

Solitary Waves for Nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell Equations

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Abstract

In this paper we study the existence of radially symmetric solitary waves for nonlinear Klein-Gordon equations and nonlinear Schrödinger equations coupled with Maxwell equations. The method relies on a variational approach and the solutions are obtained as mountain-pass critical points for the associated energy functional.

1 Introduction

This paper has been motivated by the search of nontrivial solutions for the following nonlinear equations of the Klein-Gordon type:

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + m^2 \psi - |\psi|^{p-2} \psi = 0, \quad x \in \mathbb{R}^3, \quad (1.1)$$

or of the Schrödinger type:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi - |\psi|^{p-2} \psi, \quad x \in \mathbb{R}^3, \quad (1.2)$$

where $\hbar > 0$, $m > 0$, $p > 2$, $\psi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}$.

In recent years many papers have been devoted to find standing waves of (1.1) or (1.2), i.e. solutions of the form

$$\psi(x, t) = e^{i\omega t} u(x), \quad \omega \in \mathbb{R}.$$

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With this *Ansatz* the nonlinear Klein-Gordon equation, as well as the nonlinear Schrödinger equation, is reduced to a semilinear elliptic equation and existence theorems have been established whether u is radially symmetric and real (see [8], [9]), or u is non-radially symmetric and complex (see [13], [16]). In this paper we want to investigate the existence of nonlinear Klein-Gordon or Schrödinger fields interacting with an electromagnetic field $\mathbf{E} - \mathbf{H}$; such a problem has been extensively pursued in the case of assigned electromagnetic fields (see [3], [4], [12]). Following the ideas already introduced in [5], [6], [7], [10], [11], [14], [15], we do not assume that the electromagnetic field is assigned. Then we have to study a system of equations whose unknowns are the wave function $\psi = \psi(x, t)$ and the gauge potentials $\mathbf{A}$, Φ ,

$$\mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3, \quad \Phi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$$

which are related to $\mathbf{E} - \mathbf{H}$ by the Maxwell equations

$$\begin{aligned} \mathbf{E} &= - \left(\nabla \Phi + \frac{\partial \mathbf{A}}{\partial t} \right) \\ \mathbf{H} &= \nabla \times \mathbf{A}. \end{aligned}$$

Let us first consider equation (1.1). The Lagrangian density related to (1.1) is given by

$$\mathcal{L}_{KG} = \frac{1}{2} \left[\left| \frac{\partial \psi}{\partial t} \right|^2 - |\nabla \psi|^2 - m^2 |\psi|^2 \right] + \frac{1}{p} |\psi|^p.$$

The interaction of ψ with the electromagnetic field is described by the minimal coupling rule, that is the formal substitution

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + ie\Phi, \quad \nabla \mapsto \nabla - ie\mathbf{A},$$

where e is the electric charge. Then the Lagrangian density becomes:

$$\mathcal{L}_{KGM} = \frac{1}{2} \left[\left| \frac{\partial \psi}{\partial t} + ie\psi\Phi \right|^2 - |\nabla \psi - ie\mathbf{A}\psi|^2 - m^2 |\psi|^2 \right] + \frac{1}{p} |\psi|^p.$$

If we set

$$\psi(x, t) = u(x, t)e^{iS(x, t)},$$

where u , $S : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$, the Lagrangian density takes the form

$$\mathcal{L}_{KGM} = \frac{1}{2} \left\{ u_t^2 - |\nabla u|^2 - \left[|\nabla S - e\mathbf{A}|^2 - (S_t + e\Phi)^2 + m^2 \right] u^2 \right\} + \frac{1}{p} |u|^p.$$

Now consider the Lagrangian density of the electromagnetic field $\mathbf{E} - \mathbf{H}$,

$$\mathcal{L}_0 = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{H}|^2) = \frac{1}{2}|\mathbf{A}_t + \nabla\Phi|^2 - \frac{1}{2}|\nabla \times \mathbf{A}|^2. \quad (1.3)$$

Therefore, the total action is given by

$$\mathcal{S} = \int \int \mathcal{L}_{KGM} + \mathcal{L}_0.$$

Making the variation of $\mathcal{S}$ with respect to u , S , Φ and $\mathbf{A}$ respectively, we get

$$u_{tt} - \Delta u + [|\nabla S - e\mathbf{A}|^2 - (S_t + e\Phi)^2 + m^2]u - |u|^{p-2}u = 0, \quad (1.4)$$

$$\frac{\partial}{\partial t}[(S_t + e\Phi)u^2] - \operatorname{div}[(\nabla S - e\mathbf{A})u^2] = 0, \quad (1.5)$$

$$\operatorname{div}(\mathbf{A}_t + \nabla\Phi) = e(S_t + e\Phi)u^2, \quad (1.6)$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}(\mathbf{A}_t + \nabla\Phi) = e(\nabla S - e\mathbf{A})u^2. \quad (1.7)$$

We are interested in finding standing (or *solitary*) waves of (1.4)-(1.7), that is solutions having the form

$$u = u(x), \quad S = \omega t, \quad \mathbf{A} = 0, \quad \Phi = \Phi(x), \quad \omega \in \mathbb{R}.$$

Then the equations (1.5) and (1.7) are identically satisfied, while (1.4) and (1.6) become

$$-\Delta u + [m^2 - (\omega + e\Phi)^2]u - |u|^{p-2}u = 0, \quad (1.8)$$

$$-\Delta\Phi + e^2u^2\Phi = -e\omega u^2. \quad (1.9)$$

In [6] the authors proved the existence of infinitely many symmetric solutions (u_n, Φ_n) of (1.8)-(1.9) under the assumption $4 < p < 6$, by using an equivariant version of the mountain pass theorem (see [1], [2]).

The object of the first part of this paper is to extend this result as follows.

Theorem 1.1. *Assume that one of the following two hypotheses hold: either*

$$a) \quad m > \omega > 0 \text{ and } 4 \leq p < 6,$$

or

$$b) \quad m\sqrt{p-2} > \sqrt{2}\omega > 0 \text{ and } 2 < p < 4.$$

Then the system (1.8) – (1.9) has infinitely many radially symmetric solutions (u_n, Φ_n) , $u_n \not\equiv 0$ and $\Phi_n \not\equiv 0$, with $u_n \in H^1(\mathbb{R}^3)$, $\Phi_n \in L^6(\mathbb{R}^3)$ and $|\nabla \Phi_n| \in L^2(\mathbb{R}^3)$.

In the second part of the paper we study the Schrödinger equation for a particle in a electromagnetic field.

Consider the Lagrangian associated to (1.2):

$$\mathcal{L}_S = \frac{1}{2} \left[i\hbar \frac{\partial \psi}{\partial t} \bar{\psi} - \frac{\hbar^2}{2m} |\nabla \psi|^2 \right] + \frac{1}{p} |\psi|^p.$$

By using the formal substitution

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + i\frac{e}{\hbar} \Phi, \quad \nabla \mapsto \nabla - i\frac{e}{\hbar} \mathbf{A},$$

we obtain

$$\mathcal{L}_{SM} = \frac{1}{2} \left[i\hbar \frac{\partial \psi}{\partial t} \bar{\psi} - e\Phi |\psi|^2 - \frac{\hbar^2}{2m} \left| \nabla \psi - i\frac{e}{\hbar} \mathbf{A} \psi \right|^2 \right] + \frac{1}{p} |\psi|^p.$$

Now take

$$\psi(x, t) = u(x, t) e^{iS(x, t)/\hbar}.$$

With this *Ansatz* the Lagrangian $\mathcal{L}_{SM}$ becomes

$$\mathcal{L}_{SM} = \frac{1}{2} \left[i\hbar u u_t - \frac{\hbar^2}{2m} |\nabla u|^2 - \left(S_t + e\Phi + \frac{1}{2m} |\nabla S - e\mathbf{A}|^2 \right) u^2 \right] + \frac{1}{p} |u|^p.$$

Proceeding as in [5], we consider the total action $\mathcal{S} = \int \int [\mathcal{L}_{SM} + \frac{1}{8\pi} (|\mathbf{E}|^2 - |\mathbf{H}|^2)]$ of the system “particle-electromagnetic field”. Then the Euler-Lagrange equations associated to the functional $\mathcal{S} = \mathcal{S}(u, S, \Phi, \mathbf{A})$ give rise to the following system of equations:

$$-\frac{\hbar^2}{2m} \Delta u + \left(S_t + e\Phi + \frac{1}{2m} |\nabla S - e\mathbf{A}|^2 \right) u - |u|^{p-2} u = 0, \quad (1.10)$$

$$\frac{\partial}{\partial t} u^2 + \frac{1}{m} \operatorname{div}[(\nabla S - e\mathbf{A}) u^2] = 0, \quad (1.11)$$

$$e u^2 = -\frac{1}{4\pi} \operatorname{div} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \right), \quad (1.12)$$

$$\frac{e}{2m} (\nabla S - e\mathbf{A}) u^2 = \frac{1}{4\pi} \left[\frac{\partial}{\partial t} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \right) + \nabla \times (\nabla \times \mathbf{A}) \right]. \quad (1.13)$$

If we look for solitary wave solutions in the electrostatic case, i.e.

$$u = u(x), \quad S = \omega t, \quad \Phi = \Phi(x), \quad \mathbf{A} = 0, \quad \omega \in \mathbb{R},$$

then (1.11) and (1.13) are identically satisfied, while (1.10) and (1.12) become

$$-\frac{\hbar^2}{2m}\Delta u + e\Phi u - |u|^{p-2}u + \omega u = 0, \quad (1.14)$$

$$-\Delta\Phi = 4\pi e u^2. \quad (1.15)$$

The existence of solutions of (1.14)-(1.15) was already studied for $4 < p < 6$: in [5] existence of infinitely many radial solutions was proved, while in [13] existence of a non radially symmetric solution was established. In the second part of the paper we prove the following result.

Theorem 1.2. *Let $\omega > 0$ and $4 \leq p < 6$. Then the system (1.14) – (1.15) has at least a radially symmetric solution (u, Φ) , $u \neq 0$ and $\Phi \neq 0$, with $u \in H^1(\mathbb{R}^3)$, $\Phi \in L^6(\mathbb{R}^3)$ and $|\nabla\Phi| \in L^2(\mathbb{R}^3)$.*

2 Nonlinear Klein-Gordon Equations coupled with Maxwell Equations

In this section we will prove Theorem 1.1. For sake of simplicity, assume $e = 1$ so that (1.8)-(1.9) give rise to the following system in $\mathbb{R}^3$:

$$-\Delta u + [m^2 - (\omega + \Phi)^2]u - |u|^{p-2}u = 0, \quad (2.16)$$

$$-\Delta\Phi + u^2\Phi = -\omega u^2. \quad (2.17)$$

Assume that one of the following hypotheses hold: either

a) $m > \omega > 0$, $4 \leq p < 6$,

or

b) $m\sqrt{p-2} > \sqrt{2}\omega > 0$, $2 < p < 4$.

We note that $q = 6$ is the critical exponent for the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$.

It is clear that (2.16)-(2.17) are the Euler-Lagrange equations of the functional $F : H^1 \times D^{1,2} \rightarrow \mathbb{R}$ defined as

$$F(u, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 - |\nabla \Phi|^2 + [m^2 - (\omega + \Phi)^2] u^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

Here $H^1 \equiv H^1(\mathbb{R}^3)$ denotes the usual Sobolev space endowed with the norm

$$\|u\|_{H^1} \equiv \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx \right)^{1/2} \quad (2.18)$$

and $D^{1,2} \equiv D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ with respect to the norm

$$\|u\|_{D^{1,2}} \equiv \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2}. \quad (2.19)$$

The following two propositions hold.

Proposition 2.1. *The functional F belongs to $C^1(H^1 \times D^{1,2}, \mathbb{R})$ and its critical points are the solutions of (2.16) – (2.17).*

(For the proof we refer to [6]).

Proposition 2.2. *For every $u \in H^1$, there exists a unique $\Phi = \Phi[u] \in D^{1,2}$ which solves (2.17). Furthermore*

- (i) $\Phi[u] \leq 0$;
- (ii) $\Phi[u] \geq -\omega$ in the set $\{x \mid u(x) \neq 0\}$;
- (iii) if u is radially symmetric, then $\Phi[u]$ is radial, too.

Proof. Fixed $u \in H^1$, consider the following bilinear form on $D^{1,2}$:

$$a(\phi, \psi) = \int_{\mathbb{R}^3} (\nabla \psi \nabla \psi + u^2 \phi \psi) dx.$$

Obviously $a(\phi, \phi) \geq \|\phi\|_{D^{1,2}}^2$. Observe that, since $H^1(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$, then $u^2 \in L^{3/2}(\mathbb{R}^3)$. On the other hand $D^{1,2}$ is continuously embedded in $L^6(\mathbb{R}^3)$, hence, by Hölder's inequality,

$$a(\phi, \psi) \leq \|\phi\|_{D^{1,2}} \|\psi\|_{D^{1,2}} + \|u^2\|_{L^{3/2}} \|\phi\|_{L^6} \|\psi\|_{L^6} \leq (1 + C \|u\|_{L^3}^2) \|\phi\|_{D^{1,2}} \|\psi\|_{D^{1,2}}$$

for some positive constant C , given by Sobolev inequality (see [20]). Therefore a defines an inner product, equivalent to the standard inner product in $D^{1,2}$.

Moreover $H^1(\mathbb{R}^3) \subset L^{12/5}(\mathbb{R}^3)$, and then

$$\left| \int_{\mathbb{R}^3} u^2 \psi dx \right| \leq \|u^2\|_{L^{6/5}} \|\psi\|_{L^6} \leq c \|u\|_{L^{12/5}}^2 \|\psi\|_{D^{1,2}}. \quad (2.20)$$

Therefore the linear map $\psi \in D^{1,2} \mapsto \int_{\mathbb{R}^3} u^2 \psi dx$ is continuous. By Lax-Milgram Lemma we get the existence of a unique $\Phi \in D^{1,2}$ such that

$$\int_{\mathbb{R}^3} (\nabla \Phi \nabla \psi + u^2 \Phi \psi) dx = -\omega \int_{\mathbb{R}^3} u^2 \psi dx \quad \forall \psi \in D^{1,2},$$

i.e. Φ is the unique solution of (2.17). Furthermore Φ achieves the minimum

$$\begin{aligned} & \inf_{\phi \in D^{1,2}} \int_{\mathbb{R}^3} \left(\frac{1}{2} (|\nabla \phi|^2 + u^2 |\phi|^2) + \omega u^2 \phi \right) dx \\ &= \int_{\mathbb{R}^3} \left(\frac{1}{2} (|\nabla \Phi|^2 + u^2 |\Phi|^2) + \omega u^2 \Phi \right) dx. \end{aligned}$$

Note that also $-|\Phi|$ achieves such a minimum; then, by uniqueness, $\Phi = -|\Phi| \leq 0$. Now let $O(3)$ denote the group of rotations in $\mathbb{R}^3$. Then for every $g \in O(3)$ and $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, set $T_g(f)(x) = f(gx)$. Note that T_g does not change the norms in H^1 , $D^{1,2}$ and L^p . In Lemma 4.2 of [6] it was proved that $T_g \Phi[u] = \Phi[T_g u]$. In this way, if u is radial, we get $T_g \Phi[u] = \Phi[u]$.

Finally, following the same idea of [17], fixed $u \in H^1$, if we multiply (2.17) by $(\omega + \Phi[u])^- \equiv -\min\{\omega + \Phi[u], 0\}$, which is an admissible test function, since $\omega > 0$, we get

$$-\int_{\Phi[u] < -\omega} |D\Phi[u]|^2 dx - \int_{\Phi[u] < -\omega} (\omega + \Phi[u])^2 u^2 dx = 0,$$

so that $\Phi[u] \geq -\omega$ where $u \neq 0$. $\square$

Remark 2.3. The result (ii) of Proposition 2.2 can be strengthened in some cases. Indeed, take $\bar{u}$ in $H^1(\mathbb{R}^3) \cap C^\infty$ radially symmetric such that

$$\bar{u} > 0 \text{ in } B(0, R), \bar{u} \equiv 0 \text{ in } \mathbb{R}^3 \setminus B(0, R)$$

for some $R > 0$. Then there results

$$-\omega \leq \Phi[\bar{u}](x) \leq 0 \quad \forall x \in \mathbb{R}^3.$$

In fact, since $\Phi[\bar{u}]$ solves (2.17), by standard regularity results for elliptic equations, $\bar{u} \in C^\infty$ implies $\Phi[\bar{u}] \in C^\infty$. By Proposition 2.2, $\Phi[\bar{u}]$ is radial; moreover $\Phi[\bar{u}]$ is harmonic outside $B(0, R)$. Since $\Phi[\bar{u}] \in D^{1,2}$, then

$$\Phi[\bar{u}](x) = -\frac{c}{|x|}, \quad |x| \geq R,$$

for some $c > 0$. Setting $\tilde{\Phi}(r) = \Phi[\bar{u}](x)$ for $|x| = r$, it results $\tilde{\Phi}'(R) > 0$ and $\tilde{\Phi}(r) > \tilde{\Phi}(R)$ for every $r > R$. Therefore the minimum of $\Phi[\bar{u}]$ is achieved in $B(0, R)$. Let $\bar{x}$ be a minimum point for $\Phi[\bar{u}]$. Then (2.17) implies

$$\Phi[\bar{u}](\bar{x}) = \frac{-\omega \bar{u}^2(\bar{x}) + \Delta \Phi[\bar{u}](\bar{x})}{\bar{u}^2(\bar{x})} \geq -\omega.$$

In view of proposition 2.2, we can define the map

$$\Phi : H^1 \longrightarrow D^{1,2}$$

which maps each $u \in H^1$ in the unique solution of (2.17). From standard arguments it results $\Phi \in C^1(H^1, D^{1,2})$ and from the very definition of Φ we get

$$F'_\phi(u, \Phi[u]) = 0 \quad \forall u \in H^1. \quad (2.21)$$

Now let us consider the functional

$$J : H^1 \longrightarrow \mathbb{R}, \quad J(u) := F(u, \Phi[u]).$$

By proposition 2.1, $J \in C^1(H^1, \mathbb{R})$ and, by (2.21),

$$J'(u) = F'_u(u, \Phi[u]).$$

By definition of F , we obtain

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 - |\nabla \Phi[u]|^2 + [m^2 - \omega^2]u^2 - u^2 \Phi[u]^2) \, dx \\ &\quad - \omega \int_{\mathbb{R}^3} u^2 \Phi[u] - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx. \end{aligned}$$

Multiplying both members of (2.17) by $\Phi[u]$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla \Phi[u]|^2 \, dx + \int_{\mathbb{R}^3} |u|^2 |\Phi[u]|^2 \, dx = -\omega \int_{\mathbb{R}^3} |u|^2 \Phi[u] \, dx. \quad (2.22)$$

Using (2.22), the functional J may be written as

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + [m^2 - \omega^2]u^2 - \omega u^2 \Phi[u]) \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx. \quad (2.23)$$

The next lemma states a relationship between the critical points of the functionals F and J (the proof can be found in [6]).

Lemma 2.4. *The following statements are equivalent:*

- i) $(u, \Phi) \in H^1 \times D^{1,2}$ is a critical point of F ,
- ii) u is a critical point of J and $\Phi = \Phi[u]$.

Then, in order to get solutions of (2.16)-(2.17), we look for critical points of J .

Theorem 2.5. *Assume hypotheses a) and b). Then the functional J has infinitely many critical points $u_n \in H^1$ having a radial symmetry.*

Proof. Our aim is to apply the equivariant version of the Mountain-Pass Theorem (see [1], Theorem 2.13, or [18], Theorem 9.12). Since J is invariant under the group of translations, there is clearly a lack of compactness. In order to overcome this difficulty, we consider radially symmetric functions. More precisely we introduce the subspace

$$H_r^1 = \{u \in H^1 \mid u(x) = u(|x|)\}.$$

We divide the remaining part of the proof in three steps.

Step 1. Any critical point $u \in H_r^1$ of $J|_{H_r^1}$ is also a critical point of J .

The proof can be found in [6].

Step 2. The functional $J|_{H_r^1}$ satisfies the Palais-Smale condition, i.e.

any sequence $\{u_n\}_n \subset H_r^1$ such that $J(u_n)$ is bounded and $J'|_{H_r^1}(u_n) \rightarrow 0$ contains a convergent subsequence.

For the sake of simplicity, from now on we set $\Omega = m^2 - \omega^2 > 0$. Let $\{u_n\}_n \subset H_r^1$ be such that

$$|J(u_n)| \leq M, \quad J'|_{H_r^1}(u_n) \rightarrow 0$$

for some constant $M > 0$. Then, using the form of J given in (2.23),

$$\begin{aligned} pJ(u_n) - J'(u_n)u_n &= \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2) dx \\ &\quad - \omega \left(\frac{p}{2} - 2\right) \int_{\mathbb{R}^3} u_n^2 \Phi[u_n] dx + \int_{\mathbb{R}^3} u_n^2 (\Phi[u_n])^2 dx \\ &\geq \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2) dx - \omega \left(\frac{p}{2} - 2\right) \int_{\mathbb{R}^3} u_n^2 \Phi[u_n] dx. \end{aligned} \quad (2.24)$$

We distinguish two cases: either $p \geq 4$ or $2 < p < 4$.

If $p \geq 4$, by (2.24), using Proposition 2.2, we immediately deduce

$$pJ(u_n) - J'(u_n)u_n \geq \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2) dx. \quad (2.25)$$

Moreover, by hypothesis a)

$$\left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2) dx \geq c_1 \|u_n\|^2, \quad (2.26)$$

and by assumption

$$pJ(u_n) - J'(u_n)u_n \leq pM + c_2 \|u_n\| \quad (2.27)$$

for some positive constants c_1 and c_2 .

Combining (2.25), (2.26), (2.27), we deduce that $\{u_n\}_n$ is bounded in H_r^1 .

If $2 < p < 4$, by Proposition 2.2 and (2.24) we get

$$\begin{aligned} pJ(u_n) - J'(u_n)u_n &\geq \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \Omega|u_n|^2) dx - \omega^2 \left(2 - \frac{p}{2}\right) \int_{\mathbb{R}^3} u_n^2 dx \\ &= \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \left(\frac{m^2(p-2) - 2\omega^2}{2}\right) \int_{\mathbb{R}^3} |u_n|^2 dx. \end{aligned}$$

By hypothesis b) $m^2(p-2) - 2\omega^2 > 0$, then we repeat the same argument as for $p \geq 4$ and obtain the bounedness of $\{u_n\}_n$ in H_r^1 .

On the other hand, using equation (2.17), and proceeding as in (2.20), we get

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \Phi[u_n]|^2 dx &\leq \int_{\mathbb{R}^3} |\nabla \Phi[u_n]|^2 dx + \int_{\mathbb{R}^3} |u_n|^2 |\Phi[u_n]|^2 dx = -\omega \int_{\mathbb{R}^3} u_n^2 \Phi[u_n] dx \\ &\leq c\omega \|u_n\|_{L^{12/5}}^2 \|\Phi[u_n]\|_{D^{1,2}}, \end{aligned}$$

which implies that $\{\Phi[u_n]\}_n$ is bounded in $D^{1,2}$.
Then, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } H_r^1 \\ \Phi[u_n] &\rightharpoonup \phi && \text{in } D^{1,2}. \end{aligned}$$

If $L : H_r^1 \rightarrow (H_r^1)'$ is defined as

$$L(u) = -\Delta u + \Omega u,$$

then

$$L(u_n) = \omega u_n \Phi[u_n] + |u_n|^{p-2} u_n + \varepsilon_n,$$

where $\varepsilon_n \rightarrow 0$ in $(H_r^1)'$, that is

$$u_n = L^{-1}(\omega u_n \Phi[u_n]) + L^{-1}(|u_n|^{p-2} u_n) + L^{-1}(\varepsilon_n).$$

Now note that $\{u_n \Phi[u_n]\}$ is bounded in $L_r^{3/2}$; in fact, by Hölder's inequality,

$$\|u_n \Phi[u_n]\|_{L_r^{3/2}} \leq \|u_n\|_{L_r^2} \|\Phi[u_n]\|_{L_r^6} \leq c \|u_n\|_{L_r^2} \|\Phi[u_n]\|_{D^{1,2}}.$$

Moreover $\{|u_n|^{p-2} u_n\}$ is bounded in $L_r^{p'}$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). The immersions $H_r^1 \hookrightarrow L_r^3$ and $H_r^1 \hookrightarrow L_r^p$ are compact (see [8] or [19]) and thus, by duality, $L_r^{3/2}$ and $L_r^{p'}$ are compactly embedded in $(H_r^1)'$. Then by standard arguments $L^{-1}(\omega u_n \Phi[u_n])$ and $L^{-1}(|u_n|^{p-2} u_n)$ strongly converge in H_r^1 . Then we conclude

$$u_n \rightarrow u \quad \text{in } H_r^1.$$

Step 3. The functional $J|_{H_r^1}$ satisfies the geometrical hypothesis of the equivariant version of the Mountain Pass Theorem.

First of all we observe that $J(0) = 0$. Moreover, by Proposition 2.2 and (2.23),

$$J(u) \geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\Omega}{2} \int_{\mathbb{R}^3} |u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

The hypothesis $2 < p < 6$ and the continuous embedding $H^1 \subset L^p$ imply that there exists $\rho > 0$ small enough such that

$$\inf_{\|u\|_{H^1}=\rho} J(u) > 0.$$

Since J is even, the thesis of step 3 will follow if we prove that for every finite dimensional subset V of H_r^1 it results

$$\lim_{\substack{u \in V, \\ \|u\|_{H^1} \rightarrow +\infty}} J(u) = -\infty. \quad (2.28)$$

Let V be an m -dimensional subspace of H_r^1 and let $u \in V$. By Proposition 2.2 $\Phi[u] \geq -\omega$ where $u \neq 0$, so that

$$J(u) \leq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \Omega|u|^2 + \omega^2 u^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \leq c \|u\|_{H^1}^2 - \frac{1}{p} \|u\|_{L^p}^p$$

and (2.28) follows, since all norms in V are equivalent. $\square$

Proof of Theorem 1.1. Lemma 2.4 + Theorem 2.5.

Remark 2.6. In view of Remark 2.3 the existence of one nontrivial critical point for the functional J follows from the classical mountain pass theorem: more precisely, taken $\bar{u} \in H_r^1 \cap C^\infty$ as in Remark 2.3, since $\|\Phi[\bar{u}]\|_\infty \leq \omega$, there results

$$J(t\bar{u}) \leq \frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla \bar{u}|^2 + \Omega|\bar{u}|^2 + \omega^2 \bar{u}^2) \, dx - \frac{t^p}{p} \int_{\mathbb{R}^3} |\bar{u}|^p \rightarrow -\infty \text{ as } t \rightarrow +\infty.$$

3 Nonlinear Schrödinger Equations coupled with Maxwell Equations

For sake of simplicity assume $\hbar = m = e = 1$ in (1.14)-(1.15). Then we are reduced to study the following system in $\mathbb{R}^3$:

$$-\frac{1}{2}\Delta u + \Phi u + \omega u - |u|^{p-2}u = 0, \quad (3.29)$$

$$-\Delta \Phi = 4\pi u^2. \quad (3.30)$$

We will assume

a') $\omega > 0$

b') $4 \leq p < 6$.

Of course, (3.29)-(3.30) are the Euler-Lagrange equations of the functional $\mathcal{F} : H^1 \times D^{1,2} \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} \mathcal{F}(u, \Phi) = & \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{16\pi} \int_{\mathbb{R}^3} |\nabla \Phi|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (\Phi u^2 + \omega u^2) dx \\ & - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx, \end{aligned}$$

where H^1 and $D^{1,2}$ are defined as in the previous section.

It is easy to prove the analogous of Proposition 2.1, i.e. that $\mathcal{F} \in C^1(H^1 \times D^{1,2}, \mathbb{R})$ and that its critical points are solutions of (3.29)-(3.30).

Moreover we have the following proposition.

Proposition 3.1. *For every $u \in H^1$ there exists a unique solution $\Phi = \Phi[u] \in D^{1,2}$ of (3.30), such that*

- $\Phi[u] \geq 0$;
- $\Phi[tu] = t^2 \Phi[u]$ for every $u \in H^1$ and $t \in \mathbb{R}$.

Proof. Let us consider the linear map $\phi \in D^{1,2} \mapsto \int_{\mathbb{R}^3} u^2 \phi dx$, which is continuous by (2.20). By Lax-Milgram's Lemma we get the existence of a unique $\Phi \in D^{1,2}$ such that

$$\int_{\mathbb{R}^3} \nabla \Phi \nabla \phi dx = 4\pi \int_{\mathbb{R}^3} u^2 \phi dx \quad \forall \phi \in D^{1,2},$$

i.e. Φ is the unique solution of (3.30). Furthermore Φ achieves the minimum

$$\inf_{\phi \in D^{1,2}} \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx - 4\pi \int_{\mathbb{R}^3} u^2 \phi dx \right\} = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \Phi|^2 dx - 4\pi \int_{\mathbb{R}^3} u^2 \Phi dx.$$

Note that also $|\Phi|$ achieves such minimum; then, by uniqueness, $\Phi = |\Phi| \geq 0$.

Finally,

$$-\Delta \Phi[tu] = 4\pi t^2 u^2 = -t^2 \Delta \Phi[u] = -\Delta(t^2 \Phi[u]),$$

thus, by uniqueness, $\Phi[tu] = t^2 \Phi[u]$. □

Proceeding as in the previous section we can define the map

$$\Phi : H^1 \rightarrow D^{1,2},$$

which maps each $u \in H^1$ in the unique solution of (3.30). As before, $\Phi \in C^1(H^1, D^{1,2})$ and

$$\mathcal{F}'_{\Phi}(u, \Phi[u]) = 0 \quad \forall u \in H^1.$$

Now consider the functional $\mathcal{J} : H^1 \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}(u) = \mathcal{F}(u, \Phi[u]).$$

$\mathcal{J}$ belongs to $C^1(H^1, \mathbb{R})$ and satisfies $\mathcal{J}'(u) = \mathcal{F}_u(u, \Phi[u])$. Using the definition of $\mathcal{F}$ and equation (3.30), we obtain

$$\mathcal{J}(u) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^3} |u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |u|^2 \Phi[u] dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

As before, one can prove the following lemma.

Lemma 3.2. *The following statements are equivalent:*

- i) $(u, \Phi) \in H^1 \times D^{1,2}$ is a critical point of $\mathcal{F}$,
- ii) u is a critical point of $\mathcal{J}$ and $\Phi = \Phi[u]$.

Now we are ready to prove the existence result for equations (3.29)-(3.30).

Theorem 3.3. *Assume hypotheses a') and b'). Then the functional $\mathcal{J}$ has a nontrivial critical point $u \in H^1$ having a radial symmetry.*

Proof. Let H_r^1 be defined as in theorem 2.5.

Step 1. Any critical point $u \in H_r^1$ of $\mathcal{J}|_{H_r^1}$ is also a critical point of $\mathcal{J}$.

The proof is as in theorem 2.5.

Step 2. The functional $\mathcal{J}|_{H_r^1}$ satisfies the Palais-Smale condition.

Let $\{u_n\}_n \subset H_r^1$ be such that

$$|\mathcal{J}(u_n)| \leq M, \quad \mathcal{J}'_{|H_r^1}(u_n) \rightarrow 0$$

for some constant $M > 0$. Then

$$\begin{aligned} & p\mathcal{J}(u_n) - \mathcal{J}'(u_n)u_n \\ &= \left(\frac{p}{4} - \frac{1}{2}\right) \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \left(\frac{p}{4} - 1\right) \int_{\mathbb{R}^3} \Phi[u_n] u_n^2 dx + \left(\frac{p}{2} - 1\right) \omega \int_{\mathbb{R}^3} |u_n|^2 dx \\ &\geq \left(\frac{p}{4} - \frac{1}{2}\right) \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \left(\frac{p}{2} - 1\right) \omega \int_{\mathbb{R}^3} |u_n|^2 dx \end{aligned}$$

by Proposition 3.1, since $p \geq 4$. Moreover

$$\left(\frac{p}{4} - \frac{1}{2}\right) \int_{\mathbb{R}^3} (|\nabla u|^2 + \omega|u|^2) dx \geq c_1 \|u_n\|^2$$

and by assumption

$$p\mathcal{J}(u_n) - \mathcal{J}'(u_n)u_n \leq pM + c_2 \|u_n\|_{H^1}$$

for some positive constants c_1 and c_2 .

We have thus proved that $\{u_n\}_n$ is bounded in H_r^1 .

On the other hand, $\|\Phi[u_n]\|_{D^{1,2}}^2 = 4\pi \int_{\mathbb{R}^3} u^2 \Phi[u_n] dx$, and then, using inequality (2.20), we easily deduce that $\{\Phi[u_n]\}_n$ is bounded in $D^{1,2}$.

The remaining part of the proof follows as in Step 2 of Theorem 2.5, after replacing L with $\mathcal{L} : H_r^1 \rightarrow (H_r^1)'$ defined as $\mathcal{L}(u) = -\frac{1}{2}\Delta u + \omega u$.

Step 3. The functional $\mathcal{J}|_{H_r^1}$ satisfies the three geometrical hypothesis of the mountain pass theorem.

By Proposition 3.1 it results

$$\mathcal{J}(u) \geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^3} |u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

Then, using the continuous embedding $H^1 \subset L^p$, we deduce that $\mathcal{J}$ has a strict local minimum in 0.

We introduce the following notation: if $u : \mathbb{R}^3 \rightarrow \mathbb{R}$, we set

$$u_{\lambda,\alpha,\beta}(x) = \lambda^\beta u(\lambda^\alpha x), \quad \lambda > 0, \alpha, \beta \in \mathbb{R}.$$

Now fix $u \in H_r^1$. We want to show that

$$\Phi[u_{\lambda,\alpha,\beta}] = (\Phi[u])_{\lambda,\alpha,2(\beta-\alpha)}. \quad (3.31)$$

In fact

$$\begin{aligned} -\Delta \Phi[u_{\lambda,\alpha,\beta}](x) &= 4\pi u_{\lambda,\alpha,\beta}^2(x) = 4\pi \lambda^{2\beta} u^2(\lambda^\alpha x) \\ &= -\lambda^{2\beta} (\Delta \Phi[u])(\lambda^\alpha x) = -\Delta((\Phi[u])_{\lambda,\alpha,2(\beta-\alpha)})(x). \end{aligned}$$

By uniqueness (see Proposition 3.1), (3.31) follows.

Now take $u \not\equiv 0$ in H_r^1 and evaluate

$$\mathcal{J}(u_{\lambda,\alpha,\beta}) = \frac{\lambda^{2\beta-\alpha}}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\omega}{2} \lambda^{2\beta-3\alpha} \int_{\mathbb{R}^3} u^2 dx$$

$$+ \frac{\lambda^{4\beta-5\alpha}}{4} \int_{\mathbb{R}^3} u^2 \Phi[u] dx - \frac{\lambda^{\beta p-3\alpha}}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

We want to prove that $\mathcal{J}(u_{\lambda,\alpha,\beta}) < \mathcal{J}(0)$ for some suitable choice of λ , α and β .

For example assume

$$\begin{cases} \beta p - 3\alpha < 0, \\ \beta p - 3\alpha < 2\beta - \alpha, \\ \beta p - 3\alpha < 2\beta - 3\alpha, \\ \beta p - 3\alpha < 4\beta - 5\alpha, \end{cases} \quad (3.32)$$

then it is clear that $\mathcal{J}(u_{\lambda,\alpha,\beta}) \rightarrow -\infty$ as $\lambda \rightarrow 0$.

So we look for a couple (α, β) which satisfies (3.32). From the third inequality we get $\beta < 0$. Combining the second and the fourth ones, we derive

$$4 - p < \frac{2\alpha}{\beta} < p - 2. \quad (3.33)$$

Such an inequality is satisfied by taking $\beta = 2\alpha$, which also satisfies the first inequality in (3.32).

In a similar way one can prove that if

$$\begin{cases} \beta p - 3\alpha > 0, \\ \beta p - 3\alpha > 2\beta - \alpha, \\ \beta p - 3\alpha > 2\beta - 3\alpha, \\ \beta p - 3\alpha > 4\beta - 5\alpha, \end{cases} \quad (3.34)$$

then $\mathcal{J}(u_{\lambda,\alpha,\beta}) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$ with the same choice $\beta = 2\alpha$.

Remark 3.4. Notice that the systems (3.32) or (3.34) have a solution for every $p > 3$. More precisely for every $p > 3$ there is a couple (α, β) which satisfies the inequality (3.33) and, consequently, $\mathcal{J}(u_{\lambda,\alpha,\beta}) \rightarrow -\infty$. The restriction $p \geq 4$ appears in proving the Palais-Smale condition.

Proof of Theorem 1.2 Lemma 3.2 + Theorem 3.3.

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