STABILITY OF SOME UNILATERAL FREE-DISCONTINUITY PROBLEMS IN TWO-DIMENSIONAL DOMAINS

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ABSTRACT. The purpose of this paper is to study the stability of some unilateral free-discontinuity problems in two-dimensional domains, with the density of the volume part having p-growth, with $1 < p < \infty$, under perturbations of the discontinuity sets in the Hausdorff metric.

Key words: Boundary value problems for nonlinear elliptic PDE, brittle fracture, capacity, free-discontinuity problems, Hausdorff distance, variational inequalities.

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CONTENTS

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with Lipschitz continuous boundary $\partial\Omega$ and let $\partial_D\Omega \subset \partial\Omega$ be a (non-empty) relatively open subset of $\partial\Omega$ composed of a finite number of connected components. Let $g \in W^{1,p}(\Omega)$ and let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function which satisfies the assumptions (2.1)-(2.2) below. We consider pairs (u, K) with K a compact subset of $\overline{\Omega}$ and $u \in L^{1,p}(\Omega \setminus K) := \{v \in L^p_{loc}(\Omega \setminus K),\, \nabla v \in L^p(\Omega \setminus K, \mathbb{R}^2)\}$ with $u = g$ on $\partial_D\Omega \setminus K$, which satisfy the following unilateral minimality condition:

$$\int_{\Omega\setminus K} f(x, \nabla u)\, dx + \mathcal{H}^1(K) \le \int_{\Omega\setminus H} f(x, \nabla w)\, dx + \mathcal{H}^1(H), \tag{1.1}$$

among all compact subsets H of $\overline{\Omega}$ with $H \supset K$ and all functions $w \in L^{1,p}(\Omega \setminus H)$ with $w = g$ on $\partial_D\Omega \setminus H$.

Our goal in this paper is to study the stability of the problem (1.1) under variations of the compact set K in the Hausdorff metric and of the boundary datum g in the strong topology of $W^{1,p}(\Omega)$. Precisely, let (K_h) be a sequence of compact subsets of $\overline{\Omega}$ which converges to a compact set K in the Hausdorff metric and let $(g_h) \subset W^{1,p}(\Omega)$ be a sequence which converges strongly to a function g in $W^{1,p}(\Omega)$. Let u_h be such that the pair (u_h, K_h) is a solution of (1.1) relative to the boundary data g_h. We are studying the conditions under which the sequence (∇u_h) converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$ for some function u such that the pair (u, K) is a solution of (1.1) relative to the boundary data g.

Minimization problems of the type (1.1) arise for instance in the mathematical formulation of the irreversible quasi-static growth of brittle fractures based on Griffith's theory of crack growth. In this model, the crack path is determined by the competition between bulk and surface energy. The variational model proposed by G.A. Francfort and J.-J. Marigo [15] is described as follows: from an initial crack K_0 (possibly an empty set), the crack $K(t)$ at a given time t corresponding to a loading $g(t)$ applied to $\partial_D\Omega$, will minimize the total energy (bulk energy + surface energy) among all the possible cracks K which contain the previous one $K(s)$, $s < t$. This continuum evolution of the cracks during the loading process is obtained as a limit of a discretized evolution described as a step by step unilateral minimization problem of the type (1.1).

The precise mathematical formulation of this model has been studied by G. Dal Maso and R. Toader [9, 10] in the special case of linearized elasticity for anti-plane shear and for an *a priori bound* on the number of connected component of the test cracks. In this case the reference configuration is an infinite cylinder $\Omega \times \mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, and the displacement field has the form $v := (0, 0, u)$ where u is a scalar function defined on Ω. The cracks are assumed also to be of the form $K \times \mathbb{R}$, where K is a compact subset of $\overline{\Omega}$.

Recently, a weak formulation for the variational model of fracture growth in the framework of SBV space of special functions of bounded variation, has been proposed by G.A. Francfort and C.J. Larsen [14] for anti-plane shear in higher dimensions. This approach is more natural since it is performed in any dimension and with no restrictions on the test cracks. However, the strong formulation in [9], based on the Hausdorff convergence of compact sets, is more handable and elementary in two dimensions and leads to the convergence in the Hausdorff metric of the cracks obtained in the discretized evolution.

One of the key points in [9] is the stability of (1.1) for $f(x, \xi) = |\xi|^2$, which follows from the stability of the following minimization problem:

$$\min_v \left\{ \int_{\Omega \setminus K} f(x, \nabla v)\, dx \colon \; v \in L^{1,p}(\Omega \setminus K)\, , \quad v = g \;\text{ on }\; \partial_D \Omega \setminus K \right\}. \tag{1.2}$$

Actually the stability of (1.2) holds for every $p \leq 2$ under the hypotheses of [9] (see [8]), while in the case $p > 2$ some counter-examples have been given in [8] and in [13]. The strategy to get the stability of problem (1.1) for every $1 < p < \infty$ is to obtain the stability of (1.2) using the unilateral minimality condition.

The obstruction to the stability of (1.2) when $p > 2$ is due to the fact that two connected components of the approximating sequence (K_h) can approach and touch each other in the limit fracture K, leading then to the appearance of a transmission term in the limit problem. To avoid such phenomena we joint these two connected components by curves of infinitesimal length, obtaining then a new sequence of cracks (H_h) having the properties that $K_h \subset H_h$, H_h converges to K, $\mathcal{H}^1(H_h \setminus K_h) \to 0$ and any connected component of H_h converges to a connected component of the limit fracture K. Then the stability of (1.2) along this new sequence of cracks (H_h) will follow from Proposition 4.1. Now, using the unilateral constraint, we obtain the stability of (1.2) also along the original sequence of cracks (K_h).

We prove our main results (see Theorems 4.2 and 4.3) following the duality approach, i.e., through the conjugates (see Section 3), performed in [2], [9] for linear problems, and extended recently in [8] to nonlinear problems.

2. Notation and preliminaries

Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. Let $\partial_D\Omega \subset \partial\Omega$ be a (non-empty) relatively open subset of $\partial\Omega$ composed of a finite number of connected components and $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$.

Let $\mathcal{K}(\overline{\Omega})$ be the class of compact subsets of $\overline{\Omega}$ and $\mathcal{K}_m(\overline{\Omega})$ be the subset of $\mathcal{K}(\overline{\Omega})$ whose elements have at most m connected components. We denote $\mathcal{K}_m^f(\overline{\Omega})$ the subclass of $\mathcal{K}_m(\overline{\Omega})$ whose elements have finite one-dimensional Hausdorff measure $\mathcal{H}^1$. For every $\lambda > 0$, $\mathcal{K}_m^\lambda(\overline{\Omega})$ denotes the class of sets K in $\mathcal{K}_m(\overline{\Omega})$ such that $\mathcal{H}^1(K) \leq \lambda$.

For any $x \in \Omega$ and $\rho > 0$, $B(x,\rho)$ denotes the open ball of $\mathbb{R}^2$ centered at x with radius ρ. For any subset E of $\mathbb{R}^2$, 1_E is the characteristic function of E, E^c is the complement of E, and $|E|$ is the Lebesgue measure of E. Throughout the paper p and q are real numbers, with $1 < p$, $q < +\infty$ and $p^{-1} + q^{-1} = 1$.

2.1. Deny-Lions spaces.

Given an open subset U of $\mathbb{R}^2$, the Deny-Lions space is defined by

$$L^{1,p}(U) := \{u \in L^p_{\mathrm{loc}}(U) : \nabla u \in L^p(U, \mathbb{R}^2)\}.$$

It is well-known that $L^{1,p}(U)$ coincides with the Sobolev space $W^{1,p}(U)$ whenever U is bounded and has a Lipschitz continuous boundary. It is also known that the set $\{\nabla u : u \in L^{1,p}(U)\}$ is a closed subspace of $L^p(U, \mathbb{R}^2)$. The Deny-Lions spaces $L^{1,p}$ are usually involved in minimization problems of the type (2.3) below, in non smooth domains where Poincaré inequalities do not hold in general. For further properties of the spaces $L^{1,p}$ we refer the reader to [12] and [20].

2.2. The minimization problem.

Let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function which satisfies the following assumptions: there exist positive constants α, β, γ such that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^2$

$$\alpha|\xi|^p \leq f(x,\xi) \leq \beta|\xi|^p + \gamma; \tag{2.1}$$

$$f(x,\cdot) \text{ is strictly convex and is of class } C^1. \tag{2.2}$$

Given $K \in \mathcal{K}(\overline{\Omega})$ and a function $g \in W^{1,p}(\Omega)$, we consider the following minimization problem

$$\min_v \left\{ \int_{\Omega \setminus K} f(x, \nabla v)\, dx : \; v \in L^{1,p}(\Omega \setminus K)\,, \quad v = g \;\text{ on }\; \partial_D \Omega \setminus K \right\}, \tag{2.3}$$

whose weak Euler-Lagrange equation is given by

$$\begin{cases} u \in L^{1,p}(\Omega \setminus K), & u = g \quad \text{on} \quad \partial_D\Omega \setminus K, \\ \displaystyle\int_{\Omega \setminus K} f_\xi(x, \nabla u) \cdot \nabla \varphi \; dx = 0 & \forall \varphi \in L^{1,p}(\Omega \setminus K), \quad \varphi = 0 \quad \text{on} \quad \partial_D \Omega \setminus K. \end{cases} \tag{2.4}$$

By well-known existence results for nonlinear elliptic equations involving strictly monotone operators (see e.g. Lions [19]), one can easily see that (2.4) has a unique solution in the sense that the gradient is always unique.

From now on, given $K \in \mathcal{K}_m(\overline{\Omega})$ and $u \in L^{1,p}(\Omega \setminus K)$, we set

$$E(u, K) := \int_{\Omega \setminus K} f(x, \nabla u)\, dx \; + \; \mathcal{H}^1(K). \tag{2.5}$$

Definition 2.1. *Let $g \in W^{1,p}(\Omega)$ and let m be a positive integer. We say that a pair (u, K), with $K \in \mathcal{K}_m(\overline{\Omega})$, $u \in L^{1,p}(\Omega \setminus K)$ and $u = g$ on $\partial_D\Omega \setminus K$ is an unilateral minimum of (2.5) if*

$$E(u, K) \le E(v, H) \tag{2.6}$$

among all $H \in \mathcal{K}_m(\overline{\Omega})$, $H \supset K$ and $v \in L^{1,p}(\Omega \setminus H)$ with $v = g$ on $\partial_D\Omega \setminus H$.

2.3. **Hausdorff convergence.** We recall here the *Hausdorff distance* between two closed sets K_1 and K_2 defined by

$$d_H(K_1, K_2) := \max\left\{ \sup_{x \in K_1} \operatorname{dist}(x, K_2)\,, \; \sup_{x \in K_2} \operatorname{dist}(x, K_1) \right\},$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$, so that

$$d_H(\emptyset, K) = \begin{cases} 0 & \text{if } K = \emptyset, \\ \operatorname{diam}(\Omega) & \text{if } K \neq \emptyset. \end{cases}$$

Let (K_h) be a sequence of compact subsets of $\overline{\Omega}$. We say that (K_h) converges to K in the *Hausdorff metric* if $d_H(K_h\,, K)$ converges to 0. It is well-known (see e.g., [17, Blaschke's Selection Theorem]) that $\mathcal{K}(\overline{\Omega})$ and $\mathcal{K}_m(\overline{\Omega})$ are compact with respect to the Hausdorff convergence. Moreover, using Gołąb theorem on the lower semicontinuity of the one-dimensional Hausdorff measure, we have also that $\mathcal{K}^\lambda_m(\overline{\Omega})$ is compact with respect to the Hausdorff convergence.

The following Lemma is proved in [9].

Lemma 2.2. *Let U be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Let K be a closed connected subset of $\overline{U}$. Let $\lambda > 0$ and let $(K_h) \subset \mathcal{K}^\lambda_m(\overline{U})$ be a sequence which converges to K in the Hausdorff metric. Then there exists a sequence (H_h) of closed connected subsets of $\overline{U}$ which converges to K in the Hausdorff metric, with $K_h \subset H_h$ for every h and $\mathcal{H}^1(H_h \setminus K_h) \to 0$.*

Lemma 2.3. *Let U be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary and let $(K_h) \subset \mathcal{K}_m^f(\overline{U})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let Γ be a compact subset of $\overline{U}$ with a finite number of connected components. Then there exists a sequence $(H_h) \subset \mathcal{K}_m^f(\overline{U})$ which converges to K in the Hausdorff metric, with $K_h \subset H_h$ for every h, $\mathcal{H}^1(H_h \setminus K_h) \to 0$ and such that any connected component of $H_h \cup \Gamma$ converges to a connected component of $K \cup \Gamma$ in the Hausdorff metric.*

The proof of this lemma follows the lines of [9, Lemma 3.6]. Precisely, we apply Lemma 2.2 to every connected component C of $K \cup \Gamma$ and the union of those connected components of $K_h \cup \Gamma$ whose limits in the Hausdorff metric are contained in C.

The following Lemma proved in [9] will also be useful in the proof of our main results.

Lemma 2.4. *Let p and m be two positive integers. Let (K_h) be a sequence in $\mathcal{K}_p^f(\overline{\Omega})$ which converges in the Hausdorff metric to $K \in \mathcal{K}_p^f(\overline{\Omega})$, and let $H \in \mathcal{K}_m^f(\overline{\Omega})$ with $H \supset K$. Then there exists a sequence $(H_h) \subset \mathcal{K}_m^f(\overline{\Omega})$ such that $H_h \to H$ in the Hausdorff metric, $K_h \subset H_h$, and $\mathcal{H}^1(H_h \setminus K_h) \to \mathcal{H}^1(H \setminus K)$.*

In order to study the continuity of the solution u of (2.3) with respect to the variations of the compact set K, we should be able to compare two solutions defined in two different domains. This is why, throughout this paper, given a function $u \in L^{1,p}(\Omega \setminus K)$, we extend ∇u in Ω by setting $\nabla u = 0$ in $\Omega \cap K$.

2.4. **Capacity.** Let $1 < r < \infty$ and let B be a bounded open set in $\mathbb{R}^2$. For every subset E of B, the $(1,r)$-capacity of E in B, denoted by $C_r(E,B)$, is defined as the infimum of $\int_B |\nabla u|^r \, dx$ over the set of all functions $u \in W_0^{1,r}(B)$ such that $u \geq 1$ a.e. in a neighborhood of E. If $r > 2$, then $C_r(E,B) > 0$ for every nonempty set E. On the contrary, if $r = 2$ there are nonempty sets E with $C_r(E,B) = 0$ (for instance, $C_r(\{x\},B) = 0$ for every $x \in B$).

We say that a property $\mathcal{P}(x)$ holds C_r-*quasi everywhere* (abbreviated C_r-*q.e.*) in a set E if it holds for all $x \in E$ except a subset N of E with $C_r(N,B) = 0$. We recall that the expression *almost everywhere* (abbreviated *a.e.*) refers, as usual, to the Lebesgue measure.

A function $u : E \to \overline{\mathbb{R}}$ is said to be *quasi-continuous* if for every ε there exists $A_\varepsilon \subset E$, with $C_r(A_\varepsilon, B) < \varepsilon$, such that the restriction of u to $E \setminus A_\varepsilon$ is continuous. If $r > 2$ every quasi-continuous function is continuous, while for $r = 2$ there are quasi-continuous functions that are not continuous. It is well known that, for any open subset U of $\mathbb{R}^2$, any function $u \in L^{1,r}(U)$ has a *quasi-continuous representative* $\overline{u} : U \cup \partial_L U \to \mathbb{R}$ which satisfies

$$\lim_{\rho \to 0+} \fint_{B_\rho(x) \cap U} |u(y) - \overline{u}(x)| \, dy = 0 \quad \text{for } C_r\text{-q.e. } x \in U \cup \partial_L U,$$

where $\partial_L U$ denotes the Lipschitz part of the boundary ∂U of U. We recall that if u_h converges to u strongly in $W^{1,r}(U)$, then a subsequence of $\overline{u}_h$ converges to $\overline{u}$ pointwise C_r-q.e. on $U \cup \partial_L U$. To simplify the notation we shall always identify throughout the paper each function $u \in L^{1,r}(U)$ with its quasi-continuous representative $\overline{u}$.

For these and other properties on quasi-continuous representatives the reader is referred to [16], [18], [20], [23].

The following lemma is proved in [9, Lemma 4.1] for $p=2$. The case $p \neq 2$ can be proved in the same way.

Lemma 2.5. *Let (K_h) be a sequence in $\mathcal{K}(\overline{\Omega})$ which converges to a compact set K in the Hausdorff metric. Let $u_h \in L^{1,p}(\Omega \setminus K_h)$ be a sequence such that $u_h = 0$ C_p-q.e. on $\partial_D\Omega \setminus K_h$ and (∇u_h) is bounded in $L^p(\Omega, \mathbb{R}^2)$. Then, there exists a function $u \in L^{1,p}(\Omega \setminus K)$ with $u = 0$ C_p-q.e. on $\partial_D\Omega \setminus K$ such that, up to a subsequence, ∇u_h converges weakly to ∇u in $L^p(A, \mathbb{R}^2)$ for every $A \subset\subset \Omega \setminus K$. If, in addition, $|K_h|$ converges to $|K|$, then ∇u_h converges weakly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.*

The following three lemmas will be crucial in the proof of our main result.

Lemma 2.6. *Let $(K_h) \subset \mathcal{K}_1(\overline{\Omega})$ converging to a compact set K in the Hausdorff metric. Let (v_h) be a sequence in $W^{1,q}(\Omega)$ converging weakly in $W^{1,q}(\Omega)$ to a function v, with $v_h = 0$ C_q-q.e. in K_h. Then $v = 0$ C_q-q.e. in K.*

Proof. We consider an open ball B containing $\overline{\Omega}$ and we extend both functions v_h and v to functions still denoted respectively by v_h and v such that the two extensions belong two $W_0^{1,q}(B)$ and $v_h \rightharpoonup v$ in $W^{1,q}(B)$. Let w_h and w be the solutions of the problems

$$\begin{aligned} w_h &\in W_0^{1,q}(B \setminus K_h), & \Delta_q w_h &= \Delta_q v \quad \text{in} \quad B \setminus K_h, \\ w &\in W_0^{1,q}(B \setminus K), & \Delta_q w &= \Delta_q v \quad \text{in} \quad B \setminus K. \end{aligned} \tag{2.7}$$

Using a result on the stability of Dirichlet problems by Bucur and Trebeschi [3] (see also Šverák [22] for the case $q=2$), we obtain that w_h converges to w strongly in $W_0^{1,q}(B)$. Taking $v_h - w_h$ as test function in (2.7), which is possible since $v_h - w_h \in W_0^{1,q}(B \setminus K_h)$ (see, e.g., [18, Theorem 4.5]), we obtain

$$\langle \Delta_q w_h, v_h - w_h \rangle = \langle \Delta_q v, v_h - w_h \rangle, \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{-1,p}(B)$ and $W_0^{1,q}(B)$. Passing to the limit in (2.8) we obtain

$$\langle \Delta_q w, v - w \rangle = \langle \Delta_q v, v - w \rangle,$$

which implies $v = w$ by the strict monotonicity of $-\Delta_q$. Since, by definition, $w = 0$ C_q-q.e. in K, we conclude that $v = 0$ C_q-q.e. in K. □

Lemma 2.7. *Let $(K_h) \subset \mathcal{K}_1(\overline{\Omega})$ converging to a compact set K in the Hausdorff metric. Let (v_h) be a sequence in $W^{1,q}(\Omega)$, converging weakly in $W^{1,q}(\Omega)$ to a function v. Assume that every function v_h is constant C_q-q.e. in K_h. Then v is constant C_q-q.e. in K.*

Proof. This is trivial if K contains only a single point. If K has more than one point, there exists $r > 0$ such that $\operatorname{diam}(K_h) > 2r$ for h large enough. Let us prove that the constant values

c_h taken by v_h on K_h are bounded uniformly with respect to h. To this aim let us consider a point $x_h \in K_h$. Since $\operatorname{diam}(K_h) > 2r$, we have $K_h \setminus B(x_h, r) \neq \emptyset$, and by connectedness

$$K_h \cap \partial B(x_h, \rho) \neq \emptyset \qquad \text{for every } 0 < \rho < r. \tag{2.9}$$

As $v_h = c_h$ C_q-q.e. on K_h, by using polar coordinates we deduce from (2.9) the Poincaré inequality

$$\int_{B(x_h,r)} |v_h - c_h|^q \, dx \leq M r^q \int_{B(x_h,r)} |\nabla v_h|^q \, dx,$$

where the constant M is independent of h and r. Since the sequence v_h is bounded in $W^{1,q}(\Omega)$, it follows that c_h is bounded, and so it converges (up to a subsequence) to some constant c. So, the sequence $v_h - c_h$ converges weakly to $v - c$ in $W^{1,q}(\Omega)$, and by Lemma 2.6 we get that $v = c$ C_q-q.e. on K. □

3. Conjugates and their properties

Let R be the rotation on $\mathbb{R}^2$ defined by $R(y_1, y_2) := (-y_2, y_1)$. The following proposition on the global construction of conjugates will be crucial in the proof of Theorem 4.2.

Proposition 3.1. *Let $K \in \mathcal{K}(\overline{\Omega})$ and let u be a solution of the problem (2.3). Assume that Ω is simply connected. Then there exists a function $v \in W^{1,q}(\Omega)$ such that $\nabla v = R f_\xi(x, \nabla u) 1_{K^c}$ a.e. in Ω. Moreover, v is constant C_q-q.e. on each connected component of $K \cup \partial_N \Omega$.*

Proof. Let u be a solution of (2.3). We consider the vector field $\Phi \in L^q(\Omega, \mathbb{R}^2)$ defined by

$$\Phi := f_\xi(x, \nabla u) 1_{K^c}.$$

We have that $\operatorname{div}(\Phi) = 0$ in $\mathcal{D}'(\Omega)$; hence $\operatorname{rot}(R\Phi) = 0$ in $\mathcal{D}'(\Omega)$. As Ω is simply connected and has a Lipschitz boundary, there exists $v \in W^{1,q}(\Omega)$ such that $\nabla v = R\Phi$ a.e. on Ω.

Let us now prove that v is constant C_q-q.e. on each connected component of $K \cup \partial_N \Omega$ we proceed as follows. Let C be a connected component of $K \cup \partial_N \Omega$ with $C_{1,q}(C) > 0$ and let $\varepsilon > 0$. We set

$$C_\varepsilon := \{x \in \overline{\Omega} \colon \operatorname{dist}(x\,,\, C) < \varepsilon\} \quad \text{and} \quad K_\varepsilon := (K \cup \partial_N \Omega) \cup \overline{C}_\varepsilon.$$

Let u_ε be the solution of the problem (2.3) in $\Omega \setminus K_\varepsilon$. From Lemma 2.5 applied to $u_\varepsilon - g$ and by the monotonicity of K_ε, we have that ∇u_ε converges (up to a subsequence) to ∇u^* weakly in $L^p(\Omega \setminus K, \mathbb{R}^2)$ for some $u^* \in L^{1,p}(\Omega \setminus K)$ with $u^* = g$ on $\partial_D \Omega \setminus K$.

We claim that $\nabla u^* = \nabla u$ a.e. in Ω. Indeed, by reformulating the problem (2.4) as a variational inequality in $\Omega \setminus K_\varepsilon$ and using Minty's lemma, we get

$$\int_{\Omega \setminus K_\varepsilon} f_\xi(x, \nabla z) \cdot (\nabla z - \nabla u_\varepsilon) \, dx \geq 0 \qquad \forall z \in L^{1,p}(\Omega \setminus K_\varepsilon),\ z = g \text{ on } \partial_D \Omega \setminus K_\varepsilon.$$

Now, let $z \in L^{1,p}(\Omega \setminus K)$ with $z = g$ on $\partial_D \Omega \setminus K$. By the monotonicity of K_ε, we have that $z \in L^{1,p}(\Omega \setminus K_\varepsilon)$ and $z = g$ on $\partial_D \Omega \setminus K_\varepsilon$. So,

$$\int_{\Omega \setminus K_\varepsilon} f_\xi(x, \nabla z) \cdot (\nabla z - \nabla u_\varepsilon) \, dx \geq 0.$$

Using the convention that $\nabla u_\varepsilon = 0$ in $\Omega \cap K_\varepsilon$ we obtain

$$\int_{\Omega\setminus K} f_\xi(x,\nabla z)\cdot(\nabla z - \nabla u_\varepsilon)\,dx \geq -\int_{K_\varepsilon\setminus K} f_\xi(x,\nabla z)\cdot\nabla z\,dx. \tag{3.1}$$

Now, letting $\varepsilon \to 0$ in (3.1) we obtain

$$\int_{\Omega\setminus K} f_\xi(x,\nabla z)\cdot(\nabla z - \nabla u^*)\,dx \geq 0 \qquad \forall z \in L^{1,p}(\Omega\setminus K),\, z = g \text{ on } \partial_D\Omega\setminus K.$$

which, using again Minty's lemma is equivalent to

$$\int_{\Omega\setminus K} f_\xi(x,\nabla u^*)\cdot\nabla\varphi\,dx = 0 \qquad \forall \varphi \in L^{1,p}(\Omega\setminus K),\, \varphi = 0 \text{ on } \partial_D\Omega\setminus K.$$

By the uniqueness of solution of (2.4) in $\Omega\setminus K$, we get that $\nabla u^* = \nabla u$. So, we have proved that all the sequence (∇u_ε) converges to ∇u weakly in $L^p(\Omega,\mathbb{R}^2)$. On the other hand, one can see that

$$\lim_{\varepsilon\to 0}\int_\Omega [f_\xi(x,\nabla u_\varepsilon) - f_\xi(x,\nabla u)]\cdot(\nabla u - \nabla u_\varepsilon)\,dx = 0. \tag{3.2}$$

Hence arguing as in [8, Lemma 2.4] (recall that $f_\xi(x,\cdot)$ is strictly monotone), it follows that ∇u_ε converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$.
Now, from the first part of the proof, we consider a function $v_\varepsilon \in W^{1,q}(\Omega)$ such that $\nabla v_\varepsilon = Rf_\xi(x,\nabla u_\varepsilon)1_{K_\varepsilon^c}$ a.e. in Ω. We can assume that $\int_\Omega v_\varepsilon\,dx = \int_\Omega v\,dx = 0$. So, by Poincaré inequality we obtain that v_ε converges strongly to v in $W^{1,q}(\Omega)$. By construction $\nabla v_\varepsilon = 0$ in C_ε from which it follows that v_ε is constant C_q-q.e. on $C_\varepsilon \cup \partial_L C_\varepsilon$. Hence v_ε is constant C_q-q.e. on C. Since a subsequence of v_ε converges to v C_q-q.e. on $\overline{\Omega}$, we conclude that v is constant C_q-q.e. on C and this completes the proof. ☐

Definition 3.2. *The function v in Proposition 3.1 is called a conjugate of the function u.*

The following lemma is proved like in [9, Theorem 4.3] for $f(x,\xi) = |\xi|^2$. For the reader's convenience we will give here the proof of the present version.

Lemma 3.3. *Let $K \in \mathcal{K}_m(\overline{\Omega})$ and $u \in L^{1,p}(\Omega\setminus K)$ with $u = g$ on $\partial_D\Omega\setminus K$. Assume that there exists $v \in W^{1,q}(\Omega)$ such that $\nabla v = Rf_\xi(x,\nabla u)1_{K^c}$ a.e. in Ω and that v is constant C_q-q.e. on every connected component of $K\cup\partial_N\Omega$. Then u is solution of (2.4).*

Proof. Let $C^1,\dots,C^l$ be the connected components of $K\cup\partial_N\Omega$. Since $v = c^i$ C_q-q.e on C^i, by [18, Theorem 4.5], we can approximate v strongly in $W^{1,q}(\Omega)$ by a sequence of functions $v_n \in C_c^\infty(\mathbb{R}^2)$ that are constant in a suitable neighborhood V_n^i of C^i. Let $\varphi \in L^{1,p}(\Omega\setminus K)$ with $\varphi = 0$ on $\partial_D\Omega\setminus K$ and let $\varphi_n \in W_0^{1,p}(\Omega\setminus K)$ such that $\varphi_n = \varphi$ in $\Omega\setminus\bigcup_i V_n^i$. Then we have that

$$\int_\Omega R\nabla v_n\nabla\varphi\,dx = \int_{\Omega\setminus K} R\nabla v_n\nabla\varphi_n dx = 0, \tag{3.3}$$

where the last equality follows from the fact that the vector field $R\nabla v_n$ is divergence free. Then passing to the limit in (3.3) for $n\to\infty$, we get

$$\int_{\Omega\setminus K} f_\xi(x,\nabla u)\nabla\varphi\,dx = \int_\Omega R\nabla v\nabla\varphi\,dx = 0 \quad \forall\varphi \in L^{1,p}(\Omega\setminus K) \text{ with } \varphi = 0 \text{ on } \partial_D\Omega\setminus K.$$

So u is a solution of (2.4). $\square$

The following Lemma on the local construction of conjugates will be used in the proof of Theorem 4.3.

Lemma 3.4. *Let $K \in \mathcal{K}(\overline{\Omega})$ and let u be a solution of (2.3) in $\Omega \setminus K$. Let U be an open rectangle such that $U \cap \Omega$ is a non empty simply connected set. Then there exists a function $v \in W^{1,q}(U \cap \Omega)$ such that $\nabla v = Rf_\xi(x, \nabla u)1_{K^c}$ a.e. in $U \cap \Omega$. Moreover, v is constant C_q-q.e. on each connected component of $\overline{U} \cap (K \cup \partial_N \Omega)$.*

Proof. We note that u is solution of the following problem

$$\min\left\{\int_{(U\cap\Omega)\setminus K} f(x, \nabla w)\, dx\colon\ w \in L^{1,p}(U \cap \Omega) \setminus K) \text{ and } w = u \text{ on } \partial(U \cap \Omega) \setminus K\right\}.$$

Since $U \cap \Omega$ is simply connected, we can apply Proposition 3.1 with Ω replaced by $U \cap \Omega$. So, there exists a function $v \in W^{1,q}(U \cap \Omega)$ such that $\nabla v = Rf_\xi(x, \nabla u)1_{K^c}$ a.e. in $U \cap \Omega$, with v constant C_q-q.e. on each connected component of $\overline{U} \cap (K \cup \partial_N \Omega)$. $\square$

4. The stability results relative to problem (2.6)

In this section we give the stability results relative to problem (2.6). First of all, we prove in the following proposition, the stability of problem (2.3) under the condition that any connected component of $K_h \cup \partial_N \Omega$ converges to a connected component of $K \cup \partial_N \Omega$ in the Hausdorff metric.

Proposition 4.1. *Let Ω be a simply connected and bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Assume that $\partial_N \Omega$ has M connected components. Let $\lambda > 0$ and let $(K_h) \subset \mathcal{K}^\lambda_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let (g_h) be a sequence in $W^{1,p}(\Omega)$ which converges to g strongly in $W^{1,p}(\Omega)$. Let $u_h \in L^{1,p}(\Omega \setminus K_h)$ and $u \in L^{1,p}(\Omega \setminus K)$ be the solutions of the minimization problem (2.3) with boundary data g_h and g respectively. Assume that any connected component of $K_h \cup \partial_N \Omega$ converges to a connected component of $K \cup \partial_N \Omega$ in the Hausdorff metric. Then ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.*

Proof. By the growth assumptions (2.1) on the function f, we have that ∇u_h and $f_\xi(x, \nabla u_h)$ are bounded respectively in $L^p(\Omega, \mathbb{R}^2)$ and in $L^q(\Omega, \mathbb{R}^2)$. So, applying Lemma 2.5 to $u_h - g_h$, we obtain that ∇u_h converges (up to a subsequence) to ∇u^* weakly in $L^p(\Omega, \mathbb{R}^2)$ for some function $u^* \in L^{1,p}(\Omega \setminus K)$ and $u^* = g$ on $\partial_D \Omega \setminus K$.

On the other hand, there exists a vector field $\Psi \in L^q(\Omega, \mathbb{R}^2)$ such that $f_\xi(x, \nabla u_h) \rightharpoonup \Psi$ weakly in $L^q(\Omega, \mathbb{R}^2)$. Let us prove that $\Psi = f_\xi(x, \nabla u^*)$ a.e. in Ω. Since $|K_h| = |K| = 0$ it is sufficient to prove that for every open ball $B \subset\subset \Omega \setminus K$, $\Psi = f_\xi(x, \nabla u^*)$ a.e. in B. Note that by the Hausdorff complementary convergence we have $B \subset\subset \Omega \setminus K_h$ for h large enough.

We may assume that the mean values of u_h and u^* on B are zero. Thus the Poincaré inequality and the Rellich theorem imply that $u_h \to u^*$ strongly in $L^p(B)$. Let $z \in W^{1,p}(B)$ and

$\varphi \in C_c^\infty(B)$ with $\varphi \geq 0$. For h large enough we have $B \subset\subset \Omega \setminus K_h$, thus by the monotonicity of $f_\xi(x, \cdot)$ we have

$$\int_B (f_\xi(x, \nabla z) - f_\xi(x, \nabla u_h)) \cdot (\nabla z - \nabla u_h) \varphi \, dx \geq 0. \tag{4.1}$$

We have also

$$\int_B f_\xi(x, \nabla u_h) \cdot \nabla((z - u_h)\, \varphi) \, dx = 0,$$

which, together with (4.1), gives

$$\int_B f_\xi(x, \nabla z) \cdot \nabla((z - u_h)\varphi) \, dx - \int_B (f_\xi(x, \nabla z) - f_\xi(x, \nabla u_h)) \cdot \nabla\varphi \, (z - u_h) \, dx \geq 0. \tag{4.2}$$

We can pass to the limit in each term of (4.2) and we get

$$\int_B f_\xi(x, \nabla z) \cdot \nabla((z - u^*)\varphi) \, dx - \int_B (f_\xi(x, \nabla z) - \Psi) \cdot \nabla\varphi \, (z - u^*) \, dx \geq 0. \tag{4.3}$$

As $\operatorname{div} \Psi = 0$ in $\mathcal{D}'(B)$, we have

$$\int_B \Psi \cdot \nabla((z - u^*)\varphi) \, dx = 0. \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$\int_B (f_\xi(x, \nabla z) - \Psi) \cdot (\nabla z - \nabla u^*) \, \varphi \, dx \geq 0.$$

As φ is arbitrary, we get $(f_\xi(x, \nabla z) - \Psi) \cdot (\nabla z - \nabla u^*) \geq 0$ a.e. in B. In particular, taking $z(x) := u^*(x) \pm \varepsilon\eta \cdot x$, with $\eta \in \mathbb{R}^2$ and $\varepsilon > 0$, we obtain $\pm(f_\xi(x, \nabla u^* \pm \varepsilon\eta) - \Psi) \cdot \eta \geq 0$ a.e. in B. As ε tends to zero we get $(f_\xi(x, \nabla u^*) - \Psi) \cdot \eta = 0$ a.e. in B, which implies that $f_\xi(x, \nabla u^*) = \Psi$ a.e. in B by the arbitrariness of η.
So we have proved that $f_\xi(x, \nabla u_h) \rightharpoonup f_\xi(x, \nabla u^*)$ weakly in $L^q(\Omega, \mathbb{R}^2)$. Now let us prove that u^* is a solution of (2.3) in $\Omega \setminus K$.

Now we use the assumption that K_h^i converges to K^i for every i. By Proposition 3.1 there exists $v_h \in W^{1,q}(\Omega)$ such that $\nabla v_h = R f_\xi(x, \nabla u_h)$ a.e. in Ω with v_h constant C_q-q.e. on each connected component of $K_h \cup \partial_N \Omega$. Since $f_\xi(x, \nabla u_h)$ converges to $f_\xi(x, \nabla u^*)$ weakly in $L^q(\Omega, \mathbb{R}^2)$, there exists a function $v \in W^{1,q}(\Omega)$ such that $v_h \rightharpoonup v$ weakly in $W^{1,q}(\Omega)$ and $\nabla v = R f_\xi(x, \nabla u^*)$ a.e. in Ω. Moreover, by Lemma 2.7 we get that v is constant C_q-q.e. on K^i for every i. So from Lemma 3.3 it follows that u^* is a solution of (2.3) in $\Omega \setminus K$ and hence Thus, $\nabla u^* = \nabla u$ a.e. in Ω. Therefore, all the sequence ∇u_h converges to ∇u weakly in $L^p(\Omega, \mathbb{R}^2)$.

Now let us prove that ∇u_h converges to ∇u strongly in $L^p(\Omega, \mathbb{R}^2)$. First of all, by lower semicontinuity we have that

$$\int_\Omega f(x, \nabla u) \, dx \leq \liminf_{h \to \infty} \int_\Omega f(x, \nabla u_h) \, dx. \tag{4.5}$$

By the convexity of $f(x, \cdot)$ we have also that

$$\int_\Omega f(x, \nabla u) \, dx \geq \int_\Omega f(x, \nabla u_h) \, dx + \int_\Omega f_\xi(x, \nabla u_h) \cdot (\nabla u - \nabla u_h) \, dx. \tag{4.6}$$

Since

$$\int_\Omega f_\xi(x,\nabla u_h)\cdot(\nabla u_h-\nabla g_h)\,dx=0 \quad\text{and}\quad \int_\Omega f_\xi(x,\nabla u)\cdot(\nabla u-\nabla g)\,dx=0,$$

it follows that

$$\begin{aligned}\lim_{h\to\infty}\int_\Omega f_\xi(x,\nabla u_h)\cdot(\nabla u-\nabla u_h)\,dx &= \lim_{h\to\infty}\int_\Omega f_\xi(x,\nabla u_h)\cdot(\nabla u-\nabla g_h)\,dx\\ &= \int_\Omega f_\xi(x,\nabla u)\cdot(\nabla u-\nabla g)\,dx=0.\end{aligned}$$

Hence passing to the limit in (4.6) we get

$$\int_\Omega f(x,\nabla u)\,dx\geq\limsup_{h\to\infty}\int_\Omega f(x,\nabla u_h)\,dx,$$

which together with (4.5) implies

$$\lim_{h\to\infty}\int_\Omega f(x,\nabla u_h)\,dx=\int_\Omega f(x,\nabla u)\,dx. \tag{4.7}$$

Since $\nabla u_h\rightharpoonup\nabla u$ weakly in $L^p(\Omega,\mathbb{R}^2)$, using the strict convexity of $f(x,\cdot)$, it follows from (4.7) that ∇u_h converges to ∇u strongly in $L^p(\Omega,\mathbb{R}^2)$. □

We are now in a position to prove the main results of the paper.

4.1. **The case Ω simply connected.**

Theorem 4.2. *Let Ω be a simply connected and bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Assume that $\partial_N\Omega$ has M connected components. Let $\lambda>0$ and $(K_h)\subset\mathcal{K}_m^\lambda(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let (g_h) be a sequence in $W^{1,p}(\Omega)$ which converges to g strongly in $W^{1,p}(\Omega)$. Let u_h be such that (u_h,K_h) is an unilateral minimum relative to g_h of the functional E defined in (2.5) and let $u\in L^{1,p}(\Omega\setminus K)$ be the solution of the minimization problem (2.3). Then ∇u_h converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$. Moreover, the pair (u,K) is an unilateral minimum of the functional E relative to g.*

The proof of Theorem 4.2. Step 1. Let us prove that ∇u_h converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$. Let $K_h^1,\dots,K_h^{n_h}$ be the connected components of $K_h\cup\partial_N\Omega$. As by assumption $n_h\leq m+M$, passing to a subsequence we can assume that $n_h=n$ for every h and that, for every $i\in\{1,\dots,n\}$, K_h^i converges to some compact connected set K^i in the Hausdorff metric.

If $K^i\cap K^j=\emptyset$ for every $i\neq j$, then $K^1,\dots,K^n$ are exactly the connected components of $K\cup\partial_N\Omega$. So, by Proposition 4.1 it follows that ∇u_h converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$.

Now we remove the assumption that $K^i\cap K^j=\emptyset$ for every $i\neq j$.

Applying Lemma 2.3 for $U=\Omega$ and $\Gamma=\partial_N\Omega$, we obtain a sequence $(H_h)\subset\mathcal{K}_m^f(\overline{\Omega})$ which converges to K in the Hausdorff metric, with $K_h\subset H_h$ for every h, $\mathcal{H}^1(H_h\setminus K_h)\to0$ and such that any connected component of $H_h\cup\partial_N\Omega$ converges to a connected component of $K\cup\partial_N\Omega$ in the Hausdorff metric.

We consider now the following minimization problem

$$\min_w \left\{ \int_{\Omega\setminus H_h} f(x,\nabla w)\,dx:\; w\in L^{1,p}(\Omega\setminus H_h)\,,\quad w=g_h \text{ on } \partial_D\Omega\setminus H_h \right\}. \tag{4.8}$$

Let $w_h \in L^{1,p}(\Omega\setminus H_h)$ be the solution of (4.8). From Proposition 4.1, it follows that ∇w_h converges to ∇u strongly in $L^p(\Omega,\mathbb{R}^2)$. Now using the fact that the pair (u_h, K_h) is a unilateral minimum of the functional (2.5), we get that

$$\begin{aligned} \limsup_{h\to\infty} \int_{\Omega\setminus K_h} f(x,\nabla u_h)\,dx \;&\le\; \lim_{h\to\infty}\int_{\Omega\setminus H_h} f(x,\nabla w_h)\,dx + \lim_{h\to\infty}\mathcal{H}^1(H_h\setminus K_h) \\ &=\; \int_{\Omega\setminus K} f(x,\nabla u)\,dx. \end{aligned} \tag{4.9}$$

Hence, recalling that ∇u_h converges to ∇u^* weakly in $L^p(\Omega,\mathbb{R}^2)$, we obtain

$$\begin{aligned} \int_{\Omega\setminus K} f(x,\nabla u^*)\,dx \;&\le\; \liminf_{h\to\infty}\int_{\Omega\setminus K_h} f(x,\nabla u_h)\,dx \;\le \\ \le\; \limsup_{h\to\infty}\int_{\Omega\setminus K_h} f(x,\nabla u_h)\,dx \;&\le\; \int_{\Omega\setminus K} f(x,\nabla u)\,dx, \end{aligned}$$

which implies (by the uniqueness of solution of (2.3) in $\Omega\setminus K$) that $\nabla u^* = \nabla u$ a.e. in Ω. So, all the sequence ∇u_h converges to ∇u weakly in $L^p(\Omega,\mathbb{R}^2)$ and

$$\lim_{h\to\infty}\int_\Omega f(x,\nabla u_h)\,dx = \int_\Omega f(x,\nabla u)\,dx.$$

Since $\xi\to f(x,\xi)$ is strictly convex, it follows that ∇u_h converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$ and this achieves the proof of Step 1.

Step 2. Let us prove that the pair (u,K) is an unilateral minimum of the functional E relative to g. Let $H\in\mathcal{K}_m(\overline{\Omega})$ with $K\subset H$ and let $w\in L^{1,p}(\Omega\setminus H)$ with $w=g$ on $\partial_D\Omega\setminus H$. By Lemma 2.4, there exists a sequence $(H_h)\subset\mathcal{K}_m(\overline{\Omega})$ such that $H_h\to H$ in the Hausdorff metric, $K_h\subset H_h$, and $\mathcal{H}^1(H_h\setminus K_h)\to\mathcal{H}^1(H\setminus K)$. From Lemma 2.3, we have a sequence $(\tilde{H}_h)\subset\mathcal{K}_m(\overline{\Omega})$ which converges to H in the Hausdorff metric and such that $H_h\subset\tilde{H}_h$, $\mathcal{H}^1(\tilde{H}_h\setminus H_h)\to 0$ and, every connected component of $H_h\cup\partial_N\Omega$ converges in the Hausdorff metric to a connected component of $H\cup\partial_N\Omega$. Let $z_h\in L^{1,p}(\Omega\setminus\tilde{H}_h)$ and $z\in L^{1,p}(\Omega\setminus H)$ be the solutions of (2.3) with boundary data g_h and g respectively. From Proposition 4.1 it follows that $\nabla z_h\to\nabla z$ strongly in $L^p(\Omega,\mathbb{R}^2)$.

Now using the fact that the pair (u_h,K_h) is a unilateral minimum of the functional (2.5), we get that

$$\int_{\Omega\setminus K_h} f(x,\nabla u_h)\,dx \le \int_{\Omega\setminus\tilde{H}_h} f(x,\nabla z_h)\,dx + \mathcal{H}^1(\tilde{H}_h\setminus K_h). \tag{4.10}$$

So, passing to the limit in (4.10) and using the fact that $\nabla u_h \to \nabla u$ strongly in $L^p(\Omega,\mathbb{R}^2)$ and $\nabla z_h \to \nabla z$ strongly in $L^p(\Omega,\mathbb{R}^2)$, we obtain

$$\begin{aligned} \int_{\Omega\setminus K} f(x,\nabla u)\,dx \;&=\; \lim_{h\to\infty}\int_{\Omega\setminus K_h} f(x,\nabla u_h)\,dx \;\leq \\ \leq\; \lim_{h\to\infty}\int_{\Omega\setminus \tilde H_h} f(x,\nabla z_h)\,dx \,&+\, \lim_{h\to\infty}\mathcal{H}^1(\tilde H_h\setminus K_h) \;\leq \\ \leq\; \int_{\Omega\setminus H} f(x,\nabla z)\,dx + \mathcal{H}^1(H\setminus K) \;&\leq\; \int_{\Omega\setminus H} f(x,\nabla w)\,dx + \mathcal{H}^1(H\setminus K), \end{aligned}$$

which gives Step 2 and achieves the proof of the theorem. □

4.2. **The general case.** Here we remove the assumption that Ω is simply connected and we prove the stability theorem below using the local conjugates in Lemma 3.4.

Theorem 4.3. *Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Assume that $\partial_N\Omega$ has M connected components. Let $\lambda>0$ and $(K_h)\subset\mathcal{K}_m^\lambda(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let (g_h) be a sequence in $W^{1,p}(\Omega)$ which converges to g strongly in $W^{1,p}(\Omega)$. Let u_h be such that (u_h,K_h) is an unilateral minimum relative to g_h of the functional E defined in (2.5) and let $u\in L^{1,p}(\Omega\setminus K)$ be the solution of the minimization problem (2.3). Then ∇u_h converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$. Moreover, the pair (u,K) is an unilateral minimum of the functional E relative to g.*

Proof of Theorem 4.3.
First of all let us prove that ∇u_h converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$. By the growth assumptions (2.1) on f, we have that ∇u_h is bounded in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. By Lemma 2.5 applied to u_h-g_h, we have that ∇u_h converges (up to a subsequence) to ∇u^* weakly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$ for some $u^*\in L^{1,p}(\Omega\setminus K)$ with $u^*=g$ on $\partial_D\Omega\setminus K$. We claim that $\nabla u^*=\nabla u$ a.e. in Ω.

To this aim, we fix $r>0$ such that the minimum of the diameters of the connected components of Ω^c is equal to $3r$. Using the fact that Ω has a Lipschitz continuous boundary, we may find two families of open rectangles $(Q_i)_{i=1}^n$ and $(U_i)_{i=1}^n$ such that, for every $i\in\{1,\dots,n\}$, $Q_i\subset\subset U_i$, $Q_i\cap\Omega\neq\emptyset$ and $U_i\cap\Omega$ is a Lipschitz domain and,

$$\overline{\Omega}\subset\bigcup_{i=1}^n Q_i \quad\text{and}\quad \max_{1\leq i\leq n}\operatorname{diam}(U_i)=2r.$$

We set

$$\eta:=\min_{1\leq i\leq n} d(Q_i,\partial U_i).$$

For every $i\in\{1,\dots,n\}$, the number of connected components C of $\overline{U}_i\cap K_h$ which intersect Q_i is less or equal to $m+\lambda/\eta$. Indeed, if C intersects ∂U_i, then $\mathcal{H}^1(C)\geq\eta$ and hence, their number is at most λ/η. If $C\cap\partial U_i=\emptyset$, then C is a connected component of K_h, and their number is less or equal to m. Similarly the number of connected components of $\overline{U}_i\cap\partial_N\Omega$ which intersect Q_i is less or equal to $M+\mathcal{H}^1(\partial_N\Omega)/\eta$. Let $K_h^{i,1},\dots,K_h^{i,k_h}$ be all the connected components of

$\overline{U}_i \cap K_h$ which intersect Q_i. Since $k_h \le m + \lambda/\eta$, passing to a subsequence, we can assume that $k_h = k$ for every h. We set

$$K_h^i := \bigcup_{j=1}^k K_h^{i,j}.$$

Up to a subsequence, we have that K_h^i converges in the Hausdorff metric to some compact set $K^i \in \mathcal{K}_k^\lambda(\overline{U_i \cap \Omega})$. Let Γ^i be the union of those connected components of $\overline{U}_i \cap \partial_N \Omega$ which intersect Q_i. By Lemma 2.3 applied to $U = U_i \cap \Omega$ and $\Gamma = \Gamma_i$, we get a sequence $(H_h^i) \subset \mathcal{K}_k^f(\overline{U_i \cap \Omega})$ which converges to K^i in the Hausdorff metric, with $K_h^i \subset H_h^i$ for every h, $\mathcal{H}^1(H_h^i \setminus K_h^i) \to 0$ and such that any connected component of $H_h^i \cup \Gamma^i$ converges to a connected component of $K^i \cup \Gamma^i$ in the Hausdorff metric. We set

$$H_h := \bigcup_{i=1}^n H_h^i.$$

Note that

$$(H_h) \in \mathcal{K}_m(\overline{\Omega}), \quad H_h \supset K_h, \quad \mathcal{H}^1(H_h \setminus K_h) \to 0$$

and H_h converges in the Hausdorff metric to the compact set $\tilde{K} := \bigcup_{i=1}^n K^i$. Moreover it is easy to see that $\tilde{K} = K$.

We consider now the minimization problem

$$\min_w \left\{ \int_{\Omega \setminus H_h} f(x, \nabla w)\, dx :\ w \in L^{1,p}(\Omega \setminus H_h)\,, \quad w = g_h \text{ on } \partial_D \Omega \setminus H_h \right\}. \tag{4.11}$$

Let $\tilde{u}_h \in L^{1,p}(\Omega \setminus H_h)$ be the solution of problem (4.11). Applying Lemma 2.5 to $\tilde{u}_h - g_h$, we get that $\nabla \tilde{u}_h$ converges (up to a subsequence) to $\nabla \tilde{u}^*$ weakly in $L^p(\Omega, \mathbb{R}^2)$, for some function $\tilde{u}^*$ in $L^{1,p}(\Omega \setminus K)$ with $\tilde{u}^* = g$ on $\partial_D \Omega \setminus K$. As in the proof of Proposition 4.1, we have also that $f_\xi(x, \nabla \tilde{u}_h)$ converges to $f_\xi(x, \nabla \tilde{u}^*)$ weakly in $L^q(\Omega, \mathbb{R}^2)$.

Let us prove that $\nabla \tilde{u}^* = \nabla u$ a.e. in Ω. By a localization argument, it is sufficient to prove that the function $\tilde{u}^*$ satisfies:

$$\begin{cases} \displaystyle\int_{(Q_i \cap \Omega) \setminus K} f_\xi(x, \nabla \tilde{u}^*) \cdot \nabla \varphi \, dx = 0, \\ \forall \varphi \in C_c^\infty(Q_i) \text{ with } \varphi = 0 \text{ on } (Q_i \cap \partial_D \Omega) \setminus K. \end{cases} \tag{4.12}$$

Let $\tilde{H}_h^i := H_h \cap \overline{U_i \cap \Omega}$ and $\tilde{\Gamma}^i := \partial_N \Omega \cap \overline{U_i}$. Since the diameter of U_i is strictly less than the minimum of the diameters of the connected components of Ω^c, we have that the open set $U_i \cap \Omega$ is simply connected. So, by Lemma 3.4, there exists a function $v_h^i \in W^{1,q}(U_i \cap \Omega)$ such that $\nabla v_h^i = R f_\xi(x, \nabla \tilde{u}_h)$ a.e. in $U_i \cap \Omega$ and v_h^i is constant C_q-q.e. on the connected components of $\tilde{H}_h^i \cup \tilde{\Gamma}^i$. Since $H_h^i \cup \Gamma^i \subset \tilde{H}_h^i \cup \tilde{\Gamma}^i$, we have that any connected component of $H_h^i \cup \Gamma^i$ is contained in a connected component of $\tilde{H}_h^i \cup \tilde{\Gamma}^i$. So we have also that v_h^i is constant C_q-q.e. on the connected components of $H_h^i \cup \Gamma^i$. From the fact that $f_\xi(x, \nabla \tilde{u}_h)$ converges $f_\xi(x, \nabla \tilde{u}^*)$ weakly in $L^q(\Omega, \mathbb{R}^2)$, it follows that v_h^i converges weakly to some function v^i in $W^{1,q}(U_i \cap \Omega)$ such that $\nabla v^i = R f_\xi(x, \nabla \tilde{u}^*)$ a.e. in $U_i \cap \Omega$. Since any connected component of $H_h^i \cup \Gamma^i$ converges to a connected component of $K^i \cup \Gamma^i$ in the Hausdorff metric and v_h^i is constant C_q-q.e. on the connected components of $H_h^i \cup \Gamma^i$, we get from Lemma 2.7 that v^i is constant C_q-q.e. on every connected component of $K^i \cup \Gamma^i$. Now applying Lemma 3.3 with Ω replaced by $Q_i \cap \Omega$, we get that $\tilde{u}^*$ satisfies (4.12). Therefore $\nabla \tilde{u}^* = \nabla u$ a.e. in Ω. So, $\nabla \tilde{u}_h$ converges weakly to ∇u in $L^p(\Omega, \mathbb{R}^2)$ and $f_\xi(x, \nabla \tilde{u}_h)$ converges to $f_\xi(x, \nabla u)$ weakly in $L^q(\Omega, \mathbb{R}^2)$. Thus, arguing as in the proof of Proposition 4.1, we get that $\nabla \tilde{u}_h$ converges to ∇u strongly in $L^p(\Omega, \mathbb{R}^2)$.

Now, from the minimality of the pair (u_h, K_h), we have that

$$\int_{\Omega \setminus K_h} f(x, \nabla u_h)\, dx \le \int_{\Omega \setminus H_h} f(x, \nabla \tilde{u}_h)\, dx + \mathcal{H}^1(H_h \setminus K_h). \tag{4.13}$$

So, passing to the limit in (4.13) and using the fact that $\nabla u_h \rightharpoonup \nabla u^*$ weakly in $L^p(\Omega, \mathbb{R}^2)$, we obtain

$$\begin{aligned} &\int_{\Omega \setminus K} f(x, \nabla u^*)\, dx \le \liminf_{h\to\infty} \int_{\Omega \setminus K_h} f(x, \nabla u_h)\, dx \le \\ &\le \lim_{h\to\infty} \int_{\Omega \setminus H_h} f(x, \nabla \tilde{u}_h)\, dx\, dx + \lim_{h\to\infty} \mathcal{H}^1(H_h \setminus K_h) = \int_{\Omega \setminus K} f(x, \nabla u)\, dx, \end{aligned}$$

which implies (by the uniqueness of solution of (2.3) in $\Omega \setminus K$) that $\nabla u^* = \nabla u$ a.e. in Ω and

$$\lim_{h\to\infty} \int_\Omega f(x, \nabla u_h)\, dx = \int_\Omega f(x, \nabla u)\, dx. \tag{4.14}$$

Since $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^2)$, using the strict convexity of $f(x, \cdot)$, it follows from (4.14) that ∇u_h converges to ∇u strongly in $L^p(\Omega, \mathbb{R}^2)$. This achieves the proof of the first part of the theorem.

Now let us prove that the pair (u, K) is an unilateral minimum of the functional E relative to g. Let $H \in \mathcal{K}_m(\overline{\Omega})$ with $K \subset H$ and let $w \in L^{1,p}(\Omega \setminus H)$ with $w = g$ on $\partial_D \Omega \setminus H$. It is not restrictive to assume that $H \in \mathcal{K}_m^f(\overline{\Omega})$. By Lemma 2.4, there exists a sequence $(H_h) \subset \mathcal{K}_m(\overline{\Omega})$ such that $H_h \to H$ in the Hausdorff metric, $K_h \subset H_h$, and

$$\mathcal{H}^1(H_h \setminus K_h) \to \mathcal{H}^1(H \setminus K).$$

Since $(K_h) \subset \mathcal{K}_m^\lambda(\overline{\Omega})$ and $H \in \mathcal{K}_m^f(\overline{\Omega})$, we have that $(H_h) \subset \mathcal{K}_m^{\lambda+\varepsilon}(\overline{\Omega})$ for some $\varepsilon > 0$. Arguing as in the first part of the proof, we can construct a sequence $(\tilde{H}_h) \subset \mathcal{K}_m(\overline{\Omega})$ such that $H_h \subset \tilde{H}_h$, $\mathcal{H}^1(\tilde{H}_h \setminus H_h) \to 0$, and denoting $z_h \in L^{1,p}(\Omega \setminus \tilde{H}_h)$ and $z \in L^{1,p}(\Omega \setminus H)$ the solutions of (2.3) with boundary data g_h and g respectively, we get that $\nabla z_h \to \nabla z$ strongly in $L^p(\Omega, \mathbb{R}^2)$. Now we can achieve the proof as in Step 2 of Theorem 4.2. □

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