Partial regularity for non autonomous functionals with non standard growth conditions

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ABSTRACT. We prove a $C^{1,\mu}$ partial regularity result for minimizers of a non autonomous integral funcitional of the form

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x, Du)\, dx$$

under the so-called non standard growth conditions. More precisely we assume that

$$c|z|^p \le f(x,z) \le L(1+|z|^q),$$

for $2 \le p < q$ and that $D_z f(x,z)$ is α-Hölder continuous with respect to the x-variable. The regularity is obtained imposing that $\frac{p}{q} < \frac{n+\alpha}{n}$ but without any assumption on the growth of $D_z^2 f$.

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1 Introduction

Let us consider the integral functional

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x, Du)\, dx \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^N$, $n \ge 2$ and $N \ge 1$.

Throughout the paper we shall assume that the integrand f is a $C^2(\Omega \times \mathbb{R}^{n\times N})$ function satisfying the following non standard growth condition

$$c|\xi|^p \le f(x,\xi) \le L(1+|\xi|^q), \tag{F1}$$

for some suitable $2 \le p < q$ and with c and L positive constants. Concerning the derivatives of f, in view of the controls from above and below, we have the following natural assumptions:

$$\nu(1+|\xi|^2)^{\frac{p-2}{2}}|\zeta|^2 \le \langle D_{\xi\xi} f(x,\xi)\zeta, \zeta\rangle; \tag{F2}$$

$$|D_\xi f(x_1,\xi) - D_\xi f(x_2,\xi)| \le C|x_1-x_2|^\alpha\ (1+|\xi|^{q-1}); \tag{F3}$$

for any $\xi, \zeta, \xi_1, \xi_2 \in \mathbb{R}^{nN}$, for any $x, x_1, x_2 \in \Omega$ and where C and ν are positive constants. Then by assumption (F1) we are dealing with functionals satisfying the so-called non standard growth conditions.

The study of the properties of minimizers of such functionals started with a series of seminal papers by Marcellini (see [16, 17]), in case of autonomous functionals. From the very beginning it has been clear that, even in the scalar case, no regularity can be expected if the exponents p and q are too far apart.

In fact, Marcellini himself produced an example of functional with non standard growth conditions having unbounded minimizers (see [12] and [15]).

On the other hand if the ratio

$$\frac{q}{p} \leq c(n) \to 1 \tag{1.2}$$

as $n \to +\infty$ many regularity results are available both in the scalar and in the vectorial setting. The starting issue in treating the regularity of minimizers is to show the higher integrability of the gradient. In this direction we quote [8, 9, 10, 11, 18]. We stress that, in this setting, this kind of regularity is crucial; indeed, since many apriori estimates depend on the L^q norm because of the right hand side of (F2), the first step in the analysis of the regularity of minimizers is just to improve the integrability of Du from L^p to L^q.

On the other hand $C^{1,\mu}$ partial regularity results have been established (see [4], [19]), without using higher integrability of the gradient, by means of a blow up argument. It is worth pointing out that all the quoted results concern autonomous functionals.

Only recently, the study of the regularity of non autonomous functionals with non standard growth produced both higher integrability and $C^{1,\mu}$ partial regularity. In particular, we quote the paper [10] by Esposito, Leonetti and Mingione where, under the above assumptions on f, it has been proved that a minimizer $u \in W^{1,p}_{loc}(\Omega)$ of $\mathcal{F}$ actually belongs to $W^{1,q}_{loc}(\Omega)$ if $\frac{q}{p} < \frac{n+\alpha}{n}$, provided that for the functional $\mathcal{F}$ does not occur the Lavrentiev Phenomenon. More precisely, introducing for a fixed ball $B_R \subset\subset \Omega$ and for every $u \in W^{1,p}(B_R)$ the gap functional relative to $\mathcal{F}$:

$$\mathcal{L}(u, B_R) := \bar{\mathcal{F}}(u) - \mathcal{F}(u), \qquad \mathcal{L}(u, B_R) := 0 \quad \text{if} \quad \mathcal{F}(u) = +\infty$$

where $\bar{\mathcal{F}}$ is the sequentially lower semicontinuous (s.l.s.) envelope of $\mathcal{F}$:

$$\bar{\mathcal{F}} := \sup \left\{ \mathcal{G} : W^{1,p}(B_R) \to [0, +\infty] : \mathcal{G} \quad \text{is} \quad \text{s.l.s.}, \quad \mathcal{G} \leq \mathcal{F} \quad \text{on} \quad W^{1,p}(B_R) \cap W^{1,q}(B_R) \right\},$$

the requirement is that:

$$\mathcal{L}(u, B_R) = 0, \quad \text{for any} \quad B_R \subset\subset \Omega. \tag{F4}$$

When the dependence on x is allowed, it is clear that a bound similar to (1.2) has to be assumed with $c(n)$ replaced by $c(n, \alpha)$ where α is the Hölder continuity exponent appearing in $(F3)$. More precisely Esposito, Leonetti and Mingione proved in [10] that a sufficient condition in order to have that a $W^{1,p}$ local minimizer of $\mathcal{F}$ belongs to $W^{1,q}$ is

$$\frac{q}{p} < \frac{n+\alpha}{n}. \tag{1.3}$$

Actually, by mean of a counterexample, in [10] the authors showed that (1.3) cannot be avoided in order to prove higher integrability of minimizers. In fact, if $\frac{q}{p} > \frac{n+\alpha}{n}$ there are local minimizers $u \in W^{1,p}_{loc}$ of suitable functionals such that $u \notin W^{1,q}_{loc}$.

In [6] assuming (1.3), Bildhauer and Fuchs prove $C^{1,\mu}$ partial regularity assuming that $D_\xi f$ is Lipschitz continuous with respect to x and that the second derivative of f with respect to ξ have a $(q-2)$-power type growth. These assumptions are stronger than the usual when one tries to establish $C^{1,\mu}$ partial regularity results.

The aim of the present paper is to remove these stronger assumptions on f showing that $C^{1,\mu}$ partial regularity still hold for minimizers. In fact we are able to prove the following

Theorem 1.1. *Let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ satisfy the assumptions (F1), (F2), (F3), (F4) and let $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ be a local minimizer of $\mathcal{F}$. Assume that $p \geq 2$ and* (1.3) *holds. Then there exists an open subset Ω_0 of Ω such that*

$$|\Omega \setminus \Omega_0| = 0$$

and $u \in C^{1,\mu}(\Omega_0;\mathbb{R}^N)$ for some $\mu \in (0,1)$.

Our proof is based on a decay estimate for the excess function which measures how far the gradient of minimizers is far from being constant in a ball $B_R(x_0)$. In our case the excess is defined as

$$E(x,r) = \fint_{B_r(x)} |Du-(Du)_r|^2 + |Du-(Du)_r|^p + r^\beta.$$

with $\delta < \alpha$, where α is the Hölder continuity exponent appearing in (F3).

We shall prove the decay estimate by using a standard argument consisting in blowing up the solution in small balls and reducing the problem to the study of convergence of minimizers of a suitable rescaled functionals in the unit ball. A useful tool in order to let this argument work is the higher integrability of the minimizers of the rescaled functionals. Note that we need an higher integrability result which is uniform with respect to the rescaling procedure. Hence we cannot use the result in [10] and the higher integrability result will be proved in Proposition 3.1.

Even though the result in [10] holds true for $p > 1$, here we confine ourselves tho the case $p \geq 2$ in order to avoid the heavy technicalities needed to treat the case $1 < p < 2$, which, however, will be faced into the forthcoming paper [7].

We also mention that by the method introduced in [14] we are able to estimate the Hausdorff dimension of the singular set. In fact we have the following

Theorem 1.2. *Under the same assumptions on f, p and q as in Theorem 1.1, if $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ then*

$$dim_{\mathcal{H}}(\Omega \setminus \Omega_0) < n - \frac{\alpha}{2}p \tag{1.4}$$

where α is the exponent appearing in (F3).

2 Preliminaries

In this section we recall some standard definitions and collect several Lemmas that we shall need to establish our main result.

First of all we recall the definition of local minimizer for a functional with nonstandard growth conditions.

Definition 2.1. A function $u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if $x \to f(x,Du(x)) \in L^1_{loc}(\Omega)$ and

$$\int_{supp\,\varphi} f(x,Du)\,dx \leq \int_{supp\,\varphi} f(x,Du+D\varphi)\,dx,$$

for any $\varphi \in W^{1,1}_{loc}(\Omega,\mathbb{R}^N)$ with $supp\,\varphi \subset \Omega$.

The higher integrability of the minimizers will be achieved by means of imbedding theorem in the context of fractional order Sobolev spaces, where we have to use fractional difference quotient. Therefore we introduce the following finite difference operator.

Definition 2.2. If $G:\mathbb{R}^n\to\mathbb{R}^k$ is a vector valued function the finite difference operator for G is defined by

$$\tau_{s,h}G(x)=G(x+he_s)-G(x)$$

where $h\in\mathbb{R}$, e_s is the unit vector in the x_s direction and $s\in\{1,\dots,n\}$.

The basic properties of the finite difference operator are described in the following proposition whose proof can be found, for example, in [13].

Proposition 2.3. *Let F and G be two functions such that $F,G\in W^{1,p}(\Omega)$, with $p\geq 1$, and let us consider the set*

$$\Omega_{|h|}:=\left\{x\in\Omega : dist(x,\partial\Omega)>|h|\right\}.$$

Then

(d1) $\tau_{s,h}F\in W^{1,p}(\Omega)$ *and*

$$D_i(\tau_{s,h}F)=\tau_{s,h}(D_iF).$$

(d2) If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then

$$\int_\Omega F\,\tau_{s,h}G\,dx=-\int_\Omega G\,\tau_{s,-h}F\,dx.$$

(d3) We have

$$\tau_{s,h}(FG)(x)=F(x+he_s)\tau_{s,h}G(x)+G(x)\tau_{s,h}F(x).$$

Next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 2.4. *If $0<\rho<R$, $|h|<R-\rho$, $1\leq p<+\infty$, $s\in\{1,\dots,n\}$ and $F,D_sF\in L^p(B_R)$ then*

$$\int_{B_\rho}|\tau_{s,h}F(x)|^p\,dx\leq|h|^p\int_{B_R}|D_sF(x)|^p\,dx. \tag{2.1}$$

Moreover

$$\int_{B_\rho}|F(x+he_s)|^p\,dx\leq c(n,p)\int_{B_R}|F(x)|^p\,dx. \tag{2.2}$$

Next Lemma, useful to estimate the different quotient of a function, is of particular interest for us.

Lemma 2.5. *For every $p>1$ and $G:B_R\to\mathbb{R}^k$ there exists a positive constant $c\equiv c(k,p)$ such that*

$$|\tau_{s,h}((1+|G(x)|^2)^{(p-2)/4}G(x))|^2\leq c(1+|G(x)|^2+|G(x+he_s)|^2)^{(p-2)/2}|\tau_{s,h}G(x)|^2$$

for every $x\in B_\rho$, with $|h|<\frac{R-\rho}{2}$ and every $s\in\{1,\dots,n\}$.

Now we recall the fundamental embedding properties for fractional order Sobolev spaces. (For the proof see, for example, [5]).

Lemma 2.6. *If $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^2(B_R)$ and for some $\rho \in (0, R)$, $\beta \in (0, 1]$, $M > 0$, we have that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^2 \, dx \leq M^2 |h|^{2\beta}$$

for every h with $|h| < \frac{R-\rho}{2}$, then $F \in W^{k,2}(B_\rho; \mathbb{R}^N) \cap L^{\frac{2n}{n-2k}}(B_\rho; \mathbb{R}^N)$ for every $k \in (0, \beta)$ and

$$||F||_{L^{\frac{2n}{n-2k}}(B_\rho)} \leq c \left(M + ||F||_{L^2(B_R)} \right),$$

with $c \equiv c(n, N, R, \rho, \beta, k)$.

Let us recall that the singular set Σ of a local minimizer u of the functional $\mathcal{F}$ is included in the set of non-Lebesgue points of the gradient of u. Therefore the estimate for the Hausdorff dimension of Σ is an immediate corollary of the higher integrability result stated in Proposition 3.1 in the next section through the application of the following proposition that can be found, for example, in [14].

Lemma 2.7. *Let $v \in W^{\theta,p}(\Omega, \mathbb{R}^N)$ where $\theta \in (0, 1)$, $p > 1$ and set*

$$A := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} \fint_{B(x,\rho)} |v(y) - (v)_{x,\rho}|^p \, dy > 0 \right\} \cup \left\{ x \in \Omega : \limsup_{\rho \to 0^+} |(v)_{x,\rho}| = +\infty \right\}.$$

Then $dim_{\mathcal{H}}(A) \leq n - \theta p$.

Next Lemma finds an important application in the so called hole-filling method. Its proof can be found in [13].

Lemma 2.8. *Let $h : [\rho, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \theta < 1$, $0 \leq A$, $0 < \beta$. Assume that*

$$h(r) \leq \frac{A}{(d-r)^\beta} + \theta h(d)$$

for $\rho \leq r < d \leq R_0$. Then

$$h(\rho) \leq \frac{cA}{(R_0 - \rho)^\beta},$$

where $c = c(\theta, \beta) > 0$.

Now, for our future needs, we introduce the rescaled functional on the unit ball $B \equiv B_1(0)$

$$\mathcal{I}(v) := \int_B g(y, Dv) \, dy$$

where

$$g(y, \xi) = \frac{f(x_0 + r_0 y, A + \lambda\xi) - f(x_0 + r_0 y, A) - D_\xi f(x_0 + r_0 y, A)\lambda\xi}{\lambda^2}. \tag{2.3}$$

Here A is a matrix such that $|A|$ is uniformly bounded by a positive constant M and λ is a parameter such that $0 < \lambda < 1$. Next Lemma contains the growth conditions on g.

Lemma 2.9. *Let $p \geq 2$ and let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ be a function satisfying the assumptions (F1), (F2) and (F3). Let $g(y,\xi)$ be defined by (2.3) then we have*

$$c(|\xi|^2 + \lambda^{p-2}|\xi|^p) \leq g(y,\xi) \leq c(|\xi|^2 + \lambda^{q-2}|\xi|^q); \tag{I1}$$

$$|D_\xi g(y,\xi)| \leq c(|\xi| + \lambda^{q-2}|\xi|^{q-1}); \tag{I2}$$

$$|D_\xi g(y_1,\xi) - D_\xi g(y_2,\xi)| \leq c\frac{r_0^\alpha}{\lambda}\ (1 + \lambda^{q-1}|\xi|^{q-1})|y_1 - y_2|^\alpha; \tag{I3}$$

$$c(1 + \lambda^2|\xi|^2)^{\frac{p-2}{2}}|\zeta|^2 \leq \langle D_{\xi\xi} g(y,\xi)\zeta, \zeta\rangle \tag{I4}$$

where the constant c depends on M and on q.

Proof. The (I1) can be proved as in Lemma 2.3 of [3] and the (I2) is an immediate consequence of the convexity of g.

Now we prove (I3). Thanks to the definition of g we have that

$$D_\xi g(y,\xi) = \frac{1}{\lambda}[D_\xi f(x_0 + r_0 y, A + \lambda\xi) - D_\xi f(x_0 + r_0 y, A)].$$

So by (F3) we get

$$\begin{aligned} |D_\xi g(y_1,\xi) - D_\xi g(y_2,\xi)| \leq & \frac{1}{\lambda}|D_\xi f(x_0 + r_0 y_1, A + \lambda\xi) - D_\xi f(x_0 + r_0 y_2, A + \lambda\xi)| \\ & + \frac{1}{\lambda}|D_\xi f(x_0 + r_0 y_1, A) - D_\xi f(x_0 + r_0 y_2, A)| \\ \leq & \frac{r_0^\alpha}{\lambda}|y_1 - y_2|^\alpha[(1 + |A + \lambda\xi|^{q-1}) + (1 + |A|^{q-1})] \\ \leq & \frac{r_0^\alpha}{\lambda}|y_1 - y_2|^\alpha(c(M) + \lambda^{q-1}|\xi|^{q-1}) \leq c\frac{r_0^\alpha}{\lambda}|y_1 - y_2|^\alpha(1 + \lambda^{q-1}|\xi|^{q-1}). \end{aligned}$$

where the constant c depends on M and on q.

To prove the (I4) it is enough to develop the second derivatives of g with respect to ξ and to observe that

$$D_{\xi\xi} g(y,\xi) = D_{\xi\xi} f(x_0 + r_0 y, A + \lambda\xi).$$

So we are led to the ellipticity condition (F2) on f. □

We shall denote by MF the Hardy-Littlewood maximal function of a function $F \in L^1_{loc}$, which is defined as

$$MF(x) = \sup_{x\in Q} \fint_Q |F(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, with sides parallel to coordinate axes.

The following Lemma can be found in [2].

Lemma 2.10. *Let $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ and $p \geq 1$. For every $K > 0$, if we set*

$$H_K = \{x \in \mathbb{R}^n \,:\, M(|Du|) \leq K\},$$

then there exists $v \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^N)$ such that $||Dv||_\infty \leq c\,K$, $v = u$ on H_K, and

$$meas(\mathbb{R}^n \setminus H_K) \leq \frac{c\,||Dv||^p_{L^p}}{K^p}.$$

3 Higher integrability

The first step in the proof of Theorem 1.1 is to obtain an higher integrability result for minimizers of the rescaled functional $\mathcal{I}$. To be more precise we need this result for the following perturbation or $\mathcal{I}$

$$\mathcal{J}(v) := \int_{B_{\tilde{R}}} g(y, Dv)\ dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda}(Dw - Dv)\, dy.$$

where $v \in w + W_0^{1,q}(B_{\tilde{R}})$ and x_0, r_0, A are the same appearing in the definition of g, therefore $|A| \leq M$.
We obtain the higher integrability with the following

Proposition 3.1. *Let us suppose that $g \in C^2(B_1(0), \mathbb{R}^{n\times N})$ satisfies the assumptions $(I1)$, $(I2)$, $(I3)$ and $(I4)$ with $2 \leq p \leq q < p\left(\frac{n+\alpha}{n}\right)$. If the function $v \in W^{1,q}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\mathcal{J}$ then there exist $\delta > 0$, $\sigma > 0$ such that*

$$\int_{B_\rho} (|Dv(y)|^2 + \lambda^{q-2}|Dv(y)|^q)^{1+\delta}\ dy \leq c\left(\int_{B_R} (1 + |Dv(y)|^2 + \lambda^{p-2}|Dv(y)|^p)\ dy\right)^\sigma \tag{3.1}$$

for every $B_R \subset\subset \Omega$, $\rho < R$ and for a positive constant c which depends on ρ and R but does not depend on v and it is also independent of the parameters λ, r_0 and of the point x_0 appearing in the definition of $g(y, \xi)$.

Proof. Let us fix a ball $B_{\tilde{R}} \subset\subset \Omega$; by the minimality of $v \in W^{1,q}(\Omega; \mathbb{R}^N)$ we have

$$\int_{B_{\tilde{R}}} g(y, Dv)\ dy \leq \int_{B_{\tilde{R}}} g(y, Dv + D\varphi)\ dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} D\varphi\, dy. \tag{3.2}$$

for every $\varphi \in W_0^{1,q}(B_{\tilde{R}}; \mathbb{R}^N)$. For a fixed $\varepsilon \in (0, 1)$ we can write (3.2) as follows

$$\int_{B_{\tilde{R}}} [g(y, Dv + \varepsilon\, D\varphi) - g(y, Dv)]\ dy + \int_{\tilde{R}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} \varepsilon D\varphi\, dy \geq 0$$

which is equivalent to

$$\int_{B_{\tilde{R}}} \int_0^1 D_\xi g(y, Dv + \varepsilon t D\varphi)\, \varepsilon\, D\varphi\ dt\, dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} \varepsilon D\varphi\, dy \geq 0.$$

Dividing the previous inequality by ε, changing φ in $-\varphi$ and taking the limit as $\varepsilon \to 0^+$, thanks to the assumption of continuity of the function $D_\xi g$, we get the Euler-Lagrange equations

$$\int_{B_{\tilde{R}}} D_\xi g(y, Dv)\, D\varphi\, dy + \int_{B_{\tilde{R}}} \frac{D_\xi f(x_0 + r_0 y, A)}{\lambda} D\varphi\, dy = 0. \tag{3.3}$$

Let us pick $0 < \rho \leq r < d \leq \tilde{R} \leq 1$ and let η be a cut-off function in $C_0^\infty(B_{\frac{d+r}{2}})$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_r and $|D\eta| < 4/(d - r)$. Let us consider the function $\varphi = \tau_{s,-h}(\eta^2 \tau_{s,h} v)$ with s fixed in $\{1, \ldots, n\}$ (which from now on we shall omit for the sake of simplicity) and $0 \leq |h| < (d - r)/4$. Now we plug such function φ into (3.3) and use $(d1)$ and $(d2)$ of Propostion 2.3 to get

$$- \int_{B_{\tilde{R}}} \tau_h\left(D_\xi g(y, Dv)\right) D(\eta^2 \tau_h v)\, dy$$
$$- \frac{1}{\lambda} \int_{B_{\tilde{R}}} [D_\xi f(x_0 + r_0(y + he_s), A) - D_\xi f(x_0 + r_0 y, A)] \cdot D(\eta^2 \tau_h v)\ dy = 0.$$

We develop the derivatives inside the first integral and use the Hölder continuity condition (F3) and the bound $|A| \leq M$ into the second one obtaining the estimate

$$\int_{B_{\tilde{R}}} \eta^2 \, \tau_h \left(D_\xi g(y, Dv)\right) \, \tau_h Dv \, dy \leq -2 \int_{B_{\tilde{R}}} \eta \, \tau_h (D_\xi g(y, Dv)) \, D\eta \otimes \tau_h v \, dy$$

$$+ c \, \frac{r_0^\alpha}{\lambda} \, |h|^\alpha \int_{B_{\tilde{R}}} |D(\eta^2 \tau_h v)| \, dy \tag{3.4}$$

with the constant c depending on M. Observing that

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \eta^2 \, \tau_h \left(D_\xi g(y, Dv)\right) \, \tau_h Dv \, dy \\
&\quad = \int_{B_{\tilde{R}}} \eta^2 \, \left[D_\xi g(y + he_s, Dv(y + he_s)) - D_\xi g(y, Dv(y))\right] \, \tau_h Dv \, dy \\
&\quad = \int_{B_{\tilde{R}}} \eta^2 \, \left[D_\xi g(y + he_s, Dv(y + he_s)) - D_\xi g(y, Dv(y + he_s))\right] \, \tau_h Dv \, dy \\
&\qquad + \int_{B_{\tilde{R}}} \eta^2 \, \left[D_\xi g(y, Dv(y + he_s)) - D_\xi g(y, Dv(y))\right] \, \tau_h Dv \, dy
\end{aligned}$$

we can write (3.4) as

$$\begin{aligned}
&\int_{B_R} \int_0^1 \eta^2 [D_{\xi\xi} g(y, Dv + t\tau_h Dv)] D(\tau_h v) D(\tau_h v) \, dt \, dy \\
&\quad \leq - \int_{B_{\tilde{R}}} \eta^2 \, \left[D_\xi g(y + he_s, Dv(y + he_s)) - D_\xi g(y, Dv(y + he_s))\right] \cdot \tau_h Dv \, dy \\
&\qquad - 2 \int_{B_{\tilde{R}}} \eta \, \tau_h (D_\xi g(y, Dv)) \, D\eta \otimes \tau_h v \, dy + c \, \frac{r_0^\alpha}{\lambda} \, |h|^\alpha \int_{B_{\tilde{R}}} |D(\eta^2 \tau_h v)| \, dy.
\end{aligned}$$

Now we use ellipticity condition (I4) in the left hand side and the growth conditions (I2) and (I3) in the right hand side. Thus the following estimate holds:

$$\begin{aligned}
&\int_{B_{\tilde{R}}} \eta^2 (1 + \lambda^2 |Dv(y)|^2 + \lambda^2 |Dv(y + he_s)|^2)^{\frac{p-2}{2}} |\tau_h Dv|^2 \, dy \\
&\leq c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_{\tilde{R}}} \eta^2 (1 + \lambda^{q-1} |Dv(y + he_s)|^{q-1}) \, |\tau_h Dv| \, dy \\
&\quad + c \int_{B_{\tilde{R}}} \eta \, |D\eta| (|Dv(y)| + |Dv(y + he_s)| + \lambda^{q-2} |Dv(y)|^{q-1} + \lambda^{q-2} |Dv(y + he_s)|^{q-1}) \, |\tau_h v| \, dy \\
&\quad + c \, \frac{r_0^\alpha}{\lambda} \, |h|^\alpha \int_{B_{\frac{d+r}{2}}} |D(\eta^2 \tau_h v)| \, dy := (I) + (II) + (III),
\end{aligned} \tag{3.5}$$

with $c \equiv c(n, N, p, q, L, \nu, M)$.

The use of Lemma 2.5 in the left hand side of (3.5) yields

$$\int_{B_{\tilde{R}}} \eta^2 |\tau_h ((1 + \lambda^2 |Dv|^2)^{\frac{p-2}{4}} Dv)|^2 \, dy \leq (I) + (II) + (III). \tag{3.6}$$

We have to estimate the integrals (I), (II) and (III). For (I) we simply use the definition of $\tau_h Dv$ and remember how we choose $|h|$ so that we can apply (2.2) of Lemma 2.4 as follows

$$(I) \leq c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_d} (|Dv(y)| + \lambda^{q-1} |Dv(y)|^q) \, dy$$

where $c \equiv c(n, N, p, q, L, \nu, M)$.

To estimate (II) we remember the assumptions on $|D\eta|$ and use triangle inequality which yields

$$\begin{aligned}(II) \leq & \frac{c}{(d-r)} \int_{B_{\frac{d+r}{2}}} (|Dv(y)| + |Dv(y+he_s)|) \, |\tau_h v| \, dy \\ & + \frac{c}{(d-r)} \lambda^{q-2} \int_{B_{\frac{d+r}{2}}} (|Dv(y)|^{q-1} + |Dv(y+he_s)|^{q-1}) \, |\tau_h v| \, dy\end{aligned}$$

then we apply Hölder inequality to each integral and we use (2.1) of Lemma 2.4 in each of the resulting addend, thus getting:

$$\begin{aligned}(II) \leq & \frac{c}{(d-r)} |h| \left(\int_{B_d} |Dv(y)|^2 \, dy \right)^{\frac{1}{2}} \left(\int_{B_d} |Dv(y)|^2 \, dy \right)^{\frac{1}{2}} \\ & + \frac{c}{(d-r)} |h| \, \lambda^{q-2} \left(\int_{B_d} |Dv(y)|^q \, dy \right)^{1-\frac{1}{q}} \left(\int_{B_d} |Dv(y)|^q \, dy \right)^{\frac{1}{q}}\end{aligned}$$

that is

$$(II) \leq \frac{c}{(d-r)} |h| \left(\int_{B_d} (|Dv(y)|^2 + \lambda^{q-2} |Dv(y)|^q) \, dy \right)$$

To estimate (III) we develop the derivative inside the integral and use triangle inequality, the assumptions on η and $|D\eta|$ and (2.1) of Lemma (2.4):

$$(III) \leq c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_{\frac{d+r}{2}}} |\tau_h Dv(y)| \, dy + \frac{c}{(d-r)} |h|^\alpha \frac{r_0^\alpha}{\lambda} \int_{B_{\frac{d+r}{2}}} |\tau_h v(y)| \, dy \leq c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_d} |Dv(y)| \, dy$$

where we also used the assumption $|h| < \frac{d-r}{4}$.

Collecting the estimates for (I), (II) and (III) and summing up on $s \in \{1, \dots, n\}$ we get, in place of (3.6), the following estimate

$$\begin{aligned}\int_{B_{\tilde{R}}} \sum_{s=1}^{n} \eta^2 |\tau_{s,h}((1+\lambda^2|Dv|^2)^{\frac{p-2}{4}} Dv)|^2 \, dy \leq & c \, |h|^\alpha \, \frac{r_0^\alpha}{\lambda} \int_{B_d} (|Dv(y)| + \lambda^{q-1} |Dv(y)|^q) \, dy \\ & + \frac{c}{(d-r)} |h| \int_{B_d} (|Dv(y)|^2 + \lambda^{q-2} |Dv|^q) \, dy,\end{aligned}$$

with $c \equiv c(n, N, r, d, p, q, L, \nu, M)$ independent of v, λ, r_0 and x_0. Notice that, in what follows, we shall have $\frac{r_0^\alpha}{\lambda} < 1$. Now we apply Lemma 2.6 and find that

$$(1+\lambda^2|Dv|^2)^{\frac{p-2}{4}} Dv \in L^{\frac{2n}{n-2\theta}}(B_r), \qquad \forall \, \theta \in \left(0, \frac{\alpha}{2}\right) \tag{3.7}$$

and

$$\int_{B_r} ((1+\lambda^2|Dv|^2)^{\frac{p-2}{2}} |Dv|^2)^{\frac{n}{n-2\theta}} \, dy \leq c \left(\int_{B_d} (1 + |Dv(y)|^2 + \lambda^{q-2} |Dv(y)|^q) \, dy \right)^{\frac{n}{n-2\theta}},$$

where

$$\frac{n}{n-2\theta} > 1.$$

But, since $2 \leq p \leq q$ we have

$$(J) := \int_{B_r} (|Dv(y)|^2 + \lambda^{p-2} |Dv(y)|^p)^{\frac{n}{n-2\theta}} \, dy \leq c \left(\int_{B_d} (1 + |Dv(y)|^2 + \lambda^{q-2} |Dv(y)|^q) \, dy \right)^{\frac{n}{n-2\theta}}. \tag{3.8}$$

Now we are going to estimate (J) from below in order to have an inequality which can be used to perform the same iteration procedure of [10]. We have

$$
\begin{aligned}
(J) &= \int_{B_r} (|Dv(y)|^{2\frac{q}{p}\frac{p}{q}} + \lambda^{(p-2)\frac{q}{p}\frac{p}{q}}|Dv(y)|^{p\frac{p}{q}\frac{q}{p}})^{\frac{n}{n-2\theta}}\, dy \\
&\geq c(p,q)\int_{B_r} (|Dv(y)|^{2\frac{q}{p}} + \lambda^{(p-2)\frac{q}{p}}|Dv(y)|^{q})^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy \qquad (3.9)
\end{aligned}
$$

where we used the elementary inequality

$$
(a^p + b^p) \geq c(p)(a+b)^p, \qquad \forall\, p > 0.
$$

Thus we have

$$
\begin{aligned}
(J') &:= \int_{B_r} (1 + |Dv(y)|^{2\frac{q}{p}} + \lambda^{(p-2)\frac{q}{p}}|Dv(y)|^{q})^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy \\
&\leq c\left(\int_{B_d} (1 + |Dv(y)|^{2} + \lambda^{q-2}|Dv(y)|^{q})\, dy\right)^{\frac{n}{n-2\theta}} \qquad (3.10)
\end{aligned}
$$

But we also have

$$
\begin{aligned}
(J') &\geq c\int_{B_r} ((1 + |Dv(y)|^{2})^{\frac{q}{p}} + \lambda^{(p-2)\frac{q}{p}}|Dv(y)|^{q})^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy \\
&\geq c\int_{B_r} (1 + |Dv(y)|^{2} + \lambda^{(p-2)\frac{q}{p}}|Dv(y)|^{q})^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy \qquad (3.11)
\end{aligned}
$$

since

$$
(1 + |Dv(y)|^2)^{\frac{q}{p}} \geq 1 + |Dv(y)|^2.
$$

Now we remember that $0 < \lambda < 1$ and observe that

$$
(p-2)\frac{q}{p} \leq q - 2
$$

since $2 \leq p \leq q$, so that

$$
\lambda^{q-2} < \lambda^{(p-2)\frac{q}{p}}.
$$

Hence we conclude that

$$
c\int_{B_r} (1 + |Dv(y)|^{2} + \lambda^{(p-2)\frac{q}{p}}|Dv(y)|^{q})^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy \geq c\int_{B_r} (1 + |Dv(y)|^{2} + \lambda^{q-2}|Dv(y)|^{q})^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy. \qquad (3.12)
$$

Collecting (3.8), (3.9), (3.10), (3.11) and (3.12) we can conclude that

$$
\int_{B_r} (1 + |Dv(y)|^{2} + \lambda^{q-2}|Dv(y)|^{q})^{\frac{p}{q}\frac{n}{n-2\theta}}\, dy \leq c\left(\int_{B_d} (1 + |Dv(y)|^{2} + \lambda^{q-2}|Dv(y)|^{q})\, dy\right)^{\frac{n}{n-2\theta}}.
$$

From here we can complete the proof using exactly the same iteration scheme of [10] with the same exponents. □

4 Decay estimate

Let $u \in W^{1,p}_{loc}(\Omega)$ be a local minimizer of $\mathcal{F}$ under the assumptions (F1), (F2), (F3), (F4) and define its excess function as

$$
E(x,r) = \fint_{B_r(x)} |Du - (Du)_r|^2 + |Du - (Du)_r|^p + r^\beta \qquad (4.1)
$$

with $\beta < \alpha$.

As usual the proof of Theorem 1.1 relies on a blow up argument which is contained in the following

Proposition 4.1. *Fix $M > 0$. There exists a constant $C(M) > 0$ such that, for every $0 < \tau < \frac{1}{4}$, there exists $\varepsilon = \varepsilon(\tau, M)$ such that, if*

$$|(Du)_{x_0,r}| \le M \qquad \textit{and} \qquad E(x_0, r) \le \varepsilon,$$

then

$$E(x_0, \tau r) \le C(M)\, \tau^\beta \, E(x_0, r).$$

Proof. Step 1. Blow up

Fix $M > 0$. Assume by contradiction that there exists a sequence of balls $B_{r_j}(x_j) \subset\subset \Omega$ such that

$$|(Du)_{x_j,r_j}| \le M \qquad \text{and} \qquad \lambda_j^2 = E(x_j, r_j) \to 0 \tag{4.2}$$

but

$$\frac{E(x_j, \tau r_j)}{\lambda_j^2} > \tilde{C}(M)\tau^\beta \tag{4.3}$$

where $\tilde{C}(M)$ will be determined later. Setting $A_j = (Du)_{x_j,r_j}$, $a_j = (u)_{x_j,r_j}$ and

$$v_j(y) = \frac{u(x_j + r_j y) - a_j - r_j A_j y}{\lambda_j r_j} \tag{4.4}$$

for all $y \in B_1(0)$, one can easily check that $(Dv_j)_{0,1} = 0$ and $(v_j)_{0,1} = 0$. By the definition of λ_j at (4.2), we get

$$\fint_{B_1(0)} |Dv_j|^2 + \lambda_j^{p-2}|Dv_j|^p \, dy + \frac{r_j^\beta}{\lambda_j^2} = 1 \tag{4.5}$$

Therefore passing possibly to not relabeled sequences

$$\begin{aligned}
&v_j \rightharpoonup v && \text{weakly in } W^{1,2}(B_1(0); \mathbb{R}^N) \\
&A_j \longrightarrow A && \\
&r_j \longrightarrow 0 && \frac{r_j^\gamma}{\lambda_h^2} \longrightarrow 0, \qquad \gamma > \beta.
\end{aligned} \tag{4.6}$$

Step 2. Minimality of v_j

We normalize f around A_j as follows

$$f_j(y, \xi) = \frac{f(x_j + r_j y, A_j + \lambda_j \xi) - f(x_j + r_j y, A_j) - D_\xi f(x_j + r_j y, A_j)\lambda_j \xi}{\lambda_j^2} \tag{4.7}$$

and we consider the corresponding rescaled functionals

$$\mathcal{I}_j(w) = \int_{B_1(0)} [f_j(y, Dw)] dy. \tag{4.8}$$

Observe that Lemma 2.9 applies to each f_j thus having that (I1), (I2), (I3), (I4) hold for f_j. The minimality of u yields that

$$\int_{B_1(0)} f(x_j + r_j y, Du(x_j + r_j y))\, dy \leq \int_{B_1(0)} f(x_j + r_j y, Du(x_j + r_j y) + D\varphi(x_j + r_j y))\, dy$$

for every $\varphi \in W^{1,q}(B_{r_j}(x_j); \mathbb{R}^N)$ that is

$$\int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y))\, dy \leq \int_{B_1(0)} f(x_j + r_j y, A_j + \lambda_j Dv_j(y) + D\varphi(x_j + r_j y))\, dy$$

for every $\varphi \in W^{1,q}(B_{r_j}(x_j); \mathbb{R}^N)$. Thus by the definition of the rescaled functionals, we have

$$\mathcal{I}_j(v_j) \leq \mathcal{I}_j(v_j + \varphi) + \int_{B_1(0)} \frac{D_\xi f(x_j + r_j y, A_j) D\varphi}{\lambda_j}\, dy. \tag{4.9}$$

Hence using (I3)

$$\begin{aligned} \mathcal{I}_j(v_j) &\leq \mathcal{I}_j(v_j + \varphi) + \int_{B_1(0)} \frac{[D_\xi f(x_j + r_j y, A_j) - D_\xi f(x_j, A_j)] D\varphi}{\lambda_j}\, dy \\ &\leq \mathcal{I}_j(v_j + \varphi) + c(M) \frac{r_j^\alpha}{\lambda_j} \int_{B_1(0)} |D\varphi|\, dy. \end{aligned} \tag{4.10}$$

Step 3. Higher integrability

Since $u \in W^{1,p}_{loc}(\Omega)$ is a local minimizer of $\mathcal{F}$ under the assumptions (F1), (F2), (F3), (F4), by Theorem 4 in [10], $u \in W^{1,q}(B_{r_j}(x_j))$. Therefore, by a simple change of variables, we also have that each $v_j \in W^{1,q}(B_1)$. Moreover, since v_j satisfy (4.9) and f_j satisfy (I1), (I2), (I3) and (I4), we are legitimate to apply Theorem 3.1. Hence there exist $\delta > 0$ and $\sigma > 0$ such that for all $\rho < 1$

$$\int_{B_\rho} (|Dv_j(y)|^2 + \lambda^{q-2}|Dv_j(y)|^q)^{1+\delta}\, dy \leq c \left(\int_{B_1} (1 + |Dv_j(y)|^2 + \lambda^{p-2}|Dv_j(y)|^p)\, dy \right)^\sigma \tag{4.11}$$

with c depending on M and ρ. But (4.5) yields

$$\int_{B_\rho} (|Dv_j(y)|^2 + \lambda^{q-2}|Dv_j(y)|^q)^{1+\delta}\, dy \leq c,$$

for every ball B_ρ contained in B_1. From that we obtain

$$v_j \rightharpoonup v \qquad \text{weakly in } W^{1,2(1+\delta)}_{loc}(B_1(0); \mathbb{R}^N).$$

Step 4. v solves a linear system

Using that v_j satisfies inequality (4.10), we conclude that

$$0 \leq \frac{c}{\lambda_j} \int_{B_1(0)} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi\, dy + \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_1(0)} |D\varphi| dy. \tag{4.12}$$

Following the argument in [1, 19], let us split

$$B_1(0) = E_j^+ \cup E_j^- = \{y \in B_1 :\ \lambda_j |Dv_j| > 1\} \cup \{y \in B_1 :\ \lambda_j |Dv_j| \leq 1\}$$

By (4.5) we get

$$|E_j^+| \leq \int_{E_j^+} \lambda_j^2 |Dv_j|^2\, dy \leq \lambda_j^2 \int_{E_j^+} |Dv_j|^2\, dy \leq \lambda_j^2. \tag{4.13}$$

By assumption (F1) and the convexity of f, applying Hölder's inequality we obtain

$$\begin{aligned}
&\frac{1}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)]D\varphi\, dy\right| \\
\leq& \frac{c}{\lambda_j}|E_j^+| + c\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^{q-1}\, dy \leq \frac{c}{\lambda_j}|E_j^+| + c\lambda_j^{q-2}\left(\int_{E_j^+}|Dv_j|^q\, dy\right)^{\frac{q-1}{q}}|E_j^+|^{\frac{1}{q}} \\
\leq& c\lambda_j\left(1 + \left(\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^q\, dy\right)^{\frac{q-1}{q}}\right).
\end{aligned} \tag{4.14}$$

The last term in (4.14) vanishes as $j \to \infty$. In fact, the higher integrability at (4.11) implies that

$$\lambda_j^{q-2}\int_{E_j^+}|Dv_j|^q\, dy \leq c.$$

Hence we infer that

$$\lim_{j\to\infty}\frac{c}{\lambda_j}\left|\int_{E_j^+}[D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)]D\varphi\, dy\right| = 0. \tag{4.15}$$

On E_j^- we have

$$\begin{aligned}
&\frac{1}{\lambda_j}\int_{E_j^-}[D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)]D\varphi\, dy \\
=& \int_{E_j^-}\int_0^1 D_{\xi\xi} f(x_j + r_j y, A_j + t\lambda_j Dv_j)\, dt Dv_j D\varphi\, dy
\end{aligned} \tag{4.16}$$

Note that (4.13) yields that $\chi_{E_j^-} \to \chi_{B_1}$ in L^r, for every $r < \infty$. Moreover by (4.6) we have, at least for subsequences, that

$$\lambda_j Dv_j \to 0 \quad \text{a.e. in } B_1, \quad r_j \to 0 \quad \text{and} \quad x_j \to x_0.$$

Hence the uniform continuity of $D_{\xi\xi} f$ on bounded sets implies

$$\lim_j \frac{1}{\lambda_j}\int_{E_j^-}[D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)]D\varphi\, dy = \int_{B_1} D_{\xi\xi} f(x_0, A)DvD\varphi\, dy. \tag{4.17}$$

Observe that by (4.6)

$$\lim_j \frac{r_j^\alpha}{\lambda_j} = 0. \tag{4.18}$$

By estimates (4.15), (4.17) and (4.18), passing to the limit as $j \to \infty$ in (4.12) yields

$$0 \leq \int_{B_1} D_{\xi\xi} f(x_0, A)DvD\varphi\, dy$$

Changing φ in $-\varphi$ we finally get

$$\int_{B_1} D_{\xi\xi} f(x_0, A)DvD\varphi\, dy = 0,$$

that is v solves a linear system which is elliptic thank to the convexity of f. Classical regularity results (see [12], [13]) imply that $v \in C^\infty(B_1)$ and for any $0 < \tau < 1$

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2 \, dy \le c\tau^2 \fint_{B_1} |Dv - (Dv)_1|^2 \, dy \le c\tau^2, \tag{4.19}$$

for a constant c depending on M.

Step 5. Upper bound

Let us fix $r < \frac{1}{4}$. Passing to a subsequence, it is not restrictive to assume that

$$\lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)]$$

exists. We shall prove that

$$\lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)] \le 0 \tag{4.20}$$

Let us choose $s < r$ and a cut-off function $\eta \in C^1_0(B_r)$ such that $\eta = 1$ on B_s, $0 \le \eta \le 1$ and $|D\eta| \le \frac{c}{r-s}$. Using in (4.10) as test function $\varphi_j = \eta(v - v_j)$, we get

$$\begin{aligned}
\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v) &\le \mathcal{I}_{j,r}(v_j + \varphi_j) - \mathcal{I}_{j,r}(v) + \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_r} |D\varphi_j| dy \\
\le& \int_{B_r \setminus B_s} [f_j(y, Dv_j + D\varphi_j) - f_j(y, Dv)] \, dy + \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_r} |D\varphi_j| dy \\
\le& \; c \int_{B_r \setminus B_s} (|Dv_j|^2 + \lambda_j^{q-2} |Dv_j|^q) \, dy + c \int_{B_r \setminus B_s} (|Dv|^2 + \lambda_j^{q-2} |Dv|^q) \, dy \\
+& \; c \int_{B_r \setminus B_s} \left(\frac{|v_j - v|^2}{(r-s)^2} + \lambda_j^{q-2} \frac{|v_j - v)|^q}{(r-s)^2} \right) dy + \frac{c(M) r_j^\alpha}{\lambda_j} \int_{B_r} |Dv_j - Dv| \, dy \\
+& \; \frac{c(M) r_j^\alpha}{\lambda_j (r-s)} \int_{B_r \setminus B_s} |v_j - v| \, dy,
\end{aligned} \tag{4.21}$$

thanks to the growth conditions on f_j. Now, we use (4.5) and (4.11) in order to have

$$\int_{B_r \setminus B_s} (|Dv_j|^2 + \lambda_j^{q-2} |Dv_j|^q) dy \le \left(\int_{B_r \setminus B_s} (|Dv_j|^2 + \lambda_j^{q-2} |Dv_j|^q)^{(1+\delta)} dy \right)^{\frac{1}{1+\delta}} |B_r \setminus B_s|^{\frac{\delta}{1+\delta}} \le c(r-s)^{\frac{\delta}{1+\delta}}. \tag{4.22}$$

Moreover, since $v \in C^\infty(B_1)$, we get

$$\int_{B_r \setminus B_s} (|Dv|^2 + \lambda_j^{q-2} |Dv|^q) \, dy \le c \left[1 + \sup_{B_r} |Dv|^2 \right] (r - s). \tag{4.23}$$

For the third integral in (4.21) we have that

$$c \left(\int_{B_r \setminus B_s} \frac{|v_j - v|^2}{(r-s)^2} \, dy + \lambda_j^{q-2} \int_{B_r \setminus B_s} \frac{|v_j - v|^q}{r-s)^q \, dy} \right) = I_j + II_j. \tag{4.24}$$

Note that, by (4.6), $v_j \to v$ strongly in $L^2(B_1)$, hence

$$\lim_j I_j = 0. \tag{4.25}$$

Moreover denoting by

$$q^* = \begin{cases} \frac{nq}{n-q} & if\ q < n \\ \\ r > q & if\ q \geq n \end{cases}$$

there exists $\mu \in (0,1)$ such that $\frac{1}{q} = \frac{\mu}{q^*} + \frac{1-\mu}{2}$. Using Hölder and Sobolev Poincaré inequalities we get

$$\begin{aligned} II_j &\leq \lambda_j^{q-2} \left(\int_{B_1} |v_j - v|^2 \, dy \right)^{\frac{q(1-\mu)}{2}} \left(\int_{B_1} |v_j - v|^{q^*} \, dy \right)^{\frac{q\mu}{q^*}} \\ &\leq c\lambda_j^{q-2} \left(\int_{B_1} |v_j - v - (v_j - v)_{B_1}|^{q^*} \, dy \right)^{\frac{q\mu}{q^*}} + c\lambda_j^{q-2} \left(\int_{B_1} |(v_j - v)_{B_1}|^{q^*} \, dy \right)^{\frac{q\mu}{q^*}} \\ &\leq c\lambda_j^{q-2} \left(\int_{B_1} |Dv_j - Dv|^q \, dy \right)^{\mu} + c\lambda_j^{q-2} \leq c\lambda_j^{q-2} \left(\int_{B_1} |Dv_j|^q \, dy \right)^{\mu} + c\lambda_j^{q-2} \\ &\leq c\lambda_j^{(q-2)(1-\mu)}. \end{aligned} \tag{4.26}$$

Since $0 < \mu < 1$ we obtain

$$\lim_j II_j = 0 \tag{4.27}$$

Moreover we have that

$$\begin{aligned} & \frac{c(M)r_j^\alpha}{\lambda_j} \int_{B_r} |Dv_j - Dv| dy + \frac{c(M)r_j^\alpha}{\lambda_j(r-s)} \int_{B_r \setminus B_s} |v_j - v| \, dy \\ \leq\ & \frac{c(M)r_j^\alpha}{\lambda_j} \left(\int_{B_1} |Dv_j|^2 dy \right)^{\frac{1}{2}} + \frac{c(M)r_j^\alpha}{\lambda_j} \left(\int_{B_1} |Dv|^2 dy \right)^{\frac{1}{2}} \\ +\ & \frac{c(M)r_j^\alpha}{\lambda_j(r-s)} \left(\int_{B_r \setminus B_s} |v_j - v|^2 \, dy \right)^{\frac{1}{2}} (r-s)^{\frac{1}{2}}. \end{aligned} \tag{4.28}$$

Hence, using that $\lim_j \frac{r_j^\alpha}{\lambda_j} = 0$, the fact that $v \in C^\infty(B_1)$ and (4.5) we get that the right hand side of (4.28) vanishes as $j \to \infty$. Therefore we conclude with (4.20), taking first the limit as $j \to \infty$ and then as $s \to r$ in (4.21).

Step 6. Lower bound

We claim that for $t < r < \frac{1}{4}$ we have

$$\limsup_j \int_{B_t} |Dv_j - Dv|^2 + \lambda_j^{p-2} |Dv_j - Dv|^p \, dy \leq c \limsup_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)].$$

Let us choose a cut-off function $\phi \in C_0^1(B_{\frac{1}{2}})$ such that $\phi = 1$ on $B_{\frac{1}{4}}$, $0 \leq \phi \leq 1$ and $|D\phi| \leq c$. Set

$$\tilde{v}_j = \phi v_j \qquad \tilde{v} = \phi v.$$

We can always suppose that the higher integrability exponent δ of (4.11) is such that $2(1+\delta) < q^*$, so we may apply Sobolev-Poincaré inequality to have that

$$\int_{\mathbb{R}^n} (|D\tilde{v}_j|^2 + \lambda_j^{q-2} |D\tilde{v}_j|^q)^{1+\delta} \, dy \leq c. \tag{4.29}$$

Fix $k > 0$. By Lemma 2.10 we can find a sequence $(w_j) \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^N)$ such that if $S_{j,k} = \{y \in \mathbb{R}^n : M(|D\tilde{v}_j|) > k\}$ then

$$w_j = \tilde{v}_j \qquad \text{on } \mathbb{R}^n \setminus S_{j,k} \tag{4.30}$$

and

$$||Dw_j||_\infty \leq c(n)k. \tag{4.31}$$

Passing to a subsequence we may suppose that

$$w_j \rightharpoonup w \qquad \text{weakly}^* \text{ in } W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^N). \tag{4.32}$$

By the maximal theorem and (4.29) we deduce that

$$\int_{\mathbb{R}^n} (M(|D\tilde{v}_j|)^2 + \lambda_j^{q-2} M(|D\tilde{v}_j|)^q)^{1+\delta}\, dy \leq c, \tag{4.33}$$

hence the sequences

$$\left\{(|D\tilde{v}_j|^2 + \lambda_j^{q-2}|D\tilde{v}_j|^q)\right\}, \qquad \left\{(M(|D\tilde{v}_j|)^2 + \lambda_j^{q-2}M(|D\tilde{v}_j|)^q)\right\}$$

are uniformly bounded in $L^{1+\delta}(\mathbb{R}^n)$ and therefore also equiabsolutely continuous in $L^1(\mathbb{R}^n)$. Then

$$\lim_{k\to\infty} \int_{S_{j,k}} (|D\tilde{v}_j|^2 + \lambda_j^{q-2}|D\tilde{v}_j|^q)\, dy = \lim_{k\to\infty} \int_{S_{j,k}} (M(|D\tilde{v}_j|)^2 + \lambda_j^{q-2}M(|D\tilde{v}_j|)^q)\, dy = 0.$$

Fix $\varepsilon > 0$ and observe that

$$\exists k_\varepsilon :\ \text{if } k \geq k_\varepsilon,\ \forall j \quad \int_{S_{j,k}} (M(|D\tilde{v}_j|)^2 + \lambda_j^{q-2}M(|D\tilde{v}_j|)^q)\, dy < \varepsilon. \tag{4.34}$$

Therefore, from the definition of $S_{j,k}$, for k sufficiently large we get

$$|S_{j,k}|k^2 \leq \int_{S_{j,k}} M(|D\tilde{v}_j|)^2 \leq \varepsilon$$

and so

$$|S_{j,k}| < \frac{\varepsilon}{k^2}. \tag{4.35}$$

Let us write

$$\begin{aligned}\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v) =& [\mathcal{I}_{j,r}(\tilde{v}_j) - \mathcal{I}_{j,r}(w_j)] + [\mathcal{I}_{j,r}(w_j) - \mathcal{I}_{j,r}(w)] + [\mathcal{I}_{j,r}(w) - \mathcal{I}_{j,r}(v)] \\ =& R_j^1 + R_j^2 + R_j^3. \end{aligned} \tag{4.36}$$

Now, by (4.30) and (4.31), we have

$$\begin{aligned} |R_j^1| \leq& \int_{S_{j,k}\cap B_r} |f_j(y, D\tilde{v}_j) - f_j(y, Dw_j)|\, dy \leq \int_{S_{j,k}\cap B_r} (|D\tilde{v}_j|^2 + \lambda_j^{q-2}|D\tilde{v}_j|^q)\, dy \\ &+ \int_{S_{j,k}\cap B_r} (|Dw_j|^2 + \lambda_j^{q-2}|Dw_j|^q)\, dy \leq \int_{S_{j,k}\cap B_r} (|D\tilde{v}_j|^2 + \lambda_j^{q-2}|D\tilde{v}_j|^q)\, dy + ck^2|S_{j,k}| \end{aligned} \tag{4.37}$$

since for every $k > k_\varepsilon$ there exists $j_0 = j_0(\varepsilon)$ such that

$$j > j_0 \ \Rightarrow\ |Dw_j|^2 + \lambda_j^{q-2}|Dw_j|^q \leq 2k^2.$$

Therefore, by (4.34) and (4.35) we get

$$\lim_{k\to\infty} \sup_j |R_j^1| \leq \varepsilon. \tag{4.38}$$

Choose $s < r$ and take ζ a cut-off function between B_s and B_r. Define

$$\psi_j = \zeta(w_j - w)$$

and split R_j^2 as follows:

$$\begin{aligned} R_j^2 &= [\mathcal{I}_{j,r}(w_j) - \mathcal{I}_{j,r}(w + \psi_j)] + [\mathcal{I}_{j,r}(w + \psi_j) - \mathcal{I}_{j,r}(w) - \mathcal{I}_{j,r}(\psi_j)] + \mathcal{I}_{j,r}(\psi_j) \\ &= R_j^4 + R_j^5 + R_j^6. \end{aligned} \tag{4.39}$$

Then, by (4.31), (4.32) and the growth conditions on f_j, we have

$$\begin{aligned} |R_j^4| &\le \int_{B_r \setminus B_s} |f_j(y, Dw_j) - f_j(y, Dw + D\psi_j)| \, dy \le \int_{B_r \setminus B_s} (|Dw_j|^2 + \lambda_j^{q-2} |Dw_j|^q) \, dy \\ &+ \int_{B_r \setminus B_s} (|Dw|^2 + \lambda_j^{q-2} |Dw|^q) \, dy + \int_{B_r \setminus B_s} (|w - w_j|^2 + \lambda_j^{q-2} |w - w_j|^q) \, dy \\ &\le c(k)|B_r \setminus B_s| + \int_{B_r \setminus B_s} (|w - w_j|^2 + \lambda_j^{q-2} |w - w_j|^q) \, dy. \end{aligned} \tag{4.40}$$

Using (4.32), we conclude that

$$\limsup_j |R_j^4| \le c(k)|B_r \setminus B_s|. \tag{4.41}$$

To bound R_j^5, we use the definition of f_j in order to have

$$|R_j^5| = \int_{B_r} dy \int_0^1 \int_0^1 D^2 f(x_j + r_j y, A_j + s\lambda_j Dw + t\lambda_j D\psi_j) Dw D\psi_j \, ds \, dt. \tag{4.42}$$

Hence

$$\limsup_j |R_j^5| = 0 \tag{4.43}$$

thank to (4.32), since $D^2 f(x_j + r_j y, A_j + s\lambda_j Dw + t\lambda_j D\psi_j)$ uniformly converges to $D^2 f(x_0, A)$. On the other hand, by (I1), we get

$$|R_j^6| = \mathcal{I}_{j,r}(\psi_j) = \int_{B_r} f_j(y, D\psi_j) \, dy \ge \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2} |Dw_j - Dw|^p) \, dy. \tag{4.44}$$

Therefore, passing possibly to a subsequence, we may suppose that $\lim_j R_j^2$ exists and collecting estimates (4.41), (4.43) and (4.44), we obtain

$$\lim_j R_j^2 \ge \limsup_j \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2} |Dw_j - Dw|^p) \, dy - c(k)(r - s). \tag{4.45}$$

Setting $S = \{y \in B_r : \ v(y) \neq w(y)\}$ and $\tilde{S} = S \cap \{y \in B_r : \ v(y) \neq \lim_j v_j(y)\}$ we have $|S| = |\tilde{S}|$. We claim that

$$|S| \le \frac{2\varepsilon}{k^2}. \tag{4.46}$$

In fact, suppose by contradiction that $|S| > \frac{2\varepsilon}{k^2}$. Then by (4.35) for j large enough we would have

$$|\tilde{S} \setminus S_{j,k}| > \frac{\varepsilon}{k^2}.$$

But by Lemma 2.10 there exists $\bar{y} \in B_r$ such that $\bar{y} \in \tilde{S} \setminus S_{j,k}$ for infinitely many j and hence

$$v(\bar{y}) = w(\bar{y})$$

and this is a contradiction. Since $Dv = Dw$ in $B_r \setminus S$, we have

$$|R_j^3| \le \int_{B_r\cap S} |f_j(y, Dw) - f_j(y, Dv)|\, dy \le \int_{B_r\cap S} (|Dw|^2 + \lambda_j^{q-2}|Dw|^q)\, dy$$
$$+ \int_{B_r\cap S} (|Dv|^2 + \lambda_j^{q-2}|Dv|^q)\, dy \le c|S| \le \frac{c\varepsilon}{k^2}. \tag{4.47}$$

Estimates (4.38), (4.45) and (4.47) leads us to

$$\lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)] \ge -\frac{c\varepsilon}{k^2} - c(k)(r-s) + \limsup_j \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2}|Dw_j - Dw|^p)\, dy. \tag{4.48}$$

Now, if $t < s < r$ we have that

$$\int_{B_t} (|Dv_j - Dv|^2 + \lambda_j^{p-2}|Dv_j - Dv|^p)\, dy \le \int_{B_s} (|Dw_j - Dw|^2 + \lambda_j^{p-2}|Dw_j - Dw|^p)\, dy$$
$$+ \int_{B_s} (|Dw_j - Dv_j|^2 + \lambda_j^{p-2}|Dw_j - Dv_j|^p)\, dy + \int_{B_s} (|Dw - Dv|^2 + \lambda_j^{p-2}|Dw - Dv|^p)\, dy. \tag{4.49}$$

Last two integrals in (4.49) can be treated exactly as R_j^1 and R_j^3 thus leading to

$$\lim_j [\mathcal{I}_{j,r}(v_j) - \mathcal{I}_{j,r}(v)] \ge -\frac{c\varepsilon}{k^2} - c(k)(r-s) + \limsup_j \int_{B_t} (|Dv_j - Dv|^2 + \lambda_j^{p-2}|Dv_j - Dv|^p)\, dy. \tag{4.50}$$

The desired estimate follows letting first $s \to r$ and then $k \to \infty$ in (4.50).

Step 7. Conclusion

From previous two steps we can conclude that

$$\lim_j \int_{B_r} |Dv - Dv_j|^2 + \lambda_j^{p-2}|Dv - Dv_j|^p = 0. \tag{4.51}$$

The conclusion follows observing that

$$\begin{aligned}
\lim_j \frac{U(x_j, \tau r_j)}{\lambda_j^2} &= \lim_j \frac{1}{\lambda_j^2} \fint_{B_{\tau r_j}(x)} (|Du - (Du)_{\tau r_j}|^2 + |Du - (Du)_{\tau r_j}|^p)\, dy + \lim_j \frac{\tau^\beta r_j^\beta}{\lambda_j^2} \\
&\le \lim_j \fint_{B_\tau(0)} (|Dv_j - (Dv_j)_\tau|^2 + \lambda_j^{p-2}|Dv_j - (Dv_j)_\tau|^p)\, dy + \tau^\beta \\
&= \lim_j \fint_{B_\tau(0)} (|Dv_j - Dv|^2 + \lambda_j^{p-2}|Dv_j - Dv|^p)\, dy \\
&+ \lim_j \fint_{B_\tau(0)} (|(Dv_j)_\tau - (Dv)_\tau|^2 + \lambda_j^{p-2}|(Dv_j)_\tau - (Dv)_\tau|^p)\, dy \\
&+ \lim_j \fint_{B_\tau(0)} (|Dv - (Dv)_\tau|^2 + \lambda_j^{p-2}|Dv - (Dv)_\tau|^p)\, dy + \tau^\beta \\
&\le \fint_{B_\tau(0)} |Dv - (Dv)_\tau|^2\, dy \le c_M \tau^2 + \tau^\beta \le c_M \tau^\beta,
\end{aligned} \tag{4.52}$$

since the first integral vanishes as $j \to +\infty$ thanks to (4.51), the second one vanishes since $(Dv_j)_\tau \to (Dv)_\tau$ as $j \to +\infty$,

$$\lambda_j^{p-2}|Dv - (Dv)_\tau|^p \le c\lambda^{p-2}$$

vanishes as $j \to +\infty$ and thanks to (4.5)

$$\lim_{j\to+\infty} \frac{\tau^\beta r_j^\beta}{\lambda_j^2} \le \tau^\beta.$$

Estimate (4.52) is a contradiction if we choose $\tilde{c}(M) > c_M$ and this concludes the proof. □

The proof of Theorem 1.1 now follows by a standard iteration procedure, see [**?**]. The following proof of Theorem 1.2 is an immediate corollary of the higher differentiability result for the gradient of minimizers of $\mathcal{F}$ that can be inferred from the proof of the Proposition 3.1 (see (3.7)) or from the proof of Theorem 4 in [10].

Proof. (of Theorem 1.2) The singular set Σ of minimizers of $\mathcal{F}$ turns out to be contained in the set

$$\Sigma_0 := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} \fint_{B(x,\rho)} |Du(y) - (Du)_{x,\rho}|^p \, dy > 0 \right\} \cup \left\{ x \in \Omega : \limsup_{\rho \to 0^+} |(Du)_{x,\rho}| = +\infty \right\}.$$

Hence Lemma 2.7 applies in order to conclude the proof. □

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