CURVATURE VARIFOLDS WITH BOUNDARY

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ABSTRACT. We introduce a new class of nonoriented sets in $\mathbb{R}^k$ endowed with a generalized notion of second fundamental form and boundary, proving several compactness and structure properties. Our work extends the definition and some results of J. E. Hutchinson [13] and can be applied to variational problems involving surfaces with boundary.

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1. INTRODUCTION

Some problems in the calculus of variations are concerned with existence of minima for functionals defined on smooth manifolds embedded in $\mathbb{R}^k$ and involving quantities related to the geometry of the manifolds. The functionals we are interested in depend on the curvature tensor of manifolds. As usual, in order to get existence of minimizers by the so called direct methods of calculus of variations it is necessary to enlarge the space where the functional is defined and work out a compactness–semicontinuity theorem in the enlarged domain.

The aim of this paper is to introduce a new class of n–dimensional sets endowed with a weak notion of second fundamental form and boundary. We prove that this class has good compactness and structure properties.

Our work is based on the theory of integer rectifiable varifolds developed by Allard in [1], [2] (see section 2). Roughly speaking, an integer n–varifold is an n–dimensional set in $\mathbb{R}^k$ endowed with an integer multiplicity; smooth n–dimensional manifolds can be considered as unit density varifolds.

Inspired by the classical divergence formula on manifolds and by the first variation of the area functional, Allard gave a weak definition of mean curvature (see

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also [18]) and boundary for varifolds. The Allard's definition is strong enough to guarantee compactness and rectifiability properties. However Allard's varifolds do not share strong local regularity properties, because of multiplicity (see the example in [6], p. 162) and because the mean curvature does not "see" some singularity points, for instance the triple junction with equal angles of three halflines in $\mathbb{R}^2$.

Using a suitable integration by parts formula involving functions of the tangent space, Hutchinson introduced in [13] the so called curvature varifolds with second fundamental form in L^p and proved several compactness, semicontinuity and regularity results (see [12], [14]). The theory of Hutchinson provides a weak formulation of variational problems involving surfaces without boundary and functionals depending on the second fundamental form.

Motivated by variational problems involving piecewise smooth surfaces (see for instance [3]) we extend the theory of Hutchinson in order to include smooth manifolds with boundary.

We give here a brief outline of the paper.

Section 2. This is an introductory section about varifolds and basic facts we will need in the sequel.

Section 3. We give the definition of *curvature varifolds with boundary*, explaining the similarities and the differences with the definitions of Allard and Hutchinson. We also prove that the generalized second fundamental form and the generalized boundary are uniquely determined and have the same formal properties of the smooth case.

Section 4. In this section we prove that the class of curvature varifolds with boundary is stable under localization in the ambient space and in the Grassmannian. This provides a weak, local orientability property of these varifolds which is very useful from the analytic viewpoint.

Section 5. The section is devoted to the study of the tangent space function $P(x)$ of a curvature varifold with boundary, defined $\mathcal{H}^n$–almost everywhere on the support of the varifold. We prove that $P(x)$ is approximately differentiable $\mathcal{H}^n$–almost everywhere and its approximate differential is the (weak) second fundamental form. This property was not known even for Hutchinson's curvature varifolds.

Section 6. We prove in this section a compactness result in the class of varifolds with second fundamental form in L^p. We also give some examples showing the utility of curvature varifolds with boundary in the study of some variational problems involving piecewise smooth surfaces.

Section 7. Using the local orientability property of section 4 and the approximate differentiability of the tangent space function we extend the *Boundary Rectifiability Theorem* of Federer–Fleming to curvature varifolds with boundary. This provides at the end a complete description of the boundary measure.

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2. Notations and Preliminaries

Standard reference for the theory and the notations of this section is [18].

The ambient space containing all the objects we deal with is always an open set Ω in $\mathbb{R}^k$ and we will denote with $B_r(x)$ the open ball centred at x with radius r. $\mathcal{H}^n$ is the n–dimensional Hausdorff measure on $\mathbb{R}^k$.

Given an n–dimensional vector subspace P of $\mathbb{R}^k$, we can consider the $k \times k$–matrix $\{P_{ij}\}$ of the orthogonal projection over the subspace P. So we can think of the Grassmannian $G_{n,k}$ of n–spaces in $\mathbb{R}^k$, endowed with the relative metric, as a compact subset of $\mathbb{R}^{k^2}$; this identification is used throughout the paper. Moreover given a subset A of $\mathbb{R}^k$, we define the product space

$$G_n(A) = A \times G_{n,k}.$$

If $\{\mu_k\}$ and μ are Radon measure on a locally compact and separable space X we write

$$\mu_k \rightharpoonup \mu$$

to denote the weak* convergence as elements of the dual space of $C^0_c(X)$.

Given a Radon measure μ on X and a measurable function $f : X \to Y$ we canonically define the image measure $f_\#\mu$ on Y setting

$$f_\#\mu\,(B) = \mu(f^{-1}(B))$$

for every B Borel subset of Y.

We define a special subclass of Radon measures on the open set $\Omega \subset \mathbb{R}^k$, $\mathcal{R}_n(\Omega)$ to be the set of signed Radon measures μ on Ω with these properties:

- μ is supported in a countably n–rectifiable set N;
- $|\mu|$ is absolutely continuous with respect to the measure $\mathcal{H}^n \llcorner N$.

Now we introduce the terminology and some basic facts about varifolds.

A general *n–varifold* V in an open set $\Omega \subset \mathbb{R}^k$ is simply a Radon measure on $G_n(\Omega)$. The varifold convergence is the weak* convergence of measures on $G_n(\Omega)$.

We can associate to any varifold V a Radon measure μ_V on the open Ω projecting the measure V on the first factor of the product space $G_n(\Omega)$:

$$\mu_V = \pi_\# V$$

where $\pi : G_n(\Omega) \to \Omega$ is the projection. This measure is called the *weight measure* of the varifold V.

Consider now a countably n–rectifiable, $\mathcal{H}^n$–measurable set M in Ω and a non–negative function $\theta : M \to \mathbb{R}$, locally integrable with respect to $\mathcal{H}^n \llcorner M$. We give the following definition:

Definition 2.1. Let us assume that for some $x^0 \in M$ there exists an n–dimensional vector subspace T of $\mathbb{R}^k$ such that

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_M \theta(x)\varphi\left(\frac{x - x^0}{\rho}\right) d\mathcal{H}^n(x) = \theta(x^0) \int_T \varphi(y)\, d\mathcal{H}^n(y) \qquad \forall \varphi \in C^0_c(\mathbb{R}^k).$$

Then we say that T is the *approximate tangent space* to the countably n–rectifiable set M at x^0 with respect to the function θ.

It is a well known fact that for $\mathcal{H}^n$–a.e. $x \in M$ there exists the approximate tangent space apT_xM to M at x with respect to the function θ and that if we choose a different function θ' the tangent spaces are the same $\mathcal{H}^n$–almost everywhere in M.

Then we can define the *rectifiable varifold* $V \equiv V_{M,\theta}$ associated to the pair (M, θ) as above, to be the Radon measure characterized by

$$\int_{G_n(\Omega)} \varphi(x, P)\, dV(x, P) = \int_M \theta(x)\varphi(x, apT_xM)\, d\mathcal{H}^n(x)$$

for every function $\varphi \equiv \varphi(x, P) \in C^0_c(G_n(\Omega))$. We say that apT_xM is the approximate space tangent to the rectifiable varifold V.

Note 2.2. It can be proved that the function apT_xM defined before is $\mathcal{H}^n$–measurable and so are its components when we use the identification subspace–matrix of projection. Hence the formula above defines a measure on $G_n(\Omega)$, or on the larger space $\Omega \times \mathbb{R}^{k^2}$ containing $G_n(\Omega)$. Usually in the sequel we write $P(x) \equiv \{P_{ij}(x)\}$ for the tangent space function apT_xM of M.

With these definitions, the weight measure of a rectifiable varifold $V_{M,\theta}$ is $\mathcal{H}^\setminus \llcorner \theta$ (extending θ to zero outside the set M). Commonly M and θ are called respectively the *support* and the *density function* of the rectifiable varifold V.

If the density function of a rectifiable varifold V is integer valued, we say that V is an *integer varifold.*

In the following we are concerned only with this special class, so when we will write varifold we will always mean integer varifold.

Now we come to the definition of curvature. Usually the curvature tensor of an embedded manifold M is described by its *second fundamental form* which is a symmetric bilinear form defined at every $x \in M$ by (see for instance [5], [8], [13])

$$\mathbf{B} : T_xM \times T_xM \to N_xM$$
$$\mathbf{B}(v, w) = (D_v w)^\perp$$

where N_xM is the normal space to M at x and $D_v w$ denotes covariant differentiation in the Euclidean space $\mathbb{R}^k$. We can naturally extend $\mathbf{B}$ to a symmetric bilinear form on all $\mathbb{R}^k$ with values in $\mathbb{R}^k$ setting

$$\mathbf{B}(v, w) = \mathbf{B}(v^\top, w^\top)$$

where the symbol $^\top$ indicates the projection on the tangent space to M. The components of the form $\mathbf{B}$ are defined by

$$\mathbf{B}^k_{ij} = < \mathbf{B}(e_i, e_j), e_k > .$$

The mean curvature vector $\underline{\mathbf{H}}$ has then components

$$\mathbf{H}_i = \mathbf{B}^i_{jj}$$

summing on the repeated indexes from 1 to k.

We observe this convention on repeated indexes throughout all the paper.

There is another way to express the second fundamental form that is useful in our context. We define for an arbitrary function $\varphi \in C^1(M)$ its *tangential gradient*, denoted by $\nabla^M\varphi$, as the projection on the tangent space of the gradient of the function φ (it is clear that to compute the derivatives we have to extend the function in a neighbourhood of the manifold M, but it is easy to see that the tangential part of the gradient is independent of the extension).

We can consider the tangential gradients of the tangent space functions

$$A_{ijk} = \nabla_i^M P_{jk}. \tag{2.1}$$

The interesting fact is that the functions A_{ijk} are univocally related to the components of the second fundamental form $\mathbf{B}$ (see [13]).

Proposition 2.3. *For every $x \in M$ the following hold:*

- $\mathbf{B}_{ij}^k = P_{jl} A_{ikl}$
- $A_{ijk} = \mathbf{B}_{ij}^k + \mathbf{B}_{ik}^j$
- $\mathbf{H}_i = A_{jij}$.

After this classical introduction we can show the way Allard defined a distributional notion of mean curvature for a varifold $V \equiv V_{M,\theta}$ in an open $\Omega \subset \mathbb{R}^k$.

Consider the linear functional δV, defined on the space of vector fields X in $\mathbb{R}^k$ with compact support in Ω

$$\delta V(X) = \int_M \operatorname{div}^M X(x)\, d\mu_V(x)$$

where $\operatorname{div}^M X$ is the tangential divergence of the vector field X with respect to the countably rectifiable M and is defined by

$$\operatorname{div}^M X(x) = \sum_{i,j} P_{ij}(x) \nabla_j X_i(x)$$

(here $P_{ij}(x)$ are the *approximate* tangent space functions).

If δV is a locally bounded functional it can be represented, by the Riesz Theorem, by a Radon measure that we still denote by δV. Hence, using the Radon–Nikodym Theorem, we split δV in its absolutely continuous and singular part with respect to the measure μ_V, obtaining

$$\int_M \operatorname{div}^M X\, d\mu_V = -\int_M < X, \underline{\mathbf{H}} > \, d\mu_V - \int_\Omega < X, \underline{\nu} > \, d\sigma$$

for a certain Radon measure σ on Ω and functions $\underline{\mathbf{H}} \in L^1_{loc}(\mu_V, \mathbb{R}^k)$, $\underline{\nu} \in L^1_{loc}(\sigma, \mathbf{S}^{k-1})$.

Considering the analogy with the classical case (the tangential divergence formula, see [5]), Allard defined $\underline{\mathbf{H}}$, $\underline{\nu}$, σ respectively to be the *generalized mean curvature*, the *generalized inner normal* and the *generalized boundary.*

The class of varifolds such that this property holds are called *varifolds with locally bounded first variation.*

This class of sets is endowed with a distributional notion of mean curvature and boundary that generalizes the classical case of smooth manifolds. The basic compactness result in this class is the following theorem.

Theorem A (Allard's Compactness Theorem)**.** *Given for every open set $\Omega' \subset\subset \Omega$ a positive constant $c(\Omega')$, the class of integer n–varifolds V in an open set $\Omega \subset \mathbb{R}^k$ such that*

$$\mu_V(\Omega') + \|\delta V\|(\Omega') \leq c(\Omega')$$

is sequentially compact with respect to varifold convergence. Moreover in the same class the mapping $V \mapsto \delta V$ is weakly* continuous.

For a proof, see [18].

Finally we need a theorem of Brakke (see [6], Chapter 5) concerning the orthogonality of the generalized mean curvature vector and a "flattening property" for integer varifolds.

Theorem B (Brakke's Orthogonality Theorem)**.** *If V is an integer varifold with locally bounded first variation the vector $\underline{\mathbf{H}}(x)$ is orthogonal to the tangent space $P(x)$, for μ_V–almost all points $x \in \Omega$. Moreover*

$$\lim_{\rho\to 0} \rho^{-n-1} \int\limits_{B_\rho(x^0)} |P(x) - P(x^0)|^2 \, d\mu_V = 0$$

for μ_V–a.e. $x^0 \in \Omega$.

Before going on we have to introduce some tools from the *theory of currents.*

An n–current in Ω is a continuous linear functional on the vector space of n–differential forms with compact support in Ω, endowed with the usually locally convex topology of distributions.

An integral n–current T in Ω is defined by a countably n–rectifiable, $\mathcal{H}^n$–measurable set $M \subset \Omega$, an integer function $\theta \in L^1_{loc}(\mathcal{H}^n \llcorner M)$ and a $\mathcal{H}^n$-measurable field η of n–vectors defined on M. We denote this current with $T \equiv (M, \theta, \eta)$.

T acts as a linear functional on n–differential forms with compact support in Ω, by integration:

$$< T, \omega > = \int\limits_M \theta(x) < \omega(x), \eta(x) > \, d\mathcal{H}^n(x).$$

The *boundary* of an n–current T is the $(n-1)$–current ∂T acting as follows:

$$< \partial T, \omega > = < T, d\omega > .$$

We define the norm of a differential form $\omega(x)$ with compact support in Ω as

$$\|\omega\| = \sum_{0 \leq i_1 < \ldots < i_n \leq k} \sup_{x \in \Omega} | < \omega(x), e_{i_1} \wedge \ldots \wedge e_{i_n} > |$$

(compare with [9]) and the *mass* of a current T in an open $\Omega' \subset \Omega$ by duality as

$$\mathcal{M}_{\Omega'}(T) = \sup_{\operatorname{supp} \omega \subset\subset \Omega'} \frac{| < T, \omega > |}{\|\omega\|}.$$

Now we can state the famous theorem of Federer and Fleming.

Theorem C (Boundary Rectifiability Theorem)**.** *If T is an integral n–current in Ω and for every open $\Omega' \subset\subset \Omega$*

$$\mathcal{M}_{\Omega'}(T) + \mathcal{M}_{\Omega'}(\partial T) < +\infty$$

then ∂T is an integral $(n-1)$–current in Ω.

For a proof, see [18].

3. Curvature Varifolds with Boundary and Basic Properties

In this section we introduce the idea of Hutchinson and our generalization. We work out the same calculation of [13] to get an integration by parts formula. The only difference is that we consider an n–dimensional smooth manifold M *with smooth boundary* ∂M, embedded in an open set $\Omega \subset \mathbb{R}^k$, while Hutchinson assumed that the boundary was empty.

Suppose that $\varphi \equiv \varphi(x, P) : \Omega \times \mathbb{R}^{k^2} \to \mathbb{R}$ is a C^1_c function, we write respectively

$$D_i\varphi \qquad \text{and} \qquad D^*_{jk}\varphi$$

for the derivatives of φ with respect to the variables x_i and P_{jk}.

Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^k$ and $P(x) \equiv \{P_{ij}(x)\}$ the tangent space function of the manifold M. Let us consider in the classical divergence formula the smooth vector field $X(x) = \varphi(x, P(x))\pi_{P(x)}e_i$ that is the orthogonal projection of the vector field $\varphi(x, P(x))e_i$ on the tangent space.

As the mean curvature is a normal vector to the manifold,

$$\int_M \operatorname{div}^M X \, d\mathcal{H}^n = -\int_{\partial M} < X, \underline{\nu} > \, d\mathcal{H}^{n-1}$$

where $\underline{\nu}$ is the inner normal to ∂M.

Working out the calculation of the tangential divergence (see [13]) we obtain

$$\int_M P_{ij}(x)D_j\varphi(x, P(x)) + A_{ijk}(x)D^*_{jk}\varphi(x, P(x)) + A_{jij}(x)\varphi(x, P(x)) \, d\mathcal{H}^n(x)$$
$$= -\int_{\partial M} \varphi(y, P(y))\nu_i(y) \, d\mathcal{H}^{n-1}(y)$$

where the functions $A_{ijk}(x)$ that appear above, are defined by the formula (2.1) of the previous section.

Representing the manifold as a varifold $V \equiv V_{M,1}$ and introducing a Radon *boundary measure* ∂V on $G_n(\Omega)$ with values in $\mathbb{R}^k$, we can write the formula above as

$$\int_{G_n(\Omega)} P_{ij}D_j\varphi + A_{ijk}D^*_{jk}\varphi + A_{jij}\varphi \, dV = -\int_{G_n(\Omega)} \varphi \, d\,\partial V_i$$

with

$$\partial V_i = (Id \times P)_\#(\nu_i\mathcal{H}^{n-1} \llcorner \partial M).$$

This is the motivation for the following definition.

Definition 3.1. Let $V \equiv V_{M,\theta}$ be an n–dimensional varifold in $\Omega \subset \mathbb{R}^k$, with $0 < n < k$. We say that V is a *curvature varifold with boundary* if there exist functions $A_{ijk} \in L^1_{loc}(V)$ and a Radon vector measure ∂V on $G_n(\Omega)$ with values in $\mathbb{R}^k$ such that

(3.1)
$$\int_{G_n(\Omega)} P_{ij} D_j \varphi(x,P) + D^*_{jk}\varphi(x,P) A_{ijk}(x,P) + \varphi(x,P) A_{jij}(x,P)\, dV(x,P)$$
$$= -\int_{G_n(\Omega)} \varphi(x,P)\, d\,\partial V_i(x,P) \qquad \forall \varphi \equiv \varphi(x,P) \in C^1_c(\Omega \times \mathbb{R}^{k^2})$$

for every index i.

In the extreme cases $n = 0, k$ for sake the of coherence we *define* $A_{ijk}(x,P) \equiv 0$ and we look for a measure ∂V such that the formula above is true. We call ∂V the *boundary measure of the varifold* V and we denote with $AV_n(\Omega)$ the class of n–dimensional curvature varifolds with boundary in Ω. Moreover we introduce the subclasses $AV^p_n(\Omega)$ consisting of those varifolds in $AV_n(\Omega)$ such that $A_{ijk} \in L^p(V)$.

Remark 3.2. We point out that the extreme cases are not so interesting because in dimension zero the varifold consists of a discrete set of points and the measure ∂V is the zero measure. In codimension zero ($n = k$) the theory is included in the theory of sets with locally finite perimeter (developed by E. De Giorgi in [10] and [11]) because the density function turns out to be an integer BV function and the boundary measure is essentially its distributional derivative.

Note 3.3. Hutchinson's definition of curvature varifolds of is analogous but assumes that the right hand side of the formula (3.1) is identically zero. It is so clear that the curvature varifolds in the sense of Hutchinson are the elements of $AV_n(\Omega)$ with zero boundary measure.

We define the generalized second fundamental form $\mathbf{B}$ from the functions A_{ijk}, using the relations in proposition 2.3. It is then easy to see that the L^p summability of $\mathbf{B}$ and of the functions A_{ijk} are equivalent.

Now we prove a theorem asserting that there are essentially unique second fundamental form and boundary.

Proposition 3.4 (Uniqueness)**.** *The functions A_{ijk} and the measure ∂V are uniquely determined by the formula* (3.1).

Proof. Suppose there are two pairs $(A^1_{ijk}, \partial V^1)$ and $(A^2_{ijk}, \partial V^2)$ that satisfy the definition. Setting $A_{ijk} = A^1_{ijk} - A^2_{ijk}$ and $\partial V = \partial V^1 - \partial V^2$, for every function $\varphi \in C^1_c(\Omega \times \mathbb{R}^{k^2})$ we have

$$\int_{G_n(\Omega)} D^*_{jk}\varphi(x,P) A_{ijk}(x,P) + \varphi(x,P) A_{jij}(x,P)\, dV(x,P)$$
$$= -\int_{G_n(\Omega)} \varphi(x,P)\, d\,\partial V_i(x,P).$$

Then we can write

$$\int_{G_n(\Omega)} D^*_{jk}\varphi A_{ijk}\, dV = \int_{G_n(\Omega)} \varphi\, d\sigma_i \qquad \forall \varphi \in C^1_c(\Omega \times \mathbb{R}^{k^2}) \tag{3.2}$$

where $\sigma_i = -\partial V_i - A_{jij}V$ is a Radon measure on $G_n(\Omega)$.
By this formula we deduce that, for every $\phi(x) \in C^1_c(\Omega)$, the functional

$$L_\phi(\psi) = \int\limits_{G_n(\Omega)} \phi(x) A_{ijk}(x,P) D^*_{jk}\psi(P)\, dV$$

is a bounded linear functional from $C^1(G_{n,k})$ to $\mathbb{R}$, in the relative topology induced by $C^0(G_{n,k})$.

If $A_{ijk} \not\equiv 0$ we can find a Lebesgue point x^0 for the functions $P(x)$ and $A_{ijk}(x, P(x))$ such that $A_{ijk}(x^0, P(x^0)) \neq 0$, at x^0 the density and the tangent space $P(x^0)$ to the varifold V exist and

$$\limsup_{\rho\to 0} \frac{\pi_\# |\sigma_i|(B_\rho(x^0))}{\omega_n \rho^n} < +\infty. \tag{3.3}$$

Choose now $\chi(t) \in C^1_c(\mathbb{R})$, $\chi \geq 0$ not identically zero and set

$$\phi_h(x) = \frac{\chi(h|x - x^0|)}{\omega_n h^{-n}}.$$

The functionals L_{ϕ_h} pointwise converge as $h \to +\infty$ to the functional

$$L(\psi) = \theta(x^0) A_{ijk}(x^0, P(x^0)) D^*_{jk}\psi(P(x^0)) \int\limits_{P(x^0)} \chi(|y|)\, d\mathcal{H}^n(y) \tag{3.4}$$

on $C^1(G_{n,k})$. Moreover we can extend L_{ϕ_h} to equibounded functionals defined in all $C^0(G_{n,k})$, because of the equations (3.2), (3.3) and the upper estimate

$$\|L_{\phi_h}\| \leq \int\limits_\Omega \phi_h \, d\pi_\# |\sigma_i|.$$

Hence the functional L is continuous in $C^1(G_{n,k})$ with respect to convergent sequences in $C^0(G_{n,k})$, in evident contradiction with the equation (3.4).
It follows that $A \equiv 0$ and $\underline{\sigma} = 0$. The definition of $\underline{\sigma}$ implies that $\partial V = 0$ too. $\square$

We state now some propositions about the formal and geometric properties of the tensor A_{ijk} and of the boundary measure ∂V. The proofs are postponed until after theorem 5.4.

Proposition 3.5 (Singularity of $|\partial V|$). *If the pair $(A_{ijk}, \partial V)$ satisfy the definition 3.1 then the measure ∂V has support included in the support of the measure V and the projection of its total variation $|\partial V|$ is singular with respect to the weight measure μ_V of the varifold V.*

Proposition 3.6 (Formal Properties). *For $V-a.e.\ (x,P) \in G_n(\Omega)$ it is true that:*

- $A_{ijk}(x,P) = A_{ikj}(x,P)$;
- $\sum_j A_{ijj}(x,P) = 0$;
- $A_{ijk}(x,P) = P_{jr}A_{irk}(x,P) + P_{rk}A_{ijr}(x,P)$.

Proposition 3.7 (Tangential Properties). *The boundary measure ∂V is tangential, in the sense that for every index $i \in \{1,\dots,k\}$*

$$P_{il}\partial V_l(x,P) = \partial V_i(x,P)$$

as measures on $G_n(\Omega)$.

The functions $A_{ijk}(x,P)$ *satisfy the relations:*

$$P_{il}A_{ljk}(x,P) = A_{ijk}(x,P)$$

and defining

$$H_i(x,P) = \sum_j A_{jij}(x,P) \qquad \text{we have} \quad P_{il}H_l(x,P) = 0$$

for $V-$ *a.e.* $(x,P) \in G_n(\Omega)$. *That is, the functions* A_{ijk} *are tangential and the (formal) mean curvature vector is normal to the varifold.*

Note 3.8. These propositions extend to this class of varifolds formal and geometric results that hold in the classical case of a smooth manifold.

Remark 3.9. We wrote "formal" mean curvature vector because this is only the trace of the generalized second fundamental form and, at this point, has nothing in common with Allard's definition. The connection between these notions will be shown below.

We want to describe now the differences between this class of varifolds and Allard's varifolds with locally bounded first variation. First of all, it is obvious that a curvature varifold with boundary has first variation given by the Radon measure

$$\delta V_i = -\pi_\#(A_{jij}\, V + \partial V_i)$$

where π is as usual the projection from $G_n(\Omega)$ on Ω. This can be seen considering in the formula (3.1) functions φ depending only on the x variable. More precisely we can write respectively Allard's mean curvature and boundary, as we could expect, using the functions A_{ijk} and the boundary measure ∂V.

Proposition 3.10. *A curvature varifold with boundary is a varifold with locally bounded first variation. The generalized mean curvature vector is given by*

$$\mathbf{H}_i(x) = \sum_j A_{jij}(x, P(x))$$

and the generalized boundary by

$$\sigma = \pi_\# \partial V$$

where $P(x)$ *is the approximate tangent space at* x.

One of the advantages of our definition is that ∂V carries much more information on the local structure of V, while Allard's boundary, being the projection of ∂V, can be even equal to zero.

Example 3.11. Consider the varifold in $\mathbb{R}^2$ formed by three halflines from the origin, forming three angles of $120°$. According to Allard's definition this varifold has mean curvature and boundary measure equal to zero, because at the origin the sum of the three inner normals is zero. For our definition the boundary measure is the sum of three Dirac deltas supported in the points $(0, P_i)$ in $G_n(\Omega)$, where P_i denote the 1–spaces determined by the halflines in $\mathbb{R}^2$.

Another difference, as we will see in section 7, is concerned with the set where the boundary measure is supported. The only thing we can say about Allard's boundary measure is that it is singular with respect to μ_V. We will show that the projection of $|\partial V|$ is supported in a countably $(n-1)$–rectifiable subset of Ω for every n–curvature varifold with boundary V.

4. Localization

In this section we introduce a basic tool for the study of this class of varifolds that could be interesting by itself. We prove that curvature varifolds are stable under localization in (x, P).

Lemma 4.1 (Localization Lemma)**.** *Let V be an n–dimensional curvature varifold with boundary in $\Omega \subset \mathbb{R}^k$ and let $x^0 \in \Omega$, $P_0 \in G_{n,k}$ and ρ', $\delta' > 0$. Then there exist $\rho'/2 \leq \rho \leq \rho'$, $\delta'/2 \leq \delta \leq \delta'$ such that, defining $B^\rho_\delta = B_\rho(x^0) \times B_\delta(P_0) \subset G_n(\Omega)$, we have that $V^\rho_\delta = V \llcorner B^\rho_\delta$ is a curvature varifold with boundary.*

Proof. Let be given x^0, P_0, δ', ρ' as in the statement.

We study the localization in the x variable. Consider in the formula (3.1) a function $\varphi(x, P) = \psi(x, P)\chi(x)$, where ψ is an arbitrary function in $C^1_c(\Omega \times \mathbb{R}^{k^2})$ and χ is a *cut-off* function so defined: $\chi(x) = h(r)$, $r = |x - x^0|$ and $h(t)$ is a function in $C^\infty(\mathbb{R})$, with the properties, $h(t) = 1$ for $t < \rho/2$, $h(t) = 0$ for $t > \rho$, $h'(t) \leq 0$.

Computing the derivatives we get

$$\int_{G_n(\Omega)} \chi(x)P_{ij}D_j\psi(x, P) + \chi(x)D^*_{jk}\psi(x, P)A_{ijk}(x, P) + \chi(x)\psi(x, P)A_{jij}(x, P)\, dV(x, P)$$

$$= -\int_{G_n(\Omega)} \chi(x)\psi(x, P)\, d\,\partial V_i(x, P) - \int_\Omega \psi(x, P(x))P_{ij}D_j\chi(x)\, d\mu_V(x)$$

$$= -\int_{G_n(\Omega)} \chi(x)\psi(x, P)\, d\,\partial V_i(x, P) - \int_{G_n(\Omega)} \psi(x, P)h'(r)P_{ij}\frac{x_j - x^0_j}{r}\, dV(x, P).$$

We take a sequence of functions $h_m(t)$ with the properties above such that $h_m(t) = 1$ for $t < \rho - 1/m$ and $|h'(t)| < 4m$. The sequence h_m pointwise converges to the characteristic function of $(-\infty, \rho)$ as $m \to \infty$.

Defining the Radon measures on $G_n(\Omega)$

$$\sigma_m = V \llcorner \left\{ h'_m(r)P_{ij}(x)\frac{x_j - x^0_j}{r} \right\}$$

we can see that σ_m is supported in the set $G_n(B_\rho(x^0) \setminus B_{\rho-1/m}(x^0))$, so we have the following estimate for its total variation

(4.1)

$$|\sigma_m|(G_n(\Omega)) = |\sigma_m|(G_n(B_\rho(x^0) \setminus B_{\rho-1/m}(x^0)))$$
$$\leq 4m\,\mu_V(B_\rho(x^0) \setminus B_{\rho-1/m}(x^0)) = 4\frac{\mu_V(B_\rho(x^0)) - \mu_V(B_{\rho-1/m}(x^0))}{1/m}.$$

Now we note that the real function $f(\rho) = \mu_V(B_\rho(x^0))$ is monotone hence differentiable for almost every $\rho \in \mathbb{R}$. At any differentiability point it follows that the total variations of the measures σ_m are equibounded. We use the Banach–Alaoglu Theorem to infer that there exists a subsequence weakly* converging to a Radon measure σ on $G_n(\Omega)$. For these values of ρ the restricted varifold $V \llcorner B_\rho(x^0) \times G_{n,k}$ is again a curvature varifold with boundary.

The study of localization in the P variable is quite similar: using a cut–off function $\chi(P) = h(|P - P_0|)$ we get an extra boundary measure σ given by the weak* limit of a subsequence of the family

$$\sigma_m = V \llcorner \left\{ h'_m(|P - P_0|) A_{ijk}(x, P) D^*_{jk} |P - P_0| \right\}. \tag{4.2}$$

□

Remark 4.2. Note that this stability property under localization is not true in the context of Hutchinson's curvature varifolds, not even if we assume that the varifolds correspond to smooth embedded manifolds without boundary.

5. Approximate Differentiability of the Tangent Space Functions

In this section we are going to show that the functions $P_{jk}(x)$ are approximately differentiable and that their approximate gradients are precisely the functions $A_{ijk}(x, P)$ of definition 3.1, in accordance with the classical case of a regular manifold. This result implies all the formal properties of A_{ijk} stated in proposition 3.6 and leads to an estimate of the extra boundary created by the localization in lemma 4.1.

The basic result leading to the approximate differentiability of P_{jk} is the following:

Theorem 5.1. *Let $V \equiv V_{M,\theta}$ be a curvature varifold with boundary and $\psi \in C^1_c(\Omega \times \mathbb{R}^{k^2})$. Then there exists an $\mathcal{H}^n$–negligible set M_0 such that*

$$\{(x, \psi(x, P(x))) \,|\, x \in M \setminus M_0\}$$

is countably n–rectifiable in $\Omega \times \mathbb{R}$.

Proof. We first suppose that the support of the varifold $V \equiv V(M, \theta) \in AV_n(\Omega)$ is included in $\Omega \times B_{\delta/2}(P_0)$, where P_0 is the n–space generated by $e_1, ..., e_n$. If δ is small enough then any $P \in B_\delta(P_0)$ can be oriented by the unit n–vector η defined by

$$\eta = \frac{\eta^1 \wedge \ldots \wedge \eta^n}{|\eta^1 \wedge \ldots \wedge \eta^n|} \qquad \eta^i = \pi_P e_i = \sum_j P_{ij} e_j.$$

We take a nonnegative function $\psi(x, P) \in C^1_c(\Omega \times \mathbb{R}^{k^2})$ and we consider the $(n + 1)$–integral current $T \equiv (T, \theta', \eta')$ in the space $\Omega \times \mathbb{R}$, where T is the set $\{(x, y) \,|\, x \in M \quad 0 \leq y \leq \psi(x, P(x))\}$, $\theta'(x, y) = \theta(x)$ and, calling ε the unit vertical vector, $\eta' = \varepsilon \wedge \eta$. It is clear that T is a $\mathcal{H}^{n+1}$–measurable set so the current is well defined. Now we prove that this current has a boundary of finite mass, hence it can be represented as an integral n–current.

To do this we have to test two kind of differential forms:

$$\omega_1(x,y) = \varphi_1(x,y)\,dy \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n}$$

$$\omega_2(x,y) = \varphi_2(x,y)\,dx^{j_1} \wedge \ldots \wedge dx^{j_n}.$$

For multi–indexes $I \equiv (i_2, \ldots, i_n)$ and $J \equiv (j_1, \ldots, j_n)$ we define the functions

$$\beta^I_s(P) = < dx^{i_2} \wedge \ldots \wedge dx^{i_n}, \frac{\eta^1 \wedge \ldots \wedge \widehat{\eta^s} \wedge \ldots \wedge \eta^n}{|\eta^1 \wedge \ldots \wedge \eta^n|} > \tag{5.1}$$

$$\beta^J(P) = < dx^{j_1} \wedge \ldots \wedge dx^{j_n}, \frac{\eta^1 \wedge \ldots \wedge \eta^n}{|\eta^1 \wedge \ldots \wedge \eta^n|} > \tag{5.2}$$

that belong, by our choice of δ, to $C^\infty(B_\delta(P_0))$.

Then for ω_1 we have,

$$\partial T(\omega_1) = T(d\omega_1) = -\sum_{i=1}^k T\left(\frac{\partial\varphi_1}{\partial x_i}(x,y)\,dy \wedge dx^i \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n}\right)$$

$$\begin{aligned}\partial T(\omega_1) &= -\sum_{i=1}^k \int\limits_M \Bigg(\int\limits_0^{\psi(x,P(x))} \frac{\partial\varphi_1}{\partial x_i}(x,y)\,dy\Bigg) < dy \wedge dx^i \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_n}, \varepsilon \wedge \eta > \,d\mu_V(x) \\ &= \sum_{i=1}^k \sum_{s=1}^n (-1)^{s+1} \int\limits_{G_n(\Omega)} P_{is} \Bigg(\int\limits_0^{\psi(x,P)} \frac{\partial\varphi_1}{\partial x_i}(x,y)\,dy\Bigg) \beta^I_s(P)\,dV(x,P).\end{aligned}$$

Now we extend the functions $\beta^I_s(P)$ and $\beta^J(P)$ to smooth functions on $G_{n,k}$ without modifying them in $\overline{B}_{\delta/2}(P_0)$. Carrying the derivative out of the integral, using the formula (3.1) and taking into account that the support of V is contained in $B_{\delta/2}(P_0)$, we obtain

$$\begin{aligned}
\partial T(\omega_1) = &\sum_{i=1}^{k}\sum_{s=1}^{n}(-1)^{s+1}\int\limits_{G_n(\Omega)} P_{is}\frac{\partial}{\partial x_i}\Bigg(\int\limits_0^{\psi(x,P)}\varphi_1(x,y)\,dy\Bigg)\beta_s^I(P)\,dV(x,P)\\
&-\sum_{i=1}^{k}\sum_{s=1}^{n}(-1)^{s+1}\int\limits_{G_n(\Omega)} P_{is}\frac{\partial\psi}{\partial x_i}(x,P)\,\varphi_1(x,\psi(x,P))\,\beta_s^I(P)\,dV(x,P)\\
=&\sum_{s=1}^{n}(-1)^s\Bigg\{\int\limits_{G_n(\Omega)} A_{sjk}\,D^*_{jk}\Bigg(\beta_s^I(P)\int\limits_0^{\psi(x,P)}\varphi_1(x,y)\,dy\Bigg)\,dV(x,P)\\
&+\int\limits_{G_n(\Omega)} A_{jsj}\,\beta_s^I(P)\Bigg(\int\limits_0^{\psi(x,P)}\varphi_1(x,y)\,dy\Bigg)\,dV(x,P)\\
&+\int\limits_{G_n(\Omega)} \beta_s^I(P)\Bigg(\int\limits_0^{\psi(x,P)}\varphi_1(x,y)\,dy\Bigg)\,d\,\partial V_s(x,P)\\
&+\sum_{i=1}^{k}\int\limits_{G_n(\Omega)} P_{is}\frac{\partial\psi}{\partial x_i}(x,P)\,\varphi_1(x,\psi(x,P))\,\beta_s^I(P)\,dV(x,P)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
=&\sum_{s=1}^{n}(-1)^s\Bigg\{\int\limits_{G_n(\Omega)} A_{sjk}\,\varphi_1(x,\psi(x,P))\,D^*_{jk}\psi(x,P)\,\beta_s^I(P)\,dV(x,P)\\
&+\int\limits_{G_n(\Omega)} A_{sjk}\Bigg(\int\limits_0^{\psi(x,P)}\varphi(x,y)\,dy\Bigg)\,D^*_{jk}\beta_s^I(P)\,dV(x,P)\\
&+\int\limits_{G_n(\Omega)} A_{jsj}\,\beta_s^I(P)\Bigg(\int\limits_0^{\psi(x,P)}\varphi_1(x,y)\,dy\Bigg)\,dV(x,P)\\
&+\int\limits_{G_n(\Omega)} \beta_s^I(P)\Bigg(\int\limits_0^{\psi(x,P)}\varphi_1(x,y)\,dy\Bigg)\,d\,\partial V_s(x,P)\\
&+\sum_{i=1}^{k}\int\limits_{G_n(\Omega)} P_{is}\frac{\partial\psi}{\partial x_i}(x,P)\,\varphi_1(x,\psi(x,P))\,\beta_s^I(P)\,dV(x,P)\Bigg\}.
\end{aligned}$$

As the functions β_s^I and ψ are bounded with their derivatives, it is now clear that we can have an estimate

$$|\partial T(\omega_1)| \leq c\|\varphi_1\|$$

with c depending only on δ and ψ.

Now we test the differential form ω_2

$$\begin{aligned}\partial T(\omega_2) = T(d\omega_2) &= \sum_{i=1}^{k} T\left(\frac{\partial\varphi_2}{\partial x_i}(x,y)\,dx^i \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_n}\right) \\ &\quad + T\left(\frac{\partial\varphi_2}{\partial y}(x,y)\,dy \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_n}\right) \\ &= T\left(\frac{\partial\varphi_2}{\partial y}(x,y)\,dy \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_n}\right)\end{aligned}$$

because T has a vertical orientation. Hence

(5.3)

$$\begin{aligned}\partial T(\omega_2) &= \int_{\Omega}\left(\int_0^{\psi(x,P(x))} \frac{\partial\varphi_2}{\partial y}(x,y)\,dy\right) < dy \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_n}, \varepsilon \wedge \eta > \, d\mu_V(x) \\ &= \int_{G_n(\Omega)} (\varphi_2(x,\psi(x,P)) - \varphi_2(x,0))\,\beta^J(P)\,dV(x,P)\end{aligned}$$

so, also in this case we have the estimate

$$|\partial T(\omega_2)| \leq c\|\varphi_2\|.$$

This calculation shows that T is an integral current with integral boundary ∂T (by the Boundary Rectifiability Theorem), so ∂T is represented by (N,τ,ξ), where N is a countably n–rectifiable set, τ is an integer valued function defined on N and $\mathcal{H}^n$–measurable, ξ is a simple unit n–vector field orienting N.

Now we consider the sets of points $N_1 \equiv \{(x,y) \in N \,|\, \xi(x,y) \wedge \varepsilon = 0\}$ and $N_2 = N \setminus N_1$, that is, N_1 is the set of points of N where the tangent space contains a vertical vector. Defining the integral current $G = (N_2,\tau,\xi) + (M,\theta,\eta)$, it is clear that G can be represented as an integration on a countably n–rectifiable set. Now, by (5.3) we get

$$\begin{aligned}G(\omega_2) &= \int_M \varphi_2(x,\psi(x,P(x)))\beta^J(P(x))\,d\mu_V(x) \\ &= \int_{N_2} \tau(x,y)\varphi_2(x,y) < dx^{j_1} \wedge, \ldots \wedge dx^{j_n}, \xi(x,y) > \, d\mathcal{H}^n(x,y).\end{aligned}$$

Since φ_2 is arbitrary, arguing as in [4] we can show that

$$\{(x,\psi(x,P(x)))\,|\,x \in M \setminus M_0\} \subset N_2 \tag{5.4}$$

for a suitable $\mathcal{H}^n$–negligible set $M_0 \subset M$.

Consider now a curvature varifold with boundary $V \equiv V_{M,\theta}$ without conditions on its support; we can apply the localization lemma 4.1 to find out a countable family of curvature varifolds with boundary $V^i \equiv V^i_{M_i,\theta_i}$ satisfying (up to a rotation) the hypotheses at the beginning of the proof and such that $\sum_i V^i \geq V$. Applying (5.4) to all the varifolds V^i we infer the theorem. □

Now we introduce the approximate differentiability property.

Definition 5.2. Suppose $V \equiv V_{M,\theta}$ is an n–varifold with weight measure μ_V and $f : M \to \mathbb{R}$ is a μ_V–measurable function. We say that f is *approximately differentiable* at $x^0 \in M$ with *approximate gradient* $\nabla^M f(x^0) = v$ if:

- at x^0 there exists the tangent space $T_{x^0}M$ to the varifold and $v \in T_{x^0}M$;
- for every $\varepsilon > 0$ the set

$$L_\varepsilon = \left\{ x \in M \setminus \{x^0\} \,|\, \frac{|f(x) - f(x^0) - < v, x - x^0 > |}{|x - x^0|} > \varepsilon \right\}$$

has zero density at x^0:

$$\lim_{\rho \to 0} \frac{\mu_V(L_\varepsilon \cap B_\rho(x^0))}{\rho^n} = 0.$$

For this definition and basic properties we refer to [9].

It is not hard to show the following lemma.

Lemma 5.3. *Let $V \equiv V_{M,\theta}$ and f as in the definition above. Let us assume that there exists an $\mathcal{H}^n$–negligible set M_0 such that*

$$\{(x, f(x)) \,|\, x \in M \setminus M_0\}$$

is countably n–rectifiable in $\Omega \times \mathbb{R}$. Then f is approximately differentiable μ_V–almost everywhere in Ω.

The proof of the lemma basically follows covering the graph of f on $M \setminus M_0$ with C^1 manifolds Γ_i of dimension n and taking the nonvertical parts of Γ_i.

Now we can state the main result of this section.

Theorem 5.4 (Approximate Differentiability)**.** *If $V \equiv V_{M,\theta}$ is a curvature varifold with boundary, then the components of the tangent space function $P_{jk}(x)$ are approximately differentiable for μ_V–almost all points $x^0 \in M$, with approximate gradients*

$$\nabla_i^M P_{jk}(x^0) = A_{ijk}(x^0, P(x^0)).$$

Proof. By theorem 5.1 we know that $f(x) = P_{jk}(x)$ satisfies the assumptions of lemma 5.3. Hence, we know that the functions P_{jk} are approximately differentiable μ_V–almost everywhere in Ω.

Let $B_{ijk} = \nabla_i^M P_{jk}$; we will show that $B_{ijk} = A_{ijk}$ by a blow up argument.

We define as usual two cut-off functions χ, $\tau \in C_c^1(\mathbb{R})$ with the properties $\chi(t)$, $\tau(t) = 1$ for $|t| \leq 1/2$, $\chi(t)$, $\tau(t) = 0$ for $|t| \geq 1$. We consider in (3.1) a function

$$\varphi_\rho(x, P) = \chi\left(\frac{|x - x^0|}{\rho}\right) \tau\left(\frac{P_{jk} - P_{jk}(x^0)}{\rho}\right)$$

so that

$$
\begin{aligned}
&\frac{1}{\rho}\int_{G_n(\Omega)} P_{il}\frac{x_l - x_l^0}{|x-x^0|}\chi'\left(\frac{|x-x^0|}{\rho}\right)\tau\left(\frac{P_{jk}-P_{jk}(x^0)}{\rho}\right)dV(x,P)\\
&+\frac{1}{\rho}\int_{G_n(\Omega)} A_{ijk}(x,P)\chi\left(\frac{|x-x^0|}{\rho}\right)\tau'\left(\frac{P_{jk}-P_{jk}(x^0)}{\rho}\right)dV(x,P)\\
=&-\int_{G_n(\Omega)} A_{lil}(x,P)\chi\left(\frac{|x-x^0|}{\rho}\right)\tau\left(\frac{P_{jk}-P_{jk}(x^0)}{\rho}\right)dV(x,P)\\
&+\int_{G_n(\Omega)} \chi\left(\frac{|x-x^0|}{\rho}\right)\tau\left(\frac{P_{jk}-P_{jk}(x^0)}{\rho}\right)d\,\partial V_i(x,P).
\end{aligned}
$$

Dividing each side by ρ^{n-1}, if x^0 is chosen in such a way that

- x^0 is a point where the tangent space $P(x)$ to the varifold V exists.
- x^0 is a point of approximate differentiability of the function $P_{jk}(x)$ and the approximate gradient has components B_{ijk}.
- x^0 is a Lebesgue point for all the functions $A_{ijk}(x, P(x))$ with respect to the measure μ_V.
- $\pi_\#|\partial V|(B_\rho(x^0))$ tends to zero faster than ρ^{n-1}.

We remark that this happens for μ_V–almost all points $x^0 \in M$. Under these conditions we have

$$
\begin{aligned}
\lim_{\rho\to 0}\frac{1}{\rho^n}\Bigg\{&\int_{G_n(\Omega)} P_{il}\frac{x_l - x_l^0}{|x-x^0|}\chi'\left(\frac{|x-x^0|}{\rho}\right)\tau\left(\frac{P_{jk}-P_{jk}(x^0)}{\rho}\right)dV(x,P)\\
&+\frac{1}{\rho^n}\int_{\Omega} A_{ijk}(x,P(x))\chi\left(\frac{|x-x^0|}{\rho}\right)\tau'\left(\frac{P_{jk}(x)-P_{jk}(x^0)}{\rho}\right)d\mu_V(x)\Bigg\}=0.
\end{aligned}
$$

As x^0 is a Lebesgue point for the functions A_{ijk} it is clear that we can replace $A_{ijk}(x, P(x))$ with $A_{ijk}(x^0, P(x^0))$ in the second term of the limit above. Moreover because of the existence of the approximate tangent plane $P(x^0)$ and the fact that the function $P_{jk}(x)$ is approximately differentiable at x^0 with gradient B_{ijk} we have

$$
\begin{aligned}
&\theta(x^0)\int_{P(x^0)} P_{il}(x^0)\frac{x_l}{|x|}\chi'(|x|)\tau(B_{ijk}x_i)\,d\mathcal{H}^n(x)\\
&+A_{ijk}(x^0,P(x^0))\theta(x^0)\int_{P(x^0)}\chi(|x|)\tau'(B_{ijk}x_i)\,d\mathcal{H}^n(x)=0.
\end{aligned}
\tag{5.5}
$$

The first term in (5.5) is equal to

$$\theta(x^0)\int\limits_{P(x^0)} P_{il}(x^0)\frac{\partial}{\partial x_l}\left\{\chi(|x|)\tau(B_{ijk}x_i)\right\}\,d\mathcal{H}^n(x)$$
$$-\theta(x^0)\int\limits_{P(x^0)} P_{il}(x^0)\chi(|x|)\tau'(B_{ijk}x_i)B_{ljk}\,d\mathcal{H}^n(x). \tag{5.6}$$

We note that the first term in (5.6) is zero, being a divergence on the tangent space. In the second $P_{il}(x^0)B_{ljk} = B_{ijk}$ because the B_{ijk}–vector is tangent to the varifold, hence substituting in (5.5) and adding we have

$$(A_{ijk}(x^0, P(x^0)) - B_{ijk})\theta(x^0)\int\limits_{P(x^0)} \chi(|x|)\tau'(B_{ijk}x_i)\,d\mathcal{H}^n(x) = 0.$$

We can always choose χ and τ in such a way that the integral is different from zero, therefore $A_{ijk}(x^0, P(x^0)) = B_{ijk}$.

We remark that this also proves that the functions $A_{ijk}(x, P)$ are tangential. □

Now we can prove the propositions stated in section 3.

Proof of proposition 3.6. The thesis follows immediately by the linear properties of the approximate gradient. □

Proof of proposition 3.7. The tangential properties of the functions A_{ijk} are in the final part of the proof of the theorem 5.4.

We now see that the "formal" mean curvature vector $H_i(x) = A_{jij}(x, P(x))$ is orthogonal to the tangent space $P(x)$ for $\mu_V - a.e.\, x \in \Omega$. As we know that

$$A_{ijk}(x, P(x)) = P_{il}(x)\nabla^M_l P_{jk}(x)$$

for $\mu_V - a.e.\, x \in \Omega$, using the linear properties of the approximate gradient the following holds

$$\begin{aligned}
P_{hi}(x)H_i(x) &= P_{hi}(x)P_{jl}(x)\nabla^M_l P_{ij}(x)\\
&= P_{jl}(x)\nabla^M_l(P_{hi}(x)P_{ij}(x)) - P_{ij}(x)P_{jl}(x)\nabla^M_l P_{hi}(x)\\
&= A_{jhj}(x, P(x)) - A_{ihi}(x, P(x)) = 0
\end{aligned}$$

summing over the indexes i and j. That is, the projection on $P(x)$ of the (formal) mean curvature vector $H(x)$ is zero for $\mu_V - a.e.\, x \in \Omega$, hence the thesis.

The fact that ∂V is tangent is a consequence of the orthogonality of H and of the uniqueness theorem 3.4. Indeed, considering in the formula (3.1) a function $\psi(x, P) = P_{si}\varphi(x, P)$ and summing over the index i we obtain

(5.7)
$$\int\limits_{G_n(\Omega)} P_{sj}D_j\varphi(x,P) + D^*_{jk}\varphi(x,P)A_{sjk}(x,P) + A_{isi}\varphi(x,P) + \varphi(x,P)P_{si}H_i(x,P)\,dV$$
$$= -\int\limits_{G_n(\Omega)} \varphi(x,P)P_{is}\,d\,\partial V_i(x,P).$$

because of the orthogonality of H we see that $\widetilde{\partial V} = \sum_s P_{is}\partial V_s$ is a measure satisfying the definition too. Applying the uniqueness theorem we have the thesis. □

Proof of propositions 3.5 and 3.10. The fact that $\operatorname{supp}\partial V \subset \operatorname{supp} V$ is obvious.

Setting $\lambda = \pi_\#|\partial V|$, it is well known that there exist suitable vector measures σ_x with values in $\mathbb{R}^k$ such that $|\sigma_x|(G_{n,k}) = 1$ and

$$\int\limits_{G_n(\Omega)} \varphi(x,P)\, d\,\partial V = \int\limits_{\Omega}\Bigg(\int\limits_{G_{n,k}} \varphi(x,P)\, d\sigma_x(P)\Bigg)\, d\lambda(x)$$

for every bounded Borel function $\varphi(x,P)$. The result that ∂V is tangential implies that

$$P_{ij}(\sigma_x)_j = (\sigma_x)_i \tag{5.8}$$

as measures, for λ–almost every $x \in \Omega$. Now let $\lambda \llcorner A$ be the absolutely continuous part of λ with respect to μ_V. Using test functions depending only on the x variable we see that the varifold V has generalized mean curvature vector given by

$$H_i(x)\mu_V = A_{jij}(x, P(x))\mu_V + (\sigma_x)_i(G_{n,k})\lambda \llcorner A \tag{5.9}$$

and generalized boundary

$$(\sigma_x)_i(G_{n,k})\, \lambda \llcorner (\Omega \setminus A). \tag{5.10}$$

The orthogonality of A_{jij} (see 3.7) and Brakke's Theorem B imply that

$$P_{ij}(x)(\sigma_x)_j(G_{n,k}) = 0$$

for $\lambda \llcorner A$–almost all points $x \in \Omega$. If $|\sigma_x|$ were supported in $\{P(x)\}$ for $\lambda \llcorner A$–almost every $x \in \Omega$ (or equivalently for μ_V–almost all points $x \in \Omega$) then the equation above would be in contradiction with (5.8) yielding $\lambda \llcorner A = 0$.

To prove that $|\sigma_x|$ is an atomic measure we consider in the formula (3.1) a function

$$\varphi(x,P) = |P - P(x^0)|^2 \xi(P) \frac{\chi(\rho^{-1}|x - x^0|)}{\omega_n \rho^n}$$

where $\xi \in C^1(G_{n,k})$, $\chi \in C^1_c(\mathbb{R})$ is positive and the following properties hold:

- At x^0 the approximate tangent space $P(x^0)$ to the varifold exists.
- The flattening property holds at x^0:

$$\lim_{\rho\to 0} \rho^{-n-1} \int\limits_{B_\rho(x^0)} |P(x) - P(x^0)|^2\, d\mu_V = 0.$$

- At x^0 the measure $\lambda \llcorner (\Omega \setminus A)$ has zero density with respect to μ_V.
- x^0 is a Lebesgue point with respect to the measure μ_V for all the functions

$$x \mapsto \int\limits_{G_{n,k}} \psi(P)\, d\sigma_x(P) \qquad \psi \in C^0(G_{n,k})$$

By Brakke's Theorem B these conditions hold for $\mu_V - a.e.\, x^0 \in \Omega$.

Taking the limit as $\rho \to 0$ in equation (3.1) we find

$$\left(\theta(x^0) \int_{P(x^0)} \chi(|y|)\, d\mathcal{H}^n(y)\right) \int_{G_{n,k}} |P - P(x^0)|^2 \xi(P)\, d\sigma_{x^0}(P) = 0.$$

Since $\xi(P)$ is an arbitrary function this implies that the support σ_{x^0} is $\{P(x^0)\}$. $\square$

The proposition 3.10 easily follows by formulas (5.9), (5.10) and by the fact that the measure λ is singular with respect to μ_V.

Using the approximate differentiability property and a Lipschitz approximation argument of Federer, we are now able to show that the extra boundary created by the localization is an $(n-1)$–dimensional measure.

Proposition 5.5. *In the thesis of lemma 4.1 we can require that the extra boundary measure* σ

$$\sigma = \partial\big(V \llcorner (B_\rho(x^0) \times B_\delta(P_0))\big) - \partial V \llcorner (B_\rho(x^0) \times B_\delta(P_0))$$

has the property that

$$\pi_\# |\sigma| \in \mathcal{R}_{n-1}(\Omega)$$

Proof. We need two lemmas.

Lemma 5.6. *Let* $\mu = \mathcal{H}^n \llcorner \tau$ *be a Radon measure,* $f : M \to \mathbb{R}^m$ *be* μ*–apdifferentiable* μ*–almost everywhere. Then there exist a sequence of pairwise disjoint, compact subsets* K_h *of* M *such that*

$$\mathcal{H}^n(M \setminus \bigcup_h K_h) = 0$$

and $f|_{K_h}$ *is Lipschitz for every index* h*.*

The proof can be found in the book of Federer [9], Chapter 3.

Now given a finite positive measure μ on Ω, a Borel function $f : \Omega \to \mathbb{R}^m$, $\rho \in \mathbb{R}^+$ and a generic point $y^0 \in \mathbb{R}^m$, we define $\theta_\rho(\mu, f, y^0)$ as the class of weak* limits of the family of Radon measures

$$\frac{\mu \llcorner f^{-1}(B_\rho(y^0) \setminus B_{\rho-\varepsilon}(y^0))}{\varepsilon} \tag{5.11}$$

as ε tends to zero.

Remark 5.7. It is clear that for $\mathcal{L}^1 - a.e.\, \rho \in \mathbb{R}^+$ the set $\theta_\rho(\mu, f, y^0)$ is not empty. In fact, this is true for every ρ such that the real monotone function $M(\rho) = \mu(f^{-1}(B_\rho(y^0)))$ is differentiable at ρ, because of the fact that the family of Radon measures above is equibounded.

Now the second lemma.

Lemma 5.8. *Given* μ *and* f *as in the lemma 5.6, let* K_h *be the compact sets we obtain. We set* $F(x) = |f(x) - y^0|$ *with* $y^0 \in \mathbb{R}^m$*. Then for* $\mathcal{L}^1 - a.e.\, \rho \in \mathbb{R}^+$ *we have that*

$$N = \bigcup_h K_h \cap f^{-1}(\partial B_\rho(y^0))$$

is a countably $(n-1)$*–rectifiable set and*

$$\theta_\rho(|\nabla^M F|\mu, f, y^0) \equiv \left\{\tau\mathcal{H}^{n-1} \llcorner N\right\}. \tag{5.12}$$

Proof. Firstly we suppose that f is a Lipschitz function. With an abuse of notation we denote with $\theta_\rho(|\nabla^M F|\mu, f, y^0)$ one of the weak* limits defined above (we have seen that we can suppose the existence of at least one of them). The first part of the thesis follows immediately by the general coarea formula (see [9]), moreover it is clear that any measure in $\theta_\rho(|\nabla^M F|\mu, f, y^0)$ is supported in N, that in this case is a relatively closed set in Ω. It remains to prove formula (5.12).

Applying the coarea formula to the Lipschitz function F we get that for every positive Borel function $\varphi : M \to \mathbb{R}$

$$\int_M \varphi(x)|\nabla^M F(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}} \int_{F^{-1}(t)\cap M} \varphi(x)\, d\mathcal{H}^{n-1}(x)\, d\mathcal{H}^1(t).$$

Considering $\varphi(x) = \psi(x)\tau(x)$ if $\rho - \varepsilon \leq F(x) < \rho$ and zero otherwise, we obtain

$$\int_{F^{-1}([\rho-\varepsilon,\rho))} \psi(x)|\nabla^M F(x)|\, d\mu(x) = \int_{\rho-\varepsilon}^{\rho} \int_{F^{-1}(t)\cap M} \psi(x)\tau(x)\, d\mathcal{H}^{n-1}(x)\, d\mathcal{H}^1(t) \tag{5.13}$$

where ψ is an arbitrary positive Borel function. We take in the formula above a *dense* countable family $\{\psi_i\}$ of nonnegative continuous functions with compact support and we choose ρ to be a Lebesgue point for all the real functions

$$g_i(t) = \int_{F^{-1}(t)\cap M} \psi_i(x)\tau(x)\, d\mathcal{H}^{n-1}(x)$$

(the fact that the functions g_i belongs to $L^1(\mathbb{R})$ is given again by the coarea formula). Dividing by ε each side of (5.13) and taking the limit as $\varepsilon \to 0$, we get

$$\int_\Omega \psi_i\, d\theta_\rho(|\nabla^M F|\mu, f, y^0) = \int_{F^{-1}(\rho)\cap M} \psi_i \tau\, d\mathcal{H}^{n-1}.$$

By a density argument we can conclude that

$$\theta_\rho(|\nabla^M F|\mu, f, y^0) = \tau\mathcal{H}^{n-1} \llcorner f^{-1}(\partial B_\rho(y^0))$$

for $\mathcal{L}^1 - a.e.\, \rho \in \mathbb{R}^+$. It is clear this implies the thesis.

In the case of a function f which is only μ–apdifferentiable the thesis similarly follows once we know (5.13).

To achieve this we use lemma 5.6 and consider the measures $\mu^h = \mu \llcorner K_h$. They satisfy the hypotheses of lemma and adding them together, by linearity in (5.13), we prove also this case. □

Now we use the two lemmas to prove the proposition 5.5. We remind that the localization in the x variable creates an extra boundary measure σ given by the weak* limit of

$$\sigma_m = h'_m(r)P_{ij}(x)\frac{x_j - x_j^0}{r}\mu_V$$

as $m \to +\infty$. We can suppose that the total variations of σ_m converge to a Radon measure λ, hence $|\sigma| \leq \lambda$. By the equation (5.11) we deduce that $\pi_\#\lambda \ll \theta_\delta(\mu_V, Id_M, x^0)$, with the notations of lemma 5.8. As the last one belongs to $\mathcal{R}_{n-1}(\Omega)$ the same holds for the measure $\pi_\#|\sigma|$.

The localization in P is a bit more involved. Let

$$\sigma_m = V \llcorner \left\{h'_m(|P - P_0|)A_{ijk}(x, P)D^*_{jk}|P - P_0|\right\}$$

be converging to σ and let us suppose that $|\sigma_m|$ converge to a positive Radon measure λ on $G_n(\Omega)$. It is then evident that $\pi_\#|\sigma_m| \to \pi_\#\lambda$ and $|\sigma| \leq \lambda$. We prove the thesis showing that $\pi_\#\lambda \in \mathcal{R}_{n-1}(\Omega)$.

Indeed the equality holds

$$\pi_\#|\sigma_m| = \mu_V \llcorner \left\{\left|h'_m(|P - P_0|)A_{ijk}(x, P(x))\frac{P_{jk}(x) - P_{0jk}}{|P(x) - P_0|}\right|\right\}.$$

We know that the tangent space function $P : M \to G_{n,k}$ is μ_V–apdifferentiable hence we can estimate

$$\pi_\#|\sigma_m| \leq 4|\nabla^M F|\frac{\mu_V \llcorner P^{-1}(B_\rho(P_0) \setminus B_{\rho-1/m}(P_0))}{1/m}$$

where $F(x) = |P(x) - P_0|$.

It is now clear that applying lemma 5.8 with $f(x) = P(x)$, $\mu = \mu_V$ we get that for $\mathcal{L}^1 - a.e.\, \rho$ the weak* limit of any subsequence of $\pi_\#|\sigma_m|$ belongs to $\mathcal{R}_{n-1}(\Omega)$. $\square$

Remark 5.9. Performing at the same time localizations in x and P it turns out that for any $(x^0, P_0) \in G_n(\Omega)$ we have

$$V \llcorner B_\rho(x^0) \times B_\rho(P_0) \in AV_n(\Omega)$$

$$\sigma = \partial\big(V \llcorner (B_\rho(x^0) \times B_\rho(P_0))\big) - \partial V \llcorner (B_\rho(x^0) \times B_\rho(P_0)) \in \mathcal{R}_{n-1}(\Omega)$$

for arbitrarily small $\rho > 0$.

6. Compactness Properties

In this section we prove a compactness–semicontinuity theorem in the class of curvature varifolds with boundary such that the generalized second fundamental form belongs to $L^p_{loc}(V)$ with $p > 1$.

Theorem 6.1. *Let V_l be a sequence of curvature varifolds with boundary in $AV^p_n(\Omega)$, with $p > 1$, such that for every open set $W \subset\subset \Omega$*

$$\mu_{V_l}(W) + \int_{G_n(W)} \|A^{(l)}\|^p\, dV_l + |\partial\, V^{(l)}|(G_n(W)) \leq c(W) \qquad \forall l$$

where $c(W)$ is a real constant and $\|A^{(l)}\| = \sum_{i,j,k} |A^{(l)}_{ijk}|$. Then

(1) *There exists a subsequence V_{l_h} converging to a curvature varifold with boundary V, with $A_{ijk}^{(l_h)} \llcorner V_{l_h}$ weakly* converging to $A_{ijk} \llcorner V$ and $\partial V^{(l_h)}$ weakly* converging to ∂V.*
(2) *For every convex and lower semicontinuous function $f : \mathbb{R}^{k^3} \to [0, +\infty]$ we have the inequality*

$$\int_{G_n(\Omega)} f(A_{ijk})\, dV \le \liminf_{h\to\infty} \int_{G_n(\Omega)} f(A_{ijk}^{(l_h)})\, dV_{l_h}.$$

Proof. We remark that the hypotheses imply that the first variations of the varifolds V_l are locally equibounded. Hence we can use Allard's compactness theorem to get a subsequence V_{l_h} converging to an integer rectifiable varifold V.

By the Banach–Alaoglu theorem we can suppose that the measures $\partial V^{(l_h)}$ weakly* converge to a Radon measure ∂V and the measures $V_{l_h} \llcorner A_{ijk}^{(l_h)}$ weakly* converge to Radon measures σ_{ijk}.

To conclude the proof we apply the following theorem (see [7], compare with the measure function pairs of [13]).

Definition 6.2. Let $f : \mathbb{R}^s \to [0, +\infty]$ be a convex lower semicontinuous function with a more than linear growth at infinity, i.e.

$$\lim_{|z|\to+\infty} \frac{f(z)}{|z|} = +\infty.$$

We define a functional G on pairs of Radon measures (ν, μ) on the open subset Ω of a locally Euclidean space where μ is a positive measure and ν a vector measure with values in $\mathbb{R}^s$, setting

$$G(\nu, \mu) = \int_{\Omega} f\left(\frac{d\nu}{d\mu}(x)\right) d\mu(x)$$

if $\nu \ll \mu$ and $G(\nu, \mu) = +\infty$ otherwise ($d\nu/d\mu$ denotes the Radon–Nikodym derivative).

Theorem 6.3. *The functional G is sequentially lower semicontinuous with respect to the weak* convergence of measures, that is*

$$\nu_h \to \nu, \quad \mu_h \to \mu \qquad \Longrightarrow \qquad G(\nu, \mu) \le \liminf_{h\to\infty} G(\nu_h, \mu_h).$$

By this theorem (with $f(z) = |z|^p$) we infer the existence of functions $A_{ijk} \in L^p_{loc}(V)$ such that $\sigma_{ijk} = V \llcorner A_{ijk}$, so we obtain that V is a curvature varifold with boundary.

The lower semicontinuity of the curvature depending functionals follows again by the theorem above if f is superlinear. In the general case we approximate f by $f_\varepsilon(z) = f(z) + \varepsilon|z|^p$. □

This theorem can be used to find weak minima of several functionals depending on curvature of regular manifolds. We show an example of application which explains how this approach can be applied to study even more complex functionals, involving also the curvature of the boundary.

Let K be a compact subset of $\Omega \subset \mathbb{R}^3$, $p_1, p_2 > 1$. Setting

$$\mathcal{A} = \{(V_1, V_2) \mid V_i \in AV_i^{p_i}(\Omega),\ supp\, \mu_{V_i} \subset K,\ \partial V_1 = 0,\ \pi_\#(|\partial V_2|) \leq \mu_{V_1}\}$$

we can consider the problem

$$\min_{(V_1,V_2)\in\mathcal{A}} \int_\Omega |A_1|^{p_1}\, d\mu_{V_1} + \int_\Omega |A_2|^{p_1}\, d\mu_{V_2} + \|\mu_{V_2} - \nu\|$$

for a fixed Radon measure ν on Ω. Notice that if V_2 is a C^2 surface M and V_1 is its C^2 boundary ∂M, the functional essentially takes into account the difference between the measure ν and the measure associated to the surface M, penalizing the curvatures of M and of ∂M. Similar problems concerning stratified sets were considered by F. Morgan in [17] and [15].

We want to prove the existence of minima using the compactness theorem. If (V_1^n, V_2^n) is a minimizing sequence, the masses and the L^p integrals of the second fundamental forms of V_2^n are obviously equibounded, moreover the fact that $\partial V_1^n = 0$ and that the curvatures of V_1^n are equibounded in L^p gives, by the isoperimetric inequality for varifolds with equibounded supports (see [18]), a uniform bound on $\mu_{V_1^n}$, hence on $\|\partial V_2^n\|$. This, with the compactness theorem, imply that passing to a subsequence, we can suppose that $V_1^n \to V_1$ and $V_2^n \to V_2$ in varifold sense. Every term of the functional is lower semicontinuous so the pair (V_1, V_2) gives a minimum (notice that this pair belongs to the class $\mathcal{A}$).

We remark that the condition $\pi_\#(|\partial V_2|) \leq \mu_{V_1}$ is a weak formulation of the relation holding between a manifold and its boundary, applied to the two varifolds V_1 and V_2. Moreover it is simple to see that we could study the problem also in the enlarged class of pairs (V_1, V_2) with $\partial V_1 \neq 0$, adding to the functional a penalization depending on the mass of the boundary of V_1. This example can be obviously generalized considering chains of varifolds longer than two.

Finally we notice here that the iteration of the operation of taking the boundary, behaves particularly well when applied to polyhedral sets, considered as curvature varifolds with zero second fundamental form. Infact for a polyhedral set, if we take k–times the operation of boundary, we get (with a suitable weight) the $(n-k)$–skeleton.

7. A Boundary Rectifiability Result

In this section we prove that the boundary measure ∂V of a n–dimensional curvature varifold V is supported in $N \times G_{n,k}$ for a suitable countably $(n-1)$–rectifiable set N. To this aim we fix in this section a curvature varifold V and we denote with σ the positive Radon measure $\pi_\#|\partial V|$ on Ω.

Theorem 7.1 (Boundary Rectifiability)**.** *The measure σ belongs to $\mathcal{R}_{n-1}(\Omega)$, i.e., there exists a countably $(n-1)$–rectifiable set N in Ω and a positive Borel function $\tau : N \to \mathbb{R}$ such that $\sigma = \tau\mathcal{H}^{n-1} \llcorner N$.*

Remark 7.2. This property of the boundary measure is not shared by Allard's varifolds. For instance, if $u : [0,1] \to \mathbb{R}$ is the Cantor function and U is a primitive of u then the unit density varifold associated to the graph of U has a singular, non atomic mean curvature in $(0,1) \times \mathbb{R}$ supported in the part of the graph which projects on the Cantor set.

In the end of the section we will describe the complete structure of the measure ∂V.

Proof of Theorem 7.1. We first need a lemma.

Lemma 7.3. *Let $V \equiv V_{M,\theta}$ be a curvature varifold with boundary supported in $\Omega \times B_\delta(P_0)$, where δ is smaller than a dimensional constant $C = C(n,k)$ and suppose that $F : G_{n,k} \to \mathbb{R}$ is a C^1 function in a neighbourhood of $B_\delta(P_0)$. If we take an orthonormal basis $\{v_i\}$ of P_0, the n–rectifiable current*

$$T \equiv T(M, \theta(x)F(P(x)), \eta(x))$$

where

$$\eta(x) = \frac{\eta^1(x) \wedge \ldots \wedge \eta^n(x)}{|\eta^1(x) \wedge \ldots \wedge \eta^n(x)|} \qquad \eta^i(x) = \pi_{P(x)} v_i$$

is well defined and has a boundary of locally finite mass.

Proof. We can suppose that $P_0 =< e_1, \ldots, e_n >$. If C is small enough, $|P - P_0| < C$ implies that, denoting with $\pi_P : \mathbb{R}^k \to P$ the orthogonal projection on P, the vectors $\eta^i(P) = \pi_P(e_i)$ $i = 1, \ldots, n$ are a basis of P and

$$\beta(P) = \frac{< dx^1 \wedge \ldots \wedge dx^n, \eta^1(P) \wedge \ldots \wedge \eta^n(P) >}{|\eta^1(P) \wedge ... \wedge \eta^n(P)|} > 1/2. \tag{7.1}$$

It is hence clear that for $\mu_V - a.e.\, x \in \Omega$ the vectors $\eta^i(P(x))$ are a basis of $P(x)$, so the current T is well defined.

We consider the differential forms

$$\omega(x) = \varphi(x) dx^{i_2} \wedge \ldots \wedge dx^{i_n}.$$

Setting $I \equiv (i_2, \ldots, i_n)$ we have

$$\begin{aligned}
< \partial T, \omega > &=< T, d\omega >=< T, \frac{\partial \varphi(x)}{\partial x_i} dx^i \wedge dx^I > \\
&= \sum_{i=1}^{k} \int_M \theta(x) F(P(x)) \frac{\partial \varphi(x)}{\partial x_i} < dx^i \wedge dx^I, \eta(x) > \, d\mathcal{H}^n(x) \\
&= \sum_{i=1}^{k} \int_{\Omega \times B_\delta(P_0)} \frac{\partial \varphi(x)}{\partial x_i} F(P) < dx^i \wedge dx^I, \frac{\eta^1(P) \wedge \ldots \wedge \eta^n(P)}{|\eta^1(P) \wedge \ldots \wedge \eta^n(P)|} > \, dV(x,P) \\
&= \sum_{s=1}^{n} \sum_{i=1}^{k} (-1)^{s-1} \int_{\Omega \times B_\delta(P_0)} \frac{\partial \varphi(x)}{\partial x_i} F(P) dx^i(\eta^s(P)) \beta^I_s(P) \, dV(x,P)
\end{aligned}$$

recalling the functions β^I_s defined in (5.1). As $dx^i(\eta^s(P)) = P_{is}$ we get

$$< \partial T, \omega >= \sum_{s=1}^{n} \sum_{i=1}^{k} (-1)^{s-1} \int_{\Omega \times B_\delta(P_0)} P_{si} \frac{\partial \varphi(x)}{\partial x_i} F(P) \beta^I_s(P) \, dV(x,P). \tag{7.2}$$

It is now simple to see that extending the functions $\beta^I_s(P)$ to C^1 functions all over $G_{n,k}$ without modifying them on $B_C(P_0)$ and using the formula (3.1) we can state the inequality

$$| < \partial T, \omega > | \leq K \|\varphi\|_\infty$$

where K is a positive constant dependent only on the support of the form ω. $\square$

Now we take balls $B^\rho_\delta = B_\rho(x^0) \times B_\delta(P_0)$ in $G_n(\Omega)$ such that $V^\rho_\delta = V \llcorner B^\rho_\delta$ are again curvature varifolds with boundary in Ω (by the localization lemma 4.1) and δ is smaller than the constant C in the lemma above. We can suppose as usual that $P_0 =< e_1, \dots, e_n >$, so the current T^ρ_δ associated to V^ρ_δ with $F(P) \equiv 1$ is an integral current with boundary ∂T^ρ_δ of locally finite mass. Applying the boundary rectifiability theorem C, ∂T^ρ_δ is an $(n-1)$–integral current in Ω, $\partial T^\rho_\delta \equiv (N, \tau, \xi)$.

We now recall and continue the computation of lemma 7.3. Starting from (7.2) and using the formula (3.1) we get

$$\begin{aligned} < \partial T^\rho_\delta, \varphi\, dx^I > = \sum_{s=1}^n (-1)^{s-1} \Bigg\{ & \int_{B^\rho_\delta} \varphi(x) A_{sjk}(x,P) D^*_{jk} \beta^I_s(P)\, dV(x,P) \\ & + \int_{B^\rho_\delta} \varphi(x) \beta^I_s(P) A_{jsj}(x,P)\, dV(x,P) \\ & + \int_{B^\rho_\delta} \varphi(x) \beta^I_s(P)\, d\partial V_s(x,P) \\ & + \int_{G_n(\Omega)} \varphi(x) \beta^I_s(P)\, d\sigma_s(x,P) \Bigg\} \\ = \int_N \varphi(x)\tau(x) <dx^I, \xi> \, d\mathcal{H}^{n-1}(x) & \end{aligned}$$

where σ is the extra boundary measure given by the localization lemma 4.1. We suppose now that $\pi_\#|\sigma|$ belongs to $\mathcal{R}_{n-1}(\Omega)$ (proposition 5.5). Since $\pi_\#|\partial V|$ is singular with respect to μ_V (proposition 3.5), the sum of the first two integrals is zero. Hence the formula reduces to

$$\sum_{s=1}^n (-1)^{s-1} \Bigg\{ \int_{B^\rho_\delta} \varphi(x)\beta^I_s(P)\, d\partial V_s(x,P) + \int_{G_n(\Omega)} \varphi(x)\beta^I_s(P)\, d\sigma_s(x,P) \Bigg\}$$

$$= \int_N \varphi(x)\tau(x) < dx^I, \xi > \, d\mathcal{H}^{n-1}(x).$$

Again since φ is arbitrary we deduce that

$$\pi_\# \Bigg(\sum_{s=1}^n (-1)^s \beta^I_s\, \partial V_s \llcorner B^\rho_\delta + \beta^I_s \sigma_s \Bigg) = \tau < dx^I, \xi > \mathcal{H}^{n-1} \llcorner N$$

so that, recalling that $\pi_\#|\sigma| \in \mathcal{R}_{n-1}(\Omega)$, it follows

$$\pi_\#\left(\sum_{s=1}^{n}(-1)^s\beta_s^I\,\partial V_s \llcorner B_\delta^\rho\right) \in \mathcal{R}_{n-1}(\Omega). \tag{7.3}$$

We have proved that for any choice of $(x^0, P_0) \in G_n(\Omega)$ the formula (7.3) holds for arbitrary small ρ and δ.

We denote now with $\nu(x, P)$ the Radon–Nikodym derivative of ∂V with respect to its total variation $|\partial V|$, the fact that ∂V is tangential (lemma 3.7) implies that $\nu(x, P) \in P$ for $|\partial V|$–almost every $(x, P) \in G_n(\Omega)$, hence

$$\sum_{j=1}^{n}|\alpha_j|(x, P) > 0 \qquad |\partial V| - a.e.\ in\ G_n(\Omega) \tag{7.4}$$

where $\alpha_1, \dots, \alpha_n$ are the components of $\nu(x, P)$ in the basis $\eta^1(P), \dots, \eta^n(P)$.

We fix $j \in \{1, \dots, n\}$ and choose I such that $I \cup \{j\} = \{1, \dots, n\}$. Since $\partial V_s = \sum\limits_{i=1}^{n}\alpha_i\eta_s^i|\partial V|$, the formula (7.3) can be written as

$$\sum_{i=1}^{n}\pi_\#\left(\sum_{s=1}^{n}(-1)^s\alpha_i\eta_s^i\beta_s^I|\partial V| \llcorner B_\delta^\rho\right) \in \mathcal{R}_{n-1}(\Omega). \tag{7.5}$$

Noticing that $\eta_s^i = \eta_i^s$ and that

$$\sum_{s=1}^{n}(-1)^s\eta_i^s\beta_s^I = - < dx^i \wedge dx^I, \eta(P) > \tag{7.6}$$

the only term different from zero in (7.6) is the one with $i = j$, and it equals $(-1)^j\beta(P)$ (see (7.1)). Hence we obtain

$$\pi_\#\left(\alpha_j\beta|\partial V| \llcorner B_\delta^\rho\right) \in \mathcal{R}_{n-1}(\Omega).$$

By the next lemma, remark 5.9 and the fact that $\beta \neq 0$ on $B_\delta(P_0)$ we deduce that $\pi_\#|\alpha_j||\partial V| \in \mathcal{R}_{n-1}(\Omega)$ and by (7.4) we get $\pi_\#|\partial V| \in \mathcal{R}_{n-1}(\Omega)$. □

Lemma 7.4. *If μ is a signed Radon measure on $G_n(\Omega)$ such that, for every pair (x^0, P_0)*

$$\pi_\#\left(\mu \llcorner B_\rho(x^0) \times B_\rho(P_0)\right) \in \mathcal{R}_{n-1}(\Omega)$$

for arbitrarily small ρ, then

$$\pi_\#|\mu| \in \mathcal{R}_{n-1}(\Omega).$$

Proof. Let $A \subset G_n(\Omega)$ be a Borel set such that $\mu \llcorner A = \mu^+$ and let $K \subset A$ be an arbitrary compact set. The family of balls $B_\rho = B_\rho(x^0) \times B_\rho(P_0)$ of the hypothesis is a fundamental covering of $G_n(\Omega)$, so by the Besicovitch covering theorem (see [16] p. 14) it is possible to find, for every $\varepsilon > 0$, a sequence of pairwise disjoint balls $B_{\rho_i}^\varepsilon$ included in the ε–neighbourhood of K such that their union covers $|\mu|$–almost all K. The measures

$$\mu_\varepsilon = \sum_{i=1}^{\infty} \mu \llcorner B^\varepsilon_{\rho_i}$$

strongly converge to $\mu \llcorner K$ as $\varepsilon \to 0$. Since $\pi_\# \mu_\varepsilon \in \mathcal{R}_{n-1}(\Omega)$ it is clear that $\pi_\# \mu \llcorner K$ belongs to $\mathcal{R}_{n-1}(\Omega)$ too. Since $K \subset A$ is arbitrary we obtain that the projection of the positive part of μ belongs to $\mathcal{R}_{n-1}(\Omega)$. A similar argument for the negative part concludes the proof. □

Let N and τ be given by theorem 7.1. By standard measure theoretical arguments it is known that we can represent ∂V as

$$\int_{G_n(\Omega)} \varphi(x,P)\,\partial V = \int_N \left(\int_{G_{n,k}} \varphi(x,P)\, d\tau_x(P) \right) d\mathcal{H}^{n-1}(x) \tag{7.7}$$

where τ_x are Radon measures on $G_{n,k}$, univocally defined $\mathcal{H}^{n-1} \llcorner N$–almost everywhere such that $|\tau_x|(G_{n,k}) = \tau(x)$. Our next goal is the study of these measures; to this aim we have to analyse the density properties of V.

Lemma 7.5. *If V is a curvature n–varifold with boundary in Ω, the density ratios of V*

$$\frac{\mu_V(B_\rho(x))}{\rho^n}$$

are bounded for $x \in \Omega \setminus L$ where L is an $(n-1)$–purely unrectifiable set in Ω (see [9]*, Chapter 3). In the special case $n = 1$ the density ratio is bounded for* every *point of Ω.*

Proof. We first suppose that the varifold $\widetilde{V} \equiv \widetilde{V}_{M,\theta}$ has support contained in $B_\delta(x^0) \times B_\delta(P_0)$ where δ is smaller than the constant C of lemma 7.3, we also suppose $P_0 \equiv < e_1, \dots, e_n >$ to simplify the calculation.

We consider the rectifiable (not necessarily integral) current $T \equiv T(M, \theta(x)F(P(x)), \eta(x))$ of lemma 7.3. We have seen that this current has a boundary of locally finite mass.

Let $\pi : \mathbb{R}^k \to \mathbb{R}^n$ be the projection map on the first n coordinates and $S = \pi_\# T$, so S is an n–current in $\mathbb{R}^n$ with compact support and boundary of finite mass. We study now the current S.

Consider a differential form $\omega(y) = \varphi(y)\, dy^1 \wedge, \dots, \wedge dy^n$ on $\mathbb{R}^n$ and remind that the function $\beta(P)$, defined in formula (7.1), represents the Jacobian of the projection map π.

$$
\begin{aligned}
< S, \omega > &= < T, \pi^{\#}\omega > \\
&= \int_M \varphi(x)\theta(x)F(P(x)) \frac{< dx^1 \wedge, \dots, \wedge dx^n, \eta^1(x)\wedge, \dots, \wedge \eta^n(x) >}{|\eta^1(x)\wedge, \dots, \wedge\eta^n(x)|} \, d\mathcal{H}^n(x) \\
&= \int_M \varphi(x)\theta(x)F(P(x))\beta(P(x)) \, d\mathcal{H}^n(x) \\
&= \int_{\mathbb{R}^n} \varphi(y)\Big(\int_{\pi^{-1}(y)\cap M} \theta(x)F(P(x)) \, d\mathcal{H}^0(x) \Big) \, d\mathcal{H}^n(y) \\
&= \int_{\mathbb{R}^n} \varphi(y)\psi(y) \, d\mathcal{H}^n(y)
\end{aligned}
$$

using the coarea formula and defining the function

$$
\psi(y) = \int_{\pi^{-1}(y)\cap M} \theta(x)F(P(x)) \, d\mathcal{H}^0(x).
$$

The fact that the current S has a boundary of finite mass implies that ψ is a function in $BV(\mathbb{R}^n)$. Choosing $F(P) = \beta(P)^{-1}$ we have that the function

$$
\phi(y) = \int_{\pi^{-1}(y)\cap M} \theta(x) \frac{1}{\beta(P(x))} \, d\mathcal{H}^0(x)
$$

belongs to $BV(\mathbb{R}^n)$. We use this fact to give an upper estimate to the density ratios: indeed

$$
\begin{aligned}
\mu_V(B_\rho(x^0)) &\leq \mu_V(\pi^{-1}(B_\rho(\pi(x^0))) \\
&= \int_{B_\rho(\pi(x^0))} \Big(\int_{\pi^{-1}(y)\cap M} \theta(x) \frac{1}{\beta(P(x))} \, d\mathcal{H}^0(x) \Big) \, d\mathcal{H}^n(y) \\
&= \int_{B_\rho(\pi(x^0))} \phi(y) \, d\mathcal{H}^n(y).
\end{aligned}
$$

So when at $\pi(x^0)$ the last term is bounded, we have an upper estimate for the density ratios at x^0.

We apply now the following theorem about BV functions in $\mathbb{R}^n$.

Theorem D. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a BV function, in $\mathcal{H}^{n-1}$–almost every point $x \in \mathbb{R}^n$ the ratio*

$$
\frac{1}{\rho^n} \int_{B_\rho(x)} |f(y)| \, d\mathcal{H}^n(y)
$$

is bounded.

For a proof of this fact see [9], Theorem 4.5.9.

Going back to an arbitrary varifold V,we can choose a finite family of sets $S^i = \Omega \times B_{\delta_i}(P_i)$ such that their union is $G_n(\Omega)$, $V^i = V \llcorner S^i$ is again a curvature varifold with boundary in Ω and $\delta_i < C$.

Let us suppose by contradiction that there exists a $(n-1)$–dimensional embedded C^1 manifold M' and a subset of positive $\mathcal{H}^{n-1}$–measure M where the density ratios are not bounded. There exists a restriction varifold V^i whose density ratios are not bounded in a subset A of positive measure of M. Varying possibly a little bit the projection space P_i, B is mapped on a set of positive $\mathcal{H}^{n-1}$ measure in $\mathbb{R}^n$ and this is a contradiction. So we proved the lemma. □

Before going on we need a definition.

Definition 7.6. Given a point $x^0 \in \Omega$ we define the set $VarTan(V, x^0)$ as the collection of the weak limits (as varifolds in $\mathbb{R}^k$) when ρ goes to zero of the family of rescaled varifolds $V_{x^0,\rho} \equiv \rho^{-n}(\frac{x-x^0}{\rho} \times Id)_\# V$ (see [18]). Sometimes, with an abuse of notation, when $VarTan(V, x)$ consists of an unique element T we denote it with $VarTan(V, x)$.

With this definition, Lemma 7.5 implies that $VarTan(V, x) \neq \emptyset$ for $x \in \Omega \setminus S$.

We can now describe the complete structure of the boundary measure ∂V.

Proposition 7.7. *Recalling the formula* (7.7) *the measures* τ_x *are described by*

$$\tau_x = \sum_{i=1}^{k_x} \nu_i^x m_i^x \delta_{P_i^x}(P)$$

where $\delta_{P_i^x}$ *is the Dirac delta measure supported in some* n*–subspace* P_i^x *on the Grassmannian* $G_{n,k}$*,* m_i^x *are positive integers and* ν_i^x *are unit vectors of* $\mathbb{R}^k$*. Moreover the subspace* P_i^x *contains the tangent space to* N *at* x *and it is generated by the linear combinations of its elements with the vector* ν_i^x*.*

Proof. By the lemma 7.5 for σ–a.e. $x^0 \in \Omega$ the following conditions hold:

- at x^0 there exist the density and the approximate tangent space S to the $(n-1)$–varifold defined by σ with support N.
- The density ratios are bounded at x^0 so there exists a sequence $\rho_i \to 0$ such that $V_{x^0,\rho_i} \to T$, and $T \in VarTan(V, x^0)$ is a curvature varifold with boundary in $\mathbb{R}^k$.
- The limit holds

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \int_{B_\rho(x^0)} |A| \, d\mu_V = 0.$$

- Along the sequence above the measures $\partial V_{x^0,\rho_i}$ converge to the measure ∂T that has the form

$$\int_{G_n(\Omega)} \varphi(x, P) \, \partial T = \int_S \left(\int_{G_{n,k}} \varphi(x, P) \, d\tau_{x^0}(P) \right) d\mathcal{H}^{n-1}(x)$$

  where S is the $(n-1)$–vector subspace of $\mathbb{R}^k$ defined above and τ_x are univocally defined at $\sigma \llcorner N$–almost all $x \in \Omega$ by

$$\int_{G_n(\Omega)} \varphi(x,P)\,\partial V = \int_N \left(\int_{G_{n,k}} \varphi(x,P)\, d\tau_x(P) \right) d\mathcal{H}^{n-1}(x).$$

Considering T as a varifold in $\mathbb{R}^k \setminus S$, T is a curvature varifold without boundary with zero second fundamental form. By a result of Hutchinson (see [12], p. 292) T consists of an union (with multiplicities) of three kind of sets: 1) affine n–subspaces not including the origin, 2) n–halfspaces H_i with boundary S and 3) n–affine subspaces for the origin intersecting transversally S.

It is simple to see that the subspaces of kind 2) and 3) are finite because of the upper bound of the density ratios.

From this we see that the boundary measure of T is described by

$$\sum_i \left(\mathcal{H}^{n-1} \llcorner S\right) \times m_i \delta_{P_i} \nu_i$$

where P_i are the subspaces determined by the halfspaces H_i, m_i are their *integer* multiplicities, and ν_i are the inner normal vectors to S with respect to H_i.

It follows that

$$\tau_{x^0}(P) = \sum_{i=1}^{k_{x^0}} \nu_i^{x^0} m_i^{x^0} \delta_{P_i^{x^0}}$$

hence the thesis. □

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