

BOUNCE ON A p -LAPLACIAN*

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ABSTRACT. The existence of nontrivial solutions for reversed variational inequalities involving p -Laplace operators is proved. The solutions are obtained as limits of solutions of suitable penalizing problems.

1. Introduction. Several interesting problems can be described by variational inequalities in which the sign of the inequality is opposite with respect to the sign of classical ones *à la* Lions–Stampacchia ([13]). This is the reason why they are called *reversed* variational inequalities. For instance the bounce problem gives rise to a reversed variational inequality (see [5], [7] and [14]), as well as the *jumping* problem (see [15]).

In this paper we consider a non-Hilbert version of the notion of reversed variational inequality, introduced for the Hilbert case in [16], in connection with a fourth order elliptic problem (see also [14]). In the present case the strategy used in [16] to prove the existence of solutions cannot be adapted directly and needs some refined tools of nonlinear analysis.

We study the existence of a solution u of the following problem:

$$(P) \quad \begin{cases} \exists u \in K_\phi := \{u \in W_0^{1,p}(\Omega) : u \geq \phi\} \text{ such that} \\ \int_\Omega |Du|^{p-2} Du \cdot D(v-u) dx - \alpha \int_\Omega |u|^{p-2} u(v-u) dx \leq 0 \\ \forall v \in K_\phi, \end{cases}$$

where Ω is a smooth bounded domain of \mathbb{R}^N , $1 \leq N < p$, ϕ is a measurable function, with $\sup_\Omega \phi < 0$, and $\alpha < \lambda_1$, where

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |Du|^p dx}{\int_\Omega |u|^p dx}. \quad (1.1)$$

By a *solution* we mean a function $u \in K_\phi$ which solves (P) and which is *not* a solution of the associated equation $\Delta_p u + \alpha|u|^{p-2}u = 0$ in Ω .

If $p = 2$ and $N = 1$ problem (P) admits bounce trajectories as solutions (see [7], [14] or [15]). In fact (P) is not a “problem with obstacle” (described by classical variational inequalities), but is a “bounce problem”.

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It is apparent that, taking $v = u + \psi$ in (P) , with $\psi \in \mathcal{D}(\Omega)$ and $\psi \geq 0$, any solution of (P) is a nontrivial solution of the problem

$$\begin{cases} -\Delta_p u - \alpha|u|^{p-2}u \leq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_p is the p -Laplace operator. Moreover, since $\alpha < \lambda_1$, any solution of (P) is nonpositive, so that $v = -u$ solves

$$-\Delta_p v - \alpha v^{p-1} \geq 0, \quad v \geq 0 \quad \text{in } \Omega. \tag{1.2}$$

As a corollary of a result of Serrin and Zou ([17, Theorem I']), if $\Omega \subset \mathbb{R}^N$, $1 \leq N < p$, is an exterior domain and v solves (1.2) with $\alpha = 1$, then $v \equiv 0$. Moreover for the problem

$$-\Delta_m v - v^{p-1} \geq 0, \quad v \geq 0 \quad \text{in } \Omega \tag{1.3}$$

they prove that:

- if Ω is an exterior domain, inequality (1.3) has a nontrivial solution if and only if $m \in (1, N)$ and $p > \frac{m(N-1)}{N-m}$ ([17, Corollary I]);
- if $\Omega = \mathbb{R}^N$ and $m \in (1, N)$, then (1.3) has a nontrivial solution if and only if $p > \frac{m(N-1)}{N-m}$ ([17, Corollary II (iii)]).

Having in mind these results, we want to show that in bounded domains Ω problem (P) , and so inequality (1.2), admits bounded solutions.

As a consequence of the previous discussion, it is clear that $\phi \leq 0$ is a necessary condition for existence of solutions to (P) when $\alpha < \lambda_1$. On the other hand, the assumptions $\alpha < \lambda_1$ and $\sup_{\Omega} \phi < 0$ are essential in all the proofs of the paper, while the case $\alpha \geq \lambda_1$ is still open. Indeed, if $\alpha \geq \lambda_1$, the geometrical structures of f_{ω} change and it seems hard to find critical points of f_{ω} . For example, the Palais-Smale condition (see Proposition 2.4) might not hold: in fact, if $\alpha = \lambda_1$ and $u_n = t_n e_1$, where $t_n \rightarrow \infty$ and e_1 is the eigenvalue associated to λ_1 (see the proof of Theorem 2.8), u_n has no converging subsequences.

Due to the lack of a general theory for "reversed" inequalities, as Stampacchia's Lemma for linear (classical) variational inequalities, a natural way to face problem (P) is to study the following family of approximating problems:

$$(P_{\omega}) \quad \begin{cases} \Delta_p u + \alpha|u|^{p-2}u - \omega((u - \phi)^-)^{k-1} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ω is a positive parameter approaching $+\infty$, $k > p$ and k is subcritical (see Section 2). First the existence of a nontrivial solution u_{ω} of (P_{ω}) is established for each ω and for each $N \geq 1$ and then we show that the family $\{u_{\omega} \mid \omega \geq \omega_0 > 0\}$ of such nontrivial solutions u_{ω} is bounded in $W_0^{1,p}(\Omega)$. Finally, when $p > N$, any limit u of sequences of solutions u_{ω_n} of (P_{ω_n}) satisfies the reversed variational inequality (P) . Moreover, the crucial part of the proof is that any solution u of (P) constructed by the limiting process satisfies $\Delta_p u + \alpha|u|^{p-2}u = \mu$ for a suitable nontrivial nonnegative Radon measure μ depending on u .

2. The Approximating Problems. Let Ω be a bounded and smooth domain of \mathbb{R}^N , $N \geq 1$, $\alpha < \lambda_1$, $\omega \geq \omega_0 > 0$, $1 < p < k$ (and $k < pN/(N-p)$ if $p < N$). Assume ϕ is a measurable function defined in Ω with $\sup_{\Omega} \phi < 0$ and consider the following problems:

$$(P_{\omega}) \quad \begin{cases} \Delta_p u + \alpha |u|^{p-2} u - \omega((u - \phi)^-)^{k-1} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2} Du)$, $u^- = \max\{-u, 0\}$ and $u \in W_0^{1,p}(\Omega)$. We endow $W_0^{1,p}(\Omega)$ with the standard norm $\|u\| = (\int_{\Omega} |Du|^p dx)^{1/p}$ and we use the standard notation p' to denote the real number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Consider $f_{\omega} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined as follows:

$$f_{\omega}(u) = \frac{1}{p} \int_{\Omega} |Du|^p dx - \frac{\alpha}{p} \int_{\Omega} |u|^p dx - \frac{\omega}{k} \int_{\Omega} ((u - \phi)^-)^k dx.$$

We observe that f_{ω} is a C^1 functional on $W_0^{1,p}(\Omega)$ and that its critical points are solutions of (P_{ω}) .

Note that, in general, the function $g(x, s) = \alpha |s|^{p-2} s - \omega((s - \phi(x))^-)^{k-1}$ does not satisfy either the classical Ambrosetti-Rabinowitz condition (see [1]) or the generalized Ambrosetti-Rabinowitz condition introduced in [9] and [10], namely $\exists \Theta$ and $\exists s_0 > 0$ such that

$$0 < \Theta G(x, s) \leq sg(x, s), \quad \forall s, |s| \geq s_0, \quad (2.1)$$

where $G(x, s) = \int_0^s g(x, \sigma) d\sigma$ and either $\Theta > 2$ ([1]) or $\Theta > p$ ([9],[10]).

In fact equation (2.1) reads as

$$0 < \Theta \frac{\alpha}{p} |s|^p + \Theta \frac{\omega}{k} ((s - \phi(x))^-)^k \leq \alpha |s|^p - \omega((s - \phi(x))^-)^{k-1} s.$$

Of course, if $\alpha > 0$ and $\Theta > p$, the last inequality is not satisfied for any $s > 0$, and it is not satisfied for any $s > 0$ when $\alpha \leq 0$, whatever $\Theta > 0$ is. See also [11] and [12] for other classes of nonlinearities.

Nevertheless we still apply the Mountain Pass Theorem to get a nontrivial critical point u_{ω} for f_{ω} , i.e. a solution of problem (P_{ω}) .

We first recall the following definition.

DEFINITION 2.1. Let $c \in \mathbb{R}$. We say that a C^1 functional $f : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition at level c , or that $(PS)_c$ holds, if every sequence $(u_n)_n$ in $W_0^{1,p}(\Omega)$ such that $f(u_n) \rightarrow c$ and $f'(u_n) \rightarrow 0$, has a strongly converging subsequence.

Sequences $(u_n)_n$ such that $f(u_n) \rightarrow c$ and $f'(u_n) \rightarrow 0$ are called Palais-Smale sequences at level c , or $(PS)_c$ -sequences.

As in the Hilbert case, in order to prove $(PS)_c$, it is sufficient to check that a Palais-Smale sequence has a bounded subsequence. In fact, one can show that the following property holds (see [4]).

PROPOSITION 2.2 ((S_+) property). Let $\Phi : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be defined as $\Phi(u) = \frac{1}{p} \|u\|_{W_0^{1,p}(\Omega)}^p$. If a sequence $(u_n)_n$ weakly converges in $W_0^{1,p}(\Omega)$ to u and

$$\limsup_{n \rightarrow \infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0,$$

then $u_n \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$.

Thus, in order to prove that f_ω satisfies $(PS)_c$, we first prove that $(PS)_c$ holds if any $(PS)_c$ -sequence has a bounded subsequence, and then we prove that any $(PS)_c$ -sequence is bounded.

LEMMA 2.3. *Let $c \in \mathbb{R}$ and let $(u_n)_n$ be a $(PS)_c$ -sequence for f_ω which has a bounded subsequence. Then $(u_n)_n$ admits a strongly convergent subsequence.*

Proof. The proof reminds the proof given in [9] for some nonlinear problems in presence of the p -Laplace operator, under the generalized Ambrosetti Rabinowitz condition.

We recall that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ compactly $\forall q < p^*$, where $p^* = \frac{pN}{N-p}$ if $p < N$ and $p^* = \infty$ if $p \geq N$.

Therefore $\int |u_n|^{p-2} u_n (u_n - u) \rightarrow 0$ and $\int ((u_n - \phi)^-)^{k-1} (u_n - u) \rightarrow 0$. Then we get

$$\begin{aligned} \langle \Phi'(u_n), u_n - u \rangle &= \int_{\Omega} |Du_n|^{p-2} Du_n \cdot (Du_n - Du) \, dx = f'_\omega(u_n)(u_n - u) \\ &\quad + \alpha \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) \, dx - \omega \int_{\Omega} ((u_n - \phi)^-)^{k-1} (u_n - u) \, dx \rightarrow 0. \end{aligned}$$

By the (S_+) property, we conclude that $u_n \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$. □

PROPOSITION 2.4 (Palais-Smale). *Suppose $\sup_{\Omega} \phi < 0$ and $\alpha < \lambda_1$. Then f_ω satisfies $(PS)_c$ for every $c \in \mathbb{R}$.*

Proof. Let u_n be a $(PS)_c$ sequence. By the previous Lemma, it is enough to show that $(u_n)_n$ has a bounded subsequence. Thus suppose by contradiction that $\|u_n\|$ is unbounded. Then we can suppose that there exists v in $W_0^{1,p}(\Omega)$ such that $v_n = u_n / \|u_n\|$ weakly converges to the function v in $W_0^{1,p}(\Omega)$.

Of course $\frac{f'_\omega(u_n)(u_n)}{\|u_n\|} \rightarrow 0$, where

$$\begin{aligned} \frac{f'_\omega(u_n)(u_n)}{\|u_n\|} &= \frac{1}{\|u_n\|} \left\{ \int_{\Omega} |Du_n|^p \, dx - \alpha \int_{\Omega} |u_n|^p \, dx \right. \\ &\quad \left. + \omega \int_{\Omega} ((u_n - \phi)^-)^{k-1} u_n \, dx \right\} = \frac{1}{\|u_n\|} \left\{ p f_\omega(u_n) \right. \\ &\quad \left. + \left(\frac{p}{k} - 1\right) \omega \int_{\Omega} ((u_n - \phi)^-)^k \, dx + \omega \int_{\Omega} ((u_n - \phi)^-)^{k-1} \phi \, dx \right\}. \end{aligned}$$

Passing to the limit we get, since $p < k$ and $\phi < 0$,

$$\lim_{n \rightarrow \infty} \frac{\int_{\Omega} ((u_n - \phi)^-)^k \, dx}{\|u_n\|} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\int_{\Omega} ((u_n - \phi)^-)^{k-1} \phi \, dx}{\|u_n\|} = 0. \tag{2.2}$$

Since $\frac{f'_\omega(u_n)(u_n)}{\|u_n\|^p} \rightarrow 0$, one has

$$1 - \alpha \int_{\Omega} |v|^p \, dx + \frac{\omega \int_{\Omega} ((u_n - \phi)^-)^{k-1} u_n \, dx}{\|u_n\|^p} \rightarrow 1 - \alpha \int_{\Omega} |v|^p \, dx = 0.$$

In this way we get an immediate contradiction if $\alpha \leq 0$. If $\alpha > 0$, we would have

$$0 = 1 - \alpha \int_{\Omega} |v|^p \, dx \geq 1 - \frac{\alpha}{\lambda_1} \int_{\Omega} |Dv|^p \, dx \geq 1 - \frac{\alpha}{\lambda_1} > 0.$$

□

We now observe that (P_ω) admits the trivial solution for every α and ω . We now want to prove the existence of a particular nontrivial solution, namely a solution which is below ϕ on a set of positive measure.

DEFINITION 2.5. *We say that u in $W_0^{1,p}(\Omega)$ is a forcing solution of problem (P_ω) if it is a solution such that $\text{meas}(\{x \in \Omega \mid (u(x) - \phi(x))^- \neq 0\}) > 0$.*

The reason for this definition lies in the fact that if u_{ω_n} is a solution of (P_{ω_n}) and $u_{\omega_n} \rightarrow u$, then $u \geq \phi$ (see Lemma 3.2) and if, moreover, u_{ω_n} is a forcing solution of (P_{ω_n}) , $u_{\omega_n} \rightarrow u$ uniformly and ϕ is continuous, then the coincidence set, or “contact set”, $\{x \in \Omega \mid u(x) = \phi(x)\}$ is not empty (see Proposition 3.3), so u is “forced” by the sequence u_{ω_n} to be over ϕ and to touch it somewhere. In this way ϕ works as a bounce wall.

REMARK 2.6. *A solution u of (P_ω) is a forcing solution if and only if $f_\omega(u) > 0$. In fact*

$$0 = f'_\omega(u)(u) = pf_\omega(u) + \left(\frac{p}{k} - 1\right)\omega \int_\Omega ((u - \phi)^-)^k dx + \omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} \phi dx.$$

An application of Sobolev’s inequality and the fact $\phi < 0$ give the following lemma.

LEMMA 2.7. *Suppose $\sup_\Omega \phi < 0$. Then*

$$\int_\Omega ((u - \phi)^-)^k dx = o(\|u\|^k)$$

as $u \rightarrow 0$.

THEOREM 2.8 (Existence Theorem). *Let $\alpha < \lambda_1$ and $\sup_\Omega \phi < 0$. Then for every ω there exists a forcing solution u_ω of problem (P_ω) such that $\sup_{\omega \geq \omega_0} f_\omega(u_\omega) < +\infty$.*

Proof. By Lemma 2.7, given $\varepsilon < 1 - \alpha/\lambda_1$, there exists $\rho > 0$ such that

$$\inf_{\|u\|=\rho} f_\omega(u) \geq \frac{1}{p} \left(1 - \varepsilon - \frac{\alpha}{\lambda_1}\right) \rho^p > 0.$$

Let e_1 be the function which minimizes the Rayleigh quotient of (1.1). We can suppose $e_1 > 0$ in Ω (see [2]).

But $f_\omega(0) = 0$ and $\lim f_\omega(-te_1) = -\infty$. Therefore there exists $t_\omega > 0$ such that $f_\omega(-t_\omega e_1) < 0$. By the Mountain Pass Theorem ([1]) there exists a nontrivial critical point u_ω of f_ω for every ω . Moreover the following estimates hold:

$$\frac{\rho^p}{p} \left(1 - \varepsilon - \frac{\alpha}{\lambda_1}\right) \leq f_\omega(u_\omega) \leq \sup_{t \in [0, t_\omega]} f_\omega(-te_1) \leq \sup_{t \geq 0} f_{\omega_0}(-te_1) < +\infty$$

for every $\omega \geq \omega_0$. □

REMARK 2.9. *If $p > N$ we can take $\varepsilon = 0$ in the inequalities above. In fact in this case $W_0^{1,p}(\Omega) \hookrightarrow C_0^0(\Omega)$, so there exists $\bar{\rho}$ such that, if $\rho \leq \bar{\rho}$ and $\|u\| \leq \rho$, then $u - \phi \geq 0$ (since $\sup_\Omega \phi < 0$). Moreover in such a case $\liminf_{\omega \rightarrow +\infty} f_\omega(u_\omega) > 0$.*

3. Bounce Equation. The aim of this section is to study the problem obtained from (P_ω) when ω tends to $+\infty$. We will essentially follow the approach of [16], but now there are some complications due to the Banach setting. In particular, in order to prove the following Theorem 3.4, we will need a theorem by Boccardo and Murat (see [3]).

From now on we will consider sequences of real positive numbers $(\omega_n)_n$ such that $\omega_n \rightarrow +\infty$ as $n \rightarrow \infty$ and we will investigate the asymptotic behavior of forcing solutions $(u_{\omega_n})_n$ of $(P_{\omega_n})_n$ as n goes to infinity. For the sake of simplicity we will write ω in place of ω_n and $\omega \rightarrow +\infty$ in place of “ $\omega_n \rightarrow +\infty$ as $n \rightarrow \infty$ ”.

THEOREM 3.1. *If $\alpha < \lambda_1$, $\sup_\Omega \phi < 0$ and u_ω is a solution of (P_ω) such that $\sup_\omega f_\omega(u_\omega) < +\infty$, then*

- a) $\sup_\omega \omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} dx < +\infty$;
- b) $(u_\omega)_\omega$ is bounded.

Proof. Suppose by contradiction that there exists a subsequence, which we still denote by $(u_\omega)_\omega$, such that $\|u_\omega\|$ diverges. Then, up to a subsequence, there exists v in $W_0^{1,p}(\Omega)$ such that $v_\omega = u_\omega/\|u_\omega\| \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω . Observe that $f'_\omega(u_\omega)(u_\omega) = 0$, that is

$$\begin{aligned} \int_\Omega |Du_\omega|^p dx - \alpha \int_\Omega |u_\omega|^p dx + \omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} u_\omega dx &= p f_\omega(u_\omega) \\ + \left(\frac{p}{k} - 1\right) \omega \int_\Omega ((u_\omega - \phi)^-)^k dx + \omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} \phi dx &= 0. \end{aligned} \tag{3.1}$$

Since $\sup_\Omega \phi < 0$, dividing by $\|u_\omega\|$ and passing to the limit, we get

$$\lim_{\omega \rightarrow +\infty} \frac{\omega \int_\Omega ((u_\omega - \phi)^-)^k dx}{\|u_\omega\|} = 0 \text{ and } \lim_{\omega \rightarrow +\infty} \frac{\omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} \phi dx}{\|u_\omega\|} = 0. \tag{3.2}$$

In this way, since $\frac{f'(u_\omega)(u_\omega)}{\|u_\omega\|^p} = 0$, we get

$$1 - \alpha \int_\Omega |v|^p dx + \frac{\omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} u_\omega dx}{\|u_\omega\|^p} \rightarrow 1 - \alpha \int_\Omega |v|^p dx = 0,$$

and one concludes as at the end of Proposition 2.4.

It is now easy to prove a) from (3.1). □

It is also easy to prove the following lemma.

LEMMA 3.2. *If u_ω solves (P_ω) and $u_\omega \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then $u - \phi \geq 0$ a.e. in Ω .*

It is readily seen that the following Proposition holds.

PROPOSITION 3.3. *Suppose ϕ is continuous, u_ω is a forcing solution of (P_ω) and $u \in W_0^{1,p}(\Omega)$ is such that u_ω converges to u uniformly in Ω . Then $\{x \in \Omega \mid u(x) = \phi(x)\} \neq \emptyset$.*

If u_ω is a forcing solution of (P_ω) , let us define \mathcal{A}_ω as (a set equivalent to)

$$\mathcal{A}_\omega = \{x \in \Omega \mid u_\omega(x) < \phi(x)\},$$

and

$$\mathcal{A} = \{x \in \Omega \mid \exists \text{ a ngrhd } U \text{ of } x, \exists \omega_0 \text{ s. t. } \forall \omega \geq \omega_0 \ m(U \cap \mathcal{A}_\omega) = 0\}.$$

We observe that \mathcal{A} is an open subset of Ω , and so its complementary set

$$\mathfrak{B} = \{x \in \Omega \mid \forall \text{ ngrhd } U \text{ of } x, \forall \omega_0, \exists \omega \geq \omega_0 \text{ s. t. } m(U \cap \mathcal{A}_\omega) > 0\}$$

is closed. We also remark that \mathfrak{B} is, in some sense, the set of points in which u touches ϕ , or the *contact set*; actually, if u and ϕ are continuous, \mathfrak{B} is the set of points x 's of Ω where $u(x) = \phi(x)$.

THEOREM 3.4. *Suppose $(u_\omega)_\omega$ is a sequence of solutions of $(P_\omega)_\omega$ and $u_\omega \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Then there exists a nonnegative Radon measure μ such that*

$$\begin{cases} \Delta_p u + \alpha |u|^{p-2} u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of distributions.

Such a measure μ is supported in \mathfrak{B} , that is $\mu(\mathcal{A}) = 0$.

Moreover, if $p > N$, $\sup \phi < 0$ and $\mathfrak{B} = \{x \in \Omega \mid u(x) = \phi(x)\} \neq \emptyset$ (for example if u_ω is a forcing solution $\forall \omega$) and ϕ is continuous, then $\mu(\mathfrak{B}) > 0$.

Proof. Consider the following linear and continuous functionals on $W_0^{1,p}(\Omega)$:

$$L_\omega(v) = - \int_\Omega |Du_\omega|^{p-2} Du_\omega \cdot Dv \, dx + \alpha \int_\Omega |u_\omega|^{p-2} u_\omega v \, dx.$$

We want to show that $\forall v$ in $W_0^{1,p}(\Omega)$, $L_\omega(v)$ converges to $L(v)$ defined as

$$L(v) = - \int_\Omega |Du|^{p-2} Du \cdot Dv \, dx + \alpha \int_\Omega |u|^{p-2} uv \, dx.$$

To this aim we apply the following theorem by Boccardo and Murat (see [3]).

THEOREM 3.5 (Boccardo-Murat). *Suppose $1 < p < \infty$, $v_\omega \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$, strongly in $L_{loc}^p(\Omega)$ and a.e. in Ω , $h_\omega \rightarrow h$ in $W^{-1,p'}(\Omega)$ and g_ω weakly $-*$ converges to g in the space $\mathfrak{M}(\Omega)$ of Radon measures. If*

$$-\Delta_p v_\omega = h_\omega + g_\omega \quad \text{in } \mathcal{D}'(\Omega),$$

then

$$Dv_\omega \rightarrow Dv \quad \text{strongly in } L^q(\Omega, \mathbb{R}^N) \text{ for any } q < p.$$

Moreover u solves

$$-\Delta_p v = h + g \quad \text{in } \mathcal{D}'(\Omega).$$

Actually the authors proved this theorem for a more general operator of Leray-Lions type, but for our goals this version is enough.

In the problem under investigation, it results $h_\omega := \alpha |u_\omega|^{p-2} u_\omega$ and $g_\omega := -\omega((u_\omega - \phi)^-)^{k-1}$, which weakly $-*$ converges to a suitable measure μ in $\mathfrak{M}(\Omega)$.

The rest of the proof of Theorem 3.4 can be made following the proof of Theorems 7.1 and 8.7 in [16]. Anyway, for the convenience of the reader, we give a sketch of it.

If $x_0 \in \mathcal{A}$, there exists a neighborhood U of x_0 and ω_0 such that $u_\omega \geq \phi$ in U for any $\omega \geq \omega_0$. Take $\psi \in C_c^\infty(U)$; then

$$\int_\Omega |Du_\omega|^{p-2} Du_\omega \cdot D\psi \, dx - \alpha \int_\Omega |u_\omega|^{p-2} u_\omega \psi \, dx = 0,$$

and passing to the limit, we get that x_0 doesn't belong to the support of μ .

Now assume $p > N$ and $\mathfrak{B} \neq \emptyset$. Assume by contradiction that $\mu \equiv 0$. Then u would solve $\Delta_p u + \alpha|u|^{p-2}u = 0$, and since $\alpha < \lambda_1$, we would have $u \equiv 0$. But $\mathfrak{B} = \{x \in \Omega : u(x) = \phi(x)\} \neq \emptyset$ and a contradiction arises, since $\sup_\Omega \phi < 0$. \square

We now want to show that the limit of a weakly convergent sequence has some finer properties.

PROPOSITION 3.6. *If $p > N$, $\sup_\Omega \phi < 0$ and u_ω is a solution of (P_ω) such that $u_\omega \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then $u_\omega \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$.*

Proof. We have $f'_\omega(u_\omega)(u - u_\omega) = 0$, that is

$$\begin{aligned} \int_\Omega |Du_\omega|^{p-2} Du_\omega \cdot D(u - u_\omega) dx - \alpha \int_\Omega |u_\omega|^{p-2} u_\omega (u - u_\omega) dx \\ + \omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} (u - u_\omega) dx = 0. \end{aligned} \quad (3.3)$$

But $p > N$, so $u_n \rightarrow u$ uniformly and then, by **a)** of Theorem 3.1,

$$\omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} (u - u_\omega) dx \rightarrow 0.$$

Then (3.3) gives

$$\int_\Omega |Du_\omega|^{p-2} Du_\omega \cdot D(u - u_\omega) dx \rightarrow 0,$$

and then $u_\omega \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$ by the (S_+) property. \square

THEOREM 3.7. *Assume $p > N$, $\sup_\Omega \phi < 0$ and u_ω is a solution of (P_ω) which (weakly) converges to u in $W_0^{1,p}(\Omega)$. Then $\forall v \in K_\phi$*

$$\int_\Omega |Du|^{p-2} Du \cdot D(v - u) dx - \alpha \int_\Omega |u|^{p-2} u (v - u) dx \leq 0. \quad (3.4)$$

Proof. Let v belong to $W_0^{1,p}(\Omega)$ and evaluate $f'_\omega(u_\omega)(v - u_\omega)$. We get

$$\begin{aligned} \int_\Omega |Du_\omega|^{p-2} Du_\omega \cdot D(v - u_\omega) dx - \alpha \int_\Omega |u_\omega|^{p-2} u_\omega (v - u_\omega) dx \\ + \omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} (v - u_\omega) dx = 0. \end{aligned}$$

But $((u_\omega - \phi)^-)^{k-1} (v - u_\omega) = ((u_\omega - \phi)^-)^{k-1} (v - \phi) + ((u_\omega - \phi)^-)^k \geq 0$.

The thesis follows from Proposition 3.6. \square

THEOREM 3.8 (Reversed Variational Inequality). *Suppose that $p > N$, $\sup_\Omega \phi < 0$ and $\alpha < \lambda_1$. Then there exists at least one nontrivial solution of problem (P) .*

Proof. Theorem 2.8 + Theorem 3.1 + Lemma 3.2 + Proposition 3.3 + Theorem 3.4 + Theorem 3.7. \square

REMARK 3.9. *We remark again that (3.4) has quite a different nature with respect to classical variational inequalities (see [13]): in the latter case we would have $-\Delta_p + l.o.t. \geq 0$ in $\mathcal{D}'(\Omega)$, while in the former case we get $-\Delta_p + l.o.t. \leq 0$ in $\mathcal{D}'(\Omega)$.*

We also note that Theorem 3.7 states that the u is an "upper critical point" (see for example [6] or [8]) for the functional f defined on $W_0^{1,p}(\Omega)$ as

$$f(u) = \begin{cases} \frac{1}{p} \int_{\Omega} |Du|^p dx - \frac{\alpha}{p} \int_{\Omega} |u|^p dx & \text{if } u \in K_{\phi}, \\ -\infty & \text{otherwise,} \end{cases}$$

and so Theorem 3.8 shows the existence of at least one nontrivial critical point of f which belongs to ∂K_{ϕ} . Such a result seems quite interesting: in fact f is upper semicontinuous and it is quite difficult to find critical points for it directly, since it is not clear how to define curves of steepest descent for such a functional.

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