ON THE VARIATIONAL APPROXIMATION OF FREE-DISCONTINUITY PROBLEMS IN THE VECTORIAL CASE

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We provide a variational approximation for quasiconvex energies defined on vector valued special functions with bounded variation. We extend the Ambrosio-Tortorelli's construction to the vectorial case.

1. Introduction

Many mathematical problems arising from Computer Vision Theory and Fracture Mechanics (see for instance[33,10]) involve energies consisting of two parts, the first taking into account a volume energy and the second a surface energy. The variational formulation of the problem leads to the minimization of functionals represented by

$$\mathcal{E}\left(u,K\right)=\int_{\Omega\setminus K}f\left(x,u,\nabla u\right)dx+\int_{K}\varphi\left(x,u^{-},u^{+},\nu\right)d\mathcal{H}^{n-1}, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a fixed domain, K is a (sufficiently regular) closed subset of Ω and $u:\Omega\setminus K\to\mathbb{R}^N$ belongs to a (sufficiently regular) class of functions with traces $u^{\pm}$ defined on K.

Since the closed subsets of Ω cannot be endowed with a topology which ensures that the direct methods apply, a weak formulation of the problem is needed. To do this, De Giorgi[23] proposed to interpret K as the set of discontinuity points of u. This idea motivates the terminology "free-discontinuity problem" for the minimization of (1.1), to underline the fact that one looks for a function whose discontinuities are not assigned a priori.

Thus, it is natural to set the problem in the space BV of function with bounded variation, i.e., functions u which are summable and whose first order distributional derivative is representable by a measure Du with finite total variation. Actually, since free-discontinuity problems deal with volume and surface energies, it is natural to allow in these problems only BV functions whose distributional derivative has the same structure. Indeed, De Giorgi and Ambrosio[24] relaxed the problem in the space SBV of special functions with bounded variation, i.e., function u in BV such that the singular part of Du with respect to the Lebesgue measure is supported in the complement of the set of Lebesgue points for u, denoted by S_u. Thus, setting $K=S_u$ in (1.1) and defining $\mathcal{F}\left(u\right)=\mathcal{E}\left(u,S_u\right)$, the free-discontinuity problem reduces to

$$\min_{u\in SBV}\mathcal{F}\left(u\right). \tag{1.2}$$

The abstract theory for such problems has been developed in the last years: Ambrosio[6,9] established the existence theory, and many authors studied the regularity of solutions (see[25,18,13,11]), thus solving the original problem in (1.1).

The numerical approximation for solutions of the problem (1.2) revealed to be a hard task because of the use of spaces of discontinuous functions. The idea to overcome this difficulty is to perform a preliminary variational approximation of the functional $\mathcal{F}$ in the sense of De Giorgi's Γ-convergence[26] via simpler functionals defined on Sobolev spaces, easier to be handled numerically, and then to discretize each of the approximating functionals. Many approaches have been proposed for the approximation problem in the scalar case (see[3,14,15,17,19,29] and the book[16] for an exhaustive treatment of the subject), while the vectorial case had not been treated, yet.

Here we attack the vectorial problem, extending the Ambrosio-Tortorelli's approximation.[14,15]

The energies we deal with have the form

$$\int_{\Omega} f\left(x, u, \nabla u\right) dx + \int_{S_u} \varphi\left(\nu_u\right) d\mathcal{H}^{n-1}, \tag{1.3}$$

where f is a positive Carathéodory integrand with superlinear growth and quasiconvex in the gradient variable (see Section 2 for definitions) and φ is a norm on $\mathbb{R}^n$. Since the general form of the volume term in (1.3), the slicing methods no longer applies to obtain a lower bound for it in the Γ-limit. The idea, then, is to deduce lower estimates on the bulk term and on the surface term separately: the first thanks to truncations and the lower semicontinuity Theorem 2.15, the second using the slicing techniques. The upper bound for the Γ-limit is obtained reducing ourselves to an explicit construction only for functions with a polyhedral discontinuity set S_u.

The plan of the paper is the following: in Section 2 we introduce the notation and recall the many results we need concerning Γ-convergence, SBV, GSBV functions; in Section 3 we state and prove the main result of the paper Theorem 3.1; finally Section 4 is devoted to a convergence result for the minimizers of the approximating functionals.

2. Notation and Preliminary Results

2.1. *Basic notation*

Let $n, k, N \in \mathbb{N}$, we use standard notations for Lebesgue and Sobolev spaces, $\mathcal{L}^n$ denotes the Lebesgue measure and $\mathcal{H}^k$ denotes the k dimensional Hausdorff measure in $\mathbb{R}^n$.

With Ω we will denote a bounded and open set of $\mathbb{R}^n$ with Lipschitz boundary, and with $\mathcal{A}(\Omega)$ the family of open sets of Ω. Moreover, let

$$\mathcal{B}\left(\Omega, \mathbb{R}^N\right) = \left\{u : \Omega \to \mathbb{R}^N \,:\, u \text{ is a Borel function}\right\}.$$

The space $\mathcal{B}\left(\Omega, \mathbb{R}^N\right)$ can be endowed with a metric which induces the convergence in measure.

Let $A \subset\subset B \subset \mathbb{R}^n$ be open sets, a *cut-off function between A and B* will be a function β satisfying

$$\beta \in C_c^\infty(B), 0 \le \beta \le 1, \beta = 1 \text{ on } A, \|\nabla\beta\|_\infty \le \frac{2}{d(A,B)},$$

where $d(A,B) = \inf\{|a-b| : a \in A, b \in B\}$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ induced by the scalar product $\langle a,b\rangle = \sum_{i=1}^n a_i b_i$.

Let $\varphi : \mathbb{R}^n \to [0,+\infty)$ be a norm, set M, m for $\max_{\mathbb{S}^{n-1}} \varphi$ and $\min_{\mathbb{S}^{n-1}} \varphi$, respectively. Notice that $m > 0$, thus for every $\nu \in \mathbb{R}^n$ there holds

$$m\,|\nu| \leq \varphi(\nu) \leq M\,|\nu|. \tag{2.1}$$

Let $g \in C_c^1([0,+\infty))$ be such that $g(t) = t$ if $0 \leq t \leq 1$, $g(t) = 0$ if $t \geq 2$ and $\|g\|_\infty \leq 2$, fix $k \in \mathbb{N}$ and define $g_k(t) = kg(t/k)$, then consider the radial maps $\Psi_k : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\Psi_k(w) = \begin{cases} g_k(|w|)\,\frac{w}{|w|} & w \neq 0 \\ 0 & w = 0, \end{cases} \tag{2.2}$$

notice that $\Psi_k \in C_c^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and $Lip(\Psi_k) = Lip(g_k) = Lip(g)$.

2.2. Γ-*convergence*

We recall some definitions and properties related to Γ-convergence, the main reference will be the book[22].

Definition 2.1. *Let (X,d) be a metric space, let $Y \subseteq X$ and let be given $f_h : Y \to [-\infty,+\infty]$. We say that f_h Γ-converges to $f : X \to [-\infty,+\infty]$ on X, and we write $f_h \xrightarrow{\Gamma} f$, if the following two conditions hold:*

(LB) Lower Bound inequality*: for every $x \in X$ and every sequence $(x_h) \xrightarrow{d} x$ there holds*

$$f(x) \leq \liminf_{h\to+\infty} f_h(x_h). \tag{2.3}$$

(UB) Upper Bound inequality*: there exists a sequence $(x_h) \xrightarrow{d} x$ such that*

$$f(x) \geq \limsup_{h\to+\infty} f_h(x_h). \tag{2.4}$$

We call *recovery sequence* any sequence satisfying (2.4); for such a sequence, combining (2.3) and (2.4), there holds

$$f(x) = \lim_{h\to+\infty} f_h(x_h).$$

The function f is uniquely determined by (LB) and (UB) and is called the Γ-*limit* of (f_h). Moreover, given a family of functions (f_ε) labelled by a continuous parameter $\varepsilon > 0$, we say that f_ε Γ-*converges* to f on X as $\varepsilon \to 0^+$ if f is the Γ-limit of (f_{ε_h}) for every sequence $\varepsilon_h \to 0$.

The main properties of Γ-convergence are listed below.

Lemma 2.2. *(i)* Lower semicontinuity*: the Γ-limit is lower semicontinuous on X;*

(ii) Stability under continuous perturbations: *if $g : X \to \mathbb{R}$ is continuous and $f_\varepsilon \xrightarrow{\Gamma} f$ then $f_\varepsilon + g \xrightarrow{\Gamma} f + g$;*

(iii) Stability of minimizing sequences: *if* $f_\varepsilon \xrightarrow{\Gamma} f$ *and* (x_ε) *is asymptotically minimizing, i.e.,*

$$\lim_{\varepsilon\to 0^+}\left(f_\varepsilon\left(x_\varepsilon\right)-\inf_Y f_\varepsilon\right)=0,$$

then every cluster point x *of* (x_ε) *minimizes* f *over* X, *and*

$$\lim_{\varepsilon\to 0^+}\inf_Y f_\varepsilon = f(x).$$

2.3. *Functions of bounded variation*

We recall some definitions and basic results on functions with bounded variation, our main reference is the book[12] (see also[27],[28]).

Let $u:\Omega\to\mathbb{R}^N$ be a measurable function, let $S=\mathbb{R}^N\cup\{\infty\}$ be the one point compactification of $\mathbb{R}^N$, fix $x\in\Omega$, we say that $z\in S$ is the *approximate limit* of u at x with respect to Ω, we write $z = ap-\lim_{\substack{y\to x\\ y\in\Omega}} u\left(y\right)$, if for every neighbourhood U of z in S there holds

$$\lim_{\rho\to 0}\frac{1}{\rho^n}\mathcal{L}^n\left(\left\{y\in\Omega:|y-x|<\rho, u\left(y\right)\notin U\right\}\right)=0.$$

If $z\in\mathbb{R}^N$ we say that x is a *Lebesgue point* of u and we denote by S_u the complement of the set of Lebegue point of u. It is known that $\mathcal{L}^n\left(S_u\right)=0$, thus u coincides $\mathcal{L}^n$ a.e. with the function $\tilde{u}:\Omega\setminus S_u\to\mathbb{R}^N$ defined by

$$\tilde{u}(x) = ap-\lim_{\substack{y\to x\\ y\in\Omega}} u(y).$$

Moreover, we say that u is *approximately differentiable* at a Lebesgue point x such that $\tilde{u}(x)\neq\infty$, if there exists a matrix $L\in\mathbb{R}^{N\times n}$ such that

$$ap-\lim_{\substack{y\to x\\ y\in\Omega}}\frac{|u(y)-\tilde{u}(x)-L(y-x)|}{|y-x|}=0. \tag{2.5}$$

If u is approximately differentiable at a Lebesgue point x, the matrix L uniquely determined by (2.5), will be denoted by $\nabla u(x)$ and will be called the *approximate gradient* of u at x.

Definition 2.3. *Let* $u\in L^1\left(\Omega,\mathbb{R}^N\right)$, *we say that* u *is a function with* Bounded Variation *in* Ω, *we write* $u\in BV\left(\Omega,\mathbb{R}^N\right)$, *if the distributional derivative* Du *of* u *is representable by a* $N\times n$ *matrix valued measure on* Ω *with finite total variation* $|Du|\left(\Omega\right)$ *whose entries are denoted by* $D_i u^\alpha$, *i.e., if* $u=\left(u^1,\dots,u^N\right)$ *and* $\varphi\in C_c^1\left(\Omega,\mathbb{R}^N\right)$ *then*

$$\sum_{\alpha=1}^N\int_\Omega u^\alpha div\varphi^\alpha dx = -\sum_{\alpha=1}^N\sum_{i=1}^n\int_\Omega \varphi_i^\alpha dD_i u^\alpha. \tag{2.6}$$

Moreover, given E *a subset of* Ω, *we say that* E *is a* Set of Finite Perimeter *in* Ω *if* $\mathcal{X}_E\in BV\left(\Omega\right)$ *and we denote its total variation* $|D\mathcal{X}_E|\left(\Omega\right)$ *by* $Per\left(E\right)$.

If $u \in BV\left(\Omega, \mathbb{R}^N\right)$, then u is approximately differentiable $\mathcal{L}^n$ a.e., S_u turns out to be countably $\left(\mathcal{H}^{n-1}, n-1\right)$ rectifiable, i.e.,

$$S_u = N \cup \bigcup_{i \geq 1} K_i,$$

where $\mathcal{H}^{n-1}(N) = 0$ and each K_i is a compact subset of a C^1 manifold. Hence, for $\mathcal{H}^{n-1}$ a.e. $y \in S_u$ we can define an *exterior unit normal* ν_u to S_u as well as *inner* and *outer traces* of u on S_u by

$$u^{\pm}(x) = ap - \lim_{\substack{y \to x \\ y \in \pi^{\pm}(x, \nu_u(x))}} u(y)$$

where $\pi^{\pm}(x, \nu_u(x)) = \{y \in \mathbb{R}^n : \pm \langle y - x, \nu_u(x) \rangle > 0\}$.

Let us consider the Lebesgue's decomposition of Du with respect to $\mathcal{L}^n$, then $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part and $D^s u$ is the singular one. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides $\mathcal{L}^n$ a.e. with the approximate gradient ∇u of u. Define the *jump part* of Du, $D^j u$, to be the restriction of $D^s u$ to S_u and the *Cantor part*, $D^c u$, to be the restriction of $D^s u$ to $\Omega \setminus S_u$, thus we have

$$Du = D^a u + D^j u + D^c u.$$

Moreover, it holds $D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \lfloor S_u$, where given $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ $a \otimes b$ is the matrix with entries equal to $a_i b^j$, $1 \leq i \leq N$ and $1 \leq j \leq n$.

Definition 2.4. *Let $u \in BV\left(\Omega, \mathbb{R}^N\right)$, we say that u is a* Special Function with Bounded Variation *in Ω, we write $u \in SBV\left(\Omega, \mathbb{R}^N\right)$, if $D^c u = 0$.*

Functionals involved in free-discontinuity problems are often not coercive in $SBV\left(\Omega, \mathbb{R}^N\right)$, then it is useful to consider the following wider class (see[24],[7]).

Definition 2.5. *Given a Borel function $u : \Omega \to \mathbb{R}^N$, we say that u is a* Generalized Special Function with Bounded Variation *in Ω, and we write $u \in GSBV\left(\Omega, \mathbb{R}^N\right)$, if $g(u) \in SBV(\Omega)$ for every $g \in C^1\left(\mathbb{R}^N\right)$ such that ∇g has compact support.*

Notice that $GSBV \cap L^\infty\left(\Omega, \mathbb{R}^N\right) = SBV \cap L^\infty\left(\Omega, \mathbb{R}^N\right)$.
Functions $u \in GSBV\left(\Omega, \mathbb{R}^N\right)$ are approximately differentiable $\mathcal{L}^n$ a.e. in Ω, S_u turns out to be $\left(\mathcal{H}^{n-1}, n-1\right)$ rectifiable and it is possible to define $\mathcal{H}^{n-1}$ a.e. in S_u the exterior normal ν_u and the one side traces $u^{\pm}$ (see[7]).

The main features of the space $GSBV\left(\Omega, \mathbb{R}^N\right)$ are the following closure and compactness theorems (see[6], see also[2]).

Theorem 2.6. *Let $\phi : [0, +\infty) \to [0, +\infty)$ be a convex non-decreasing function such that $\frac{\phi(t)}{t} \to +\infty$ as $t \to +\infty$, let $\theta : [0, +\infty) \to [0, +\infty]$ be a concave function such that $\frac{\theta(t)}{t} \to +\infty$ as $t \to 0^+$.*

Let $(u_h) \subset GSBV\left(\Omega, \mathbb{R}^N\right)$ and assume that

$$\sup_h \left\{ \int_\Omega \phi\left(|\nabla u_h|\right) dx + \int_{S_{u_h}} \theta\left(\left|u_h^+ - u_h^-\right|\right) d\mathcal{H}^{n-1} \right\} < +\infty. \tag{2.7}$$

If u_h converges to u $\mathcal{L}^n$ a.e. Ω, then $u \in GSBV\left(\Omega, \mathbb{R}^N\right)$ and *(i)* $\nabla u_h \to \nabla u$ *weakly in* $L^1\left(\Omega, \mathbb{R}^{N\times n}\right)$*;*

(ii) $D^j u_{h_k}$ *converges weakly in the sense of measures to* $D^j u$*;*

(iii) $\int_\Omega \phi\left(|\nabla u|\right) dx \le \liminf_{h\to+\infty} \int_\Omega \phi\left(|\nabla u_h|\right) dx$*;*

(iv) $\int_{S_u} \theta\left(|u^+ - u^-|\right) d\mathcal{H}^{n-1} \le \liminf_{h\to+\infty} \int_{S_{u_h}} \theta\left(\left|u_h^+ - u_h^-\right|\right) d\mathcal{H}^{n-1}$.

Theorem 2.7. *Let* ϕ, θ *be as in Theorem 2.6. Consider* $(u_h) \subset GSBV\left(\Omega, \mathbb{R}^N\right)$ *satisfying (2.7) and assume, in addition, that* $\|u_h\|_{\infty,\Omega}$ *is uniformly bounded in* h*, then there exists a subsequence* (u_{h_k}) *and a function* $u \in SBV\left(\Omega, \mathbb{R}^N\right)$*, such that* $u_{h_k} \to u$ $\mathcal{L}^n$ *a.e. in* Ω*.*

The original proofs of Theorem 2.6 and Theorem 2.7 make use of the one dimensional sections of BV functions which turned out to provide a useful tool for the study of variational approximations of free-discontinuity problems.

Before recalling the Slicing Theorem (see[6]) let us fix some notations. Let $\xi \in \mathbb{S}^{n-1}$, let Π^ξ be the orthogonal space to ξ, i.e., $\Pi^\xi = \{y \in \mathbb{R}^n : \langle \xi, y\rangle = 0\}$. If $y \in \Pi^\xi$ and $E \subset \mathbb{R}^n$ define $E_{\xi,y} = \{t \in \mathbb{R} : y + t\xi \in E\}$; moreover, given $u : E \to \mathbb{R}^N$ set $u_{\xi,y} : E_{\xi,y} \to \mathbb{R}^N$ by $u_{\xi,y}(t) = u(y + t\xi)$.

Theorem 2.8. *Let* $u \in GSBV(\Omega)$*, then* $u_{\xi,y} \in GSBV(\Omega_{\xi,y})$ *for all* $\xi \in \mathbb{S}^{n-1}$ *and* $\mathcal{H}^{n-1}$ *a.e.* $y \in \Pi^\xi$*. For such* y *we have*

(i) $u'_{\xi,y}(t) = \langle \nabla u\left(y + t\xi\right), \xi\rangle$ *for* $\mathcal{L}^1$ *a.e.* $t \in \Omega_{\xi,y}$*;*

(ii) $S_{u_{\xi,y}} = \{t \in \mathbb{R} : y + t\xi \in S_u\}$*;*

(iii) $u^\pm_{\xi,y}(t) = u^\pm(y + t\xi)$ *or* $u^\pm_{\xi,y}(t) = u^\mp(y + t\xi)$ *according to the cases* $\langle \nu_u, \xi\rangle > 0$*,* $\langle \nu_u, \xi\rangle < 0$ *(the case* $\langle \nu_u, \xi\rangle = 0$ *being negligible).*

Moreover, for every open set $A \subseteq \Omega$ *there holds*

$$\int_{\Pi^\xi} \mathcal{H}^0\left(S_{u_{\xi,y}} \cap A\right) d\mathcal{H}^{n-1}(y) = \int_{S_u \cap A} |\langle \nu_u(y), \xi\rangle| d\mathcal{H}^{n-1}(y). \tag{2.8}$$

Let us now introduce a useful sub-class of SBV functions.

Definition 2.9. *Let* $\mathcal{W}\left(\Omega, \mathbb{R}^N\right)$ *be the space of all* $u \in SBV\left(\Omega, \mathbb{R}^N\right)$ *such that*

(i) S_u *is* essentially closed*, i.e.,* $\mathcal{H}^{n-1}\left(\overline{S_u} \setminus S_u\right) = 0$*;*

(ii) $\overline{S_u}$ *is a* polyhedral set*, i.e.,* $\overline{S_u}$ *is the intersection of* Ω *with the union of a finite number of* $(n-1)$ *dimensional simplexes;*

(iii) $u \in W^{k,\infty}\left(\Omega \setminus \overline{S_u}, \mathbb{R}^N\right)$ *for every* $k \in \mathbb{N}$*.*

The following theorem proved by Cortesani and Toader[21] provides a density result of the class $\mathcal{W}\left(\Omega, \mathbb{R}^N\right)$ in $SBV \cap L^\infty\left(\Omega, \mathbb{R}^N\right)$ with respect to anisotropic surface energies.

Theorem 2.10. *Let $u \in SBV \cap L^{\infty}\left(\Omega, \mathbb{R}^N\right)$ be such that*

$$\mathcal{H}^{n-1}\left(S_u\right) < +\infty \text{ and } \nabla u \in L^p\left(\Omega, \mathbb{R}^{N\times n}\right),$$

for some $p \geq 1$, then there exists a sequence $(u_h) \subset \mathcal{W}\left(\Omega, \mathbb{R}^N\right)$ such that

(i) $u_h \to u$ *strongly in* $L^1\left(\Omega, \mathbb{R}^N\right)$;

(ii) $\nabla u_h \to \nabla u$ *strongly in* $L^p\left(\Omega, \mathbb{R}^{N\times n}\right)$;

(iii) $\limsup_{h\to+\infty} \|u_h\|_{\infty} \leq \|u\|_{\infty}$;

(iv) for every $A \subset\subset \Omega$ and for every upper semicontinuous function $\varphi : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0, +\infty)$ such that $\varphi(x, a, b, \nu) = \varphi(x, b, a, -\nu)$ for every $x \in \Omega$, $a, b \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ there holds

$$\limsup_{h\to+\infty} \int_{\overline{A}\cap S_{u_h}} \varphi\left(x, u_h^-, u_h^+, \nu_{u_h}\right) d\mathcal{H}^{n-1} \leq \int_{\overline{A}\cap S_u} \varphi\left(x, u^-, u^+, \nu_u\right) d\mathcal{H}^{n-1}. \tag{2.9}$$

Remark 2.11. *The sequence (u_h) can be chosen such that (2.9) holds for every open set $A \subseteq \Omega$ if the following additional condition is satisfied*

$$\limsup_{\substack{\left(y, a', b', \mu\right)\to(x,a,b,\nu)\\ y\in\Omega}} \varphi\left(y, a', b', \mu\right) < +\infty$$

for every $x \in \partial\Omega$, $a, b \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$. In this case, $\overline{A}$ must be replaced by the relative closure of A in Ω (see[21] Remark 3.2).

Eventually, we state the following result which will be useful in the sequel (see for instance[16]).

Lemma 2.12. *Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be a superadditive function on disjoint open sets, let λ be a positive measure on Ω, let $\psi_h : \Omega \to [0, +\infty]$ be a countable family of Borel functions such that $\mu(A) \geq \int_A \psi_h d\lambda$ for every $A \in \mathcal{A}(\Omega)$.*

Set $\psi = \sup_{h\in\mathbb{N}} \psi_h$, then

$$\mu(A) \geq \int_A \psi d\lambda$$

for every $A \in \mathcal{A}(\Omega)$.

2.4. *Lower semicontinuity in GSBV*

Let us first recall some definitions.

Definition 2.13. *We say that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0, +\infty)$ is a* Carathéodory integrand *if $f(\cdot, s, z)$ is Borel measurable for every $(s, z) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ and $f(x, \cdot, \cdot)$ is continuous for $\mathcal{L}^n$ a.e. $x \in \Omega$.*

Definition 2.14. *We say that a Carathéodory integrand* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0,+\infty)$ *is* quasiconvex *in z if for $\mathcal{L}^n$ a.e. $x \in \Omega$ and for every $s \in \mathbb{R}^N$*

$$f(x,s,z)\,\mathcal{L}^n(\Omega) \le \int_\Omega f(x,s,z+D\varphi(y))\,dy \tag{2.10}$$

for every $\varphi \in C_c^1(\Omega,\mathbb{R}^N)$.

We recall the following result, proved by Kristensen[30] in a more general version (see also[9]), which ensures lower semicontinuity for variational integrals exactly in the setting prescribed by the GSBV Compactness Theorem 2.7.

Theorem 2.15. *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0,+\infty)$ be a Carathéodory integrand quasiconvex in z satisfying*

$$c_1\big(|z|^p + b(s) - a(x)\big) \le f(x,s,z) \le c_2\big(|z|^p + b(s) + a(x)\big)$$

for every $(x,s,z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ with $p > 1$, c_1 and c_2 positive constants, $a \in L^1(\Omega)$, and $b \in C^0(\mathbb{R}^N)$ a non negative function.

Let u_h, $u \in GSBV(\Omega,\mathbb{R}^N)$ be such that

(i) $u_h \to u$ *$\mathcal{L}^n$ a.e. in* Ω;

(ii) $\nabla u_h \to \nabla u$ *weakly in* $L^1(\Omega,\mathbb{R}^{N\times n})$;

(iii) $\sup_h \|\nabla u_h\|_{p,\Omega} < +\infty$;

(iv) there exists a concave function $\theta : [0,+\infty) \to [0,+\infty]$ satisfying $\frac{\theta(t)}{t} \to +\infty$ as $t \to 0^+$, such that

$$\sup_h \int_{S_{u_h}} \theta\left(\left|u_h^+ - u_h^-\right|\right) d\mathcal{H}^{n-1} < +\infty.$$

Then

$$\int_\Omega f(x,u,\nabla u)\,dx \le \liminf_{h\to+\infty} \int_\Omega f(x,u_h,\nabla u_h)\,dx. \tag{2.11}$$

Eventually, we end this subsection stating the following result concerning the lower semicontinuity of surface integrals which follows straightforward from a more general theorem proved by Ambrosio.[7]

Theorem 2.16. *Let $\varphi : \mathbb{R}^n \to [0,+\infty)$ be a norm, let u_h, $u \in GSBV(\Omega,\mathbb{R}^N)$ be such that*

(i) $u_h \to u$ *in measure on* Ω;

(ii) there exists $p > 1$ such that

$$\sup_h \|\nabla u_h\|_{p,\Omega} < +\infty.$$

Then

$$\int_{S_u} \varphi(\nu_u)\, dx \le \liminf_{h\to+\infty} \int_{S_{u_h}} \varphi(\nu_{u_h})\, dx.$$

3. Γ-Convergence Result

In this section we prove a variational approximation for functionals defined on $GSBV\left(\Omega, \mathbb{R}^N\right)$ having the form

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1}, \tag{3.1}$$

where f is a positive function satisfying some growth and regularity condition and $\varphi : \mathbb{R}^n \to [0, +\infty)$ is a norm. To perform the approximation we add a formal extra variable v to $\mathcal{F}$, defining $F : \mathcal{B}\left(\Omega, \mathbb{R}^N\right) \times \mathcal{B}(\Omega) \to [0, +\infty]$ by

$$F(u, v, \Omega) = \begin{cases} \mathcal{F}(u) & u \in GSBV\left(\Omega, \mathbb{R}^N\right),\ v = 1\ \mathcal{L}^n \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

The approximating functionals $F_\varepsilon : \mathcal{B}\left(\Omega, \mathbb{R}^N\right) \times \mathcal{B}(\Omega) \to [0, +\infty]$ have the form

$$F_\varepsilon(u, v, \Omega) = \begin{cases} \int_\Omega \big((\psi(v) + \eta_\varepsilon)\, f(x, u, \nabla u) + \frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla v) + \frac{1}{\varepsilon p'} W(v)\big) dx \\ \qquad\qquad (u, v) \in W^{1,p}\left(\Omega, \mathbb{R}^N\right) \times W^{1,p}(\Omega),\ 0 \le v \le 1 \\ +\infty \qquad\qquad \text{otherwise.} \end{cases} \tag{3.3}$$

where $\psi : [0, 1] \to [0, 1]$ is any increasing lower semicontinuous function such that $\psi(0) = 0$, $\psi(1) = 1$, and $\psi(t) > 0$ if $t > 0$; $p \in (1, +\infty)$ and $p' = \frac{p}{p-1}$; η_ε is any positive infinitesimal faster than ε^{p-1} for $\varepsilon \to 0^+$; $W(t) = \frac{1}{\alpha}(1-t)^p$, with $\alpha = \left(2 \int_0^1 (1-s))^{\frac{p}{p'}} ds\right)^{p'}$, so that defining the auxiliary function $\Phi : [0, 1] \to [0, +\infty)$ by

$$\Phi(t) = \int_0^t (W(s))^{\frac{1}{p'}}\, ds, \tag{3.4}$$

we have $\Phi(0) = 0$ and $\Phi(1) = \frac{1}{2}$.

Let us state and prove the main result of the paper.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0, +\infty)$ be a Carathéodory integrand, quasiconvex in z, satisfying*

$$c_1\big(|z|^p + b(s) - a(x)\big) \le f(x, s, z) \le c_2\big(|z|^p + b(s) + a(x)\big) \tag{3.5}$$

for every $(x, s, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ with $p > 1$, c_1 and c_2 positive constants, $a \in L^1(\Omega)$, and $b \in C^0\left(\mathbb{R}^N\right)$ a non negative function.

Then

$$\Gamma\left(\mathcal{B}\left(\Omega, \mathbb{R}^N\right) \times \mathcal{B}(\Omega)\right)\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u, v, \Omega) = F(u, v, \Omega).$$

We divide the proof of Theorem 3.1 into two parts, each corresponding to the (LB) and (UB) inequality of Definition 2.1.

3.1. *Lower bound inequality*

We first derive a lower bound for the surface term in dimension $n = 1$. In such a case $\varphi(t) = \varphi(1)|t|$, then arguing like in[14] (see also[15],[16]) we get the following result. We outline the proof for the convenience of the reader.

Lemma 3.2. *Let $I \subset \mathbb{R}$ be a bounded and open set, then for every sequence $(u_h, v_h) \to (u, v)$ in measure on I such that*

$$\liminf_{h\to+\infty} F_{\varepsilon_h}(u_h, v_h, I) < +\infty, \tag{3.6}$$

it follows

$$\liminf_{h\to+\infty} \int_I \left(\frac{\varepsilon_h^{p-1}}{p} \varphi^p(v_h') + \frac{1}{\varepsilon_h p'} W(v_h) \right) dx \geq \varphi(1)\, \mathcal{H}^0(S_u \cap I). \tag{3.7}$$

Proof. Condition (3.6) implies $v = 1$ for $\mathcal{L}^1$ a.e. $x \in I$. Moreover, we may extract a subsequence, not relabelled for convenience, such that $(u_h, v_h) \to (u, v)$ $\mathcal{L}^1$ a.e. in I and the inferior limit in (3.7) is a limit. Notice that we may assume S_u not empty, since otherwise (3.7) is trivial.

Let $\{t_1, ..., t_r\}$ be an arbitrary subset of S_u, then consider $I_i = (a_i, b_i)$, $1 \leq i \leq r$, pairwise disjoint intervals such that $t_i \in I_i$, $I_i \subset\subset I$ and $\bigcup_{i=1}^r I_i \subset I$. We claim that

$$s_i = \limsup_{h\to+\infty} \left(\inf_{I_i} \psi(v_h) \right) = 0.$$

Indeed, if $s_j > 0$ for some $j \in \{1, ..., r\}$, there exists a subsequence (v_{h_k}) for which it holds

$$\inf_{I_j} \psi(v_{h_k}) \geq \frac{s_j}{2}.$$

The growth condition (3.5) yields

$$\frac{s_j}{2} \liminf_{k\to+\infty} \int_{I_j} |u_{h_k}'|^p \leq c,$$

thus there exists a subsequence of (u_{h_k}) converging to u weakly in $W^{1,1}(I_j, \mathbb{R}^N)$, so that $u \in W^{1,1}(I_j, \mathbb{R}^N)$, which is a contradiction.

So let $t_h^i \in I_i$ be such that

$$\lim_{h\to+\infty} v_h(t_h^i) = 0,$$

and α_i, $\beta_i \in I_i$, with $\alpha_i < t_h^i < \beta_i$, be such that

$$\lim_{h\to+\infty} v_h(\alpha_i) = \lim_{h\to+\infty} v_h(\beta_i) = 1.$$

Using the auxiliary function Φ introduced in (3.4), by Young's inequality we get

$$
\begin{aligned}
& \int_{I_i}\left(\frac{\varepsilon_h^{p-1}}{p}\varphi^p\left(v_h'\right)+\frac{1}{\varepsilon_h p'}W\left(v_h\right)\right)dx \\
\geq\ & \varphi\left(1\right)\left|\int_{\alpha_i}^{t_h^i} v_h'\left(W\left(v_h\right)\right)^{\frac{1}{p'}}dt\right|+\varphi\left(1\right)\left|\int_{t_h^i}^{\beta_i} v_h'\left(W\left(v_h\right)\right)^{\frac{1}{p'}}dt\right| \\
=\ & \varphi(1)\left|\Phi\left(v_h\left(t_h^i\right)\right)-\Phi\left(v_h\left(\alpha_i\right)\right)\right|+\varphi(1)\left|\Phi\left(v_h\left(\beta_i\right)\right)-\Phi\left(v_h\left(t_h^i\right)\right)\right|,
\end{aligned}
$$

from which we deduce

$$\liminf_{h\to+\infty}\int_{I_i}\left(\frac{\varepsilon_h^{p-1}}{p}\varphi^p\left(v_h'\right)+\frac{1}{\varepsilon_h p'}W\left(v_h\right)\right)dx\geq\varphi(1).$$

Adding the last inequality on i, and using the arbitrariness of r we get inequality (3.7). $\square$

We are now ready to prove (LB) inequality.

Lemma 3.3. *Let* $(u_h,v_h)\in\mathcal{B}\left(\Omega,\mathbb{R}^N\right)\times\mathcal{B}\left(\Omega\right)$ *be such that* $(u_h,v_h)\to(u,v)$ *in measure on* Ω*, then*

$$\liminf_{h\to+\infty}F_{\varepsilon_h}\left(u_h,v_h,\Omega\right)\geq F\left(u,v,\Omega\right). \tag{3.8}$$

Proof. Without loss of generality we may suppose

$$\liminf_{h\to+\infty}F_{\varepsilon_h}\left(u_h,v_h,\Omega\right)<+\infty. \tag{3.9}$$

Notice that condition (3.9) implies the convergence of v_h to 1 in measure on Ω, hence $v=1$ $\mathcal{L}^n$ a.e. in Ω.

We further divide the proof of the lower bound inequality (3.8) into two steps corresponding to the estimate on the bulk term and on the surface term, respectively.

STEP 1:*(Bulk energy inequality)* We prove the following inequality

$$\liminf_{h\to+\infty}\int_\Omega\psi\left(v_h\right)f\left(x,u_h,\nabla u_h\right)dx\geq\int_\Omega f\left(x,u,\nabla u\right)dx. \tag{3.10}$$

First suppose to extract a subsequence, not relabelled for convenience, such that $(u_h,v_h)\to(u,1)$ $\mathcal{L}^n$ a.e. in Ω and

$$\liminf_{h\to+\infty}\int_\Omega\psi\left(v_h\right)f\left(x,u_h,\nabla u_h\right)dx=\lim_{h\to+\infty}\int_\Omega\psi\left(v_h\right)f\left(x,u_h,\nabla u_h\right)dx.$$

Consider the auxiliary function Φ introduced in (3.4), we claim that $\left(\Phi\left(v_h\right)\right)$ is bounded in $BV(\Omega)$. Indeed, (2.1), Young's inequality and (3.9) yield

$$
\begin{aligned}
& \sup_h\left|D\Phi\left(v_h\right)\right|\left(\Omega\right)=\sup_h\int_\Omega\left|\nabla\Phi\left(v_h\right)\right|dx \\
\leq\ & \frac{1}{m}\sup_h\int_\Omega\left(\frac{\varepsilon_h^{p-1}}{p}\varphi^p\left(v_h'\right)+\frac{1}{\varepsilon_h p'}W\left(v_h\right)\right)dx<+\infty,
\end{aligned} \tag{3.11}
$$

where m is the constant defined in (2.1).

To prove that $u \in GSBV\left(\Omega, \mathbb{R}^N\right)$, let $0 < \gamma < \gamma' < \Phi(1)$ and set $U_{h,t} = \{x \in \Omega : \Phi(v_h(x)) > t\}$, then by the Fleming-Rishel Coarea Formula (see[12],[27],[28]) $U_{h,t}$ has finite perimeter for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$. Set $p_h(t) = Per(U_{h,t})$, by the Mean Value Theorem there exists $t_h \in (\gamma, \gamma')$ such that

$$(\gamma' - \gamma)\, p_h(t_h) \leq \int_{\gamma}^{\gamma'} p_h(t)\, dt \leq \int_0^{\Phi(1)} p_h(t)\, dt = |D\Phi(v_h)|(\Omega). \tag{3.12}$$

Let $U_h = U_{h,t_h}$, and $g \in C^1\left(\mathbb{R}^N\right)$ such that ∇g has compact support. Define the functions $g_h = g(u_h)\,\mathcal{X}_{U_h}$, then $g_h \in SBV(\Omega)$ with $\mathcal{H}^{n-1}(S_{g_h}) \leq p_h(t_h)$ and $\nabla g_h = \nabla(g(u_h))\,\mathcal{X}_{U_h}$ (see[34], see also Chapter 3 of[12]).
Thus by (3.5), (3.9), (3.11), (3.12), $inf_h t_h \geq \gamma$, and since (g_h) is equi-bounded in $L^\infty(\Omega)$, (g_h) satisfies all the assumptions of the GSBV Compactness Theorem 2.7, so that we can extract a subsequence, not relabelled for convenience, converging $\mathcal{L}^n$ a.e. in Ω to $w \in SBV(\Omega)$.
Moreover, since $g(u_h) \to g(u)$ $\mathcal{L}^n$ a.e. in Ω, the whole sequence (g_h) converges to $w = g(u)$ $\mathcal{L}^n$ a.e. in Ω and then $g(u) \in SBV(\Omega)$ so that $u \in GSBV(\Omega, \mathbb{R}^N)$.

To prove (3.10) define $w_h = u_h \mathcal{X}_{U_h}$, thus $w_h \in GSBV\left(\Omega, \mathbb{R}^N\right)$ and by (3.5), (3.9), (3.11) and (3.12) the sequence (w_h) satisfies all the assumptions of the GSBV Closure Theorem 2.6 with $w_h \to u$ $\mathcal{L}^n$ a.e. in Ω. Then $\nabla w_h \to \nabla u$ weakly in $L^1\left(\Omega, \mathbb{R}^{N\times n}\right)$, and so (w_h) satisifies all the assumptions of the GSBV Lower Semicontinuity Theorem 2.15, thus we deduce

$$\begin{aligned}
& \liminf_{h\to+\infty} \int_\Omega \psi(v_h)\, f(x, u_h, \nabla u_h)\, dx \\
\geq\ & \liminf_{h\to+\infty} \psi\left(\Phi^{-1}(\gamma)\right) \int_{U_h} f(x, u_h, \nabla u_h)\, dx \\
=\ & \liminf_{h\to+\infty} \psi\left(\Phi^{-1}(\gamma)\right) \int_{U_h} f(x, w_h, \nabla w_h)\, dx \\
=\ & \liminf_{h\to+\infty} \psi\left(\Phi^{-1}(\gamma)\right) \int_{\Omega} f(x, w_h, \nabla w_h)\, dx \\
\geq\ & \psi\left(\Phi^{-1}(\gamma)\right) \int_\Omega f(x, u, \nabla u)\, dx,
\end{aligned}$$

where the last equality follows from (3.5).
The lower semicontinuity of ψ yields inequality (3.10), since, letting $\gamma \to \Phi(1)$ we have $\Phi^{-1}(\gamma) \to 1$.

<u>STEP 2:</u>*(Surface energy inequality)* We prove the following inequality

$$\liminf_{h\to+\infty} \int_\Omega \left(\frac{\varepsilon_h^{p-1}}{p} \varphi^p(\nabla v_h) + \frac{1}{\varepsilon_h p'} W(v_h) \right) dx \geq \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1}. \tag{3.13}$$

Without loss of generality we may assume that the inferior limit in (3.13) is a limit. Fix $\xi \in \mathbb{S}^{n-1}$, by using the notations of Theorem 2.8 we have that $(u_h)_{\xi,y} \to u_{\xi,y}$ and $(v_h)_{\xi,y} \to v_{\xi,y}$ in measure on $\Omega_{\xi,y}$ for $\mathcal{H}^{n-1}$ a.e. $y \in \Pi^\xi$.

Consider the dual norm $\varphi_\circ$ of φ defined as

$$\varphi_\circ(\nu) = \sup_{\xi \in \mathbb{S}^{n-1}} \left(\frac{1}{\varphi(\xi)} \langle \nu, \xi \rangle \right), \tag{3.14}$$

then $(\varphi_\circ)_\circ \equiv \varphi$.
Notice that by conditions (3.5) and (3.9) it follows

$$\liminf_{h\to+\infty}\int_\Omega\left(\left(\psi\left(v_h\right)+\eta_{\varepsilon_h}\right)\left|\nabla u_h\right|^p+\frac{{\varepsilon_h}^{p-1}}{p}\varphi^p\left(\nabla v_h\right)+\frac{1}{\varepsilon_h p'}W\left(v_h\right)\right)dx<+\infty, \tag{3.15}$$

thus (3.14), (3.15) and Fatou's lemma yield

$$\begin{aligned}\liminf_{h\to+\infty}\int_{\Omega_{\xi,y}}&\left(\left(\psi\left(\left(v_h\right)_{\xi,y}\right)+\eta_{\varepsilon_h}\right)\left|\left(u_h\right)'_{\xi,y}\right|^p\right.\\&\left.+\frac{\varepsilon_h^{p-1}}{p}\frac{1}{\varphi_\circ^p\left(\xi\right)}\left|\left(v_h\right)'_{\xi,y}\right|^p+\frac{1}{\varepsilon_h p'}W\left(\left(v_h\right)_{\xi,y}\right)\right)dt<+\infty,\end{aligned} \tag{3.16}$$

for $\mathcal{H}^{n-1}$ a.e. $y\in\Pi^\xi$.

Now we introduce local functionals depending only on the v variable. Indeed, let $G_\varepsilon : W^{1,p}\left(\Omega\right)\times\mathcal{A}\left(\Omega\right)\to[0,+\infty]$ be defined by

$$G_\varepsilon\left(v,A\right)=\int_A\left(\frac{\varepsilon^{p-1}}{p}\varphi^p\left(\nabla v\right)+\frac{1}{\varepsilon p'}W\left(v\right)\right)dx,$$

then (3.14) yields

$$\begin{aligned}G_\varepsilon\left(v,A\right)&\geq\int_A\left(\frac{\varepsilon^{p-1}}{p}\frac{1}{\varphi_\circ^p\left(\xi\right)}\left|\langle\nabla v,\xi\rangle\right|^p+\frac{1}{\varepsilon p'}W\left(v\right)\right)dx\\&=\int_{\Pi^\xi}d\mathcal{H}^{n-1}\left(y\right)\int_{A_{\xi,y}}\left(\frac{\varepsilon^{p-1}}{p}\frac{1}{\varphi_\circ^p\left(\xi\right)}\left|\left(v\right)'_{\xi,y}\right|^p+\frac{1}{\varepsilon p'}W\left(v_{\xi,y}\right)\right)dt.\end{aligned} \tag{3.17}$$

Define $\mu:\mathcal{A}\left(\Omega\right)\to[0,+\infty)$ by

$$\mu\left(A\right)=\liminf_{h\to+\infty}G_{\varepsilon_h}\left(v_h,A\right),$$

then by Fatou's lemma and by Lemma 3.2, which we can apply thanks to (3.16) and the convergence in measure of the one dimensional sections, (3.17) yields

$$\begin{aligned}&\mu\left(A\right)\\\geq\;&\int_{\Pi^\xi}d\mathcal{H}^{n-1}\left(y\right)\liminf_{h\to+\infty}\int_{A_{\xi,y}}\left(\frac{\varepsilon_h^{p-1}}{p}\frac{1}{\varphi_\circ^p\left(\xi\right)}\left|\left(v_h\right)'_{\xi,y}\right|^p+\frac{1}{\varepsilon_h p'}W\left(\left(v_h\right)_{\xi,y}\right)\right)dt\\\geq\;&\frac{1}{\varphi_\circ\left(\xi\right)}\int_{\Pi^\xi}\mathcal{H}^0\left(S_{u_{\xi,y}}\cap A\right)d\mathcal{H}^{n-1}\left(y\right)\\=\;&\frac{1}{\varphi_\circ\left(\xi\right)}\int_{S_u\cap A}\left|\langle\nu_u(y),\xi\rangle\right|d\mathcal{H}^{n-1}\left(y\right),\end{aligned} \tag{3.18}$$

where the last equality holds by (2.8).
Moreover, since μ is a superadditive set function on disjoint open sets contained in Ω, by Lemma 2.12 and the very definition of the dual norm we get

$$\mu\left(\Omega\right)\geq\int_{S_u}\varphi\left(\nu_u\right)d\mathcal{H}^{n-1}, \tag{3.19}$$

passing to the sup in (3.18) on a sequence $(\xi_h)_{h\in\mathbb{N}}$ dense in $\mathbb{S}^{n-1}$.
Notice that (3.19) is exactly inequality (3.13).

Eventually, Step 1, Step 2 and $\eta_{\varepsilon_h} > 0$ yield (LB). $\square$

3.2. *Upper bound inequality*

To prove the upper bound inequality (UB), we have to construct a recovery sequence for any function u in $GSBV\left(\Omega, \mathbb{R}^N\right)$.

First notice that using an approximation procedure we can reduce ourselves to consider the case in which the limit u belongs to $\mathcal{W}\left(\Omega, \mathbb{R}^N\right)$. Indeed, without loss of generality we may assume $v = 1$ $\mathcal{L}^n$ a.e. in Ω and $\mathcal{H}^{n-1}(S_u) < +\infty$, the cases $v \not\equiv 1$ and $\mathcal{H}^{n-1}(S_u) = +\infty$ being trivial, and suppose inequality (UB) proven for functions in $\mathcal{W}\left(\Omega, \mathbb{R}^N\right)$.

Let u belong to $SBV \cap L^\infty\left(\Omega, \mathbb{R}^N\right)$, take $(u_h) \subset \mathcal{W}\left(\Omega, \mathbb{R}^N\right)$ to be the sequence provided by Theorem 2.10, then (2.9), Remark 2.11 and Theorem 2.16 yield

$$\lim_{h\to+\infty} \int_{S_{u_h}} \varphi(\nu_{u_h})\, d\mathcal{H}^{n-1} = \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1},$$

moreover, Theorem 2.15 and Fatou's lemma yield

$$\lim_{h\to+\infty} \int_{\Omega} f(x, u_h, \nabla u_h)\, dx = \int_{\Omega} f(x, u, \nabla u)\, dx.$$

By a simple diagonal argument (UB) inequality then follows for any u in $SBV \cap L^\infty\left(\Omega, \mathbb{R}^N\right)$.

Eventually, if u belongs to $GSBV\left(\Omega, \mathbb{R}^N\right)$, fix $k \in \mathbb{N}$ and consider the auxiliary functions Ψ_k defined in (2.2), notice that $u^k = \Psi_k(u)$ belongs to $SBV \cap L^\infty\left(\Omega, \mathbb{R}^N\right)$. Lebesgue's Dominated Convergence Theorem yields

$$\lim_{k\to+\infty} \int_{S_{u^k}} \varphi(\nu_{u^k})\, d\mathcal{H}^{n-1} = \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1},$$

moreover, Theorem 2.15 and Fatou's lemma yield

$$\lim_{k\to+\infty} \int_{\Omega} f\left(x, u^k, \nabla u^k\right) dx = \int_{\Omega} f(x, u, \nabla u)\, dx,$$

then we may use again a standard diagonal argument to conclude.

Thus, we have reduced ourselves to prove the following lemma.

Lemma 3.4. *Let $u \in \mathcal{W}\left(\Omega, \mathbb{R}^N\right)$, there exists a sequence $(u_h, v_h) \to (u, 1)$ in measure on Ω such that*

$$\limsup_{h\to+\infty} F_h(u_h, v_h, \Omega) \leq F(u, 1, \Omega).$$

Proof. Assumption $u \in \mathcal{W}\left(\Omega, \mathbb{R}^N\right)$ implies that we can find a finite number of polyhedral sets K^i such that

(i) $\overline{S_u} = \Omega \cap \bigcup_{i=1}^r K^i$;

(ii) for every $1 \le i \le r$ the set K^i is contained in a $(n-1)$ dimensional hyperplane π_i and $\pi_i \neq \pi_j$ for $i \neq j$.

Let a_ε^i, b_ε, d_ε be positive infinitesimals for $\varepsilon \to 0^+$, fix $1 \le i \le r$ and denote with ν_i a normal to π_i. Let γ_ε^i be a minimizer of the one dimensional problem

$$\int_{b_\varepsilon}^{a_\varepsilon^i + b_\varepsilon} \left(\frac{\varepsilon^{p-1}}{p} \varphi^p(\nu_i) |v'|^p + \frac{1}{\varepsilon p'} W(v) \right) dt, \tag{3.20}$$

with the conditions $v(b_\varepsilon) = 0$, $v\left(a_\varepsilon^i + b_\varepsilon\right) = 1 - d_\varepsilon$, $v \in W^{1,1}\left(b_\varepsilon, a_\varepsilon^i + b_\varepsilon\right)$.
According to[32], the minimum value in (3.20) is exactly $\varphi(\nu_i)\Phi(1-d_\varepsilon)$, where Φ is the auxiliary function defined in (3.4), and it is achieved by functions for which Young's inequality holds with an equality sign, i.e., γ_ε^i is the unique solution of the Cauchy's problem

$$\begin{cases} \left(\gamma_\varepsilon^i\right)' = \frac{1}{\varphi(\nu_i)\varepsilon} \left(W\left(\gamma_\varepsilon^i\right)\right)^{\frac{1}{p}} \\ \gamma_\varepsilon^i(b_\varepsilon) = 0. \end{cases}$$

Thus $0 \le \gamma_\varepsilon^i \le 1 - d_\varepsilon$, an explicit computation yields $a_\varepsilon^i = -\varepsilon\varphi(\nu_i) \ln d_\varepsilon$, so d_ε is chosen such that $\varepsilon \ln d_\varepsilon$ is infinitesimal for $\varepsilon \to 0^+$.
Define the functions $\alpha_\varepsilon^i : [0,+\infty) \to [0, 1-d_\varepsilon]$ by

$$\alpha_\varepsilon^i(t) = \begin{cases} 0 & 0 \le t \le b_\varepsilon \\ \gamma_\varepsilon^i(t) & b_\varepsilon \le t \le a_\varepsilon^i + b_\varepsilon \\ 1 - d_\varepsilon & t \ge a_\varepsilon^i + b_\varepsilon. \end{cases} \tag{3.21}$$

Denote by $\Pi_i : \mathbb{R}^n \to \pi_i$ the orthogonal projection on π_i and set $d_i(x) = d(x, \pi_i)$; it is well known that if $x \in \mathbb{R}^n \setminus \pi_i$ there holds

$$\nabla d_i(x) = \frac{x - \Pi_i(x)}{|x - \Pi_i(x)|} = \pm\nu_i. \tag{3.22}$$

For any $\delta > 0$ set

$$K_\delta^i = \left\{ y \in \pi_i : d\left(y, K^i\right) \le \delta \right\},$$

fix $\varepsilon > 0$, let β_ε^i be a cut-off function between K_ε^i and $K_{2\varepsilon}^i$, then define

$$v_\varepsilon^i(x) = \beta_\varepsilon^i(\Pi_i(x))\, \alpha_\varepsilon^i(d_i(x)) + \left(1 - \beta_\varepsilon^i(\Pi_i(x))\right)(1 - d_\varepsilon). \tag{3.23}$$

Let

$$\begin{aligned} B_\varepsilon^i &= \left\{ x \in \mathbb{R}^n : \Pi_i(x) \in K_\varepsilon^i \text{ and } d_i(x) \le b_\varepsilon \right\}; \\ C_\varepsilon^i &= \left\{ x \in \mathbb{R}^n : \Pi_i(x) \in K_{2\varepsilon}^i \text{ and } d_i(x) \le a_\varepsilon^i + b_\varepsilon \right\}. \end{aligned}$$

By the very definition of v_ε^i there holds (see Figure 1 below)

$$v_\varepsilon^i = \begin{cases} 1 - d_\varepsilon & \Omega \setminus C_\varepsilon^i \\ 0 & B_\varepsilon^i, \end{cases} \tag{3.24}$$

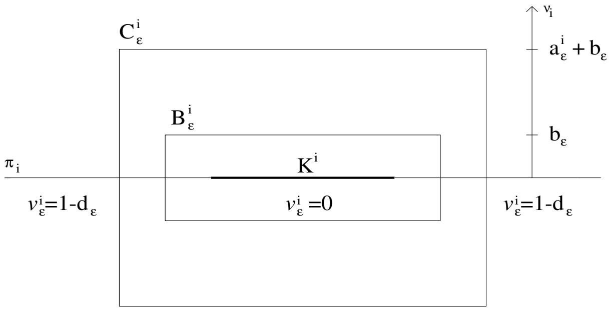

Figure 1: construction of v_ε^i

and since $\Pi_i \in Lip_1(\mathbb{R}^n, \mathbb{R}^n)$, $\left\|\nabla \beta_\varepsilon^i\right\|_\infty \leq c\varepsilon^{-1}$ and $|\nabla d_i| = 1$ $\mathcal{L}^n$ a.e., there exists a positive constant c such that

$$\left\|\nabla v_\varepsilon^i\right\|_\infty \leq \frac{c}{\varepsilon}. \tag{3.25}$$

Thus, $0 \leq v_\varepsilon^i \leq 1$, $v_\varepsilon^i \in W^{1,\infty}(\Omega)$ and $v_\varepsilon^i \to 1$ $\mathcal{L}^n$ a.e. in Ω.

Define $H_\varepsilon^i = C_\varepsilon^i \setminus B_\varepsilon^i$, let us estimate the integral

$$I_\varepsilon^i = \int_{H_\varepsilon^i} \left(\frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nabla v_\varepsilon^i\right) + \frac{1}{\varepsilon p'} W\left(v_\varepsilon^i\right) \right) dx. \tag{3.26}$$

To do this, consider the sets

$$\begin{aligned} H_\varepsilon^{i,1} &= \left\{x \in \mathbb{R}^n : \Pi_i(x) \in K_{2\varepsilon}^i \setminus K_\varepsilon^i \text{ and } d_i(x) \leq a_\varepsilon^i + b_\varepsilon\right\}, \\ H_\varepsilon^{i,2} &= \left\{x \in \mathbb{R}^n : \Pi_i(x) \in K_\varepsilon^i \text{ and } b_\varepsilon \leq d_i(x) \leq a_\varepsilon^i + b_\varepsilon\right\}, \end{aligned}$$

then $H_\varepsilon^i = H_\varepsilon^{i,1} \cup H_\varepsilon^{i,2}$, and setting

$$I_\varepsilon^{i,j} = \int_{H_\varepsilon^{i,j}} \left(\frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nabla v_\varepsilon^i\right) + \frac{1}{\varepsilon p'} W\left(v_\varepsilon^i\right) \right) dx, \tag{3.27}$$

it follows $I_\varepsilon^i = I_\varepsilon^{i,1} + I_\varepsilon^{i,2}$. We estimate the $I_\varepsilon^{i,j}$ separately.
By (3.25), and since $\mathcal{H}^{n-1}\left(K_{2\varepsilon}^i \setminus K_\varepsilon^i\right) = O(\varepsilon)$ for $\varepsilon \to 0^+$, we get

$$I_\varepsilon^{i,1} \leq \frac{c}{\varepsilon} \mathcal{L}^n\left(H_\varepsilon^{i,1}\right) = 2c \frac{a_\varepsilon^i + b_\varepsilon}{\varepsilon} \mathcal{H}^{n-1}\left(K_{2\varepsilon}^i \setminus K_\varepsilon^i\right) = o(1). \tag{3.28}$$

Moreover, by the definition of v_ε^i on $H_\varepsilon^{i,2}$ there holds

$$\nabla v_\varepsilon^i(x) = \left(\gamma_\varepsilon^i\right)'(d_i(x)) \nabla d_i(x),$$

thus, by (3.20) and (3.22) we get

$$\begin{aligned} I_\varepsilon^{i,2} &= 2 \int_{K_\varepsilon^i} d\mathcal{H}^{n-1} \int_{b_\varepsilon}^{a_\varepsilon^i + b_\varepsilon} \left(\frac{\varepsilon^{p-1}}{p} \varphi^p(\nu_i) \left|\left(\gamma_\varepsilon^i\right)'(t)\right|^p + \frac{1}{\varepsilon p'} W\left(\gamma_\varepsilon^i(t)\right) \right) dt \\ &= 2\Phi(1 - d_\varepsilon) \varphi(\nu_i) \mathcal{H}^{n-1}\left(K_\varepsilon^i\right) \leq \int_{K^i} \varphi(\nu_i) \, d\mathcal{H}^{n-1} + o(1). \end{aligned} \tag{3.29}$$

Eventually, by adding (3.28) and (3.29) we get

$$I_\varepsilon^i \leq \int_{K^i} \varphi(\nu_i)\, d\mathcal{H}^{n-1} + o(1). \tag{3.30}$$

Now we define the recovery sequence for the v variable "gluing up" together the v_ε^i as to minimize the surface energy. This will be done defining a function which, on every C_ε^i, coincides with v_ε^i up to a region of very small area.

More precisely, let

$$V_\varepsilon = \min_{1\leq i\leq r} v_\varepsilon^i, \tag{3.31}$$

then $0 \leq V_\varepsilon \leq 1$, $V_\varepsilon \in W^{1,\infty}(\Omega)$ and $V_\varepsilon \to 1$ $\mathcal{L}^n$ a.e. in Ω.
Setting $B_\varepsilon = \bigcup_{i=1}^r B_\varepsilon^i$ and $C_\varepsilon = \bigcup_{i=1}^r C_\varepsilon^i$ there holds

$$V_\varepsilon = \begin{cases} 1 - d_\varepsilon & \mathbb{R}^n \setminus C_\varepsilon \\ 0 & B_\varepsilon, \end{cases} \tag{3.32}$$

and also

$$\nabla V_\varepsilon = \nabla v_\varepsilon^i \quad \mathcal{L}^n \text{ a.e. in } \mathcal{V}_{i,\varepsilon} = \bigcap_{j\neq i} \left\{ v_\varepsilon^i \leq v_\varepsilon^j \right\}, \tag{3.33}$$

so that by (3.25) it follows

$$\|\nabla V_\varepsilon\|_\infty \leq \frac{c}{\varepsilon}. \tag{3.34}$$

Since $\Omega = (\Omega \setminus C_\varepsilon) \cup (\Omega \cap C_\varepsilon \setminus B_\varepsilon) \cup (\Omega \cap B_\varepsilon)$, (3.32) yields

$$\begin{aligned} &\int_\Omega \left(\frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla V_\varepsilon) + \frac{1}{\varepsilon p'} W(V_\varepsilon) \right) dx \\ \leq\ & c\frac{d_\varepsilon^p}{\varepsilon} \mathcal{L}^n(\Omega \setminus C_\varepsilon) + \int_{\Omega\cap(C_\varepsilon\setminus B_\varepsilon)} (\dots)\, dx + c\frac{b_\varepsilon}{\varepsilon} \\ =\ & \int_{\Omega\cap(C_\varepsilon\setminus B_\varepsilon)} (\dots)\, dx + o(1) = R_\varepsilon + o(1), \end{aligned} \tag{3.35}$$

choosing d_ε such that $d_\varepsilon^p = o(\varepsilon)$ as well as $\varepsilon \ln d_\varepsilon = o(1)$, and also $b_\varepsilon = o(\varepsilon)$.

To estimate R_ε, notice that $C_\varepsilon \setminus B_\varepsilon = \cup_{i=1}^r \left(H_\varepsilon^i \setminus B_\varepsilon\right)$, thus

$$R_\varepsilon \leq \sum_{i=1}^r \int_{\Omega\cap(H_\varepsilon^i\setminus B_\varepsilon)} \left(\frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla V_\varepsilon) + \frac{1}{\varepsilon p'} W(V_\varepsilon) \right) dx,$$

and consider the inclusion $H_\varepsilon^i \setminus B_\varepsilon \subseteq \cup_{j\neq i} \left(H_\varepsilon^i \cap H_\varepsilon^j\right) \cup \cap_{j\neq i} \left(H_\varepsilon^i \setminus C_\varepsilon^j\right)$.
Since $\cap_{j\neq i}\left(H_\varepsilon^i \setminus C_\varepsilon^j\right) \subseteq \mathcal{V}_{i,\varepsilon}$, arguing like in (3.30), by (3.33) we have

$$\begin{aligned} &\int_{\cap_{j\neq i}\left(H_\varepsilon^i\setminus C_\varepsilon^j\right)\cap\Omega} \left(\frac{\varepsilon^{p-1}}{p} \varphi^p(\nabla V_\varepsilon) + \frac{1}{\varepsilon p'} W(V_\varepsilon) \right) dx \\ =\ & \int_{\cap_{j\neq i}\left(H_\varepsilon^i\setminus C_\varepsilon^j\right)\cap\Omega} \left(\frac{\varepsilon^{p-1}}{p} \varphi^p\left(\nabla v_\varepsilon^i\right) + \frac{1}{\varepsilon p'} W\left(v_\varepsilon^i\right) \right) dx \\ \leq\ & \int_{\Omega\cap K^i} \varphi(\nu_i)\, d\mathcal{H}^{n-1} + o(1). \end{aligned} \tag{3.36}$$

Moreover, by (2.1) and (3.34) we have

$$\sum_{j\neq i}\int_{H_\varepsilon^i\cap H_\varepsilon^j}\left(\frac{\varepsilon^{p-1}}{p}\varphi^p\left(\nabla V_\varepsilon\right)+\frac{1}{\varepsilon p'}W\left(V_\varepsilon\right)\right)dx\leq\frac{c}{\varepsilon}\sum_{j\neq i}\mathcal{L}^n\left(H_\varepsilon^i\cap H_\varepsilon^j\right), \tag{3.37}$$

we claim that for every $i,j\in\{1,...,r\}$ it holds

$$\mathcal{L}^n\left(H_\varepsilon^i\cap H_\varepsilon^j\right)=o(\varepsilon). \tag{3.38}$$

Indeed, we may assume $K^i\cap K^j\neq\emptyset$, since otherwise for ε sufficiently small it follows $H_\varepsilon^i\cap H_\varepsilon^j=\emptyset$ and then $\mathcal{L}^n\left(H_\varepsilon^i\cap H_\varepsilon^j\right)=0$. Notice that

$$H_\varepsilon^i\cap H_\varepsilon^j\subseteq\left\{x\in\mathbb{R}^n:\ d_i(x)\leq a_\varepsilon^i+b_\varepsilon\right\}\cap\left\{x\in\mathbb{R}^n:\ d_j(x)\leq a_\varepsilon^j+b_\varepsilon\right\}, \tag{3.39}$$

and since condition $\pi_i\neq\pi_j$ implies that $K^i\cap K^j$ is contained in an $(n-2)$ dimensional affine subspace of $\mathbb{R}^n$, from (3.39) we deduce

$$\mathcal{L}^n\left(H_\varepsilon^i\cap H_\varepsilon^j\right)\leq c(a_\varepsilon^i+b_\varepsilon)(a_\varepsilon^j+b_\varepsilon)=c_1\varepsilon^2\ln^2 d_\varepsilon+o(\varepsilon),$$

where c, c_1 depend on $\mathcal{H}^{n-2}\left(K^i\cap K^j\right)$ and on the angle between π_i and π_j.
Thus, assertion (3.38) is proved if d_ε is such that $\varepsilon^2\ln^2 d_\varepsilon=o(\varepsilon)$; the choice $d_\varepsilon=\exp\left(-\varepsilon^{-\frac{1}{4}}\right)$ fulfills all the conditions required, i.e., $d_\varepsilon^p=o(\varepsilon)$ and $\varepsilon^2\ln^2 d_\varepsilon=o(\varepsilon)$.

Eventually, (3.35), (3.36), (3.37) and (3.38) yield

$$\begin{aligned}
&\int_\Omega\left(\frac{\varepsilon^{p-1}}{p}\varphi^p\left(\nabla V_\varepsilon\right)+\frac{1}{\varepsilon p'}W\left(V_\varepsilon\right)\right)dx\\
\leq\ &\sum_{i=1}^r\int_{\Omega\cap K^i}\varphi\left(\nu_i\right)d\mathcal{H}^{n-1}+o(1)\\
=\ &\int_{S_u}\varphi\left(\nu_u\right)d\mathcal{H}^{n-1}+o(1),
\end{aligned} \tag{3.40}$$

the last equality holding thanks to the first condition in Definition 2.9.

To prove (UB) set

$$D_\varepsilon=\bigcup_{i=1}^r\left\{x\in\mathbb{R}^n:\Pi_i(x)\in K^i_{\frac{\varepsilon}{2}}\text{ and }d_i(x)\leq\frac{b_\varepsilon}{2}\right\}, \tag{3.41}$$

and let φ_ε be a cut-off function between D_ε and B_ε. Define

$$U_\varepsilon=(1-\varphi_\varepsilon)\,u, \tag{3.42}$$

$u\in\mathcal{W}\left(\Omega,\mathbb{R}^N\right)$ implies that $U_\varepsilon\in W^{1,\infty}\left(\Omega,\mathbb{R}^N\right)$, moreover $U_\varepsilon\to u$ $\mathcal{L}^n$ a.e. in Ω.
Eventually, (3.40) and (3.42) yield

$$\begin{aligned}
&F_\varepsilon\left(U_\varepsilon,V_\varepsilon,\Omega\right)\\
=\ &\int_{\Omega\setminus B_\varepsilon}\left(\psi\left(V_\varepsilon\right)+\eta_\varepsilon\right)f\left(x,u,\nabla u\right)dx+\eta_\varepsilon\int_{B_\varepsilon}f\left(x,U_\varepsilon,\nabla U_\varepsilon\right)dx\\
&+\int_\Omega\left(\frac{\varepsilon^{p-1}}{p}\varphi^p\left(\nabla V_\varepsilon\right)+\frac{1}{\varepsilon p'}W\left(V_\varepsilon\right)\right)dx\\
\leq\ &\int_\Omega f\left(x,u,\nabla u\right)dx+\int_{S_u}\varphi\left(\nu_u\right)d\mathcal{H}^{n-1}+c\eta_\varepsilon b_\varepsilon^{-p+1}+o(1),
\end{aligned}$$

inequality (UB) follows choosing $b_\varepsilon = (\eta_\varepsilon \varepsilon)^{\frac{1}{p}}$. $\square$

Remark 3.5. *The function associating to* u *in* $\mathcal{B}(\Omega, \mathbb{R}^N)$ *the value* $\int_\Omega |u|^q \, dx$, $q \in [1, +\infty)$, *is only lower semicontinuous with respect to convergence in measure, thus we cannot deduce directly from Theorem 3.1 and statement (ii) of Lemma 2.2 the* Γ*-convergence of* $F_\varepsilon(\cdot, \cdot, \Omega) + \int_\Omega |\cdot|^q \, dx$ *to* $F(\cdot, \cdot, \Omega) + \int_\Omega |\cdot|^q \, dx$.

Nevertheless, the result still holds since all the arguments and the constructions we used to prove the (LB) and (UB) inequalities in Theorem 3.1 can be directly applied to such family of approximating functionals.

4. Convergence of Minimizers

Let us state an equicoercivity result for the approximating functionals defined in Remark 3.5. For the sake of simplicity we assume $\psi(t) = t^p$ in Definition 3.3, even though the result holds true for a larger class of functions ψ.

Lemma 4.1. *Let* $(u_h, v_h) \in \mathcal{B}(\Omega, \mathbb{R}^N) \times \mathcal{B}(\Omega)$ *be such that*

$$\liminf_{h \to +\infty} \left(F_{\varepsilon_h}(u_h, v_h, \Omega) + \int_\Omega |u_h|^q \, dx \right) < +\infty, \tag{4.1}$$

with $q \in [1, +\infty)$.

Then there exist a subsequence (u_{h_k}, v_{h_k}) *and a function* $u \in GSBV(\Omega, \mathbb{R}^N)$ *such that* $(u_{h_k}, v_{h_k}) \to (u, 1)$ *in measure on* Ω.

Proof. Condition (4.1) implies that, up to a subsequence not relabelled for convenience, $v_h \to 1$ and hence $\Phi(v_h) \to \Phi(1) = \frac{1}{2}$ $\mathcal{L}^n$ a.e. in Ω.

Fix $k \in \mathbb{N}$, consider the sequence $\left(\Phi(v_h) u_h^k\right) \subset W^{1,1}(\Omega, \mathbb{R}^N)$, where $u_h^k = \Psi_k(u_h)$ with Ψ_k the auxiliary functions defined by (2.2). Arguing as in the proof of Lemma 3.3, $(\Phi(v_h))$ is bounded in $BV(\Omega)$, moreover, since $\Phi(t) \le ct$, $\psi(t) = t^p$, Young's inequality yields

$$\int_\Omega \left|\nabla\left(\Phi(v_h) u_h^k\right)\right| dx \le ck\big(1 + F_{\varepsilon_h}(u_h, v_h, \Omega)\big).$$

By (4.1), by applying the BV Compactness Theorem (see[12],[27],[28]) and a diagonal argument we may suppose that, up to a subsequence not relabelled for convenience, for every $k \in \mathbb{N}$ there exists $s^k : \Omega \to \mathbb{R}^N$, with $\|s^k\|_\infty \le 2k$, such that $\Phi(v_h) u_h^k \to s^k$ $\mathcal{L}^n$ a.e. in Ω. Hence, we deduce that for $\mathcal{L}^n$ a.e. x in Ω

$$\lim_{h \to +\infty} u_h^k(x) = 2 s^k(x), \tag{4.2}$$

for every $k \in \mathbb{N}$.
Let us prove that for $\mathcal{L}^n$ a.e. x in Ω there exists $u : \Omega \to \mathbb{R}^N$ such that

$$\lim_{k \to +\infty} 2 s^k(x) = u(x). \tag{4.3}$$

Indeed, let $x \in \Omega$ be such that (4.2) holds, then either $|u_h(x)| \to +\infty$ or there exist $w \in \mathbb{R}^N$ and $(u_{h_j}) \subset (u_h)$ such that $u_{h_j}(x) \to w$. In the first case $s^k(x) = 0$ for every $k \in \mathbb{N}$, and then (4.3) holds with $u(x) = 0$; while in the second case $u_{h_j}^k(x) \to w$ for every $k > |w|$ as $j \to +\infty$ and thus $u(x) = w$ by (4.2).

Let us prove the convergence of (u_h) to u in measure on Ω. Indeed, condition (4.1) yields

$$\mathcal{L}^n\big(\{x \in \Omega : |u_h(x)| > k\}\big) \le ck^{-q},$$

thus for every $\varepsilon > 0$, since the decomposition

$$\begin{aligned}&\{x \in \Omega : |u_h(x) - u(x)| > \varepsilon\} = \{x \in \Omega : |u_h^k(x) - u(x)| > \varepsilon\} \cup \\ &\big(\{x \in \Omega : |u_h(x) - u(x)| > \varepsilon\} \cap \{x \in \Omega : |u_h(x)| > k\}\big),\end{aligned}$$

we have

$$\mathcal{L}^n\big(\{x \in \Omega : |u_h(x) - u(x)| > \varepsilon\}\big) \le \mathcal{L}^n\big(\{x \in \Omega : |u_h^k(x) - u(x)| > \varepsilon\}\big) + ck^{-q},$$

and the claimed convergence follows by (4.2) and (4.3).
Eventually, by (4.1) and by applying the same argument used in Step 1 of Lemma 3.3, we deduce that $u \in GSBV\left(\Omega, \mathbb{R}^N\right)$. $\square$

We are now able to state the following result on the convergence of minimum problems.

Theorem 4.2. *For every $g \in L^q\left(\Omega, \mathbb{R}^N\right)$, $q \in [1, +\infty)$, and every $\gamma > 0$, there exists a minimizing pair $(u_\varepsilon, v_\varepsilon)$ for the problem*

$$m_\varepsilon = \inf\left\{F_\varepsilon(u, v, \Omega) + \gamma \int_\Omega |u - g|^q\, dx : (u, v) \in \mathcal{B}\left(\Omega, \mathbb{R}^N\right) \times \mathcal{B}(\Omega)\right\} \tag{4.4}$$

Moreover, every cluster point of (u_ε) is a solution of the minimum problem

$$m = \inf\left\{\mathcal{F}(u) + \gamma \int_\Omega |u - g|^q\, dx : u \in GSBV\left(\Omega, \mathbb{R}^N\right)\right\} \tag{4.5}$$

and $m_\varepsilon \to m$ as $\varepsilon \to 0^+$.

Proof. The existence of $(u_\varepsilon, v_\varepsilon)$ for every $\varepsilon > 0$ follows by (3.5) and the very definition of F_ε which ensure its coercivity and lower semicontinuity with respect to convergence in measure.
Assumption $g \in L^q\left(\Omega, \mathbb{R}^N\right)$ yields

$$\sup_\varepsilon \left\{F_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega) + \gamma \int_\Omega |u_\varepsilon - g|^q\, dx\right\} < +\infty,$$

thus Lemma 4.1 ensures the existence of a subsequence $(u_{\varepsilon_h}, v_{\varepsilon_h})$ converging in measure on Ω to $(u, 1)$ with $u \in GSBV\left(\Omega, \mathbb{R}^N\right)$.
Eventually, statement (iii) of Lemma 2.2 and Remark 3.5 yield the conclusion. $\square$

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