\int_{\Phi(K,s)} |\nabla w_k(q) - \nabla w(q)|\,dq\,ds.\end{aligned}$$

The curve $s\mapsto w_k(\Phi(q,s))$ is Lipschitz, for all $k\in\mathbb{N}$ and for $\mathcal{L}^2$-a.e. $q\in K$, with derivative $s\mapsto \nabla w_k(\Phi(q,s))v(\Phi(q,s))$. Passing to a subsequence and relabelling if necessary, we conclude that

$$\lim_{k\to\infty} w_k(\Phi(q,\cdot)) = w(\Phi(q,\cdot)) \quad \text{in } W^{1,1}((-\varrho,\varrho);\mathbb{R}^2)$$

for $\mathcal{L}^2$-a.e. $q\in K$, and that the weak derivative of $w(\Phi(q,\cdot))$ is $\nabla w(\Phi(q,\cdot))v(\Phi(q,\cdot))$. Combining this observation, (4.21) and (4.23), we get for any $\varphi\in C_c^\infty((\sigma_q(-\varrho),\sigma_q(\varrho));\mathbb{R}^2)$,

$$\begin{aligned}\int_{\sigma_q(-\varrho)}^{\sigma_q(\varrho)} w(\gamma_q(s))\dot\varphi(s)\,ds &= \int_{-\varrho}^{\varrho} w(\Phi(q,s))\dot\varphi(\sigma_q(s))\dot\sigma_q(s)\,ds\\ &= -\int_{-\varrho}^{\varrho} \nabla w(\Phi(q,s))v(\Phi(q,s))\,\varphi(\sigma_q(s))\,ds\\ &= -\int_{\sigma_q(-\varrho)}^{\sigma_q(\varrho)} \nabla w(\gamma_q(s))\dot\gamma_q(s)\,\varphi(s)\,ds.\end{aligned}$$

Hence $w\circ\gamma_q\in W^{1,1}((\sigma_q(-\varrho),\sigma_q(\varrho));\mathbb{R}^2)$, and its weak derivative is $(\nabla w\circ\gamma_q)\dot\gamma_q$.

□

Proof of Theorem 4.1. Let $f: D\to\mathbb{R}$ be the convex function in (1.5). We consider the vector fields $v(z) = 2z - \nabla f^\perp(z)$ and $u(z) = v^\perp(z) = \nabla f(z) + 2z^\perp$ which are both in $BV_{\mathrm{loc}}(\mathrm{int}(D);\mathbb{R}^2)\cap L^\infty_{\mathrm{loc}}(\mathrm{int}(D);\mathbb{R}^2)$. We have

$$\operatorname{div} v = 4 + f_{yx} - f_{xy} = 4. \tag{4.24}$$

If $K\subset \mathrm{int}(D)\setminus\Sigma(f)$ is a compact set and $\Omega\subset\subset \mathrm{int}(D)\setminus\Sigma(f)$ is a suitable open neighborhood of K, the existence of a regular Lagrangian flow $\Phi: K\times[-\varrho,\varrho]\to\Omega$ for some $\varrho>0$ is guaranteed by Theorem 4.3.

The function $\lambda : \Omega \to \mathbb{R}$, $\lambda = |v|$, satisfies $0 < c_1 \le \lambda \le c_2$ $\mathcal{L}^2$-a.e. in Ω by Proposition 6.2 in the Appendix. By Corollary 3.2, the vector field $w = v/\lambda$ belongs to $W^{1,1}(\Omega;\mathbb{R}^2)$, and $\operatorname{div} w^\perp = H$ $\mathcal{L}^2$-a.e. in Ω by (2.3), with $H = 3P(E)/4|E|$.

Let $\gamma_z : (\sigma_z(-\varrho), \sigma_z(\varrho)) \to \Omega$ be the integral curve of the vector field w defined in (4.22) and satisfying (4.23). We claim that γ_z parameterizes an arc of circle with curvature H. Since $\Phi(z,\cdot)$ is a reparameterization of γ_z, proving the claim concludes the proof of Theorem 4.1. The following identities hold a.e. in $(\sigma_z(-\varrho), \sigma_z(\varrho))$ for $\mathcal{L}^2$-a.e. $z \in K$:

(i) $|w \circ \gamma_z| = 1$;
(ii) $(w \cdot \partial_x w) \circ \gamma_z = (w \cdot \partial_y w) \circ \gamma_z = 0$;
(iii) $(\operatorname{div} w^\perp) \circ \gamma_z = H$.

Here, we are using the fact that if $F, G : \Omega \to \mathbb{R}$ are $\mathcal{L}^2$-measurable functions such that $F = G$ $\mathcal{L}^2$-a.e. in Ω, then it is $F \circ \Phi = G \circ \Phi$ $\mathcal{L}^3$-a.e. in $K \times [-\varrho, \varrho]$, which is a consequence of (4.17).

By Theorem 4.4, it is $\gamma_z \in W^{2,1}((\sigma_z(-\varrho), \sigma_z(\varrho));\mathbb{R}^2)$ with $\ddot\gamma_z = (\nabla w \circ \gamma_z)\dot\gamma_z$. Using (ii) and (iii), we compute

$$\ddot\gamma_z \cdot \dot\gamma_z^\perp = (-\operatorname{div} w^\perp) \circ \gamma_z = -H.$$

Moreover, by (i) we also have $\ddot\gamma_z \cdot \dot\gamma_z = 0$ a.e. in $(\sigma_z(-\varrho), \sigma_z(\varrho))$. Then $\ddot\gamma_z = -H\dot\gamma_z^\perp$ a.e. and this implies that γ_z is of class C^∞. The claim follows. □

Proof of Theorem 4.2. Let $h : F \to \mathbb{R}$ be the convex function in (1.10). We use the variables $\zeta = (y,t) \in F$. The vector field $v : F \to \mathbb{R}^2$

$$v(\zeta) = \big(1 - 2yh_t, 2yh_y - 2h\big), \tag{4.25}$$

is in $BV_{\mathrm{loc}}(\mathrm{int}(F);\mathbb{R}^2) \cap L^\infty_{\mathrm{loc}}(\mathrm{int}(F);\mathbb{R}^2)$, and

$$\operatorname{div} v = -4h_t - 2yh_{ty} + 2yh_{yt} = -4h_t \in L^\infty_{\mathrm{loc}}(\mathrm{int}(F)). \tag{4.26}$$

The vector field v is the projection onto the yt-plane of the vector field

$$(y,t) \mapsto (h_y - 2hh_t)\partial_x + (1 - 2yh_t)\partial_y + (2yh_y - 2h)\partial_t, \tag{4.27}$$

which is both horizontal and tangent to the graph of h at $\mathcal{H}^2$-a.e. point. We denote by $u = (h_y - 2hh_t, 1 - 2yh_t)$ the projection of the vector field (4.27) onto the xy-plane. The relation between v and u is

$$v = \begin{pmatrix} 0 & 1 \\ 2y & -2h \end{pmatrix} u. \tag{4.28}$$

In (4.28), we think of v and u as column vectors. The function $\lambda : \Omega \to \mathbb{R}$, $\lambda = |u|$, satisfies $0 < c_1 \le \lambda \le c_2$ $\mathcal{L}^2$-a.e. in Ω by Proposition 6.3 in the Appendix.

Let K and Ω be as in the statement of Theorem 4.2. The existence of a regular Lagrangian flow $\Phi : K \times [-\varrho, \varrho] \to \Omega$ of v follows from Theorem 4.3.

Let $F \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ be a mapping which satisfies (3.6) (with c_1 in place of δ). We consider the vector fields v/λ and $w = F \circ u$ in Ω. Then $w = u/|u|$ a.e. in Ω. We have $w \in W^{1,1}(\Omega; \mathbb{R}^2)$ by Corollary 3.3. Observing that $2yh_y - 2h = 2y(h_y - 2hh_t) + 2h(2yh_t - 1)$, we also get that $v/\lambda \in W^{1,1}(\Omega; \mathbb{R}^2)$. We claim that

$$\nabla w \, \frac{v}{\lambda} = -Hw^\perp \qquad \mathcal{L}^2\text{-a.e. in } \Omega, \tag{4.29}$$

with $H = 3P(E)/4|E|$. Indeed, using (3.7), we compute

$$\begin{aligned} \nabla w \, \frac{v}{\lambda} &= (\nabla F \circ u) \, \nabla u \, \frac{v}{\lambda} = \frac{1}{|u|^4} \big(u^\perp \otimes u^\perp\big) \nabla u \, v \\ &= \frac{1}{|u|^3} \big(u^\perp \cdot (\nabla u \, v)\big) w^\perp. \end{aligned}$$

On the other hand, by (2.8) we have $\mathcal{M}w = H$ in Ω, where $\mathcal{M}$ is the differential operator appearing in (3.1) with $a = 2h$ and $b = -2y$. Let B be the 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ -2h & -2y \end{pmatrix},$$

Using (3.7) we find

$$\mathcal{M}w = \operatorname{tr}\big((\nabla F \circ u)\nabla u B\big) = \frac{1}{|u|^3} \operatorname{tr}\big((u^\perp \otimes u^\perp)\nabla u B\big) = \frac{1}{|u|^3} u^\perp \cdot (u^\perp \nabla u B),$$

where, by a short computation based on (4.28), we have $u^\perp \cdot (u^\perp \nabla u B) = -u^\perp \cdot (\nabla u \, v)$. This ends the proof of (4.29).

Denote by $\gamma_\zeta : [\sigma_\zeta(-\varrho), \sigma_\zeta(\varrho)] \to \Omega$ the integral curve of v/λ defined in (4.22). Then the curve $s \mapsto w(\gamma_\zeta(s))$ belongs to $W^{1,1}((\sigma_\zeta(-\varrho), \sigma_\zeta(\varrho)); \mathbb{R}^2)$ for $\mathcal{L}^2$-a.e. $\zeta \in K$, and its weak derivative is equal to $(\nabla w \circ \gamma_\zeta)\dot{\gamma}_\zeta$, by Theorem 4.4. Identity (4.29) holds along $\mathcal{L}^2$-a.e. curve γ_ζ. Hence, by Theorem 4.4 applied to w, we have

$$\frac{d}{ds}(w \circ \gamma_\zeta) = -H(w^\perp \circ \gamma_\zeta) \tag{4.30}$$

in the sense of weak derivatives.

The projection of the vector field $\zeta = (y, t) \mapsto (h(\zeta), \zeta)$ onto the xy-plane belongs to $W^{1,1}(\Omega; \mathbb{R}^2)$. Denote by $\kappa_\zeta : [\sigma_\zeta(-\varrho), \sigma_\zeta(\varrho)] \to \mathbb{R}^2$ the projection of the curve $(h(\gamma_\zeta), \gamma_\zeta)$ onto the xy-plane. This projection is a reparameterization of the curve in (4.18). We claim that κ_ζ parameterizes an arc of circle with curvature H oriented clockwise. By Theorem 4.4, for $\mathcal{L}^2$-a.e. $\zeta \in K$ we have $\kappa_\zeta \in W^{1,1}((\sigma_\zeta(-\varrho), \sigma_\zeta(\varrho)); \mathbb{R}^2)$, and a short computation shows that the weak derivative of κ_ζ is $\dot{\kappa}_\zeta = w \circ \gamma_\zeta$. From (4.30), we deduce that $\kappa_\zeta \in W^{2,1}((\sigma_\zeta(-\varrho), \sigma_\zeta(\varrho)); \mathbb{R}^2)$ and $\ddot{\kappa}_\zeta = -H\dot{\kappa}_\zeta^\perp$. This implies that κ_ζ is of class C^∞ and the claim follows. □

5. Characterization of convex isoperimetric sets

In this section, we use Theorems 4.1 and 4.2 to characterize convex isoperimetric sets. We also use Theorem 6.1 in the Appendix on the structure of the characteristic set of bounded convex sets. Here, we recall some known facts about Carnot-Carathéodory geodesics in the Heisenberg group.

An absolutely continuous path $\gamma : [0,1] \to \mathbb{H}^1$ is said to be horizontal if $\dot\gamma(s) \in \mathrm{span}_{\mathbb{R}}\{X(\gamma(s)), Y(\gamma(s))\}$ for a.e. $s \in [0,1]$. The curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is horizontal if and only if

$$\gamma_3(s) = \gamma_3(0) + 2\int_0^s \big(\dot\gamma_1\gamma_2 - \dot\gamma_2\gamma_1\big)\, d\sigma. \tag{5.1}$$

The plane curve $\kappa = (\gamma_1, \gamma_2)$ is the horizontal projection of γ. If $\kappa = (\gamma_1, \gamma_2)$ is a given plane absolutely continuous curve, then a curve $\gamma = (\kappa, \gamma_3)$ with γ_3 given by (5.1) for some $\gamma_3(0)$ is called a horizontal lift of κ. The standard sub-Riemannian length of the horizontal curve γ is

$$\mathrm{Length}(\gamma) = \int_0^1 |\dot\kappa(s)|\, ds, \tag{5.2}$$

where κ is the horizontal projection of γ and $|\dot\kappa|$ is the Euclidean length of $\dot\kappa \in \mathbb{R}^2$. The distance $d : \mathbb{H}^1 \times \mathbb{H}^1 \to [0, +\infty)$ is defined as the minimum of Length(γ) over all horizontal curves connecting given points. Geodesic curves (i.e. lenght minimizing curves) can be computed explicitly. A geodesic from $0 \in \mathbb{H}^1$ to $p = (z,t) \in \mathbb{H}^1$ is of the following form:

1) If $t = 0$ the geodesic is the line segment $\gamma(s) = (sz, 0)$, $s \in [0,1]$.
2) If $t > 0$ and $z \neq 0$, the geodesic connecting 0 to $p = (z,t)$ is the horizontal lift starting at 0 of the arc of circle from 0 to z in the xy-plane oriented clockwise, such that the plane region bounded by the arc and the segment joining 0 to z has area equal to $t/4$. This geodesic is unique.
3) If $t > 0$ and $z = 0$, the geodesic from 0 to $p = (0,t)$ is not unique. Take any full circle oriented clockwise passing through 0 and with area equal to $t/4$. The horizontal lift of the circle starting from 0 is a geodesic.

The case $t < 0$ is similar. All other geodesics are obtained by left translation. If the arc of circle in 2) and 3) has radius $0 < R < +\infty$, we say that the geodesic has curvature $H = 1/R$.

The geodesics joining $(0, -\pi/2)$ to $(0, \pi/2)$ are of the type described in 3) above (up to a vertical translation). Their union bounds the isoperimetric set conjectured by Pansu. The horizontal lift of the plane circle $\kappa(s) = \frac{1}{2}\big(1 + \cos s, -\sin s\big)$, $s \in [-\pi, \pi]$, passing through the point $(1,0,0) \in \mathbb{H}^1$ at time $s = 0$ is the curve $\gamma : [-\pi, \pi] \to \mathbb{H}^1$

$$\gamma(s) = \frac{1}{2}\big(1 + \cos s, -\sin s, s + \sin s\big). \tag{5.3}$$

The third coordinate can be computed using formula (5.1). The curve γ is a geodesic with curvature $H = 2$, starting from $\gamma(-\pi) = (0, 0, -\pi/2)$ and reaching $\gamma(\pi) = (0, 0, \pi/2)$. If $(z, t) = \gamma(s) \in \gamma([-\pi, \pi])$ is a point on the curve, then we have

$$|z| = \Big(\frac{1 + \cos s}{2}\Big)^{1/2} \quad \text{and} \quad t = \frac{1}{2}\big(s + \sin s\big),$$

and we obtain the relation $|t| = \arccos |z| + |z|\sqrt{1 - |z|^2}$. The region $|t| \leq \arccos |z| + |z|\sqrt{1 - |z|^2}$ is the set E_{isop} in (1.4).

Corollary 5.1 (Foliation by geodesics I). *Under the assumptions of Theorem 4.1, for all $z \in K$, there is a geodesic $\gamma_z : [-\varrho, \varrho] \to \partial E$ with curvature $H = 3P(E)/4|E|$, such that $\gamma_z(0) = (z, f(z))$. Moreover, the length of γ_z is bounded from below by a positive constant depending on K.*

Proof. The vector field $v(z) = 2z - \nabla f^{\perp}(z)$ is the projection onto the xy-plane of the vector field

$$(x, y) \mapsto (f_y + 2x)\partial_x + (2y - f_x)\partial_y + (2xf_x + 2yf_y)\partial_t,$$

which is both horizontal and tangent to the graph of f at $\mathcal{H}^2$-a.e. point. Then the horizontal lift of the plane curve $s \mapsto \Phi(z, s)$ given by Theorem 4.1 is the curve

$$\gamma_z(s) = (\Phi(z, s), f(\Phi(z, s))), \quad s \in [-\varrho, \varrho].$$

By Theorem 4.1, this curve is a geodesic with curvature H and with length bounded from below by a positive constant depending on K. This curve exists for all $z \in K_0 \subset K$ with $\mathcal{L}^2(K \setminus K_0) = 0$. If $z \in K \setminus K_0$, there are points $z_n \in K_0$, $n \in \mathbb{N}$, such that $z_n \to z$ as $n \to +\infty$. We can use the theorem of Ascoli–Arzelà to extract a subsequence, such that, after relabeling, the curves γ_{z_n} converge uniformly to a curve $\gamma : [-\varrho, \varrho] \to \partial E$. This curve is a geodesic passing through $(z, f(z))$ at time $s = 0$, with curvature H and with length bounded from below by the same positive constant as above. □

Analogously, we have:

Corollary 5.2 (Foliation by geodesics II). *Under the assumptions of Theorem 4.2, for all $\zeta \in K$, there is a geodesic $\gamma_\zeta : [-\varrho, \varrho] \to \partial E$ with curvature H, such that $\gamma_\zeta(0) = (h(\zeta), \zeta)$. Moreover, the length of γ_ζ is bounded from below by a positive constant depending on K.*

We identify the horizontal plane spanned by the vector fields X and Y at the point $p = 0$ with the xy-plane

$$\mathrm{H}_0 = \big\{(z, 0) \in \mathbb{H}^1 \mid z \in \mathbb{C}\big\}.$$

In general, if $p=(z,t)\in\mathbb{H}^1$ we let

$$\mathrm{H}_p = p\mathrm{H}_0 = \big\{(z+z', t+2\mathrm{Im} z\bar z')\in\mathbb{H}^1 \mid z'\in\mathbb{C}\big\}.$$

The plane H_p is the boundary of the halfspaces

$$\begin{aligned}\mathrm{H}_p^- &= \big\{(z',t')\in\mathbb{H}^1 \mid t'\le t+2\mathrm{Im} z\bar z'\big\},\\ \mathrm{H}_p^+ &= \big\{(z',t')\in\mathbb{H}^1 \mid t'\ge t+2\mathrm{Im} z\bar z'\big\}.\end{aligned}\tag{5.4}$$

We say that a line $\ell\subset\mathbb{H}^1$ is horizontal if $\ell\subset \mathrm{H}_p$ for one (equivalently: all) $p\in\ell$. A horizontal segment is a segment of a horizontal line. A plane $\pi\subset\mathbb{H}^1$ is a supporting plane for a set $E\subset\mathbb{H}^1$ at a point $p\in\partial E$ if $\pi\cap\overline{E}\subset\partial E$.

Definition 5.3 (Characteristic set)**.** The characteristic set of a convex set $E\subset\mathbb{H}^1$ is $\Sigma(E)=\big\{p\in\partial E \mid \mathrm{H}_p \text{ is a supporting plane for } E \text{ at } p\big\}$.

The structure of $\Sigma(E)$ is described in Theorem 6.1 in the Appendix.

Proof of Theorem 1.1. The curvature $H=3P(E)/4|E|$ of a (convex) isoperimetric set E is homogeneous with degree -1 w.r.t. the dilations $\delta_\lambda(z,t)=(\lambda z,\lambda^2 t)$, $\lambda>0$. Thus, we can assume $H=2$ without loss of generality.

Claim 1. For all $p\in\partial E\setminus\Sigma(E)$, there is a curve of maximal length passing through p and contained in ∂E whose projection onto the xy-plane is an arc of circle with curvature H. One endpoint of the curve belongs to Σ^-, the other one to Σ^+.

Let $f: D\to\mathbb{R}$ be the function in (1.5), and let $h: F\to\mathbb{R}$ be the function in (1.10). Using a rotation of $\mathbb{R}^3$ fixing the t-axis, which is an isometry for the Carnot-Carathéodory metric, and a left translation in the y direction if necessary, we can assume that $p\in\partial E\setminus\Sigma(E)$ is either of the form

1) $p=(z,f(z))$ for some $z\in\mathrm{int}(D)\setminus\Sigma(f)$, or of the form
2) $p=(h(\zeta),\zeta)$ for some $\zeta\in\mathrm{int}(F)\setminus\big(\{y=0\}\cup\Sigma(h)\big)$.

The existence of a piece of geodesic with curvature H passing through p and contained in ∂E follows either from Corollary 5.1 or from Corollary 5.2. The existence of a curve with maximal length whose projection onto the xy-plane is an arc of circle with curvature H follows from a compactness argument. If one endpoint of the maximal curve does not belong to $\Sigma(E)$, then a continuation argument provides a proper extension of the curve. To prove this continuation, notice that two geodesics contained in ∂E, with the same curvature and meeting at one point in $\partial E\setminus\Sigma(E)$ coincide in a neighborhood of the point. Finally, the endpoints of the maximal curve cannot both belong to Σ^- or Σ^+.

Claim 2. The sets Σ^- and Σ^+ are points.

Assume by contradiction that Σ^- contains more than one point. According to Theorem 6.1, we have two cases:

1) Σ^- is a closed horizontal segment of positive length;
2) $\Sigma^- = \Sigma_1^- \cup \Sigma_2^-$ with Σ_1^- and Σ_2^- disjoint closed horizontal segments lying on the same horizontal line.

In case 1), after a left translation and a rotation fixing the t-axis, we have $\Sigma^- = \{(0, y, 0) \in \mathbb{H}^1 \mid |y| \le y_0\}$ for some $y_0 > 0$. Moreover, letting $p_0 = (0, y_0, 0)$, by the discussion of case 1) in the proof of Theorem 6.1, we have

$$E \subset \mathrm{H}^+_{p_0} \cap \mathrm{H}^+_{-p_0}. \tag{5.5}$$

Claim 1 combined with an approximation-compactness argument shows that there is a geodesic $\gamma : [0, L] \to \partial E$ parameterized proportionally to arc-length, with curvature $H = 2$ and such that $\gamma(0) = 0$. We have $\dot\gamma(0) = (\alpha, \beta, 0)$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1/4$. Assume $\alpha = 0$ and $\beta = 1/2$ (if $\alpha \ne 0$, the proof is easier). Then, by (5.3), we have

$$\gamma(s) = \frac{1}{2}\big(1 - \cos s, \sin s, s - \sin s\big) \in \partial E \quad \text{for all } s \in [0, L]. \tag{5.6}$$

However, for any $c_0 > 0$, there is $\delta > 0$ with $s - \sin s < c_0(1 - \cos s)$ for all $s \in (0, \delta)$. This contradicts (5.5).

Case 2) cannot occur. In fact, denoting by $\widehat{\Sigma}$ the nonempty open segment between Σ_1^- and Σ_2^-, we have $\widehat{\Sigma} \subset \partial E$. By Claim 1, for any $p \in \widehat{\Sigma}$ there is a geodesic contained in ∂E with curvature 2 which passes through $p \in \widehat{\Sigma}$ and has one endpoint in Σ^-. This is impossible.

Claim 3. Up to a left translation, we have $E = E_{\mathrm{isop}}$ as in (1.4).

After a left translation, we can assume that $\Sigma^- = \{(0, -\pi/2)\}$ and $\Sigma^+ = \{(z, t)\}$ with $t > -\pi/2$. This forces $z = 0$ and $t = \pi/2$, otherwise there would be at most one geodesic with curvature $H = 2$ connecting Σ^- to Σ^+.

We claim that $0 \in \mathrm{int}(D)$. If, by contradiction, $0 \in \partial D$, then $\{(0, t) \in \mathbb{H}^1 \mid |t| \le \pi/2\}$ is contained in ∂E. But there is no geodesic with curvature $H = 2$ which starts from Σ^- and passes through a point $(0, t)$ with $|t| < \pi/2$. This contradicts Claim 1.

For all $z \in \partial D$, there is a geodesic contained in ∂E, with endpoints $(0, -\pi/2)$, $(0, \pi/2)$, and which passes through $(z, f(z))$. The projection of this geodesic onto the xy-plane is a (full) circle with radius $1/2$ passing through the points 0 and z. Hence ∂E contains the surface of rotation around the t-axis generated by the curve (5.3), and the claim follows. □

6. Appendix

We describe the structure of the characteristic set $\Sigma(E)$ of a compact convex set $E \subset \mathbb{H}^1$ with nonempty interior. The notation is fixed before Definition 5.3.

Theorem 6.1 (Characteristic set)**.** *If $E \subset \mathbb{H}^1$ is a compact convex set with nonempty interior, then $\Sigma(E)$ admits the decomposition $\Sigma(E) = \Sigma^- \cup \Sigma^+$, where either*

1) *Σ^- (resp. Σ^+) is a compact horizontal segment (possibly one point); or,*
2) *$\Sigma^- = \Sigma_1^- \cup \Sigma_2^-$ (resp. $\Sigma^+ = \Sigma_1^+ \cup \Sigma_2^+$), where Σ_1^- and Σ_2^- (resp. Σ_1^+ and Σ_2^+) are disjoint compact horizontal segments lying on the same horizontal line.*

Moreover, if $0 \in \mathrm{int}(E)$, then H_0 separates Σ^- from Σ^+.

Proof. The proof is divided into a number of steps.

Claim 1. If $E \subset \mathbb{H}^1$ is a bounded strictly convex set, then $\Sigma(E)$ contains at most two points.

Let $p_0, p_1 \in \Sigma(E)$ be such that $p_0 \neq p_1$. After a left translation, we can assume that $p_0 = 0$ and $p_1 = (z_1, t_1)$ with $t_1 > 0$. E is contained between the horizontal planes H_{p_0} and H_{p_1}. By strict convexity, the segment $\{p_s = (sz_1, st_1) \in \mathbb{H}^1 \mid 0 < s < 1\}$ is contained in the interior of E. Moreover, we have

$$\partial E = \{p_0, p_1\} \cup \bigcup_{0<s<1} \partial E \cap \mathrm{H}_{p_s}.$$

By strict convexity, $\partial E \cap \mathrm{H}_{p_0} = \{p_0\}$ and $\partial E \cap \mathrm{H}_{p_1} = \{p_1\}$. Using the property $p \in \mathrm{H}_{p'}$ if and only if $p' \in \mathrm{H}_p$, we deduce that if $p \in \partial E \cap \mathrm{H}_{p_s}$ with $0 < s < 1$, then $p_s \in \mathrm{H}_p$ and H_p is not a supporting plane of E. It follows that $\Sigma(E) = \{p_0, p_1\}$.

Claim 2. If $E \subset \mathbb{H}^1$ is a bounded convex set with C^∞ boundary, then $\Sigma(E) \neq \emptyset$.

For $p \in \partial E$, let us denote by $V(p) \in \mathrm{T}_p\partial E$ the orthogonal projection of the vector field T onto $\mathrm{T}_p\partial E$ with respect to the left invariant inner product which makes X, Y, T orthonormal. Since ∂E is diffeomorphic with the sphere S^2, there exists $p \in \partial E$ such that $V(p) = 0$ and therefore $p \in \Sigma(E)$.

Claim 3. If $E \subset \mathbb{H}^1$ is a bounded strictly convex set with C^∞ boundary, then $\Sigma(E)$ contains exactly two points.

By *Claim 2* there exists $p_0 \in \Sigma(E)$. Without loss of generality we can assume that $p_0 = 0$ and $E \subset \mathrm{H}_0^+$. Let V be the vector field defined in the proof of *Claim 2*. It is $V(0) = 0$ and using the strict convexity of E it is not difficult to check that $\mathrm{index}(V, 0) = 1$ (we omit the details). If by contradiction V has no other zero on ∂E, then the Poincaré–Hopf index theorem would give $\mathrm{index}(V, 0) = \chi(\partial E) = \chi(S^2) = 2$, where χ denotes the Euler–Poincaré characteristic. This is not possible.

Claim 4. If $E \subset \mathbb{H}^1$ is a bounded convex set with nonempty interior, then $\Sigma(E)$ contains at least two points.

This follows from *Claim 3* by an approximation argument. Details are omitted.

Claim 5. We have $\Sigma(E) = \Sigma^- \cup \Sigma^+$ as in the statement of Theorem 6.1.

Without loss of generality, we assume that $0 \in \operatorname{int}(E)$. We define the convex sets

$$E^- = E \cap \mathrm{H}_0^- \quad \text{and} \quad E^+ = E \cap \mathrm{H}_0^+.$$

If $p \in \partial E^- \cap \mathrm{H}_0 = \partial E^+ \cap \mathrm{H}_0$ with $p \neq 0$, then $0 \in \mathrm{H}_p \neq \mathrm{H}_0$ and consequently $\mathrm{H}_p \cap \operatorname{int}(E^\pm) \neq \emptyset$. It follows that $\Sigma(E^\pm) \cap \mathrm{H}_0 = \{0\}$. By *Claim 4* there exist points p^-, p^+ such that $p^\pm \in \Sigma(E^\pm) \cap \operatorname{int}\left(\mathrm{H}_0^\pm\right)$. The relative interior of the line segment connecting 0 with p^-, respectively p^+, is contained in the interior of E^-, respectively E^+. A similar reasoning as in the proof of *Claim 1* gives

$$\Sigma(E^-) = \{0\} \cup \left(\Sigma(E^-) \cap \mathrm{H}_{p^-}\right) \quad \text{and} \quad \Sigma(E^+) = \{0\} \cup \left(\Sigma(E^+) \cap \mathrm{H}_{p^+}\right). \tag{6.1}$$

We consider E^- and we let $p^- = p_1$. Assume there exists $p_2 \in \Sigma(E^-) \cap \mathrm{H}_{p_1}$ with $p_2 \neq p_1$. Since the equality on the left hand side of (6.1) must hold with p_2 instead of $p_1 = p^-$, it follows that $\Sigma(E^-) \setminus \{0\}$ is contained in the horizontal line ℓ passing through p_1 and p_2. By a maximality argument, we can in fact assume that $\Sigma(E^-) \setminus \{0\}$ is contained in the compact sub-interval of ℓ with endpoints p_1 and p_2. Moreover, possibly interchanging p_1 and p_2, there are only two possible cases: either

1) $E^- \subset \mathrm{H}_{p_1}^+ \cap \mathrm{H}_{p_2}^+$; or,
2) $E^- \subset \mathrm{H}_{p_1}^+ \cap \mathrm{H}_{p_2}^-$.

In the first case, all points $p \in \ell$ between p_1 and p_2 are in $\Sigma(E^-)$, because H_p does not intersect the interior of $\mathrm{H}_{p_1}^+ \cap \mathrm{H}_{p_2}^+$. We conclude that there exists a compact, horizontal segment Σ^- such that $\Sigma(E^-) \setminus \{0\} = \Sigma^-$.

In the second case, there exists a maximal compact sub-interval Σ_1^- (resp. Σ_2^-) of ℓ such that $p_1 \in \Sigma_1^-$ and $p \in \Sigma(E^-)$ for all $p \in \Sigma_1^-$ (resp. $p_2 \in \Sigma_2^-$ and $p \in \Sigma(E^-)$ for all $p \in \Sigma_2^-$). On the other hand, there exists a point $p \in \ell$ between p_1 and p_2 such that $\mathrm{H}_p \cap \operatorname{int}(E^-) \neq \emptyset$. Hence $\Sigma_1^- \cap \Sigma_2^- = \emptyset$ and we let $\Sigma^- = \Sigma_1^- \cup \Sigma_2^-$.

Analogously, we get $\Sigma(E^+) \setminus \{0\} = \Sigma^+$ with Σ^+ as in the statement of the theorem. Note that neither Σ^- nor Σ^+ intersects H_0. Clearly, it is $\Sigma(E) = \Sigma^- \cup \Sigma^+$.

□

Let E be a compact convex set in $\mathbb{H}^1$ with nonempty interior of the form (1.5). We have the convex function $f : D \to \mathbb{R}$, where D is a compact and convex subset of $\mathbb{R}^2$ with nonempty interior. We define the characteristic set $\Sigma(f)$ of f as the set of points $z \in D$ such that the horizontal plane H_p at the point $p = (z, f(z)) \in \mathbb{R}^2 \times \mathbb{R}$ is a supporting plane for E.

An equation for the horizontal plane H_p, $p = (z, f(z))$, is $t' = f(z) - 2z^\perp \cdot z'$. Then, we have for $z \in \mathrm{int}(D)$

$$z \in \Sigma(f) \iff -2z^\perp \in \partial f(z), \tag{6.2}$$

where $\partial f(z)$ denotes the subdifferential of f at z.

Proposition 6.2 (Lower bounds I)**.** *Let $\Omega \subset \mathrm{int}(D)$ be an open set. Then, for any compact set $K \subset \Omega \setminus \Sigma(f)$ there is a constant $\delta > 0$ such that*

$$|\nabla f(z) + 2z^\perp| \geq \delta \tag{6.3}$$

for $\mathcal{L}^2$-a.e. $z \in K$.

Proof. We show that (6.3) holds at differentiability points $z \in K$ of f. By contradiction, assume that there exists a sequence $z_k \in K$, $k \in \mathbb{N}$, of differentiability points of f such that $|\nabla f(z_k) + 2z_k^\perp| \to 0$ as $k \to +\infty$. Possibly taking a subsequence, we have $z_k \to z$ for some $z \in K$. It follows that $-2z^\perp \in \partial f(z)$ and this implies $z \in \Sigma(f)$, contradicting the assumption $K \cap \Sigma(f) = \emptyset$. □

Now consider a convex set E of the form (1.10). We have the convex function $h : F \to \mathbb{R}$, with $F \subset \mathbb{R}^2$ convex set. We define the characteristic set $\Sigma(h)$ of h as the set of points $\zeta \in F$ such that the horizontal plane H_p at the point $p = (h(\zeta), \zeta) \in \mathbb{R} \times \mathbb{R}^2$ is a supporting plane for E.

An equation for the horizontal plane H_p, $p = (h(\zeta), \zeta)$ with $y \neq 0$, is

$$x' = h(\zeta) + \frac{h(\zeta)}{y}(y' - y) + \frac{1}{2y}(t' - t).$$

Then, we have for $\zeta = (y, t) \in \mathrm{int}(F)$ with $y \neq 0$

$$\zeta \in \Sigma(h) \iff \Big(\frac{h(\zeta)}{y}, \frac{1}{2y}\Big) \in \partial h(\zeta). \tag{6.4}$$

Proposition 6.3 (Lower bounds II)**.** *Let $\Omega \subset \mathrm{int}(F)$ be an open set. Then, for any compact set $K \subset \Omega \setminus \Sigma(h)$ there is a constant $\delta > 0$ such that*

$$|h_y - 2hh_t| + |1 - 2yh_t| \geq \delta \tag{6.5}$$

$\mathcal{L}^2$-a.e. in K.

Proof. If $r > 0$ is sufficiently small, we have a.e. in $K \cap \{|y| \leq r\}$

$$|1 - 2yh_y| \geq 1/2.$$

In fact, the derivatives of h are locally bounded. We claim that at differentiability points of h in $K \cap \{|y| \geq r\}$ we have

$$|h_y - h/y| + |h_t - 1/(2y)| \geq \delta_1$$

for some $\delta_1 > 0$. If this is not the case, arguing as in the proof of Proposition 6.2, we find $\zeta \in K \cap \{|y| \geq r\}$ such that $(h(\zeta)/y, 1/2y) \in \partial h(\zeta)$, which is not possible by (6.4). Hence there exists $\delta_2 > 0$ such that the condition $|h_t - 1/(2y)| \leq \delta_2$ implies $|h_y - h/y| \geq \delta_1/2$ and $|2h(h_t - 1/(2y))| \leq \delta_1/4$, whence

$$|h_y - 2hh_t| \geq |h_y - h/y| - |2h(h_t - 1/(2y))| \geq \delta_1/4.$$

The claim follows with $\delta = \min\{1/2,\, \delta_1/4,\, 2r\delta_2\}$. □

References

[A] Ambrosio, L.: Transport equation and Cauchy problem for BV vector fields. Invent. Math. **158** (2004), 227–260

[AFP] Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs, Oxford Science Publications, Clarendon Press, Oxford, 2000

[CDST] Capogna, L., Danielli D., Scott D. P., Tyson, J. T.: An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. Birkhäuser, 2007

[CHY1] Cheng, J.-H., Hwang, J.-F., Yang, P.: Existence and uniqueness for p-area minimizers in the Heisenberg group. Math. Ann. **337** (2007), no. 2, 253–293

[CHY2] Cheng, J.-H., Hwang, J.-F., Yang, P.: Regularity of C^1 smooth surfaces with prescribed p-mean curvature in the Heisenberg group, Preprint, 2007.

[CHMY] Cheng, J.-H., Hwang, J.-F., Malchiodi, A., Yang, P.: Minimal surfaces in pseudohermitian geometry. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 129–177

[DGN] Danielli, D., Garofalo, N., Nhieu, D.-M.: A partial solution of the isoperimetric problem for the Heisenberg group. To appear in Forum Math.

[FGW] Franchi, B., Gallot, S., Wheeden, R. L.: Sobolev and isoperimetric inequalities for degenerate metrics. Math. Ann. **300** (1994), 557–571

[FSSC] Franchi, B., Serapioni, R., Serra Cassano, F.: Rectifiability and perimeter in the Heisenberg group. Math. Ann. **321** (2001), 479–531

[GN] Garofalo, N., Nhieu, D.-M.: Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces. Comm. Pure Appl. Math. **49** (1996), 1081–1144

[LM] Leonardi, G. P., Masnou, S.: On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$. Ann. Mat. Pura Appl. (4) **184** (2005), 533–553

[LR] Leonardi, G. P., Rigot, S.: Isoperimetric sets on Carnot groups. Houston J. Math. **29** (2003), 609–637

[Mo1] Monti, R.: Brunn–Minkowski and isoperimetric inequality in the Heisenberg group. Ann. Acad. Sci. Fenn. Math. **28** (2003), 99–109

[Mo2] Monti, R.: Heisenberg isperimetric problems. The axial case. Adv. Calc. Var. **1** (2008), 93–121

[MM] Monti, R., Morbidelli, D.: Isoperimetric inequality in the Grushin plane. J. Geom. Anal. **14** (2004), 355–368

[MSC] Monti, R., Serra Cassano, F.: Surface measures in Carnot–Carathéodory spaces. Calc. Var. Partial Differ. Equ. **13** (2001), 339–376

[P1] Pansu, P.: Une inégalité isopérimétrique sur le groupe de Heisenberg. C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 127–130

[P2] Pansu, P.: An isoperimetric inequality on the Heisenberg group. Conference on differential geometry on homogeneous spaces (Turin, 1983). Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1983), 159–174

[Pa1] Pauls, S. D.: Minimal surfaces in the Heisenberg group. Geom. Dedicata **104** (2004), 201–231

[Pa2] Pauls, S. D.: H-minimal graphs of low regularity in $\mathbb{H}^1$. Comment. Math. Helv. **81** (2006), 337–381

[R] Ritoré, M.: A proof by calibration of an isoperimetric inequality in the Heisenberg group, work in progress

[RR1] Ritoré, M., Rosales, C.: Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group $\mathbb{H}^n$. J. Geom. Anal. **16** (2006), no. 4, 703–720

[RR2] Ritoré, M., Rosales, C.: Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. Preprint, 2005

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