Optimal Riemannian distances preventing mass transfer

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Abstract

We consider an optimization problem related to mass transportation: given two probabilities f^+ and f^- on an open subset $\Omega \subset \mathbb{R}^N$, we let vary the cost of the transport among all distances associated with conformally flat Riemannian metrics on Ω which satisfy an integral constraint (precisely, an upper bound on the L^1-norm of the Riemannian coefficient). Then, we search for an optimal distance which prevents as much as possible the transfer of f^+ into f^-: higher values of the Riemannian coefficient make the connection more difficult, but the problem is non-trivial due to the presence of the integral constraint. In particular, the existence of a solution is a priori guaranteed only on the relaxed class of costs, which are associated with possibly non-Riemannian Finsler metrics. Our main result shows that a solution does exist in the initial class of Riemannian distances.

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1 Introduction

The classical mass transport problem, introduced by Monge in [17], and reformulated by Kantorovich in [15, 16], has been widely investigated in recent years with a renewed interest (see, for instance, references [2, 3, 4, 8, 10, 11, 12, 13, 14, 18, 19]). It can be roughly described as follows: given two mass distributions, find the most efficient way to move one on the other. By efficiency it is intended that the mass transportation plan must minimize some average cost. In the original problem suggested by Monge, a pile of soil (which can be represented as a Borel probability measure f^+ on $\mathbb{R}^N$) was to be transported to some final configuration (given through a probability measure f^-). Monge wondered about the existence of a transportation map $T : \mathbb{R}^N \to \mathbb{R}^N$ minimizing the average work performed

$$\int_{\mathbb{R}^N} |x - T(x)| \, df^+(x),$$

among all the admissible transport maps T which send f^+ into f^-, *i.e.* $T_\# f^+ = f^-$, where $T_\#$ denotes the push-forward operator between measures.

Kantorovich's reformulation of the mass transportation problem consists in the following relaxation procedure: the minimum is now sought in the larger class of *admissible transport plans* (also known as stochastic transport maps). These are Borel probability measures ν defined on the product $\mathbb{R}^N \times \mathbb{R}^N$ whose marginals are precisely (f^+, f^-), that is,

$$f^+(E) = \nu(E \times \mathbb{R}^N), \qquad f^-(E) = \nu(\mathbb{R}^N \times E)$$

for every Borel subset E of $\mathbb{R}^N$. One then tries to minimize

$$\int\int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y| \, d\nu(x, y),$$

among such admissible plans. An admissible transport map T corresponds to a transport plan ν concentrated on the graph of T. Since the constraint appearing now in this relaxed version is linear, an optimal transport plan can always be shown to exist.

This problem finds a natural setting in a metric space (X, d): for a given pair (f^+, f^-) of Borel probability measures on X, the Kantorovich formulation of the mass transport problem reads as

$$\min \left\{ \int\int_{X\times X} c(x,y)\, d\nu(x,y) \ : \ \nu \text{ admissible plan} \right\} , \tag{1}$$

where $c(x, y)$ is a given nonnegative continuous function on $X \times X$, which represents the cost of transporting a point mass from x into y. The most studied situation is when the cost density $c(x, y)$ is a function of the distance d:

$$c(x,y) = \Phi(d(x,y)) \ , \qquad (x,y) \in X \times X \ ,$$

where $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and continuous. It is by now well known that the minimum (1) is realized by an optimal admissible plan. With the choice $\Phi(t) = t^p$, the quantity (1) (to the power $1/p$) is known as the p-Wasserstein distance between the measures f^+ and f^-. The case $p = 1$, the classical one considered by Monge, is related to several results in shape optimization theory (see [3, 4]); the case $p = 2$ is also widely studied for its implications in fluid mechanics (see [2]); the case $p < 1$, or more generally the case when $\Phi(t)$ is a concave function, seems to be the most realistic for several applications, and has been studied in [14].

In the present paper, we want to investigate an optimization problem which occurs when we are allowed to vary the distance d in a suitable admissible class. More precisely, we consider as X the closure $\overline{\Omega}$ of an open bounded subset Ω of the Euclidean space $\mathbb{R}^N$ with Lipschitz boundary. We let d vary among the distances generated by a conformally flat Riemannian metric in the following sense:

$$d_a(x,y) := \inf \left\{ \int_0^1 a(\gamma)|\gamma'|\, dt \ : \ \gamma \in \mathrm{Lip}(]0,1[;\Omega) \ , \ \gamma(0^+) = x \ , \ \gamma(1^-) = y \right\} . \tag{2}$$

The problem we are interested in is the following: for fixed marginals f^+ and f^-, we consider the cost functional

$$F(a) := \min \left\{ \int\int_{\overline{\Omega}\times\overline{\Omega}} \Phi(d_a(x,y))\, d\nu(x,y) \ : \ \nu \text{ admissible plan} \right\} , \tag{3}$$

defined for every nonnegative Borel coefficient $a(x)$. We want to prevent as much as possible the transportation of f^+ into f^-, by maximizing the cost $F(a)$ among all a belonging to the class

$$\mathcal{A} := \left\{ a(x) \text{ Borel measurable} \ : \ \alpha \le a(x) \le \beta \ , \ \int_\Omega a(x)\, dx \le m \right\} , \tag{4}$$

the constants α, β, m being positive numbers, satisfying the compatibility conditions

$$\alpha\,|\Omega| \le m \le \beta\,|\Omega| \ .$$

In the case when $\Phi(t) = t$, and $f^+ = \delta_x$, $f^- := \delta_y$ are Dirac masses concentrated on two fixed points $x, y \in \Omega$, the problem of maximizing F is nothing else than that of proving the existence of a conformally flat Euclidean metric whose length-minimizing geodesics joining x and y are as long as possible.

This problem seems to be unexplored in the literature on Calculus of Variations, though its study can be supported by natural motivations. Indeed, in many concrete examples, one can be interested in making as difficult as possible the communication between some masses f^+ and f^-. For instance, it is easy to imagine that this situation may arise in economics, or in medicine, or simply in traffic planning, each time the connection between two "enemies" is undesired. Of course, the problem is made non trivial by the integral constraint in (4), which has a physical meaning: it prescribes the quantity of material at one's disposal to solve the problem; in particular, it expresses that such quantity is finite. (On the other hand, the pointwise constraint in (4) is somehow of technical nature, as it is used to get compactness).

We would also like to point out that the similar problem of minimizing the cost functional $F(a)$ over the class $\mathcal{A}$, which corresponds to favor the transportation of f^+ into f^-, is trivial, since

$$\inf \Big\{ F(a) \ : \ a \in \mathcal{A} \Big\} = F(\alpha) \ .$$

In fact, it is enough to approximate f^+ and f^- by finite sums of weighted Dirac masses $f_n^+ = \sum_{i=1}^n p_i \delta_{x_i}$ and $f_n^- = \sum_{i=1}^n q_i \delta_{y_i}$, and to put $a(x) = \alpha$ in all Euclidean geodesic lines connecting every x_i to every y_j, with $a(x) = m$ elsewhere.

On the other hand, the existence of a solution for the maximization problem

$$\sup \Big\{ F(a) \ : \ a \in \mathcal{A} \Big\} \tag{5}$$

is a delicate matter. Indeed, maximizing sequences $\{a_n\} \subset \mathcal{A}$ could develop an oscillatory behavior producing only a relaxed solution. This phenomenon has been first pointed out in [1], and later investigated in more detail in [5, 7]. These works reveal that the traditional approach to attack the maximization problem (5), namely the direct methods of the Calculus of Variations, cannot be used to obtain the existence of a solution. Basically, the reason is that the class $\mathcal{A}$ is *not closed* with respect to the natural convergence which ensures the continuity of the functional F. Indeed, given a maximizing sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, it is not difficult to prove (see for instance [7]) that d_{a_n} converge uniformly on $\overline{\Omega} \times \overline{\Omega}$ to some distance d, and there holds

$$\lim_{n\to\infty} F(a_n) = \min \Big\{ \int\!\!\int_{\overline{\Omega}\times\overline{\Omega}} \Phi(d(x,y))\, d\nu(x,y) \ : \ \nu \text{ admissible plan } \Big\} \ .$$

Thus, if we could write $d = d_a$ for some $a \in \mathcal{A}$, we would have $\lim_{n\to\infty} F(a_n) = F(a)$, and a would be a solution to problem (5). The point is that the limit distance d in general cannot be associated with a Riemannian coefficient in the class $\mathcal{A}$. For instance, consider in dimension $N = 2$ a sequence of periodic coefficients $(a_n)_{n\in\mathbb{N}}$ of the form $a_n(x) = a(nx)$, where the function a takes only two different values $\beta > \alpha > 0$ respectively on the white and black squares of a chessboard. It has been shown in [1] that, for fixed points x, y, there holds

$$\lim_{n\to\infty} d_{a_n}(x,y) = \inf \Big\{ \int_0^1 \varphi(\gamma')\, dt \ : \ \gamma \in \mathrm{Lip}([0,1];\Omega) \ , \ \gamma(0) = x \ , \ \gamma(1) = y \Big\} \ ,$$

where φ is a Finsler metric independent of the position (namely it suits Definition 1 below with $\varphi(x,\xi) = \varphi(\xi)$). Moreover, when the quotient β/α is sufficiently large, the unit ball $B_\varphi := \{\xi \in \mathbb{R}^2 \ : \ \varphi(\xi) \le 1\}$ is a polytope (precisely, a regular octagon). Thus φ is non-Riemannian, and in this case the uniform limit of d_{a_n} cannot be written under the form d_a with $a \in \mathcal{A}$.

In view of these considerations, it is natural to relax problem (5), enlarging the class of admissible competitors to all Finsler metrics arising as limits of sequences $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. The existence of a solution in such a relaxed class may be easily deduced. Then, in order to understand whether a solution exists for the original problem, the effect produced by the integral constraint $\int_\Omega a_n(x)dx \le m$ on the Finsler limit of a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ must be clarified. To this aim, we embed the class $\mathcal{A}$ into a family $\mathcal{M}$ of Finsler metrics, where the functional F admits a natural extension $\overline{F}$ (see Section 2). We also endow $\mathcal{M}$ with a suitable topology τ, that guarantees both the compactness of $\mathcal{M}$ and the continuity of $\overline{F}$ on $\mathcal{M}$ (*cf.* respectively Propositions 4 and 5). Then in Section 3 we show that the crucial condition satisfied by the Finsler metrics belonging to the τ-closure of the class $\mathcal{A}$ in the wider class $\mathcal{M}$ is an integral inequality for their largest eigenvalue Λ_φ:

$$\Lambda_\varphi(x) := \max\Big\{\varphi(x,\xi) \ : \ \xi \in \mathbb{R}^N \ , \ |\xi| \le 1\Big\} \tag{6}$$

(see Theorem 7). As a consequence of this fact, we can prove that the optimization problem (5) of preventing the mass transfer of f^+ into f^- admits at least a solution in the original class $\mathcal{A}$ (see Theorem 6). By similar arguments, we also are able to treat more general maximization problems of the form (5), when F is replaced by an arbitrary cost functional satisfying suitable monotonicity and semicontinuity properties (see Theorem 8).

In some sense, our result may be read as a regularity theorem, as it ensures the existence of a solution to the relaxed problem within the smaller class $\mathcal{A}$ of Riemannian coefficients, which is considerably more manageable than $\mathcal{M}$. (In particular, in the concrete frameworks mentioned above, the optimal metric turns out to be easier to manufacture.) However, let us stress that the uniqueness of solution for the relaxed problem when the cost function Φ is strictly increasing is, at present, an open question which, in our opinion, deserves further investigation.

Notation. Throughout the paper Ω will denote a bounded, connected open subset of $\mathbb{R}^N$ with Lipschitz boundary. If E is a Lebesgue measurable subset of $\mathbb{R}^N$, we will denote by $|E|$ its N-dimensional Lebesgue measure, and E will be said to be *negligible* whenever $|E| = 0$. Finally, the characteristic function of a set E will be denoted by $\mathbf{1}_E$.

2 Preliminaries on Finsler distances

In this section we review some basic facts of the theory of Finsler metrics and their associated distances that we shall need in the sequel.

Definition 1 A Borel function $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty)$ is said to be a *Finsler metric* on $\mathbb{R}^N$ if the function $\varphi(x, \cdot)$ is positively one-homogeneous for every $x \in \mathbb{R}^N$ and convex for a.e. $x \in \mathbb{R}^N$.

Any such Finsler metric φ defines a distance d_φ on $\mathbb{R}^N$ through the formula:

$$d_\varphi(x,y) := \inf\big\{L_\varphi(\gamma) : \ \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^N), \ \gamma(0) = x \ , \ \gamma(1) = y\big\} \tag{7}$$

for every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, where the Finslerian length functional L_φ is defined by

$$L_\varphi(\gamma) := \int_0^1 \varphi(\gamma(t), \gamma'(t))\, dt, \qquad \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^N) \ . \tag{8}$$

In what follows, we shall say that a distance deriving from a Finsler metric through (7) is of *Finsler type*. Notice that, as $\varphi(x, \cdot)$ may be non-even, the distance d_φ may be non-symmetric (i.e. in general the identity $d_\varphi(x, y) = d_\varphi(y, x)$ fails to hold on $\mathbb{R}^N \times \mathbb{R}^N$). In the sequel, the word *distance* will always denote a possibly non-symmetric distance function. We stress that the proofs of the results recalled in this section are given in literature considering usual symmetric distances, but can be easily adapted to our framework by minor changes.

We shall say that a distance function is of *geodesic type* if it satisfies the following identity:

$$d(x,y) = \inf\Big\{L_d(\gamma) \ : \ \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^N), \ \gamma(0) = x \ , \ \gamma(1) = y\Big\} \ ,$$

for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $L_d(\gamma)$ denotes the classical d-length of γ, obtained as the supremum of the d-lengths of inscribed polygonal curves:

$$L_d(\gamma) := \sup\Big\{ \sum_i d\big(\gamma(t_i), \gamma(t_{i+1})\big) : \ 0 = t_0 < t_1 < .. < t_r = 1 \Big\} \ . \tag{9}$$

We stress that, for any distance d, the length functional L_d admits the integral representation

$$L_d(\gamma) = \int_0^1 \varphi_d\left(\gamma(t), \gamma'(t)\right) dt, \qquad \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^N) \ , \tag{10}$$

where φ_d is the Finsler metric associated to d by derivation, namely

$$\varphi_d(x,\xi) := \limsup_{t\to 0^+} \frac{d(x, x+t\xi)}{t} \qquad (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \ . \tag{11}$$

For the proofs of (10) and of the fact that φ_d is a Finsler metric, we refer to [9].

Lemma 2 *A distance function is of geodesic type if and only if it is of Finsler type.*

Proof. Assume that d is of geodesic type. Then, using (10), we have

$$d(x,y) = \inf\Big\{ L_{\varphi_d}(\gamma) \ : \ \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^N), \ \gamma(0) = x \ , \ \gamma(1) = y \Big\} \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \ .$$

Thus, $d = d_{\varphi_d}$, and d is of Finsler type. Conversely, assume that d is of Finsler type, say $d = d_\varphi$. Using the definitions of the length functionals (8) and (9), it is straightforward to check that the inequality $L_{d_\varphi}(\gamma) \le L_\varphi(\gamma)$ holds for all Lipschitz curves γ in $\mathbb{R}^N$. So we have

$$\begin{aligned} d_\varphi(x,y) \ &= \inf\Big\{ L_\varphi(\gamma) \ : \ \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^N), \ \gamma(0) = x \ , \ \gamma(1) = y \Big\} \\ &\ge \inf\Big\{ L_{d_\varphi}(\gamma) \ : \ \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^N), \ \gamma(0) = x \ , \ \gamma(1) = y \Big\} \ . \end{aligned}$$

The converse inequality is a straightforward consequence of the triangle inequality; thus we deduce that d_φ is of geodesic type. ☐

Remark 3 When a distance d is of geodesic type, we have seen in the above proof that $d = d_{\varphi_d}$. However, starting from a Finsler metric φ, it is generally not true that $\varphi_{d_\varphi} = \varphi$. In particular, it is possible to construct a Finsler metric φ of the form $a(x)|\xi|$ such that the corresponding φ_{d_φ} is non-Riemannian. This is due to the possible lack of regularity of Finsler metrics. An example of this singular behavior is the following: let $E := \big\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{Q} \text{ or } x_2 \in \mathbb{Q}\big\}$ and define $\varphi(x,\xi) := a(x)|\xi|$, being the coefficient $a(x)$ given by

$$a(x) = \mathbf{1}_E(x) + \beta \mathbf{1}_{\mathbb{R}^2 \setminus E}(x) \ .$$

If $\beta > 0$ is sufficiently large (i.e., such that $\beta\sqrt{|x_1|^2 + |x_2|^2} \ge |x_1| + |x_2|$ for every $x \in \mathbb{R}^2$) then the induced distance d_φ is precisely $d_\varphi(x,y) = |x_1 - y_1| + |x_2 - y_2|$. Consequently, we have $\varphi_{d_\varphi}(x,\xi) = |\xi_1| + |\xi_2|$, so that φ_{d_φ} is everywhere different from φ.

Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$ with Lipschitz boundary and let us fix two positive constants α, β, with $\beta > \alpha$. Recalling that $\mathcal{A}$ is the class of isotropic Riemannian metrics on Ω defined by (4), we set

$$\mathcal{D}(\mathcal{A}) := \Big\{ d_a \text{ distances on } \overline{\Omega} \text{ given by (2)} : \ a \in \mathcal{A} \Big\}, \tag{12}$$

Denote by d_Ω the Euclidean geodesic distance in $\overline{\Omega}$, that is $d_\Omega := d_a$ with a identically equal to 1 (when Ω is convex, d_Ω is simply the Euclidean distance). As $\partial\Omega$ is Lipschitz, it is easy to show that there exists a constant $C \geq 1$ such that $d_\Omega(x,y) \leq C|x-y|$ for all $x, y \in \overline{\Omega}$. We set:

$$\mathcal{M} := \Big\{ \varphi \text{ Finsler metrics on } \mathbb{R}^N \ : \ \alpha\,|\xi| \leq \varphi\,(x,\xi) \leq C\beta\,|\xi| \Big\}$$
$$\mathcal{D} := \Big\{ d_\varphi \ \text{ distances on } \mathbb{R}^N \text{ given by (7)} \ : \ \varphi \in \mathcal{M} \ \Big\}.$$

The class $\mathcal{D}(\mathcal{A})$ can be embedded into $\mathcal{D}$ in a natural way. Indeed, for $a \in \mathcal{A}$, d_a satisfies the inequalities $\alpha d_\Omega(x,y) \leq d_a(x,y) \leq \beta d_\Omega(x,y)$ for all $x, y \in \overline{\Omega}$. Moreover, d_a can be extended to a Finsler distance defined on the whole $\mathbb{R}^N$. This can be performed by setting $\varphi(x,\xi) := \big(a(x)\mathbf{1}_\Omega(x) + C\beta\,\mathbf{1}_{\mathbb{R}^N\setminus\Omega}(x) \big)\,|\xi|$ and by considering the Finsler distance d_φ on $\mathbb{R}^N$ defined through (7). With such a choice it is easy to see that $d_\varphi \equiv d_a$ on $\overline{\Omega} \times \overline{\Omega}$: in fact, when connecting two points of $\overline{\Omega}$ in $\mathbb{R}^N$, if one is interested in minimizing the Finslerian length L_φ there is no advantage to choosing a path which gets out of Ω, as φ is "high" outside Ω. We endow $\mathcal{D}$ with the topology of uniform convergence on compact subsets of $\mathbb{R}^N \times \mathbb{R}^N$, and $\mathcal{M}$ with the topology τ defined as follows:

$$\varphi_n \xrightarrow{\tau} \varphi \quad \Longleftrightarrow \quad d_{\varphi_n} \text{ converge uniformly to } d \text{ on compact subset of } \mathbb{R}^N \times \mathbb{R}^N, \text{ and } \varphi = \varphi_d \ .$$

Notice that any distance in $\mathcal{D}$ gives rise, through (11), to a Finsler metric in $\mathcal{M}$. However we stress that, if $d_{\varphi_n} \to d_\varphi$ uniformly on compact subset of $\mathbb{R}^N \times \mathbb{R}^N$, the τ-limit of φ_n is φ_{d_φ}, which, in view of Remark 3, is in general different from φ. For some topological equivalence results on the class $\mathcal{D}$, related to the Γ-convergence of different kinds of variational functionals, we refer to [7]. In particular, next theorems follow essentially from the results of [7].

Proposition 4 *We have:*

(i) the class $\mathcal{D}$ is compact with respect to the uniform convergence on compact subset of $\mathbb{R}^N \times \mathbb{R}^N$;

(ii) the class $\mathcal{M}$ is τ-compact.

Proof. Claim (i) has been proved in [7, Theorem 3.1] considering usual symmetric distances. It is enough to observe that this result still holds in the non-symmetric case, the proof being the same. To prove assertion (ii), let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in the class $\mathcal{M}$. Then the associated distances d_{φ_n} lie in the class $\mathcal{D}$. Up to extracting a subsequence, there exists a distance $d \in \mathcal{D}$ such that $d_{\varphi_n} \to d$ uniformly on compact subset of $\mathbb{R}^N \times \mathbb{R}^N$. Then we have by definition $\varphi_n \xrightarrow{\tau} \varphi_d$. □

The functional F defined by (3) may be extended in a natural way to the class $\mathcal{M}$ by setting, for φ in $\mathcal{M}$,

$$\overline{F}(\varphi) := \min\Big\{ \int\!\!\int_{\overline{\Omega}\times\overline{\Omega}} \Phi(d_\varphi(x,y))\, d\nu(x,y) \ : \ \nu \text{ admissible plan } \Big\}. \tag{13}$$

Proposition 5 *The functional $\overline{F}$ is τ-continuous on the class $\mathcal{M}$.*

Proof. Assume that $\varphi_n \xrightarrow{\tau} \varphi$. Then, by definition, the distances d_{φ_n} converge uniformly on compact subset of $\mathbb{R}^N \times \mathbb{R}^N$ to $d = d_\varphi$, and $\varphi = \varphi_d$. Next we observe that, for any sequence $(\nu_n)_{n\in\mathbb{N}}$ of nonnegative measures defined on $\overline{\Omega} \times \overline{\Omega}$ and weakly converging to some measure ν, there holds:

$$\lim_{n\to\infty} \int\!\!\int_{\overline{\Omega}\times\overline{\Omega}} \Phi\,(d_{\varphi_n}(x,y))\, d\nu_n\,(x,y) = \int\!\!\int_{\overline{\Omega}\times\overline{\Omega}} \Phi(d(x,y)) d\nu\,(x,y)\,. \tag{14}$$

Now, for every n, let σ_n be a plan that realizes the minimum $\overline{F}(\varphi_n)$ according to definition (13); then there exists a subsequence $(\sigma_{n_i})_{i\in\mathbb{N}}$ weakly converging to some admissible plan σ and such that $\lim_i \overline{F}(\varphi_{n_i}) = \liminf_n \overline{F}(\varphi_n)$. Then, using (14) and the identity $d = d_\varphi$, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \overline{F}(\varphi_n) &= \lim_{i\to\infty} \int\int_{\overline{\Omega}\times\overline{\Omega}} \Phi\left(d_{\varphi_{n_i}}(x,y)\right) d\sigma_{n_i}(x,y) = \int\int_{\overline{\Omega}\times\overline{\Omega}} \Phi(d(x,y)) d\sigma(x,y) \\
&= \int\int_{\overline{\Omega}\times\overline{\Omega}} \Phi(d_\varphi(x,y)) d\sigma(x,y) \geq \overline{F}(\varphi) \ .
\end{aligned}$$

To show that $\overline{F}(\varphi) \geq \limsup_n \overline{F}(\varphi_n)$, we may argue in a similar way: we apply (14) taking as $(\nu_n)_{n\in\mathbb{N}}$ a constant sequence equal to a measure σ that realizes the minimum $\overline{F}(\varphi)$ in (13). $\square$

3 The main results

Our main existence result is stated as follows.

Theorem 6 *Let $\mathcal{A}$ be the class of Borel coefficients given by (4), and let F be the functional defined by (3). Under the assumption that the cost density Φ is non-decreasing on $\mathbb{R}^+$, there exists at least an element $\overline{a} \in \mathcal{A}$ such that*

$$F(\overline{a}) = \sup\{F(a) : a \in \mathcal{A}\} .$$

The main tool for the proof of the above existence result is the next theorem. It states that the largest eigenvalue of Finsler metrics belonging to the "τ-adherence" of the class $\mathcal{A}$ must satisfy the same integral constraint as the elements of $\mathcal{A}$.

Theorem 7 *Let $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, and set $\varphi_n(x,\xi) := \left(a_n(x)\mathbf{1}_\Omega(x) + C\beta\,\mathbf{1}_{\mathbb{R}^N\setminus\Omega}(x)\right)|\xi|$. If $\varphi_n \xrightarrow{\tau} \varphi$, then we have*

$$\int_\Omega \Lambda_\varphi(x)\,dx \leq m \ ,$$

where $\Lambda_\varphi(x)$ is the largest eigenvalue of $\varphi(x,\cdot)$ defined by (6).

We now prove Theorem 6 using Theorem 7, whose proof is postponed.

Proof of Theorem 6. Let $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be a maximizing sequence for the functional F, and set $\varphi_n(x,\xi) := \left(a_n(x)\mathbf{1}_\Omega(x) + C\beta\,\mathbf{1}_{\mathbb{R}^N\setminus\Omega}(x)\right)|\xi|$. By Proposition 4, up to subsequences we have $\varphi_n \xrightarrow{\tau} \varphi$, and, by Proposition 5, we have

$$\overline{F}(\varphi) = \lim_{n\to\infty} \overline{F}(\varphi_n) = \lim_{n\to\infty} F(a_n) = \sup\{F(a) \ : \ a \in \mathcal{A}\} \ .$$

We are thus reduced to show that there exists at least an element $\overline{a} \in \mathcal{A}$ such that $F(\overline{a}) \geq \overline{F}(\varphi)$. We set

$$\overline{a}(x) := \Lambda_\varphi(x) \qquad \text{for} \ \ x \in \Omega.$$

We first remark that the coefficient $\overline{a}$ is Borel measurable. Indeed $\varphi = \varphi_d$ for some distance $d \in \mathcal{D}$ by definition (since φ is the τ-limit of a sequence of metrics $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$). From the definition of φ_d, one can easily deduce that

$$|\varphi_d(x,\xi) - \varphi_d(x,\eta)| \leq \beta|\xi - \eta| \quad \text{for all } x \in \Omega \text{ and all } \xi, \eta \in \mathbb{R}^N,$$

hence, if $(\xi_k)_{k\in\mathbb{N}}$ is a dense sequence in $\mathbb{S}^{N-1}$, we have that $\Lambda_\varphi(x) = \sup_k \varphi(x,\xi_k)$. That implies that $\overline{a}$ is Borel measurable and satisfies the bounds $\alpha \le \overline{a}(x) \le \beta$. By Theorem 7, it satisfies also the integral constraint $\int_\Omega \overline{a}(x)dx \le m$. Hence $\overline{a} \in \mathcal{A}$. Now, since

$$\overline{a}\,(x)\,|\xi| \ge \varphi\,(x,\xi) \qquad (x,\xi) \in \Omega\times\mathbb{R}^N \ ,$$

for all x,$y \in \overline{\Omega}$ we have:

$$d_{\overline{a}}(x,y) \ge \inf\left\{ \int_0^1 \varphi(\gamma,\gamma')\,dt \ : \ \gamma \in \mathrm{Lip}(]0,1[;\Omega)\ ,\ \gamma(0^+) = x\ ,\ \gamma(1^-) = y \right\} \ge d_\varphi(x,y),$$

and then, by the monotonicity of Φ, $F\,(\overline{a}) \ge \overline{F}\,(\varphi)$. □

Arguing as in the proof of Theorem 6, we may obtain the following formulation of the existence result for functionals defined on distances.

Theorem 8 *Let $\mathcal{F}$ be a functional defined on $\mathcal{D}\big|_{\overline{\Omega}} := \{\, d_{|\overline{\Omega}\times\overline{\Omega}} \ : \ d \in \mathcal{D}\,\}$. We assume that*

(i) $\mathcal{F}$ is upper semicontinuous for the uniform convergence;

(ii) $\mathcal{F}$ is non-decreasing for the usual order on distances.

Then the maximization problem

$$\max\left\{\mathcal{F}(d) \ : \ d \in \mathcal{D}(\mathcal{A})\right\}$$

admits at least a solution.

The remaining part of this section is devoted to the proof of Theorem 7. It is based on the auxiliary Propositions 9 and 12 below.

Proposition 9 *Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{M}$, with $\varphi_n \xrightarrow{\tau} \varphi$. Then, for every bounded Borel set $\omega \subset \mathbb{R}^N$ and every $\xi \in \mathbb{R}^N$, we have*

$$\int_\omega \varphi\,(x,\xi)\,dx \le \liminf_{n\to\infty} \int_\omega \varphi_n\,(x,\xi)\,dx \ .$$

Proof. By the homogeneity property of φ, it is not restrictive to assume that $|\xi| = 1$. Thus, let us fix an element $\xi \in \mathbb{S}^{N-1}$. We claim that it is possible to find a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ and a sequence of positive numbers $t_n \to 0$ such that, for a.e. $x \in \omega$,

$$\varphi\,(x,\xi) = \lim_{n\to\infty} \frac{d_{\varphi_n}\,(x, x+t_n\xi)}{t_n}. \tag{15}$$

Indeed, we first remark that, almost everywhere in x, the limsup appearing in the right hand side of (11) is actually a limit (see [9, Corollary 2.7]). Thus, denoting by d the uniform limit of d_{φ_n}, we have

$$\varphi\,(x,\xi) = \lim_{t\to 0^+} \frac{d\,(x,x+t\xi)}{t} \qquad \text{for a.e. } x \in \mathbb{R}^N \ . \tag{16}$$

Next we observe that, by uniform convergence, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ tending to zero such that

$$|d_{\varphi_n}\,(x,x+t\xi) - d\,(x,x+t\xi)| \le \varepsilon_n$$

for every $x \in \omega$ and every $t \in (0,1)$. Therefore, for a.e. $x \in \omega$ and any $t_n \to 0$, we have

$$\left| \varphi(x,\xi) - \frac{d_{\varphi_n}(x, x + t_n \xi)}{t_n} \right| \le \frac{\varepsilon_n}{t_n} + \left| \varphi(x,\xi) - \frac{d(x, x + t_n \xi)}{t_n} \right| .$$

Then (15) follows choosing $t_n := \sqrt{\varepsilon_n}$ and taking into account (16). Now, integrating (15) over ω and using Fatou's lemma we get:

$$\int_\omega \varphi(x,\xi)\, dx \le \liminf_{n\to\infty} \int_\omega \frac{d_{\varphi_n}(x, x+t_n\xi)}{t_n} dx. \tag{17}$$

Since $d_{\varphi_n}(x, x+t_n\xi)$ is less than or equal to the (Finslerian) length of the straight line segment joining x and $x + t_n\xi$, we have

$$d_{\varphi_n}(x, x+t_n\xi) \le \int_0^1 \varphi_n(x + st_n\xi, t_n\xi)\, ds. \tag{18}$$

Combining (17) and (18), we obtain

$$\begin{aligned}
\int_\omega \varphi(x,\xi)\, dx &\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \mathbf{1}_\omega(x) \int_0^1 \varphi_n(x + st_n\xi, \xi)\, ds dx \\
&= \liminf_{n\to\infty} \int_0^1 \int_{\mathbb{R}^N} \mathbf{1}_\omega(x - st_n\xi)\, \varphi_n(x,\xi)\, dx ds \\
&\le \liminf_{n\to\infty} \int_\omega \varphi_n(x,\xi)\, dx \ ,
\end{aligned}$$

the last inequality being a consequence of

$$\int_{\mathbb{R}^N} \left| \mathbf{1}_\omega(x - st_n\xi) - \mathbf{1}_\omega(x) \right| dx \to 0 \quad \text{as } n \to \infty \quad \text{for every } s \in (0,1).$$

□

We next state and prove two lemmas which will be used in the proof of Proposition 12. For every $\delta > 0$, denote by $\{Q_i^\delta\}_{i \in \mathcal{I}_\delta}$ a finite family of pairwise disjoint cubes in $\mathbb{R}^N$ of the kind $Q_i^\delta = x_i + [-\delta, \delta)^N$, such that $\Omega = \bigcup_{i \in \mathcal{I}_\delta} (\Omega \cap Q_i^\delta)$. Set then $D_i^\delta := \Omega \cap Q_i^\delta$.

Lemma 10 *Let $\varphi \in \mathcal{M}$ be a continuous Finsler metric. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_{D_i^\delta} \Lambda_\varphi(x) dx \le \sup_{|\xi|=1} \int_{D_i^\delta} [\varphi(x,\xi) + \varepsilon]\, dx \qquad \textit{for all } i \in \mathcal{I}_\delta.$$

Proof. Since φ is uniformly continuous on $\overline{\Omega} \times \mathbb{S}^{N-1}$, given $\varepsilon > 0$ it is possible to find $\delta > 0$ in such a way that

$$|\varphi(x,\xi) - \varphi(y,\xi)| < \varepsilon \qquad \text{for every } \xi \in \mathbb{S}^{N-1},\ x, y \in D_i^\delta,\ i \in \mathcal{I}_\delta .$$

As the function Λ_φ is continuous, there exist points $x_i^\delta \in D_i^\delta$ such that

$$\int_{D_i^\delta} \Lambda_\varphi(x) dx = \int_{D_i^\delta} \sup_{|\xi|=1} \varphi\left(x_i^\delta, \xi\right) dx .$$

Therefore,

$$\int_{D_i^\delta} \Lambda_\varphi(x) dx = \sup_{|\xi|=1} \int_{D_i^\delta} \varphi\left(x_i^\delta, \xi\right) dx \le \sup_{|\xi|=1} \int_{D_i^\delta} \varphi(x,\xi)\, dx + \sup_{|\xi|=1} \int_{D_i^\delta} \left[\varphi\left(x_i^\delta, \xi\right) - \varphi(x,\xi)\right] dx ,$$

and the statement of the lemma follows. □

Lemma 11 *Let $\varphi \in \mathcal{M}$ such that $\varphi(x,\cdot)$ is convex for every x. Then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \overline{\Omega}$ such that $\left|\overline{\Omega} \setminus K_\varepsilon\right| < \varepsilon$ and φ is continuous on $K_\varepsilon \times \mathbb{R}^N$.*

Proof. Let us take a sequence of vectors $(\xi_k)_{k\in\mathbb{N}}$ dense in $\mathbb{S}^{N-1}$. For every fixed k, Lusin's Theorem ensures the existence of a compact set $C_k \subset \overline{\Omega}$ such that $\varphi(\cdot,\xi_k)$ is continuous on C_k, and $\left|\overline{\Omega} \setminus C_k\right| < \varepsilon 2^{-k}$. Define

$$K_\varepsilon := \bigcap_{k\in\mathbb{N}} C_k.$$

Obviously, $\left|\overline{\Omega} \setminus K_\varepsilon\right| < \varepsilon$ and $\varphi(\cdot,\xi_k)$ is continuous on K_ε for all k. We claim that φ is actually continuous on $K_\varepsilon \times \mathbb{S}^{N-1}$ (and hence on $K_\varepsilon \times \mathbb{R}^N$ by the homogeneity property of φ). In fact, since for fixed x the function $\varphi(x,\cdot)$ is a norm, for every $\xi, \eta \in \mathbb{S}^{N-1}$ it satisfies

$$\varphi(x,\xi) - \varphi(x,\eta) \leq \varphi(x,\xi-\eta) \leq \beta C\left|\xi - \eta\right|,$$

the last inequality resulting from the fact that $\varphi \in \mathcal{M}$. Thus, for $\xi \in \mathbb{S}^{N-1}$ and $x, y \in K_\varepsilon$ we get

$$\left|\varphi(x,\xi) - \varphi(y,\xi)\right| \leq 2\beta C\left|\xi - \xi_k\right| + \left|\varphi(x,\xi_k) - \varphi(y,\xi_k)\right|.$$

We conclude by the density of $(\xi_k)_{k\in\mathbb{N}}$ and the continuity of $\varphi(\cdot,\xi_k)$ on K_ε. □

Proposition 12 *Let $\varphi \in \mathcal{M}$. Assume that, for a sequence $(\mu_n)_{n\in\mathbb{N}}$ of nonnegative Borel measures on Ω, the following property holds:*

$$\sup_{|\xi|=1} \int_\omega \varphi(x,\xi)\,dx \leq \liminf_{n\to\infty} \mu_n(\omega) \qquad \textit{for every Borel set } \omega \subset \Omega\ . \tag{19}$$

Then

$$\int_\Omega \Lambda_\varphi(x)dx \leq \liminf_{n\to\infty} \mu_n(\Omega)\,. \tag{20}$$

Proof. We proceed in three steps.

Step 1. We prove the result for φ continuous. Fix $\varepsilon > 0$ and take $\delta > 0$ given by Lemma 10. We have:

$$\int_\Omega \Lambda_\varphi(x)dx = \sum_{i\in\mathcal{I}_\delta} \int_{D_i^\delta} \Lambda_\varphi(x)dx \leq \sum_{i\in\mathcal{I}_\delta} \sup_{|\xi|=1} \int_{D_i^\delta} \left[\varphi(x,\xi) + \varepsilon\right] dx\ .$$

By assumption:

$$\sum_{i\in\mathcal{I}_\delta} \sup_{|\xi|=1} \int_{D_i^\delta} \left[\varphi(x,\xi) + \varepsilon\right] dx \leq \sum_{i\in\mathcal{I}_\delta} \left[\liminf_{n\to\infty} \mu_n\left(D_i^\delta\right) + \int_{D_i^\delta} \varepsilon\, dx\right] \leq \liminf_{n\to\infty} \mu_n(\Omega) + \varepsilon\left|\Omega\right|,$$

and since ε is arbitrary (20) follows.

Step 2. We show that (20) holds when $\varphi(x,\cdot)$ is convex and $\varphi(\cdot,\xi)$ is lower semicontinuous for every fixed x and ξ. Indeed in this case, thanks to Lemma 2.2.3 of [6], and since $\varphi(x,\cdot)$ is positively one-homogeneous, there exists a sequence of continuous functions $a_j : \mathbb{R}^N \to \mathbb{R}^N$ such that, for $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\varphi(x,\xi) = \sup_{j\in\mathbb{N}} \left\{a_j(x) \cdot \xi\right\}$$

Then, defining

$$\varphi_k(x,\xi) := \sup_{j\leq k} \left\{a_j(x) \cdot \xi\right\} \vee \alpha\left|\xi\right|\ ,$$

we obtain a sequence of continuous elements of $\mathcal{M}$ which converges increasingly to φ. Each of the metrics φ_k satisfies the property (19) because, for every Borel set $\omega \subset \Omega$ and every $\xi \in \mathbb{S}^{N-1}$, we have

$$\int_\omega \varphi_k(x,\xi)\,dx \le \int_\omega \varphi(x,\xi)\,dx \le \liminf_{n\to\infty} \mu_n(\omega).$$

Therefore, by Step 1, we get

$$\sup_{k\in\mathbb{N}} \int_\Omega \Lambda_{\varphi_k}(x)dx \le \liminf_{n\to\infty} \mu_n(\Omega). \tag{21}$$

Now, let us take a dense set $(\xi_h)_{h\in\mathbb{N}}$ in $\mathbb{S}^{N-1}$. We have:

$$\int_\Omega \Lambda_\varphi(x)dx = \int_\Omega \sup_{h\in\mathbb{N}} \varphi(x,\xi_h)\,dx = \int_\Omega \sup_{h\in\mathbb{N}}\sup_{k\in\mathbb{N}} \varphi_k(x,\xi_h)\,dx = \int_\Omega \sup_{k\in\mathbb{N}}\sup_{h\in\mathbb{N}} \varphi_k(x,\xi_h)\,dx.$$

By the Monotone Convergence Theorem and (21), we finally obtain

$$\int_\Omega \sup_{k\in\mathbb{N}}\sup_{h\in\mathbb{N}} \varphi_k(x,\xi_h)\,dx = \sup_{k\in\mathbb{N}} \int_\Omega \sup_{h\in\mathbb{N}} \varphi_k(x,\xi_h)\,dx = \sup_{k\in\mathbb{N}} \int_\Omega \Lambda_{\varphi_k}(x)dx \le \liminf_{n\to\infty} \mu_n(\Omega)\ .$$

Step 3: We finally prove the result in its full generality: let φ be only Borel measurable.
First observe that we may assume that $\varphi(x,\cdot)$ is convex for all $x \in \mathbb{R}^N$. Indeed, if this is not the case, take a negligible Borel set $E \subset \mathbb{R}^N$ which contains the points x where $\varphi(x,\cdot)$ is not convex. Then we can replace $\varphi(x,\xi)$ with $\varphi(x,\xi)\mathbf{1}_{\mathbb{R}^N\setminus E}(x) + C\beta\,\mathbf{1}_E(x)|\xi|$ without affecting the validity of (19).
Hence φ suits the assumptions of Lemma 11: we deduce that, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \overline{\Omega}$ such that $\left|\overline{\Omega} \setminus K_\varepsilon\right| < \varepsilon$ and $\varphi|_{K_\varepsilon\times\mathbb{R}^N}$ is continuous. We define

$$\varphi^\varepsilon(x,\xi) := \begin{cases} \varphi(x,\xi) & \text{if } x \in K_\varepsilon \\ \\ \beta C\,|\xi| & \text{otherwise.} \end{cases}$$

Notice that, as K_ε is closed in $\mathbb{R}^N$, φ^ε is lower semicontinuous and

$$\varphi^\varepsilon(x,\xi) \ge \varphi(x,\xi) \qquad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Moreover, for every Borel set $\omega \subset \Omega$,

$$\begin{aligned} \sup_{|\xi|=1} \int_\omega \varphi^\varepsilon(x,\xi)\,dx &\le \sup_{|\xi|=1} \int_\omega \varphi(x,\xi)\,dx + \beta C\,|\omega \setminus K_\varepsilon| \\ &\le \liminf_{n\to\infty} \mu_n(\omega) + \beta C\,|\omega \setminus K_\varepsilon| \end{aligned}$$

Applying Step 2 with $\tilde{\mu}_n := \mu_n + \beta C\,\mathbf{1}_{\Omega\setminus K_\varepsilon} dx$, we get

$$\int_\Omega \Lambda_\varphi(x)dx \le \int_\Omega \Lambda_{\varphi^\varepsilon}(x)dx \le \liminf_{n\to\infty} \tilde{\mu}_n(\Omega) \le \liminf_{n\to\infty} \mu_n(\Omega) + \beta C\,\varepsilon.$$

The claim follows since ε was arbitrarily chosen. $\square$

We are finally in position to give the

Proof of Theorem 7. Let a_n, φ_n and φ be as in the statement. Then, by Proposition 9, the limit metric φ satisfies condition (19) if we take as a sequence $(\mu_n)_{n\in\mathbb{N}}$ the Lebesgue measure on Ω with densities a_n, namely,

$$\mu_n(\omega) = \int_\omega a_n(x)dx \qquad \text{for every Borel set } \omega \subset \Omega \ .$$

Applying Proposition 12, we infer

$$\int_\Omega \Lambda_\varphi(x)dx \leq \liminf_{n\to\infty} \mu_n(\Omega) = \liminf_{n\to\infty} \int_\Omega a_n(x)dx \leq m \ .$$

□

Remark 13 With regard to Theorem 7, we point out that an analogous result holds if the integral constraint considered in the definition of $\mathcal{A}$ is replaced by one of more general kind. For example, another possible constraint could be $\int_\Omega a(x)^N \, dx \leq m$, which bounds the Riemannian volume of Ω corresponding to the metric $a(x)|\xi|$. In particular, Theorems 6 and 8 hold in this case too.

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