A Γ-convergence result for the two-gradient theory of phase transitions

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Abstract

The generalization to gradient vector fields of the classical double-well, singularly perturbed functionals,

$$I_\varepsilon(u;\Omega) := \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx,$$

where $W(\xi) = 0$ if and only if $\xi = A$ or $\xi = B$, and $A - B$ is a rank-one matrix, is considered. Under suitable constitutive and growth hypotheses on W it is shown that I_ε Γ-converge to

$$I(u;\Omega) = \begin{cases} K^* \mathcal{H}^{N-1}(S(\nabla u) \cap \Omega) & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d),\ \nabla u \in BV(\Omega;\{A,B\}), \\ +\infty & \text{otherwise,} \end{cases}$$

where K^* is the (constant) interfacial energy per unit area.

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Contents

1 Introduction

The theories of phase transitions and minimal surfaces have led to extensive study of singularly-perturbed, nonconvex functionals of the form

$$J_\varepsilon(v;\Omega) := \int_\Omega \frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2 \, dx, \tag{1.1}$$

where W is a nonnegative potential with multiple minima. This functional was first studied by Modica and Mortola [31], and subsequently it was applied by Modica [30] to the van der Waals–Cahn-Hilliard theory of fluid-fluid phase transitions to solve an "optimal design" problem proposed by Gurtin [25]:

$$\text{Minimize } \int_\Omega W(u)\,dx, \qquad \text{subject to a density constraint } \quad \frac{1}{|\Omega|}\int_\Omega u\,dx = \theta a + (1-\theta)b,$$

for some $\theta \in (0,1)$, and where W is a nonnegative bulk energy density with $\{W=0\} = \{a,b\}$, $a,b \in \mathbb{R}$, $a<b$. The striking nonuniqueness of solutions (minimizers) is due to the fact that nucleation of phases may occur without an increase in energy. In order to select physically preferred solutions, the van der Waals–Cahn-Hilliard theory adds a gradient term which upon rescaling leads to (1.1). Using De Giorgi's notion of Γ-convergence ([19]; see also [18, 15, 1]), it was shown in [31, 30], that

$$\Gamma - \lim_{\varepsilon\to 0^+} J_\varepsilon(u_0;\Omega) = \begin{cases} K_0 \mathrm{Per}_\Omega(E) & \text{if } \quad u = \chi_E a + (1-\chi_E)b,\ |E| = \theta|\Omega|,\ u \in BV(\Omega;\{a,b\}), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2}$$

where $K_0 := \int_a^b \sqrt{W(s)}ds$. We conclude, therefore, that in the limit as $\varepsilon \to 0$ partitions with minimal interfacial area and given volume fraction θ are selected.

Generalizations of (1.1)-(1.2) were obtained by Bouchitté [14] and by Owen and Sternberg [34] for the undecoupled problem, in which the integrand in J_ε has the form $\varepsilon^{-1} f(x, v(x), \varepsilon\nabla v(x))$. We refer also to the work of Kohn and Sternberg [28] where the study of local minimizers for (1.1) was undertaken.

The vector-valued setting, where $u : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}^N$, $d, N > 1$, was considered in [23, 10], where K_0 is replaced by

$$K_1 := \inf\left\{ \int_{-L}^{L} W(g(s)) + |g'(s)|^2 \, ds : L>0,\ g \text{ piecewise } C^1,\ g(-L)=a,\ g(L)=b \right\}. \tag{1.3}$$

The case where W has more than two wells was addressed by Baldo [7] (see also Sternberg [35]), and later generalized by Ambrosio [2].

The corresponding problem for gradient vector fields, where in place of J_ε we introduce

$$I_\varepsilon(u;\Omega) := \begin{cases} \displaystyle\int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx & \text{if } u \in W^{2,2}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

arises naturally in the study of elastic solid-to-solid phase transitions [9, 17, 27, 32], and it has defied a considerable mathematical effort during the past decade. Here $u : \Omega \to \mathbb{R}^d$ stands for the deformation, and taking into account frame-indifference we assume that $\{W=0\}=SO(N)A\cup SO(N)B$, where $SO(N)$ is the set of rotations in $\mathbb{R}^N$. In order to guarantee the existence of "classical" (as opposed to measure-valued) non affine solutions for the limiting problem, and in view of Hadamard's compatibility condition for layered deformations (see also Ball and James [9]), the two wells must be rank-one connected. Without loss of generality, we then assume that $A-B=a\otimes\nu$ for some $a\in\mathbb{R}^N$ and $\nu\in S^{N-1}:=\partial B(0,1)\subset\mathbb{R}^N$. We are now able to construct gradients taking values only on $\{A,B\}$ and layered perpendicularly to ν.

As a first simplification of the problem, we remove the frame-indifference constraint and we assume simply that

$$\{W=0\}=\{A,B\}\,,\quad A-B=a\otimes\nu.$$

Here interfaces of minimizers must be planar with normal ν (see [9]), therefore at first glance the analysis may seem to be greatly simplified as compared with the initial problem (1.1) which requires handling minimal surfaces. However, it turns out that the PDE constraint curl $=0$ imposed on the admissible fields presents numerous difficulties to the characterization of the Γ-limsup. Precisely, if, say, ∇u has a layered structure with two interfaces then it is possible to construct a "realizing" (effective) sequence nearby each interface, but the task of gluing together the two sequences on a suitable low-energy intermediate layer is very delicate. This is where specific constitutive hypotheses placed on W will come into play (see Sections 5 and 6 below).

An intermediate case between (1.1) and (1.4), where the nonconvex potential depends on u and the singular perturbation on $\nabla^2 u$, has been recently studied by Fonseca and Mantegazza [22] (for other generalizations see [21]). Also, in the two-dimensional case and when W vanishes on the unit circle (1.4) reduces to the so-called Eikonal functional which arises in the study of liquid crystals [5] as well as in blistering of delaminated thin films [33]. Recently, the Eikonal problem has received considerable mathematical attention, but in spite of substantial partial progress (see [3, 6, 26, 20]) its Γ-limit remains to be identified.

In this work, and under the standing hypothesis

(H_1) W is continuous, $W(\xi)=0$ if and only if $\xi\in\{A,B\}$, where $A-B=a\otimes\nu$ for some $a\in\mathbb{R}^d\setminus\{0\}$ and $\nu\in S^{N-1}$;

and additional assumptions on W, we show that as $\varepsilon\to 0^+$ the functionals I_ε Γ-converge to

$$I(u;\Omega)=\begin{cases} K^*\mathcal{H}^{N-1}(S(\nabla u)\cap\Omega) & \text{if } u\in W^{1,1}(\Omega;\mathbb{R}^d),\, \nabla u\in BV(\Omega;\{A,B\}),\\ +\infty & \text{otherwise,}\end{cases}$$

where $S(\nabla u)$ is the singular set of ∇u, i.e. the collection of interfaces,

$$\begin{aligned} K^* &:= \Gamma-\liminf_{\varepsilon\to 0^+} I_\varepsilon(u_0;Q_\nu)\\ &= \inf\left\{\liminf_{n\to\infty} I_{\varepsilon_n}(u_n;Q_\nu) : \varepsilon_n\to 0^+,\, u_n\in W^{2,2}\left(Q_\nu;\mathbb{R}^d\right),\, u_n\to u_0 \text{ in } L^1\left(Q;\mathbb{R}^d\right)\right\},\end{aligned}$$

where Q_ν is a unit cube in $\mathbb{R}^N$ centered at the origin and with two of its faces orthogonal to ν, and

$$\nabla u_0 := \begin{cases} A & \text{if } x\cdot\nu>0,\\ B & \text{if } x\cdot\nu<0.\end{cases}$$

The main results of this paper are:

Theorem 1.1 (Compactness) *Assume that the double well potential W satisfies conditions (H_1) and*

(H_2) *there exists $C>0$ such that*

$$W(\xi)\geq C|\xi|-\frac{1}{C}$$

for all $\xi\in\mathbb{R}^{d\times N}$.

Let $\varepsilon_n\to 0^+$. If $\{u_n\}\subset W^{2,2}\left(\Omega;\mathbb{R}^d\right)$ is such that

$$\sup_n I_{\varepsilon_n}(u_n;\Omega)<\infty,$$

then there exist a subsequence $\{u_{n_k}\}$ and $u\in W^{1,1}\left(\Omega;\mathbb{R}^d\right)$, with $\nabla u\in BV(\Omega;\{A,B\})$, such that

$$u_{n_k}-\frac{1}{|\Omega|}\int_\Omega u_{n_k}\,dx\to u \text{ in } W^{1,1}\left(\Omega;\mathbb{R}^d\right).$$

Theorem 1.2 (Γ-liminf) *Assume that W satisfies condition (H_1). Let $u \in W^{1,1}(\Omega;\mathbb{R}^d)$, with $\nabla u \in BV(\Omega;\{A,B\})$. Then*

$$\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon(u;\Omega) \geq K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = (1-\chi_E(x))A + \chi_E(x)\,B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

In order to characterize the Γ-limsup, we will consider two sets of additional constitutive hypotheses on W. Without loss of generality we may assume that

$$A = -B = a \otimes e_N.$$

First consider

$(H_2)'$ $W(\xi) \to \infty$ as $|\xi| \to \infty$;

(H_3) $W(\xi) \geq W(0,\xi_N)$ where $\xi = (\xi', \xi_N) \in \mathbb{R}^{d\times(N-1)} \times \mathbb{R}^d$.

Note that (H_3) is satisfied by the prototype bulk energy density

$$W(\xi) := \min\left\{|\xi - A|^2, |\xi - B|^2\right\}.$$

Theorem 1.3 (Γ-lim) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume that W satisfies the conditions (H_1), $(H_2)'$ and (H_3). Suppose, in addition, that W is differentiable at A and B. Let $u \in W^{1,1}(\Omega;\mathbb{R}^d)$, with $\nabla u \in BV(\Omega;\{A,B\})$. Then*

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon(u;\Omega) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u = (1-\chi_E(x))A + \chi_E(x)\,B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

The hypothesis (H_3) entails a one dimensional character to the asymptotic problem. Indeed in this case the characterization of the constant K^* can be greatly simplified. It can be shown (see Proposition 5.3) that K^* reduces to the analog of the constant K_1 introduced in (1.3), precisely, $K^* = K$ where

$$K := \inf\left\{\int_{-L}^{L} W(0,g(s)) + |g'(s)|^2\,ds : L > 0,\ g \text{ piecewise } C^1,\ g(-L) = -a,\ g(L) = a\right\}.$$

Theorem 1.3 is related to work of Kohn and Müller [27] who studied the minimization of the functional

$$\int_{(0,L)\times(0,1)} \left(\frac{\partial u}{\partial x_1}\right)^2 + \varepsilon\left|\frac{\partial^2 u}{\partial x_2^2}\right| dx_1 dx_2$$

subject to the constraint $\left|\frac{\partial u}{\partial x_2}\right| = 1$ and boundary conditions.

The main effort of the present paper is devoted to the construction of a realizing or effective sequence for the Γ−limsup. It turns out that this construction is strongly hinged to the geometry of the domain. We first assume that (see Figure 1)

$$\text{for each } t \in \mathbb{R} \text{ the horizontal section } \Omega_t := \{(x', x_N) \in \Omega : x_N = t\} \text{ is connected in } \mathbb{R}^N, \tag{1.5}$$

and that

$$t \mapsto \mathcal{H}^{N-1}(\Omega_t) \text{ is continuous in } (\alpha,\beta), \tag{1.6}$$

where

$$\alpha := \inf\{x_N : x \in \Omega\}, \quad \beta := \sup\{x_N : x \in \Omega\}.$$

It is easy to see that convex domains or cylinders of the form $\omega \times (a,b)$, where $\omega \subset \mathbb{R}^{N-1}$, satisfy conditions (1.5) and (1.6). This case is particularly simple since realizing sequences are one-dimensional, and the assumption that W is differentiable at A and B is not used (see Theorem 5.5).

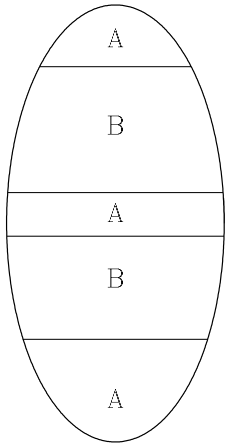

Figure 1: Example of a domain where (1.5) and (1.6) hold.

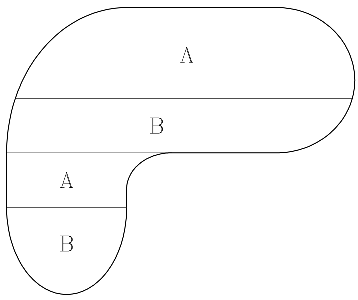

Figure 2: Example of a domain where (1.5) holds but (1.6) fails.

If we now remove the assumption (1.6) (see Figure 2), then one-dimensional sequences cease to be optimal as they would yield $K^* \operatorname{Per}_{\overline{\Omega}}(E)$ rather than $K^* \operatorname{Per}_{\Omega}(E)$ as desired. In this case, realizing sequences are one-dimensional except near horizontally flat parts of $\partial\Omega$ where $\mathcal{H}^{N-1}\left(\partial\Omega \cap \left\{(x', x_N) \in \mathbb{R}^N : x_N = t\right\}\right) > 0$ (see Theorem 5.6).

The situation becomes considerably more complicated when one drops condition (1.5) (see Figure 3) since the gradient may change abruptly when two connected components of Ω_t meet. To solve this problem we glue realizing sequences near the boundary to appropriate "mollifications" of u.

We remark that the above mentioned difficulties cannot be resolved by performing rotations and translations of Ω nearby the identity because the perimeter of the interface may change discontinuously under these transformations (see Figure 4).

As we already mentioned, the hypothesis (H_3) is quite strong as it entails a one dimensional character to the asymptotic problem. In the second part of the paper we replace it with the *isotropy assumption*:

(H_5) W is even in each variable ξ_i, $i = 1, \cdots, N-1$, that is $W(\xi_1, \cdots, -\xi_i, \cdots, \xi_N) = W(\xi_1, \cdots, \xi_i, \cdots, \xi_N)$ for each $i = 1, \cdots, N-1$,

where

$$\xi = (\xi_1, \cdots, \xi_N) \in \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N \text{ times}}, \quad \xi' = (\xi_1, \cdots, \xi_N - 1) \in \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N-1 \text{ times}},$$

so that $\xi = (\xi', \xi_N) \in \mathbb{R}^{d\times(N-1)} \times \mathbb{R}^d$.

In this case we can prove the following result

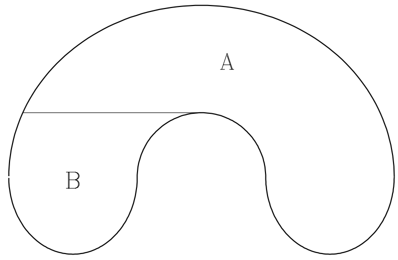

Figure 3: Example of a domain where (1.5) does not hold.

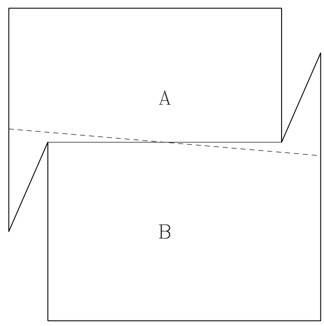

Figure 4: Example of a domain where translations of rotations of Ω cause discontinuous changes in the perimeter of the interface.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume that W satisfies the conditions (H_1), (H_5), and that there exist an exponent $p \geq 2$, constants $c, C, \rho > 0$ and a convex function $g : [0, \infty) \to [0, \infty)$, with $g(s) = 0$ if and only if $s = 0$, such that g is derivable in $s = 0$, $g(2t) \leq cg(t)$ for all $0 \leq t \leq \rho$,*

$$g(|\xi - A|) \leq W(\xi) \leq cg(|\xi - A|) \text{ if } |\xi - A| \leq \rho,$$
$$g(|\xi - B|) \leq W(\xi) \leq cg(|\xi - B|) \text{ if } |\xi - B| \leq \rho,$$

and

$$\frac{1}{C}|\xi|^p - C \leq W(\xi) \leq C(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{d\times N}$. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, with $\nabla u \in BV(\Omega; \{A, B\})$. Then

$$\Gamma - \lim_{\varepsilon \to 0^+} I_\varepsilon(u; \Omega) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = (1 - \chi_E(x)) A + \chi_E(x) B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. Moreover $K^* = K_{\mathrm{per}}$, where

$$K_{\mathrm{per}} := \inf\left\{\int_Q L\,W(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2 dx : L > 0, v \in W^{2,\infty}(Q; \mathbb{R}^d),\right.$$
$$\left.\nabla v = \pm a \otimes e_N \text{ nearby } x_N = \pm\frac{1}{2}, \; v \text{ periodic of period one in } x'\right\}.$$

It would be interesting to know if Theorem 1.4 continues to hold without assuming the isotropy assumption (H_5). We have not been able to prove this.

In the final section of this paper we exhibit an example that shows that without hypothesis (H_3), in general, we may have

$$K_{\mathrm{per}} < K.$$

Note that this is in sharp contrast with the first-order gradient theory of phase transitions modeled by (1.1), where the asymptotic problem has always a one dimensional character.

2 Preliminaries

We start with some notation. Here $\Omega \subset \mathbb{R}^N$ is an open, bounded Lipschitz domain, $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ are, respectively, the N dimensional Lebesgue measure and the $N-1$ dimensional Hausdorff measure in $\mathbb{R}^N$. We shall label the first $N-1$ coordinates of a point $x \in \mathbb{R}^N$ by x', and the N-th one by x_N, so that $x = (x', x_N)$. We define $\mathcal{A}(\Omega)$ as the class of all open subsets of Ω and $S^{N-1} := \left\{x \in \mathbb{R}^N : |x| = 1\right\}$. We let $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^N$ be the unit cube centered at the origin, and we set $Q(x_0, \varepsilon) := x_0 + \varepsilon Q$. In the sequel C and c will stand for generic real positive constants which may vary from line to line and expression to expression within the same formula.

For $\varepsilon > 0$ consider the functional

$$I_\varepsilon : L^1\left(\Omega; \mathbb{R}^d\right) \times \mathcal{A}(\Omega) \to [0, +\infty]$$

defined by

$$I_\varepsilon(u; U) := \begin{cases} \displaystyle\int_U \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx & \text{if } u \in W^{2,2}\left(\Omega; \mathbb{R}^d\right), \\ +\infty & \text{otherwise,} \end{cases}$$

where the double well potential $W : \mathbb{R}^{d \times N} \to [0, \infty)$ satisfies the following standing hypotheses:

(H_1) W is continuous, $W(\xi) = 0$ if and only if $\xi \in \{A, B\}$, where $A - B = a \otimes \nu$, for some $a \in \mathbb{R}^d \setminus \{0\}$ and $\nu \in S^{N-1}$;

(H_2) there exists $C > 0$ such that

$$W(\xi) \geq C|\xi| - \frac{1}{C}$$

for all $\xi \in \mathbb{R}^{d \times N}$.

For simplicity of notation, we shall assume that

$$A = -B = a \otimes e_N. \tag{2.1}$$

The general case may be reduced to this situation by considering in place of W a new bulk energy density

$$\hat{W}(\xi) := W((\xi + \xi_0)R^T)$$

for suitable $\xi_0 \in \mathbb{R}^{d \times N}$ and a rotation R with $R\, e_N = \nu$. We recall that Ball and James [9] have shown that there exists a non-affine Lipschitz function u such that its gradient takes only the matrix values A and B if and only if A and B are rank-one connected, i.e. $\operatorname{rank}(A - B) = 1$, in which case the jump sets (or interfaces) of ∇u are planar and orthogonal to the direction e_N. Under (2.1) the prototype blown-up macroscopic field with one interface in the unit cell $Q = (-1/2, 1/2)^N$ is

$$u(x) := |x_N|\, a. \tag{2.2}$$

We review briefly some facts about functions of bounded variation which will be useful in the sequel. A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of *bounded variation* if for all $i = 1, \cdots d$, and $j = 1, \cdots N$, there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x) \frac{\partial v}{\partial x_j}(x)\, dx = -\int_\Omega v(x)\, d\mu_{ij}$$

for every $v \in C_0^1(\Omega; \mathbb{R})$. The distributional derivative Du is the matrix–valued measure with components μ_{ij}. Given $u \in BV(\Omega; \mathbb{R}^d)$ the *approximate upper* and *lower limit* of each component u_i, $i = 1, \cdots d$, are given by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N\left(\{y \in \Omega \cap Q(x, \varepsilon) : u_i(y) > t\}\right) = 0\right\}$$

and

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\mathcal{L}^N\left(\{y \in \Omega\cap Q(x,\varepsilon) : u_i(y) < t\}\right) = 0\right\},$$

while the *jump set* of u, or *singular set*, is defined by

$$S(u) := \bigcup_{i=1}^{d}\{x \in \Omega : u_i^-(x) < u_i^+(x)\}.$$

It is well known that $S(u)$ is $N-1$ rectifiable, i.e.

$$S(u) = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface. If $x \in \Omega\backslash S(u)$ then $u(x)$ is taken to be the common value of $(u_1^+(x), \cdots, u_d^+(x))$ and $(u_1^-(x), \cdots, u_d^-(x))$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$ a.e. $x \in \Omega\backslash S(u)$. Furthermore, for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ there exist a unit vector $\nu_u(x) \in S^{N-1}$, normal to $S(u)$ at x, and two vectors $u^-(x)$, $u^+(x) \in \mathbb{R}^d$ (the traces of u on $S(u)$ at the point x) such that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{\{y\in Q(x_0,\varepsilon):\,(y-x)\cdot\nu_u(x)>0\}} |u(y) - u^+(x)|^{N/(N-1)}dy = 0$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{\{y\in Q(x_0,\varepsilon):\,(y-x)\cdot\nu_u(x)<0\}} |u(y) - u^-(x)|^{N/(N-1)}dy = 0.$$

Note that, in general, $(u_i)^+ \neq (u^+)_i$ and $(u_i)^- \neq (u^-)_i$. We denote the *jump of u across $S(u)$* by $[u] := u^+ - u^-$. The distributional derivative Du may be decomposed as

$$Du = \nabla u\,\mathcal{L}^N + (u^+ - u^-)\otimes \nu_u\,\mathcal{H}^{N-1}\lfloor S(u) + C(u),$$

where ∇u is the density of the absolutely continuous part of Du with respect to the $N-$dimensional Lebesgue measure $\mathcal{L}^N$ and $C(u)$ is the Cantor part of Du. These three measures are mutually singular.

A set $E \subset \Omega$ is of *finite perimeter* if $\chi_E \in BV(\Omega;\mathbb{R})$ and we denote by $\mathrm{Per}_\Omega(E)$ the perimeter of E in Ω.

Let $\varepsilon_n \to 0^+$. We say that a functional

$$I : L^1(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$$

is the $\Gamma-\liminf$ (resp. $\Gamma-\limsup$) of the sequence of functionals $\{I_{\varepsilon_n}\}$ with respect to the strong convergence in $L^1(\Omega;\mathbb{R}^d)$ if for every $u \in L^1(\Omega;\mathbb{R}^d)$

$$I(u;\Omega) = \inf\left\{\liminf_{n\to\infty}\ (\text{resp. } \limsup_{n\to\infty})\, I_{\varepsilon_n}(u_n;\Omega) : u_n \in L^1(\Omega;\mathbb{R}^d),\ u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d)\right\},$$

and we write

$$I = \Gamma - \liminf_{n\to\infty} I_{\varepsilon_n}\ \left(\text{resp. } I = \Gamma - \limsup_{n\to\infty} I_{\varepsilon_n}\right).$$

Since $I_\varepsilon(v;U) = \infty$ if $v \notin W^{2,2}(\Omega;\mathbb{R}^d)$, it is clear that we may write

$$I(u;\Omega) = \inf\left\{\liminf_{n\to\infty}\ (\text{resp. } \limsup_{n\to\infty})\, I_{\varepsilon_n}(u_n;\Omega) : u_n \in W^{2,2}(\Omega;\mathbb{R}^d),\ u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d)\right\}.$$

We say that the sequence $\{I_{\varepsilon_n}\}$ Γ*-converges* to I if the $\Gamma-\liminf$ and $\Gamma-\limsup$ coincide, and we write

$$I = \Gamma - \lim_{n\to\infty} I_{\varepsilon_n}.$$

The functional I is said to be the $\Gamma-\liminf$ (resp. $\Gamma-\limsup$) of the *family* of functionals $\{I_\varepsilon\}$ with respect to the strong convergence in $L^1(\Omega;\mathbb{R}^d)$ if for *every* sequence $\varepsilon_n\to 0^+$ we have that

$$I=\Gamma-\liminf_{n\to\infty} I_{\varepsilon_n}\quad\left(\text{resp. } I=\Gamma-\limsup_{n\to\infty} I_{\varepsilon_n}\right),$$

and we write

$$I=\Gamma-\liminf_{\varepsilon\to 0} I_{\varepsilon}\quad\left(\text{resp. } I=\Gamma-\limsup_{\varepsilon\to 0} I_{\varepsilon}\right).$$

Finally, we say that I is the Γ-*limit* of the *family* of functionals $\{I_\varepsilon\}$, and we write

$$I=\Gamma-\lim_{\varepsilon\to 0} I_\varepsilon,$$

if $\Gamma-\liminf$ and $\Gamma-\limsup$ coincide.

3 Compactness

The following compactness result is a direct consequence of the one obtained in [23] for the functional (1.1), and the structure of the limit has been characterized in [9]. For completeness we give here a short self-contained proof.

Theorem 3.1 (Compactness) *Assume that the double well potential W satisfies conditions (H_1) and (H_2). Let $\varepsilon_n\to 0^+$. If $\{u_n\}\subset W^{2,2}(\Omega;\mathbb{R}^d)$ is such that*

$$\sup_n I_{\varepsilon_n}(u_n;\Omega)<\infty,$$

then there exist a subsequence $\{u_{n_k}\}$ and $u\in W^{1,1}(\Omega;\mathbb{R}^d)$, with $\nabla u\in BV(\Omega;\{A,B\})$, such that

$$u_{n_k}-\frac{1}{|\Omega|}\int_\Omega u_{n_k}\,dx\to u \text{ in } W^{1,1}(\Omega;\mathbb{R}^d).$$

Proof. We claim that the sequence $\left\{u_n-\frac{1}{|\Omega|}\int_\Omega u_n\,dx\right\}$ is weakly compact in $W^{1,1}(\Omega;\mathbb{R}^d)$. Indeed, by (H_2), and with $c>0$ such that

$$\sup_n I_{\varepsilon_n}(u_n;\Omega)=:c<\infty,\tag{3.1}$$

we have

$$\varepsilon_n c\geq\int_\Omega W(\nabla u_n)\,dx\geq C\int_\Omega|\nabla u_n|\,dx-\frac{1}{C}|\Omega|,$$

and so $\{\nabla u_n\}$ is uniformly bounded in $L^1(\Omega;\mathbb{R}^{d\times N})$. By Poincaré-Friedrichs' inequality we conclude that the sequence $\left\{u_n-\frac{1}{|\Omega|}\int_\Omega u_n\,dx\right\}$ is uniformly bounded in $W^{1,1}(\Omega;\mathbb{R}^d)$. Thus, to prove the claim it remains to show that the sequence $\{\nabla u_n\}$ is equi-integrable. Fix $\epsilon>0$. By (H_2) we have

$$W(\xi)\geq\frac{1}{2}C|\xi|$$

for all $\xi\in\mathbb{R}^{d\times N}$ with $|\xi|\geq L:=\frac{2}{C^2}$, and by (3.1) we have

$$0\leq\frac{1}{2}C\int_{\{|\nabla u_n|>L\}}|\nabla u_n|\,dx\leq\int_\Omega W(\nabla u_n)\,dx\leq\varepsilon_n c\to 0\tag{3.2}$$

as $n\to\infty$. Hence there exists n_ϵ such that

$$\int_{\{|\nabla u_n|>L\}}|\nabla u_n|\,dx\leq\epsilon\quad\text{for all } n>n_\epsilon.$$

Since $\nabla u_n \in L^1\left(\Omega;\mathbb{R}^{d\times N}\right)$ for all $n=1,\cdots,n_\epsilon$, by taking L larger, if necessary, we may assume that the previous inequality holds for all n. This completes the proof of the claim.

Thus we may extract a subsequence (not relabelled) such that

$$u_n - \frac{1}{|\Omega|}\int_\Omega u_n\,dx \rightharpoonup u \text{ in } W^{1,1}\left(\Omega;\mathbb{R}^d\right) \tag{3.3}$$

and $\{\nabla u_n\}$ generates a gradient Young measure $\{\nu_x\}_{x\in\Omega}$. We claim that

$$\nu_x = (1-\theta(x))\ \delta_{\xi=A} + \theta(x)\ \delta_{\xi=B} \quad \mathcal{L}^N \text{a.e. in } \Omega,$$

where $\theta(x)\in[0,1]$. Indeed, since W is nonnegative and continuous, the Fundamental Theorem on Young Measures (see e.g. [8, 11, 36]) yields

$$0=\lim_{n\to\infty}\int_\Omega W(\nabla u_n)\,dx \geq \int_\Omega\int_{\mathbb{R}^{d\times N}} W(\xi)\,d\nu_x(\xi)\ dx;$$

hence, for $\mathcal{L}^N$ a.e $x\in\Omega$

$$\int_{\mathbb{R}^{d\times N}} W(\xi)\,d\nu_x(\xi)=0,$$

and thus by (H_1) the claim follows. In turn

$$\nabla u(x) = \int_{\mathbb{R}^{d\times N}} \xi\,d\nu_x(\xi) = (1-\theta(x))\,A + \theta(x)\,B \quad \mathcal{L}^N \text{a.e. in } \Omega. \tag{3.4}$$

Let $M>0$ and set

$$\varphi(\xi) := \inf\left\{\int_0^1 \min\left\{\sqrt{W(h(s))},M\right\}|h'(s)|\ ds:\ h:[0,1]\to\mathbb{R}^{d\times N} \text{ piecewise } C^1,\ h(0)=\xi,\ h(1)=A\right\}.$$

Then φ is Lipschitz, $\varphi(\xi)=0$ if and only if $\xi=A$, and

$$\{\varphi(\nabla u_n)\} \text{ is uniformly bounded in } W^{1,1}(\Omega;\mathbb{R}). \tag{3.5}$$

Indeed,

$$\int_\Omega |\nabla(\varphi\circ\nabla u_n)|\,dx \leq \int_\Omega \sqrt{W(\nabla u_n)}\left|\nabla^2 u_n\right|dx \leq \frac{1}{2}I_{\varepsilon_n}(u_n;\Omega) \leq \frac{1}{2}c$$

and

$$\int_\Omega |\varphi\circ\nabla u_n|\,dx \leq \int_\Omega M\,|\nabla u_n|\ dx + \varphi(0)\,|\Omega|,$$

where we have used the fact that $\varphi(\xi)\leq M\,|\xi-0|+\varphi(0)$. Hence (3.5) holds, and up to a subsequence (not relabelled)

$$\varphi(\nabla u_n) \to H \text{ in } L^1(\Omega;\mathbb{R}), \tag{3.6}$$

where $H\in BV(\Omega;\mathbb{R})$. On the other hand, the Young measure generated by $\{\varphi(\nabla u_n)\}$ is

$$\mu_x = (1-\theta(x))\ \delta_{t=\varphi(A)} + \theta(x)\ \delta_{t=\varphi(B)} \quad \mathcal{L}^N \text{ a.e. in } \Omega,$$

and the strong convergence in (3.6) now yields $\theta(x)\in\{0,1\}$ $\mathcal{L}^N$ a.e. in Ω, precisely

$$\theta(x) = \chi_E(x)$$

for some measurable set $E\subset\Omega$. By (3.4)

$$\nabla u(x) = (1-\chi_E(x))\,A + \chi_E(x)\ B$$

and

$$\begin{aligned} H &= (1-\chi_E(x))\,\varphi(A) + \chi_E(x)\,\varphi(B) \\ &= \chi_E(x)\ \varphi(B) \in BV(\Omega;\mathbb{R}), \end{aligned}$$

therefore the set E has finite perimeter and $\nabla u\in BV(\Omega;\{A,B\})$. Moreover, since $\nu_x=\delta_{\xi=\nabla u(x)}$ and by (3.2) we have that $\nabla u_n\to\nabla u$ in $L^1\left(\Omega;\mathbb{R}^{d\times N}\right)$. ■

Remark 3.2 (i) We remark that the conclusion of Theorem 3.1 still holds if we do not impose the condition (H_2) but, instead, we assume apriori that the sequence $\{u_n\}$ converges weakly in $W^{1,1}(\Omega;\mathbb{R}^d)$. Indeed, the argument follows exactly that of the latter proof once (3.3) has been established.

(ii) If we assume that

$$W(\xi) \geq C_1 |\xi|^p \text{ for all } \xi \in \mathbb{R}^{d\times N} \text{ with } |\xi| \geq L,$$

and for some $1 \leq p < \infty$, then

$$u_{n_k} - \frac{1}{|\Omega|}\int_\Omega u_{n_k}\, dx \to u \text{ in } W^{1,p}\left(\Omega;\mathbb{R}^d\right).$$

Indeed

$$C_1 \int_{\{|\nabla u_{n_k}|\geq L\}} |\nabla u_{n_k}|^p\, dx \leq \int_\Omega W(\nabla u_{n_k})\, dx \to 0.$$

On the other hand,

$$\int_{\{|\nabla u_{n_k}|\leq L\}} |\nabla u_{n_k} - \nabla u|^p\, dx \leq (L + |A| + |B|)^{p-1} \int_\Omega |\nabla u_{n_k} - \nabla u|\, dx \to 0.$$

Theorem 3.3 *Let $u \in W^{1,1}\left(\Omega;\mathbb{R}^d\right)$, with $\nabla u \in BV(\Omega;\{A,B\})$. Then the function u has the form*

$$u(x) = u(x', x_N) = \gamma_0 + a x_N - 2\psi(x)\, a,$$

where $\gamma_0 \in \mathbb{R}^d$, $\gamma_0 \cdot a = 0$, $\psi \in W^{1,\infty}(\Omega;\mathbb{R})$ satisfies $\nabla\psi(x) = \chi_E(x)\, e_N$ for some set $E \subset \Omega$ with $\mathrm{Per}_\Omega(E) < \infty$, and E is layered perpendicularly to e_N , that is

$$\partial^* E \cap \Omega = \bigcup_{i=1}^{\infty} \omega_i \times \{\alpha_i\},$$

where the sets $\omega_i \subset \mathbb{R}^{N-1}$ are connected, bounded and open, $\alpha_i \in \mathbb{R}$. Moreover, in any open subset Ω' of Ω with the property that for each $t \in \mathbb{R}$ the horizontal section

$$\{(x', x_N) \in \Omega' : x_N = t\} \text{ is connected in } \mathbb{R}^N,$$

we may write

$$u(x) = u(x', x_N) = \gamma_0 + a x_N - 2h(x_N)\, a \quad \text{a.e. in } \Omega',$$

where $h \in W^{1,\infty}(\mathbb{R};\mathbb{R})$, $h' \in BV(\mathbb{R};\{0,1\})$.

Proof. As in [9], in view of the fact that for $\mathcal{L}^N$ a.e. $x \in \Omega$.

$$\begin{aligned}\nabla u(x) &= (1 - \chi_E(x))\, A + \chi_E(x)\, B = (1 - \chi_E(x))\, a \otimes e_N - \chi_E(x)\, a \otimes e_N \\ &= a \otimes e_N - 2\chi_E(x)\, a \otimes e_N,\end{aligned}$$

we may conclude that E is layered perpendicularly to e_N , and that the function u has the form

$$u(x) = u(x', x_N) = \gamma_0 + a x_N - 2\psi(x)\, a,$$

where $\gamma_0 \in \mathbb{R}^d$, $\gamma_0 \cdot a = 0$, $\psi \in W^{1,\infty}(\Omega;\mathbb{R})$, satisfies $\nabla\psi(x) = \chi_E(x)\, e_N$. Moreover, since

$$\nabla_{x'} u(x', x_N) = \nabla_{x'}(\gamma_0 + a x_N - 2\psi(x)\, a) \equiv 0$$

we conclude that in any open subset Ω' of Ω with the property that for each $t \in \mathbb{R}$ the horizontal section

$$\{(x', x_N) \in \Omega' : x_N = t\} \text{ is connected in } \mathbb{R}^N,$$

we may represent u as

$$u(x) = \tilde{u}(x_N) = \gamma_0 + a x_N - 2h(x_N)\, a \quad \text{a.e. in } \Omega',$$

where $h \in W^{1,\infty}(\mathbb{R};\mathbb{R})$, $h' \in BV(\mathbb{R};\{0,1\})$. ∎

4 $\Gamma-$liminf: the lower bound

In view of (2.2), we define

$$\begin{aligned} K^* &:= \Gamma - \liminf_{\varepsilon \to 0^+} I_\varepsilon\left(|x_N|\, a; Q\right) \qquad (4.1)\\ &= \inf\left\{ \liminf_{n\to\infty} I_{\varepsilon_n}(u_n;Q) : \varepsilon_n \to 0^+,\ u_n \in W^{2,2}\left(Q;\mathbb{R}^d\right),\ u_n \to |x_N|\, a \text{ in } L^1\left(Q;\mathbb{R}^d\right)\right\}, \end{aligned}$$

where, we recall, $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^N$.

Theorem 4.1 *Assume that W satisfies condition (H_1). Let $u \in W^{1,1}\left(\Omega;\mathbb{R}^d\right)$, with $\nabla u \in BV\left(\Omega;\{A,B\}\right)$. Then*

$$\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon\left(u;\Omega\right) \geq K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u = \left(1-\chi_E(x)\right)A + \chi_E(x)\ B$.

The proof of Theorem 4.1 is hinged on the following lemma.

Lemma 4.2 *Let $\omega \subset \mathbb{R}^{N-1}$ be a connected, bounded, open set, with $\mathcal{H}^{N-1}(\partial\omega) = 0$, and consider the cylinder $U := \omega \times (\alpha - h, \alpha + h)$, where $\alpha \in \mathbb{R}$ and $h > 0$. If $u_0 \in W^{1,1}\left(U;\mathbb{R}^d\right)$ is such that*

$$\nabla u_0(x', x_N) = \begin{cases} A & \text{if } x_N > \alpha, \\ B & \text{if } x_N < \alpha, \end{cases}$$

then

$$\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon\left(u_0;U\right) = K^*\mathcal{H}^{N-1}(\omega).$$

Assuming that Lemma 4.2 holds (its proof is left for the remaining of Section 4), we conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. By Theorem 3.3 and since $\partial\Omega$ is Lipschitz, we may write

$$\partial^* E \cap \Omega = \bigcup_{i=1}^{\infty} \omega_i \times \{\alpha_i\},\ \text{ with } \sum_{i=1}^{\infty} \mathcal{H}^{N-1}\left(\omega_i \times \{\alpha_i\}\right) < \infty,$$

where the sets $\omega_i \subset \mathbb{R}^{N-1}$ are connected, bounded and open, with $\mathcal{H}^{N-1}(\partial\omega_i) = 0$, $\alpha_i \in \mathbb{R}$. Let $\delta > 0$ and choose $k > 1$ such that

$$\mathcal{H}^{N-1}\left(\partial^* E \cap \Omega\right) \leq \sum_{i=1}^{k} \mathcal{H}^{N-1}\left(\omega_i \times \{\alpha_i\}\right) + \delta.$$

Let $\omega_i' \subset\subset \omega_i$ be connected, bounded and open, with $\mathcal{H}^{N-1}(\partial\omega_i') = 0$, and such that

$$\mathcal{H}^{N-1}\left(\omega_i \times \{\alpha_i\}\right) \leq \mathcal{H}^{N-1}\left(\omega_i' \times \{\alpha_i\}\right) + \frac{\delta}{k}.$$

Since each $\omega_i' \times \{\alpha_i\} \subset\subset \Omega$ we may find $h > 0$ so small that

$$\bigcup_{i=1}^{k} \omega_i' \times \left(\alpha_i - h, \alpha_i + h\right) \subset\subset \Omega$$

and the sets $\omega_i' \times (\alpha_i - h, \alpha_i + h)$ are mutually disjoint. Then for any sequences $\varepsilon_n \to 0^+$ and $\{u_n\} \subset W^{2,2}(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ we have

$$\begin{aligned}\liminf_{n\to\infty} I_{\varepsilon_n}(u_n;\Omega) &\geq \liminf_{n\to\infty} I_{\varepsilon_n}\left(u_n; \bigcup_{i=1}^k \omega_i' \times (\alpha_i - h, \alpha_i + h)\right) \\ &\geq \sum_{i=1}^k \liminf_{n\to\infty} I_{\varepsilon_n}(u_n; \omega_i' \times (\alpha_i - h, \alpha_i + h)) \\ &\geq \sum_{i=1}^k K^* \mathcal{H}^{N-1}(\omega_i' \times \{\alpha_i\}) \\ &\geq \sum_{i=1}^k K^* \mathcal{H}^{N-1}(\omega_i \times \{\alpha_i\}) - \delta K^* \\ &\geq K^* \mathcal{H}^{N-1}(\partial^* E \cap \Omega) - 2\delta K^*,\end{aligned}$$

where we have used Lemma 4.2 to assert that

$$\liminf_{n\to\infty} \; I_{\varepsilon_n}(u_n; \omega_i' \times (\alpha_i - h, \alpha_i + h)) \geq K^* \mathcal{H}^{N-1}(\omega_i' \times \{\alpha_i\}).$$

Hence

$$\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon(u;\Omega) \geq \mathcal{H}^{N-1}(\partial^* E \cap \Omega) - 2\delta K^*$$

and it suffices to let $\delta \to 0^+$. ■

It remains to prove Lemma 4.2. Let $\omega \subset \mathbb{R}^{N-1}$ be a bounded, open set, with $\mathcal{H}^{N-1}(\partial\omega) = 0$, and let $h > 0$. Define

$$\begin{aligned}\mathcal{F}(\omega;h) :=& \Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon(|x_N|\, a; \omega \times (-h,h)) \\ =& \inf\Big\{\liminf_{n\to\infty} I_{\varepsilon_n}(u_n; \omega \times (-h,h)) : \varepsilon_n \to 0^+,\ u_n \in W^{2,2}(\omega \times (-h,h);\mathbb{R}^d), \\ & u_n \to |x_N|\, a \text{ in } L^1(\omega \times (-h,h);\mathbb{R}^d)\Big\}.\end{aligned} \tag{4.2}$$

Note that $K^* = \mathcal{F}\left(Q'; \frac{1}{2}\right)$, where $Q' := \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$.

Lemma 4.3 *(i)* $\mathcal{F}(x' + \omega; h) = \mathcal{F}(\omega;h)$ *for every* $x' \in \mathbb{R}^{N-1}$;

(ii) if $\omega_1 \subset \omega_2$ *then* $\mathcal{F}(\omega_1;h) \leq \mathcal{F}(\omega_2;h)$;

(iii) if $\omega_1 \cap \omega_2 = \emptyset$ *then* $\mathcal{F}(\omega_1 \cup \omega_2;h) \geq \mathcal{F}(\omega_1;h) + \mathcal{F}(\omega_2;h)$;

(iv) if $\alpha > 0$ *then* $\mathcal{F}(\alpha\omega;\alpha h) = \alpha^{N-1}\mathcal{F}(\omega;h)$, *while if* $0 < \alpha < 1$ *then* $\mathcal{F}(\alpha\omega;h) \geq \alpha^{N-1}\mathcal{F}(\omega;h)$;

(v) $\mathcal{F}(\omega;h) = \mathcal{H}^{N-1}(\omega)\,\mathcal{F}(Q';h)$;

(vi) $\mathcal{F}(\omega;h) = \mathcal{F}(\omega;\delta)$ *for each* $\delta > 0$.

We note that Lemma 4.2 is a direct consequence of Lemma 4.3 (v) and (vi).

Proof. (i) follows immediately from the translation invariance of the energies $I_{\varepsilon_n}(u_n;\cdot)$, while (ii) and (iii) are consequences of the nonnegativeness of the energy densities of I_{ε_n}, together with the fact that admissible sequences for $\omega_2 \times (-h,h)$ are still admissible for $\omega_1 \times (-h,h)$ when $\omega_1 \subset \omega_2$.

To prove (iv) let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{2,2}(\omega \times (-h,h);\mathbb{R}^d)$ be any sequence such that $u_n \to |x_N|\, a$ in $L^1(\omega \times (-h,h);\mathbb{R}^d)$. Set

$$v_n(x) := \alpha u_n(x/\alpha), \quad x \in \alpha\omega \times (-\alpha h, \alpha h).$$

Clearly $|x_N|\, a = \alpha\, |x_N/\alpha|\, a$ and so $v_n \to |x_N|\, a$ in $L^1\left(\alpha\omega \times (-\alpha h, \alpha h); \mathbb{R}^d\right)$, and we have

$$\begin{aligned}
\mathcal{F}(\alpha\omega; \alpha h) &\le \liminf_{n\to\infty} \int_{\alpha\omega\times(-\alpha h,\alpha h)} \frac{1}{\alpha\varepsilon_n} W(\nabla v_n) + \alpha\varepsilon_n |\nabla^2 v_n|^2\, dx \\
&= \liminf_{n\to\infty} \int_{\alpha\omega\times(-\alpha h,\alpha h)} \frac{1}{\alpha\varepsilon_n} W\left(\nabla u_n\left(\frac{x}{\alpha}\right)\right) + \alpha\varepsilon_n \left|\frac{1}{\alpha}\nabla^2 u_n\left(\frac{x}{\alpha}\right)\right|^2 dx \\
&= \liminf_{n\to\infty} \alpha^N \int_{\omega\times(-h,h)} \frac{1}{\alpha\varepsilon_n} W(\nabla u_n(y)) + \alpha\varepsilon_n \left|\frac{1}{\alpha}\nabla^2 u_n(y)\right|^2 dy \\
&= \alpha^{N-1} \liminf_{n\to\infty} \int_{\omega\times(-h,h)} \frac{1}{\varepsilon_n} W(\nabla u_n(y)) + \varepsilon_n \left|\nabla^2 u_n(y)\right|^2 dy.
\end{aligned}$$

Hence $\mathcal{F}(\alpha\omega; \alpha h) \le \alpha^{N-1}\mathcal{F}(\omega; h)$, and similarly

$$\mathcal{F}(\omega; h) = \mathcal{F}\left(\frac{1}{\alpha}\alpha\omega; \frac{1}{\alpha}\alpha h\right) \le \frac{1}{\alpha^{N-1}}\mathcal{F}(\alpha\omega; \alpha h).$$

This proves that $\mathcal{F}(\alpha\omega; \alpha h) = \alpha^{N-1}\mathcal{F}(\omega; h)$.

Next let $0 < \alpha < 1$; then $\mathcal{F}(\alpha\omega; h) \ge \mathcal{F}(\alpha\omega; \alpha h) = \alpha^{N-1}\mathcal{F}(\omega; h)$.

To show (v) we use Vitali's Covering Theorem to decompose

$$\omega = \bigcup_{i=1}^{\infty} (a_i + \eta_i Q') \cup N_0$$

with $\mathcal{H}^{N-1}(N_0) = 0$, $a_i + \eta_i Q'$ mutually disjoint, $0 < \eta_i < 1$, $Q' := \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$, and

$$\sum_{i=1}^{\infty} \eta_i^{N-1} = \mathcal{H}^{N-1}(\omega).$$

For all $k \in \mathbb{N}$, by (ii)-(iv)

$$\mathcal{F}(\omega; h) \ge \mathcal{F}\left(\bigcup_{i=1}^{k} (a_i + \eta_i Q'); h\right) \ge \sum_{i=1}^{k} \mathcal{F}(a_i + \eta_i Q'; h) \ge \sum_{i=1}^{k} \eta_i^{N-1}\mathcal{F}(Q'; h).$$

By letting $k \to \infty$ we conclude

$$\mathcal{F}(\omega; h) \ge \mathcal{H}^{N-1}(\omega)\,\mathcal{F}(Q'; h).$$

Conversely, with

$$Q' = \bigcup_{i=1}^{\infty} (b_i + \delta_i \omega) \cup N_1$$

with $\mathcal{H}^{N-1}(N_1) = 0$, $b_i + \delta_i\omega$ mutually disjoint, and

$$\sum_{i=1}^{\infty} \delta_i^{N-1} = \frac{1}{\mathcal{H}^{N-1}(\omega)},$$

we deduce that

$$\mathcal{F}(Q'; h) \ge \frac{1}{\mathcal{H}^{N-1}(\omega)}\mathcal{F}(\omega; h),$$

and we have concluded the proof of (v). This result will entail (vi) provided we show that

$$\mathcal{F}(Q'; h) = \mathcal{F}(Q'; \delta) \tag{4.3}$$

for every $\delta > 0$. We first claim that for all $k \in \mathbb{N}$

$$\mathcal{F}(kQ'; \delta) = k^{N-1}\mathcal{F}(Q'; \delta). \tag{4.4}$$

Indeed, write

$$kQ' = \bigcup_{i=1}^{k^{N-1}} (a_i + Q') \cup N_0$$

with $\mathcal{H}^{N-1}(N_0) = 0$, $a_i + Q'$ mutually disjoint. By (iv)

$$k^{N-1}\mathcal{F}(Q';\delta) = \mathcal{F}(kQ';k\delta) \geq \mathcal{F}(kQ';\delta) \geq \sum_{i=1}^{k^{N-1}} \mathcal{F}(a_i + Q';\delta) = k^{N-1}\mathcal{F}(Q';\delta)\ .$$

Next we show that

$$\mathcal{F}(Q';\delta) = \mathcal{F}\left(Q';\frac{\delta}{k}\right).$$

Indeed by (iv) and (4.4)

$$\mathcal{F}(Q';\delta) = \frac{1}{k^{N-1}}\mathcal{F}(kQ';k\delta) = \frac{1}{k^{N-1}}k^{N-1}\mathcal{F}(Q';k\delta) = \mathcal{F}(Q';k\delta),$$

and thus

$$\mathcal{F}(Q';\delta) = \mathcal{F}\left(Q';k\frac{\delta}{k}\right) = \mathcal{F}\left(Q';\frac{\delta}{k}\right).$$

It follows that if $p, k \in \mathbb{N}$, $k \neq 0$, then

$$\mathcal{F}\left(Q';\frac{p}{k}\delta\right) = \mathcal{F}(Q';\delta), \tag{4.5}$$

and to assert (4.3) it suffices now to establish the continuity of $\mathcal{F}(Q';\cdot)$. Let $r_n \to r$ and extract a subsequence $r'_n \to r$. Without loss of generality assume that $r'_n \to r^+$ (similarly, if $r'_n \to r^-$). Then

$$\mathcal{F}(Q';r) \leq \liminf_{n\to\infty} \mathcal{F}(Q';r'_n).$$

Let $q_n \in \mathbb{Q}$ such that $q_n r > r'_n > r$, $q_n \to 1$. Then by (4.5)

$$\mathcal{F}(Q';r'_n) \leq \mathcal{F}(Q';q_n r) = \mathcal{F}(Q';r),$$

and thus

$$\limsup_{n\to\infty} \mathcal{F}(Q';r'_n) \leq \mathcal{F}(Q';r).$$

This conclude the proof. ■

Remark 4.4 It follows immediately from Lemma 4.3(vi) that the effective energy concentrates near the interfaces. Precisely, if

$$\mathcal{F}(\omega;h) = \lim_{n\to\infty} I_{\varepsilon_n}(u_n;\omega\times(-h,h))$$

then for each $0 < \eta < h$

$$\lim_{n\to\infty} I_{\varepsilon_n}(u_n;\omega\times[(-h,h)\setminus(-\eta,\eta)]) = 0.$$

Indeed, by Lemma 4.3(vi)

$$\lim_{n\to\infty} I_{\varepsilon_n}(u_n;\omega\times(-h,h)) = \mathcal{F}(\omega;h) = \mathcal{F}(\omega;\eta) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n;\omega\times(-\eta,\eta))$$

and thus

$$\lim_{n\to\infty} I_{\varepsilon_n}(u_n;\omega\times[(-h,h)\setminus(-\eta,\eta)]) = 0.$$

5 $\Gamma-$limsup: the upper bound. Geodesic hypotheses.

In agreement with our adopted notation, in what follows the constants C and C_δ may change from line to line. Throughout this section we assume that W satisfies the following conditions:

(H_1) W is continuous, $W(\xi) = 0$ if and only if $\xi \in \{A, B\}$, where $A = -B = a \otimes e_N$ for some $a \in \mathbb{R}^d \setminus \{0\}$;

$(H_2)'$ $W(\xi) \to \infty$ as $|\xi| \to \infty$;

(H_3) $W(\xi) \geq W(0, \xi_N)$ where $\xi = (\xi', \xi_N) \in \mathbb{R}^{d\times(N-1)} \times \mathbb{R}^d$.

Note that (H_3) is verified by the prototype bulk energy density

$$W(\xi) := \min\left\{|\xi - A|^2, |\xi - B|^2\right\}.$$

5.1 Characterization of K^*

In this subsection we prove that if (H_1), $(H_2)'$, (H_3) hold then

$$K^* = \inf\left\{\int_{-L}^{L} W(0, g(s)) + |g'(s)|^2\, ds : L > 0,\ g \text{ piecewise } C^1,\ g(L) = -g(-L) = a\right\}.$$

Proposition 5.1 *Assume that W satisfies condition $(H_2)'$. Let $\{\varepsilon_n\} \subset \mathbb{R}_+$ and $\{u_n\} \subset W^{2,2}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$ be two sequences such that $\varepsilon_n \to 0^+$ and*

$$\sup_{n\in\mathbb{N}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, u_n')}{\varepsilon_n} + \varepsilon_n |u_n''|^2\, dt < +\infty.$$

Then

$$\sup_{n\in\mathbb{N}} \sup_{t\in\left(-\frac{1}{2},\frac{1}{2}\right)} |u_n'(t)| < +\infty.$$

Proof. Since $u_n' \in L^\infty((-\frac{1}{2}, \frac{1}{2}); \mathbb{R}^d)$, without loss of generality we may suppose that $0 < \varepsilon_n < \frac{1}{4}$. Let

$$c := \sup_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, u_n')}{\varepsilon_n} + \varepsilon_n |u_n''|^2\, dt < +\infty,$$

and fix $\varepsilon_n \in \left(0, \frac{1}{4}\right)$ and $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. If $t + \varepsilon_n < \frac{1}{2}$ then

$$\frac{1}{\varepsilon_n} \int_t^{t+\varepsilon_n} W(0, u_n')\, ds \leq c,$$

and so there exists $t_n \in (t, t + \varepsilon_n)$ such that

$$W(0, u_n'(t_n)) \leq c.$$

By $(H_2)'$ we may find a constant $C = C(c)$ such that $\sup_n |u_n'(t_n)| \leq C$, and Hölder's inequality now yields

$$|u_n'(t)| \leq |u_n'(t_n)| + \left(\varepsilon_n \int_t^{t+\varepsilon_n} |u_n''|^2\, ds\right)^{1/2} \leq C + \left(\varepsilon_n \int_{-\frac{1}{2}}^{\frac{1}{2}} |u_n''|^2\, ds\right)^{1/2} \leq C + \sqrt{c}. \tag{5.1}$$

If $t + \varepsilon_n \geq \frac{1}{2}$ then $t \geq \frac{1}{2} - \varepsilon_n$ and so $t - \varepsilon_n \geq \frac{1}{2} - 2\varepsilon_n > -\frac{1}{2}$. Therefore we may reason as above, using the interval $(t - \varepsilon_n, t)$ in place of $(t, t + \varepsilon_n)$ to obtain (5.1), and we conclude that $\sup_{t\in(-\frac{1}{2},\frac{1}{2})} |u_n'(t)| \leq C + \sqrt{c}$. ■

Recall that (see (4.1) and (4.2))

$$K^* = \mathcal{F}\left(Q'; \tfrac{1}{2}\right) := \inf\left\{ \liminf_{n\to\infty} \int_Q \frac{1}{\varepsilon_n} W\left(\nabla u_n\right) + \varepsilon_n \left|\nabla^2 u_n\right|^2 \, dx : \varepsilon_n \to 0^+, \ \{u_n\} \subset W^{2,2}\left(Q;\mathbb{R}^d\right), \right.$$
$$\left. u_n \to |x_N|\, a \text{ in } L^1\left(Q;\mathbb{R}^d\right) \right\},$$

where $Q := Q' \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, $Q' := \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$, and introduce the "one-dimensional" version of K^*,

$$K_* := \inf\left\{ \liminf_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W\left(0, u_n'\right)}{\varepsilon_n} + \varepsilon_n \left|u_n''\right|^2 dt : \varepsilon_n \to 0^+, u_n \in W^{2,2}\left(\left(-\tfrac{1}{2}, \tfrac{1}{2}\right); \mathbb{R}^d\right), \right.$$
$$\left. u_n \to |t|\, a \text{ in } L^1\left(\left(-\tfrac{1}{2}, \tfrac{1}{2}\right); \mathbb{R}^d\right) \right\}.$$

An immediate consequence of the latter proposition and the compactness results of Section 3 is the following corollary.

Corollary 5.2 *Assume that W satisfies conditions (H_1) and $(H_2)'$. Given any $1 \le p < \infty$ there holds*

$$K_* = \inf\left\{ \liminf_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W\left(0, u_n'\right)}{\varepsilon_n} + \varepsilon_n \left|u_n''\right|^2 dt : \varepsilon_n \to 0^+, u_n \in W^{2,2}\left(\left(-\tfrac{1}{2}, \tfrac{1}{2}\right); \mathbb{R}^d\right), \right.$$
$$\left. u_n \to |t|\, a \ \textit{in} \ W^{1,p}\left(\left(-\tfrac{1}{2}, \tfrac{1}{2}\right); \mathbb{R}^d\right) \right\}.$$

Proof. In view of Proposition 5.1, energy bounded sequences admissible for K_* are uniformly bounded in $W^{1,\infty}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$, and, in particular, must converge weakly to $|\cdot|\, a$ in $W^{1,1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$. The result now follows from Remark 3.2(i). ■

Proposition 5.3 *Assume that W satisfies conditions (H_1), $(H_2)'$ and (H_3). Then*

$$K^* = K_* = K,$$

where

$$K = \inf\left\{ \int_{-L}^{L} W\left(0, g\left(s\right)\right) + \left|g'\left(s\right)\right|^2 ds : L > 0, \ g \ \textit{piecewise} \ C^1, \ g\left(L\right) = -g\left(-L\right) = a \right\}.$$

Proof. We divide the proof in three steps.

Step 1: We prove first that $K^* = K_*$. Clearly

$$K_* \ge K^* = \mathcal{F}\left(Q'; \tfrac{1}{2}\right).$$

Indeed, if $\{\varepsilon_n\} \subset \mathbb{R}_+$ and $\{u_n\} \subset W^{2,2}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$ are two sequences such that $\varepsilon_n \to 0^+$ and $u_n \to |t|\, a$ in $L^1\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$, then the sequence $v_n\left(x\right) := u_n\left(x_N\right)$ is admissible for $\mathcal{F}\left(Q'; \frac{1}{2}\right)$. To prove the converse inequality, let $\{\varepsilon_n\} \subset \mathbb{R}_+$ and $\{u_n\} \subset W^{2,2}\left(Q; \mathbb{R}^d\right)$ be such that $\varepsilon_n \to 0^+$ and $u_n \to |x_N|\, a$ in $L^1\left(Q; \mathbb{R}^d\right)$. For $\mathcal{H}^{N-1}$ a.e. $x' \in Q'$ the function $u_n^{x'}\left(t\right) := u_n\left(x', t\right) \in W^{2,2}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$ and $u_n^{x'} \to |t|\, a$ in $L^1\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$.

Using (H_3) and Fatou's Lemma we have

$$\liminf_{n\to\infty}\int_Q \frac{1}{\varepsilon_n}W(\nabla u_n)+\varepsilon_n\left|\nabla^2 u_n\right|^2\,dx=\liminf_{n\to\infty}\int_{Q'}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\varepsilon_n}W(\nabla u_n)+\varepsilon_n\left|\nabla^2 u_n\right|^2\,dt\right)dx'$$

$$\geq \liminf_{n\to\infty}\int_{Q'}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\varepsilon_n}W\left(0,\frac{du_n^{x'}}{dt}\right)+\varepsilon_n\left|\frac{d^2u_n^{x'}}{dt^2}\right|^2\,dt\right)dx'$$

$$\geq \int_{Q'}\liminf_{n\to\infty}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\varepsilon_n}W\left(0,\frac{du_n^{x'}}{dt}\right)+\varepsilon_n\left|\frac{d^2u_n^{x'}}{dt^2}\right|^2\,dt\right)dx'$$

$$\geq K_*\int_{Q'}dx'=K_*.$$

Step 2: We now prove that $K_*=K_1$, where

$$K_1:=\inf\Bigg\{\liminf_{n\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{W(0,u_n')}{\varepsilon_n}+\varepsilon_n\left|u_n''\right|^2dt:\ \varepsilon_n\to0^+,\ u_n\in W^{2,2}\left(\left(-\tfrac{1}{2},\tfrac{1}{2}\right);\mathbb{R}^d\right),$$
$$u_n\to|t|\,a\text{ in }L^1\left(\left(-\tfrac{1}{2},\tfrac{1}{2}\right);\mathbb{R}^d\right),\ u_n'=\pm a\text{ near }t=\pm\tfrac{1}{2}\text{ (resp.)}\Bigg\}.$$

Clearly $K_1\geq K_*$. To prove the converse inequality, let $\{\varepsilon_n\}\subset\mathbb{R}_+$ and $\{u_n\}\subset W^{2,2}\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^d\right)$ be such that $\varepsilon_n\to0^+$, $u_n\to|t|\,a$ in $L^1\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^d\right)$, and

$$K_*=\liminf_{n\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{W(0,u_n')}{\varepsilon_n}+\varepsilon_n\left|u_n''\right|^2dt.$$

Without loss of generality, and up to the extraction of a subsequence, we may assume that

$$\liminf_{n\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{W(0,u_n')}{\varepsilon_n}+\varepsilon_n\left|u_n''\right|^2dt=\lim_{n\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{W(0,u_n')}{\varepsilon_n}+\varepsilon_n\left|u_n''\right|^2dt,$$

and, by Corollary 5.2, that $u_n\to|t|\,a$ in $W^{1,1}\left(\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^d\right)$. Since by Remark 4.4

$$\lim_{n\to\infty}\int_{\frac{1}{4}}^{\frac{1}{2}}\frac{W(0,u_n')}{\varepsilon_n}+\varepsilon_n\left|u_n''\right|^2+|u_n-ta|+|u_n'-a|\,dt=0,\tag{5.2}$$

there exists $t_0\in\left(\frac{1}{4},\frac{1}{3}\right)$ such that (up to the extraction of a further subsequence, if necessary)

$$\lim_{n\to\infty}\left[\frac{W(0,u_n'(t_0))}{\varepsilon_n}+\varepsilon_n\left|u_n''(t_0)\right|^2+|u_n(t_0)-t_0a|+|u_n'(t_0)-a|\right]=0.\tag{5.3}$$

Define

$$w_n(t):=\psi_n(t)\,u_n(t)+(1-\psi_n(t))\left((t-t_0)\,a+u_n(t_0)\right),$$

where ψ_n is a smooth cut-off function such that $\psi_n(t)=0$ for $t\geq t_0+\varepsilon_n$, $\psi_n(t)=1$ for $t\leq t_0$, and

$$|\psi_n'(t)|\leq C/\varepsilon_n,\quad |\psi_n''(t)|\leq C/\varepsilon_n^2.$$

Then

$$w_n'(t)=\psi_n'(t)\left(u_n(t)-u_n(t_0)-(t-t_0)\,a\right)+\psi_n(t)\,u_n'(t)+(1-\psi_n(t))\,a,\tag{5.4}$$

and

$$w_n''(t)=\psi_n''(t)\left(u_n(t)-u_n(t_0)-(t-t_0)\,a\right)+2\psi_n'(t)\left(u_n'(t)-a\right)+\psi_n(t)\,u_n''.\tag{5.5}$$

By Hölder's inequality, for $t \in (t_0, t_0 + \varepsilon_n)$

$$\left|u_n'(t) - a\right| \le \left|u_n'(t_0) - a\right| + \left(\varepsilon_n \int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds\right)^{1/2}, \tag{5.6}$$

and thus

$$\begin{aligned} \left|u_n(t) - u_n(t_0) - (t - t_0)a\right| &\le \int_{t_0}^{t_0+\varepsilon_n} \left|u_n'(s) - a\right| ds \\ &\le \varepsilon_n \left|u_n'(t_0) - a\right| + \varepsilon_n \left(\varepsilon_n \int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds\right)^{1/2}. \end{aligned} \tag{5.7}$$

Since W is continuous and $W(0, a) = 0$, we may find $\rho > 0$ and a modulus of continuity $\eta = \eta(s)$ such that

$$W(0, \xi_N) \le \eta(|\xi_N - a|) \text{ for all } |\xi_N - a| \le \rho.$$

By (5.4), (5.6) and (5.7), for $t \in (t_0, t_0 + \varepsilon_n)$ we have

$$\begin{aligned} \left|w_n'(t) - a\right| &\le \frac{C}{\varepsilon_n}\left|u_n(t) - u_n(t_0) - (t - t_0)a\right| + \left|u_n'(t) - a\right| \\ &\le C\left|u_n'(t_0) - a\right| + C\left(\varepsilon_n \int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds\right)^{1/2} \le \rho \end{aligned}$$

for ε_n sufficiently small, where we have used (5.2) and (5.3). Hence for $t \in (t_0, t_0 + \varepsilon_n)$ and since η is increasing

$$W(0, w_n') \le \eta\left(C\left|u_n'(t_0) - a\right| + C\left(\varepsilon_n \int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds\right)^{1/2}\right),$$

and thus

$$\int_{t_0}^{t_0+\varepsilon_n} \frac{W(0, w_n')}{\varepsilon_n} dt \le \eta\left(C\left|u_n'(t_0) - a\right| + C\left(\varepsilon_n \int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds\right)^{1/2}\right) \to 0.$$

Similarly, by (5.5), (5.6) and (5.7) and for $t \in (t_0, t_0 + \varepsilon_n)$,

$$\begin{aligned} \left|w_n''(t)\right| &\le \frac{C}{\varepsilon_n^2}\left|u_n(t) - u_n(t_0) - (t - t_0)a\right| + \frac{C}{\varepsilon_n}\left|u_n'(t) - a\right| + \left|u_n''\right| \\ &\le \frac{C}{\varepsilon_n}\left|u_n'(t_0) - a\right| + \frac{C}{\varepsilon_n}\left(\varepsilon_n \int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds\right)^{1/2} + \left|u_n''\right|, \end{aligned}$$

and we have

$$\varepsilon_n \left|w_n''(t)\right|^2 \le \frac{C}{\varepsilon_n}\left|u_n'(t_0) - a\right|^2 + C\int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds + C\varepsilon_n \left|u_n''\right|^2.$$

Hence, in view of (5.2) and (5.3),

$$\int_{t_0}^{t_0+\varepsilon_n} \varepsilon_n \left|w_n''(t)\right|^2 dt \le C\left|u_n'(t_0) - a\right|^2 + C\varepsilon_n \int_{t_0}^{t_0+\varepsilon_n} \left|u_n''\right|^2 ds \to 0.$$

Since $w_n(t) = u_n(t)$ for $t \le t_0$ and $w_n'(t) = a$ for $t \ge t_0 + \varepsilon_n$, we conclude that

$$\limsup_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, w_n')}{\varepsilon_n} + \varepsilon_n \left|w_n''\right|^2 dt \le \lim_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, u_n')}{\varepsilon_n} + \varepsilon_n \left|u_n''\right|^2 dt = K_*.$$

By repeating the same construction nearby $t = -\frac{1}{2}$ we ensure that the new sequence satisfies $w_n' = \pm a$ near $t = \pm\frac{1}{2}$ resp. Hence $K_1 \le K_*$, and the proof of Step 2 is complete.

Step 3: We finally prove that $K_* = K$. Let $\{\varepsilon_n\} \subset \mathbb{R}_+$ and $\{u_n\} \subset W^{2,2}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$ be such that $\varepsilon_n \to 0^+$, $u_n \to |t|\, a$ in $L^1\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$, and

$$K_* = \liminf_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, u_n')}{\varepsilon_n} + \varepsilon_n |u_n''|^2\, dt.$$

In light of Step 2, we may assume, without loss of generality, that $u_n' = a$ near $t = \frac{1}{2}$ and $u_n' = -a$ near $t = -\frac{1}{2}$. Define $v_n(s) := \frac{1}{\varepsilon_n} u_n(\varepsilon_n s)$ for $s \in \left[-\frac{1}{2\varepsilon_n}, \frac{1}{2\varepsilon_n}\right]$. Then $v_n'(s) = u_n'(\varepsilon_n s)$, $v_n''(s) = \varepsilon_n u_n''(\varepsilon_n s)$, and so

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, u_n'(t))}{\varepsilon_n} + \varepsilon_n |u_n''(t)|^2\, dt &= \int_{-\frac{1}{2\varepsilon_n}}^{\frac{1}{2\varepsilon_n}} W(0, u_n'(\varepsilon_n s)) + \varepsilon_n^2 |u_n''(\varepsilon_n s)|^2\, ds \\ &= \int_{-\frac{1}{2\varepsilon_n}}^{\frac{1}{2\varepsilon_n}} W(0, v_n'(s)) + |v_n''(s)|^2\, ds \geq K,\end{aligned}$$

therefore $K_* \geq K$. Conversely, let $g : [-L, L] \to \mathbb{R}^d$ be a piecewise C^1 curve, with $g(L) = -g(-L) = a$. Consider any sequence $\{\varepsilon_n\}$ converging to 0^+, and define

$$u_n(t) := \int_0^t v_n(s)\, ds, \quad v_n(t) := \begin{cases} -a & \text{if } t < -\varepsilon_n L, \\ g\left(\dfrac{t}{\varepsilon_n}\right) & \text{if } |t| \leq \varepsilon_n L, \\ a & \text{if } t > \varepsilon_n L. \end{cases} \tag{5.8}$$

As in [23], we have

$$v_n \to v_0 := \begin{cases} -a & \text{if } t \leq 0, \\ a & \text{if } t > 0, \end{cases} \quad \text{in } L^p\left(\left(-\tfrac{1}{2}, \tfrac{1}{2}\right); \mathbb{R}^d\right), \text{ for any } 1 \leq p < \infty,$$

and so $u_n = \int_0^t v_n(s)\, ds \to |t|\, a$ in $W^{1,p}\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^d\right)$. Moreover,

$$\begin{aligned} K_* &\leq \lim_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, u_n')}{\varepsilon_n} + \varepsilon_n |u_n''|^2\, dt = \lim_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{W(0, v_n)}{\varepsilon_n} + \varepsilon_n |v_n'|^2\, dt \\ &= \int_{-L}^{L} W(0, g(s)) + |g'(s)|^2\, ds, \end{aligned}$$

and taking the infimum over all such functions g we get the desired inequality. ■

Remark 5.4 Note that the argument of Step 3 in the latter proof, together with Propositions 5.1 and 5.3, ensures that given *any* sequence $\{\varepsilon_n\}$ converging to 0^+ there exists a sequence $\{u_n\}$ converging to $|x_N| a$ in $W^{1,p}$ for all $p \in [1, +\infty)$, and such that

$$K^* = \lim_{n\to\infty} \int_Q \frac{1}{\varepsilon_n} W(\nabla u_n) + \varepsilon_n \left|\nabla^2 u_n\right|^2\, dx\,.$$

Indeed, for each $k \in \mathbb{N}$ construct via (5.8) a sequence $\{u_{n,k}\}$ corresponding to a function g_k admissible for K and such that

$$\int_{-L}^{L} W(0, g_k(s)) + |g_k'(s)|^2\, ds \leq K^* + \frac{1}{k}.$$

Then, with $u(x) := |x_N| a$ we obtain

$$\lim_{k\to\infty} \lim_{n\to\infty} \|u_{n,k} - u\|_{W^{1,p}(Q;\mathbb{R}^d)} = 0,$$

and

$$\limsup_{k\to\infty} \lim_{n\to\infty} \int_Q \frac{1}{\varepsilon_n} W(\nabla u_{n,k}) + \varepsilon_n \left|\nabla^2 u_{n,k}\right|^2\, dx \leq K^*.$$

On the other hand, by Theorem 4.1 we always have the opposite inequality, and so

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_Q \frac{1}{\varepsilon_n}W\left(\nabla u_{n,k}\right)+\varepsilon_n\left|\nabla^2 u_{n,k}\right|^2 dx = K^*.$$

It suffices now to extract a subsequence $\{k_n\}$ of $\{k\}$ such that the subsequence $z_n := u_{n,k_n}$ satisfies (see Lemma 7.2 in [16])

$$\lim_{n\to\infty}\|z_n-u\|_{W^{1,p}(Q;\mathbb{R}^d)}=0,$$

and

$$\liminf_{n\to\infty}\int_Q \frac{1}{\varepsilon_n}W\left(\nabla z_n\right)+\varepsilon_n\left|\nabla^2 z_n\right|^2\, dx = K^*.$$

5.2 Special domains

Let

$$\alpha := \inf\{x_N:\ x\in\Omega\},\quad \beta := \sup\{x_N:\ x\in\Omega\}. \tag{5.9}$$

Throughout this subsection we assume that the domain Ω satisfies (see Figure 1):

$$\text{for each } t\in\mathbb{R} \text{ the horizontal section } \Omega_t := \{(x',x_N)\in\Omega : x_N=t\} \text{ is connected in } \mathbb{R}^N \tag{5.10}$$

and

$$t\mapsto \mathcal{H}^{N-1}\left(\Omega_t\right) \text{ is continuous in } (\alpha,\beta). \tag{5.11}$$

Theorem 5.5 *Assume that W satisfies the conditions (H_1), $(H_2)'$ and (H_3). Let $u\in W^{1,1}\left(\Omega;\mathbb{R}^d\right)$, with $\nabla u\in BV\left(\Omega;\{A,B\}\right)$. Then*

$$\Gamma-\lim_{\varepsilon\to 0^+} I_\varepsilon\left(u;\Omega\right)=K^*\operatorname{Per}_\Omega(E), \tag{5.12}$$

where $\nabla u(x)=\left(1-\chi_E(x)\right)A+\chi_E(x)\ B$ for $\mathcal{L}^N$ a.e. $x\in\Omega$.

Proof. In view of Theorem 4.1, to prove (5.12) it suffices to show that for any sequence $\{\varepsilon_n\}\subset\mathbb{R}_+$ such that $\varepsilon_n\to 0^+$ we have

$$\Gamma-\limsup_{n\to\infty} I_{\varepsilon_n}\left(u;\Omega\right)\le K^*\operatorname{Per}_\Omega(E).$$

Thus we fix a sequence $\{\varepsilon_n\}\subset\mathbb{R}_+$ converging to 0^+. For simplicity in the notation we drop the subscript n so that $\varepsilon:=\varepsilon_n$. By Theorem 3.3 and (5.10) we may assume that

$$u\left(x\right)=\tilde{u}\left(x_N\right)=\gamma_0+ax_N-2h\left(x_N\right)a\quad \text{a.e. in } \Omega,$$

where $h\in W^{1,\infty}\left(\mathbb{R};\mathbb{R}\right)$, $h'\in BV\left(\mathbb{R};\{0,1\}\right)$, with

$$S\left(\nabla u\right)\cap\Omega=\bigcup_{i=1}^{\infty}\Omega_{l_i}$$

and $\Omega_{l_i} := \{x=(x',x_N)\in\Omega:\ x_N=l_i\}$ for some $l_i\in\mathbb{R}$. We divide the proof in two steps.
Step 1: Assume that the number of interfaces is finite, that is

$$S\left(\nabla u\right)\cap\Omega=\bigcup_{i=1}^{m}\Omega_{l_i},$$

for some $m\in\mathbb{N}$, and some finite family $l_1<\cdots<l_m$. Fix $k\in\mathbb{N}$, and in view of Proposition 5.3 consider a piecewise C^1 curve $g_k:[-L,L]\to\mathbb{R}^d$, with $g_k\left(L\right)=-g_k\left(-L\right)=a$, such that

$$\int_{-L}^{L} W\left(0,g_k\left(s\right)\right)+\left|g_k'\left(s\right)\right|^2 ds\le K^*+\frac{1}{k}. \tag{5.13}$$

Let $\varepsilon > 0$ be so small that $l_i + \varepsilon L < l_{i+1} - \varepsilon L$, for $i = 1, \cdots, m-1$. Set

$$v_{\varepsilon,k}(s) := \begin{cases} g_k\left(-\operatorname{sgn}\left(u'(l_i - \varepsilon L)\cdot a\right)\frac{s-l_i}{\varepsilon}\right) & \text{if } l_i - \varepsilon L < s < l_i + \varepsilon L,\ i = 1, \cdots, m, \\ \tilde{u}'(s) & \text{otherwise in } \mathbb{R}, \end{cases}$$

where we have extended $\tilde{u}'$ constantly to $(-\infty, l_1 - \varepsilon L)$ and $(l_m + \varepsilon L, +\infty)$, and define

$$u_{\varepsilon,k}(x) = \tilde{u}_{\varepsilon,k}(x_N) := \tilde{u}(l_1) + \int_{l_1}^{x_N} v_{\varepsilon,k}(s)\, ds,\ x \in \mathbb{R}^N.$$

Then

$$\int_\Omega |\nabla u_{\varepsilon,k} - \nabla u|\, dx \le C\sum_{i=1}^m \int_{l_i - \varepsilon L}^{l_i + \varepsilon L}\left[\left|g_k\left(\pm\frac{s-l_i}{\varepsilon}\right)\right| + 1\right] ds \le 2Cm\varepsilon L + C\sum_{i=1}^m \varepsilon\int_{-L}^{L} |g_k(t)|\, dt,$$

and since $u_{\varepsilon,k}(x) = u(x)$ when $x_N = l_1$, by Poincaré's inequality we have that $u_{\varepsilon,k} \to u$ in $W^{1,1}\left(\Omega;\mathbb{R}^d\right)$ as $\varepsilon \to 0^+$. Further,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} I_\varepsilon(u_{\varepsilon,k};\Omega) &= \lim_{\varepsilon\to 0^+}\sum_{i=1}^m \int_{\Omega\cap\{l_i+\varepsilon L < x_N < l_{i+1}+\varepsilon L\}} \frac{W\left(0, g_k\left(\pm\frac{x_N - l_i}{\varepsilon}\right)\right)}{\varepsilon} + \frac{1}{\varepsilon}\left|g_k'\left(\pm\frac{x_N - l_i}{\varepsilon}\right)\right|^2 dx \\
&= \lim_{\varepsilon\to 0^+}\sum_{i=1}^m \int_{l_i-\varepsilon L}^{l_i+\varepsilon L}\left(\frac{W\left(0, g_k\left(\pm\frac{s-l_i}{\varepsilon}\right)\right)}{\varepsilon} + \frac{1}{\varepsilon}\left|g_k'\left(\pm\frac{s-l_i}{\varepsilon}\right)\right|^2\right)\mathcal{H}^{N-1}\left(\{x\in\Omega : x_N = s\}\right) ds \\
&= \lim_{\varepsilon\to 0^+}\sum_{i=1}^m \int_{-L}^{L}\left(W(0, g_k(t)) + |g_k'(t)|^2\right)\mathcal{H}^{N-1}\left(\{x\in\Omega : x_N = \varepsilon t + l_i\}\right) dt \\
&= \left(\int_{-L}^{L} W(0, g_k(t)) + |g_k'(t)|^2\, dt\right)\sum_{i=1}^m \mathcal{H}^{N-1}(\Omega_{l_i}) \le \left(K^* + \frac{1}{k}\right)\sum_{i=1}^m \mathcal{H}^{N-1}(\Omega_{l_i}),
\end{aligned}$$

where we have used (5.11) and (5.13). Hence

$$\limsup_{k\to\infty}\lim_{\varepsilon\to 0^+} I_\varepsilon(u_{\varepsilon,k};\Omega) \le K^*\sum_{i=1}^m \mathcal{H}^{N-1}(\Omega_{l_i}) = K^*\operatorname{Per}_\Omega(E),$$

and in view of Theorem 4.1 this inequality is actually an identity. As in Remark 5.4, it is possible to extract a subsequence $\{k_\varepsilon\}$ of $\{k\}$ such that the subsequence $u_\varepsilon := u_{\varepsilon,k_\varepsilon}$ satisfies

$$\lim_{\varepsilon\to 0^+}\|u_\varepsilon - u\|_{W^{1,1}(\Omega;\mathbb{R}^d)} = 0,$$

and

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(u_\varepsilon;\Omega) = K^*\operatorname{Per}_\Omega(E).$$

Step 2: Suppose now that the number of interfaces is infinite, that is

$$S(\nabla u)\cap\Omega = \bigcup_{i=1}^\infty \Omega_{l_i},$$

where $\Omega_{l_i} := \{x = (x', x_N)\in\Omega : x_N = l_i\}$, $l_i \in \mathbb{R}$. We claim that interfaces can only accumulate at the *top* or *bottom* of Ω, i.e. if l is an accumulation point of $\{l_i\}$ then $l \in \{\alpha, \beta\}$, where α and β are defined in (5.9). Indeed, if $\Omega_l \ne \emptyset$ then we may find $x = (x', l) \in \Omega$ and, in turn, an open cylinder of the form $B'(x', r)\times(l-h, l+h)\subset\Omega$. For i_k large enough $B'(x';r)\times\{l_{i_k}\}\subset\Omega$, where $\{l_{i_k}\}$ is a subsequence of $\{l_i\}$ converging to l, and so

$$\infty > \sum_{i=1}^\infty \mathcal{H}^{N-1}(\Omega_{l_i}) \ge \sum_{i_k}\mathcal{H}^{N-1}\left(\Omega_{l_{i_k}}\right) \ge \sum_{i_k}\mathcal{H}^{N-1}\left(B'(x';r)\right) = \infty,$$

and we have reached a contradiction.

Fix an integer $m \in \mathbb{N}$ and consider

$$U_m := \{x \in \Omega : \ \alpha + \delta_m < x_N < \beta - \delta_m\},$$

where $\delta_m \to 0^+$ and $\{l_i\} \cap \{\alpha + \delta_m, \beta - \delta_m\} = \emptyset$. Then $\mathcal{L}^N(\Omega \backslash U_m) \to 0$ as $m \to \infty$, and ∇u has a finite number of interfaces in U_m. By Step 1 we may construct a sequence $\{u_\varepsilon^m\}$ such that $u_\varepsilon^m \to u$ in $W^{1,1}(U_m; \mathbb{R}^d)$ as $\varepsilon \to 0^+$, $u_\varepsilon^m = u$ in $\Omega \backslash U_m$, and

$$\lim_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon^m; U_m) = K^* \operatorname{Per}_{U_m}(E).$$

We have

$$\lim_{m\to\infty} \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^m; \Omega) = \lim_{m\to\infty} \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^m; U_m) = K^* \lim_{m\to\infty} \operatorname{Per}_{U_m}(E) = K^* \operatorname{Per}_\Omega(E).$$

We can conclude as in Step 1. ■

5.3 x'-connected domains

Throughout this subsection we assume that for each $t \in \mathbb{R}$ the horizontal section

$$\Omega_t := \{(x', x_N) \in \Omega : x_N = t\} \text{ is connected in } \mathbb{R}^N. \tag{5.14}$$

Here we allow for the possibility that $t \mapsto \mathcal{H}^{N-1}(\Omega_t)$ is not continuous (see Figure 2). In this case a more careful analysis is required near the boundary under the additional hypothesis that W is smooth at the wells.

Theorem 5.6 *Assume* (5.14) *and let W satisfy the conditions (H_1), $(H_2)'$ and (H_3). Suppose, in addition, that W is differentiable at A and B. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, with $\nabla u \in BV(\Omega; \{A, B\})$. Then*

$$\Gamma - \lim_{\varepsilon \to 0^+} I_\varepsilon(u; \Omega) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = (1 - \chi_E(x)) A + \chi_E(x) B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Proof. By virtue of Theorem 4.1 it suffices to prove that given an arbitrary sequence $\{\varepsilon_n\}$ converging to 0^+ we have

$$\Gamma - \limsup_{n\to+\infty} I_{\varepsilon_n}(u; \Omega) \le K^* \operatorname{Per}_\Omega(E). \tag{5.15}$$

Fix one such sequence $\{\varepsilon_n\}$, and for simplicity of notation abbreviate $\varepsilon := \varepsilon_n$. We divide the proof of (5.15) into three steps.

Step 1: *One interface.* We assume first that u has the form

$$u(x) = |x_N|\, a \quad \text{a.e. in } \Omega,$$

so that there is only one interface $\Omega_0 := \{x = (x', x_N) \in \Omega : \ x_N = 0\}$, and we may write $\Omega_0 = \omega \times \{0\}$, where ω is an open bounded connected subset of $\mathbb{R}^{N-1}$. Let $0 < h < \frac{1}{4}\min\{\beta, -\alpha\}$, where α and β are defined in (5.9), and consider a sequence $\{\delta_m\}$ converging to 0^+ as $m \to +\infty$. We write $\delta := \delta_m$. For every δ we construct a smooth cut-off function $\psi_\delta \in C_c^\infty(\mathbb{R}^N; [0,1])$ such that $\psi_\delta = 1$ in $\omega_\delta^+ \times \left(-\frac{h}{3}, \frac{h}{3}\right)$ and $\psi_\delta = 0$ outside $\omega_{2\delta}^+ \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, where for $s > 0$ we denote $\omega_s^+ := \left\{x' \in \mathbb{R}^{N-1} : \operatorname{dist}(x', \omega) < s\right\}$ (see Figure 5). In view of Proposition 5.3 consider a piecewise C^1 curve $g : [-L, L] \to \mathbb{R}^d$, with $g(L) = -g(-L) = a$. Extend g to all of $\mathbb{R}$ by setting $g(t) = -g(-t) := a$ for all $t > L$, and define

$$u_{\varepsilon,\delta}(x) := \psi_\delta(x)\, z_\varepsilon(x) + (1 - \psi_\delta(x))\, u, \tag{5.16}$$

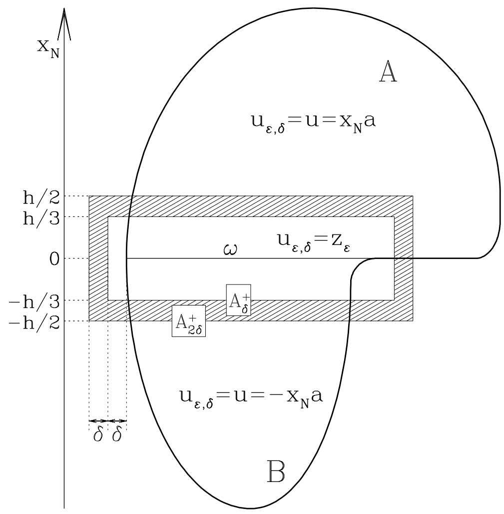

Figure 5: Construction for one interface in Step 1 of Theorem 5.6. The shaded region represents $\{0 < \psi_\delta < 1\}$, and corresponds to $A^+_{2\delta} \backslash A^+_\delta$ where $A^+_{2\delta} := \omega^+_{2\delta} \times (-\frac{h}{2}, \frac{h}{2})$ and $A^+_\delta := \omega^+_\delta \times (-\frac{h}{3}, \frac{h}{3})$. The function $u_{\varepsilon,\delta}$ coincides with u outside $A^+_{2\delta}$ and with z_ε inside A^+_δ.

where $z_\varepsilon(x) := \int_0^{x_N} g\left(\frac{s}{\varepsilon}\right) ds$ (see Figure 5). Note that $u \in W^{2,2}\left(\Omega \backslash \left(\omega^+_\delta \times \left(-\frac{h}{3}, \frac{h}{3}\right)\right); \mathbb{R}^d\right)$. Then

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon} W\left(\nabla u_{\varepsilon,\delta}\right) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx = & \int_{\{\psi_\delta = 1\}} \frac{1}{\varepsilon} W\left(\nabla z_\varepsilon\right) + \varepsilon \left|\nabla^2 z_\varepsilon\right|^2 dx \\ & + \int_{\{0<\psi_\delta<1\}} \frac{1}{\varepsilon} W\left(\nabla u_{\varepsilon,\delta}\right) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx.\end{aligned} \tag{5.17}$$

By the fact that $g(t) = -g(-t) := a$ for all $t > L$ we have

$$\int_{\{\psi_\delta=1\}} \frac{1}{\varepsilon} W\left(\nabla z_\varepsilon\right) + \varepsilon \left|\nabla^2 z_\varepsilon\right|^2 dx \leq \mathcal{H}^{N-1}\left(\omega^+_\delta\right) \int_{-L}^{L} W\left(0, g(s)\right) + \left|g'(s)\right|^2 ds. \tag{5.18}$$

Also, as W is continuous in $\mathbb{R}^{d\times N}$, $W\left(\pm a \otimes e_N\right) = 0$, and W is differentiable at $\pm a \otimes e_N$, we may find a modulus of continuity η with

$$\lim_{s \to 0^+} \frac{\eta(s)}{s} = 0 \tag{5.19}$$

and such that

$$W(\xi) \leq \min\left\{\eta\left(\left|\xi - a \otimes e_N\right|\right), \eta\left(\left|\xi + a \otimes e_N\right|\right)\right\} \text{ for all } \xi \in \mathbb{R}^{d\times N}. \tag{5.20}$$

Hence, by (5.20),

$$\int_{\{0<\psi_\delta<1\}} \frac{1}{\varepsilon} W\left(\nabla u_{\varepsilon,\delta}\right) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx \leq \int_{\{0<\psi_\delta<1\}} \frac{1}{\varepsilon} \eta\left(\left|\nabla u_{\varepsilon,\delta} - \nabla u\right|\right) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx. \tag{5.21}$$

To estimate the right hand side of (5.21) note that

$$\begin{aligned}\left|\nabla u_{\varepsilon,\delta} - \nabla u\right| &\leq \left|\nabla \psi_\delta\right| \left|z_\varepsilon - u\right| + \psi_\delta \left|\nabla z_\varepsilon - \nabla u\right|, \\ \left|\nabla^2 u_{\varepsilon,\delta}\right| &\leq \left|\nabla^2 \psi_\delta\right| \left|z_\varepsilon - u\right| + 2\left|\nabla \psi_\delta\right| \left|\nabla z_\varepsilon - \nabla u\right| + \psi_\delta \left|\nabla^2 z_\varepsilon\right|.\end{aligned} \tag{5.22}$$

Fix $L_1 \geq L$. For $|x_N| \geq \varepsilon L_1$ we have

$$z_\varepsilon - |x_N|\, a = \begin{cases} \varepsilon \int_0^L \left(g(s) - a\right) ds & \text{for } x_N \geq \varepsilon L_1, \\ -\varepsilon \int_{-L}^0 \left(g(s) + a\right) ds & \text{for } x_N \leq -\varepsilon L_1, \end{cases} \tag{5.23}$$

and so from (5.22)

$$\left|\nabla u_{\varepsilon,\delta}-\nabla u\right|\leq C_{\delta}\,\varepsilon,\ \ \left|\nabla^{2}u_{\varepsilon,\delta}\right|\leq C_{\delta}\,\varepsilon \tag{5.24}$$

for $|x_N|\geq \varepsilon L_1$, while for $|x_N|\leq \varepsilon L_1$

$$\left|\nabla\left(z_{\varepsilon}-u\right)\right|\leq\left|g\left(\frac{x_N}{\varepsilon}\right)\right|+\|\nabla u\|_{\infty}\leq C,\quad\left|\nabla^{2}z_{\varepsilon}\right|\leq\frac{1}{\varepsilon}\left|g'\left(\frac{x_N}{\varepsilon}\right)\right|\leq\frac{C}{\varepsilon}, \tag{5.25}$$

and

$$\left|z_{\varepsilon}\left(x\right)-u\left(x\right)\right|\leq\left|z_{\varepsilon}\left(x',\varepsilon L_1\right)-u\left(x',\varepsilon L_1\right)\right|+\int_{-\varepsilon L_1}^{\varepsilon L_1}\left|\nabla\left(z_{\varepsilon}-u\right)\right|dx_N\leq\varepsilon C.$$

Hence, from (5.22) we have for $|x_N|\leq \varepsilon L_1$

$$\left|\nabla u_{\varepsilon,\delta}-\nabla u\right|\leq C_{\delta}\,\varepsilon+C,\ \ \left|\nabla^{2}u_{\varepsilon,\delta}\right|\leq C_{\delta}+\frac{C}{\varepsilon}. \tag{5.26}$$

It now follows from (5.24) and (5.26) that, for $\varepsilon<h/3L_1$,

$$\begin{aligned}\int_{\{0<\psi_{\delta}<1\}}\frac{1}{\varepsilon}\eta\left(\left|\nabla u_{\varepsilon,\delta}-\nabla u\right|\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon,\delta}\right|^{2}\,dx&\leq\left(\frac{1}{\varepsilon}\eta\left(\varepsilon C_{\delta}\right)+C_{\delta}\varepsilon^{3}\right)\mathcal{H}^{N-1}\left(\omega_{2\delta}^{+}\right)h\\&\quad+\mathcal{H}^{N-1}\left(\omega_{2\delta}^{+}\backslash\omega_{\delta}^{+}\right)2L_1\varepsilon\left(\frac{1}{\varepsilon}\eta\left(\varepsilon C_{\delta}+C\right)+\varepsilon\left(C_{\delta}+\frac{C}{\varepsilon^{2}}\right)\right).\end{aligned} \tag{5.27}$$

In view of (5.17), (5.18), (5.21) and (5.27)

$$\begin{aligned}\int_{\Omega}\frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon,\delta}\right|^{2}\,dx&\leq\left(\int_{-L}^{L}W\left(0,g\left(s\right)\right)+\left|g'\left(s\right)\right|^{2}ds\right)\mathcal{H}^{N-1}\left(\omega_{\delta}^{+}\right)\\&\quad+C\left(\frac{1}{\varepsilon}\eta\left(\varepsilon C_{\delta}\right)+C_{\delta}\varepsilon^{3}\right)+C\left(\eta\left(\varepsilon C_{\delta}+C\right)+\varepsilon^{2}C_{\delta}+C\right)\mathcal{H}^{N-1}\left(\omega_{2\delta}^{+}\backslash\omega_{\delta}^{+}\right),\end{aligned} \tag{5.28}$$

and letting $\varepsilon\to0^{+}$ and then $\delta\to0^{+}$ yields, by (5.19),

$$\limsup_{\delta\to0^{+}}\limsup_{\varepsilon\to0^{+}}\int_{\Omega}\frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon,\delta}\right|^{2}\,dx\leq\left(\int_{-L}^{L}W\left(0,g\left(s\right)\right)+\left|g'\left(s\right)\right|^{2}ds\right)\mathcal{H}^{N-1}\left(\omega\right). \tag{5.29}$$

Next we claim that

$$\lim_{\delta\to0^{+}}\lim_{\varepsilon\to0^{+}}\left\|u_{\varepsilon,\delta}-u\right\|_{W^{1,p}(\Omega;\mathbb{R}^{d})}=0 \tag{5.30}$$

for any $1\leq p<\infty$. Indeed, by (5.24) and (5.26) we have

$$\int_{\Omega}\left|\nabla u_{\varepsilon,\delta}-\nabla u\right|^{p}dx\leq\varepsilon^{p}C_{\delta}\mathcal{H}^{N-1}\left(\omega_{2\delta}^{+}\right)2h+\mathcal{H}^{N-1}\left(\omega_{2\delta}^{+}\backslash\omega_{\delta}^{+}\right)2L_1\varepsilon\left|\varepsilon C_{\delta}+C\right|^{p}\to0\text{ as }\varepsilon\to0.$$

Since $u_{\varepsilon,\delta}\left(x\right)=u\left(x\right)$ for $|x_N|\geq 2h$, by Poincaré's inequality we have that $u_{\varepsilon,\delta}\to u$ in $W^{1,p}\left(\Omega;\mathbb{R}^{d}\right)$.

To conclude the proof of this step, in view of Proposition 5.3 for every $k\in\mathbb{N}$ consider a piecewise C^1 curve $g_k:[-L,L]\to\mathbb{R}^d$, with $g_k\left(L\right)=-g_k\left(-L\right)=a$, such that

$$\int_{-L}^{L}W\left(0,g_k\left(s\right)\right)+\left|g_k'\left(s\right)\right|^{2}ds\leq K^{*}+\frac{1}{k}. \tag{5.31}$$

If we denote by $u_{\varepsilon,\delta,k}$ the function defined in (5.16) and corresponding to g_k, by (5.29)–(5.31) we have

$$\limsup_{k\to\infty}\limsup_{\delta\to0^{+}}\limsup_{\varepsilon\to0^{+}}\int_{\Omega}\frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon,\delta,k}\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon,\delta,k}\right|^{2}\,dx\leq K^{*}\mathcal{H}^{N-1}\left(\omega\right), \tag{5.32}$$

and

$$\lim_{k\to\infty}\lim_{\delta\to0^{+}}\lim_{\varepsilon\to0^{+}}\left\|u_{\varepsilon,\delta,k}-u\right\|_{W^{1,p}(\Omega;\mathbb{R}^{d})}=0. \tag{5.33}$$

On the other hand, Theorem 4.1 entails

$$\liminf_{k\to\infty}\liminf_{\delta\to 0^+}\liminf_{\varepsilon\to 0^+}\int_\Omega \frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon,\delta,k}\right)+\varepsilon\left|\nabla^2 u_{\varepsilon,\delta,k}\right|^2\,dx \geq K^*\mathcal{H}^{N-1}(\omega).$$

This, together with (5.32) and (5.33), allow us to diagonalize the triple-indexed sequence $u_{\varepsilon,\delta,k}$ to obtain $v_\varepsilon := u_{\varepsilon,\delta(\varepsilon),k(\delta(\varepsilon))}$ satisfying

$$\lim_{\varepsilon\to 0^+}\|v_\varepsilon - u\|_{W^{1,p}(\Omega;\mathbb{R}^d)} = 0$$

and

$$\lim_{\varepsilon\to 0^+}\int_\Omega \frac{1}{\varepsilon}W\left(\nabla v_\varepsilon\right)+\varepsilon\left|\nabla^2 v_\varepsilon\right|^2\,dx = K^*\mathcal{H}^{N-1}(\omega).$$

The case where the function u has the form

$$u(x) = -|x_N|\,a \quad \text{a.e. in } \Omega,$$

may be treated similarly. We omit the details.

Step 2: *Finitely many interfaces.* Assume that the number of interfaces is finite, that is

$$S(\nabla u)\cap\Omega = \bigcup_{i=1}^m \Omega_{l_i},$$

where $\Omega_{l_i} := \{x=(x',x_N)\in\Omega : x_N = l_i\} = \omega_i\times\{l_i\}$, for some finite family $l_1<\cdots<l_m$.

Fix $0<h<\frac{1}{4}\min\{l_{i+1}-l_i : i=1,\cdots,m-1\}$ and consider a piecewise C^1 curve $g:[-L,L]\to\mathbb{R}^d$ as in Step 1. We may now apply the construction of Step 1 to each interface, precisely we define

$$u_{\varepsilon,\delta}(x) := \begin{cases} \psi_{\delta,n}(x)\,z_{\varepsilon,i}(x) + (1-\psi_{\delta,i}(x))\,u(x) & \text{for } |x_N - l_i| < 2h,\ i=1,\cdots,m-1, \\ u(x) & \text{otherwise,}\end{cases}$$

where now $\psi_{\delta,i}$ is a smooth cut-off function such that $\psi_{\delta,i}=1$ in $\omega^+_{\delta,i}\times\left(l_i-\frac{h}{3}, l_i+\frac{h}{3}\right)$ and $\psi_{\delta,i}=0$ outside $\omega^+_{2\delta,i}\times\left(l_i-\frac{h}{2}, l_i+\frac{h}{2}\right)$, where for $s>0$ we denote $\omega^+_{s,i} := \left\{x'\in\mathbb{R}^{N-1} : \operatorname{dist}(x',\omega_i)<s\right\}$, and

$$z_{\varepsilon,i}(x) := u(x',l_i) + \begin{cases} \int_0^{x_N-l_i} g\left(\frac{s}{\varepsilon}\right)ds & \text{if } \nabla u(x) = a\otimes e_N, \\ -\int_0^{-x_N+l_i} g\left(\frac{s}{\varepsilon}\right)ds & \text{if } \nabla u(x) = -a\otimes e_N.\end{cases}$$

As in Step 1, the argument leading to (5.28) now yields

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx &\leq \left(\int_{-L}^{L} W(0,g(s)) + |g'(s)|^2\,ds\right)\sum_{i=1}^m \mathcal{H}^{N-1}\left(\omega^+_{\delta,i}\right) \\ &+ C\left(\frac{1}{\varepsilon}\eta(\varepsilon C_\delta)+C_\delta\varepsilon^3\right) + C\left(\eta(\varepsilon C_\delta + C)+\varepsilon^2 C_\delta + C\right)\sum_{i=1}^m \mathcal{H}^{N-1}\left(\omega^+_{2\delta,i}\backslash\omega^+_{\delta,i}\right)\end{aligned}$$

and the proof is concluded as before upon letting first $\varepsilon\to 0^+$, then $\delta\to 0^+$, and finally $k\to+\infty$ where we consider in place of g a realizing sequence $\{g_k\}$ for K^*.

Step 3: *Countably many interfaces.* If the number of interfaces is infinite, we may proceed exactly as in Step 2 of Theorem 5.5. We omit the details. ■

5.4 General domains

Let $\tau:\mathbb{R}\to\mathbb{R}$ be an odd, C^∞ function, with $\tau(0)=\tau'(0)=0$, and such that $\tau(t)=t$ if $|t|\geq 1$. For $\varepsilon>0$ define

$$u_\varepsilon^{AA}(x) := \varepsilon\tau\left(\frac{x_N}{\varepsilon}\right)a,\ u_\varepsilon^{BB}(x) := -u_\varepsilon^{AA}(x),\ u_\varepsilon^{AB}(x) := \varepsilon\tau\left(\frac{|x_N|}{\varepsilon}\right)a,\ u_\varepsilon^{BA}(x) := -u_\varepsilon^{AB}(x).$$

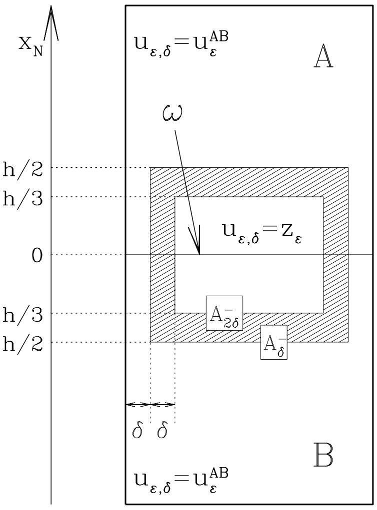

Figure 6: Construction for one interface in Step 1 of Proposition 5.7. The shaded region represents $\{0 < \psi_\delta < 1\}$, and corresponds to $A_\delta^- \backslash A_{2\delta}^-$ where $A_\delta^- := \omega_\delta^- \times (-h/2, h/2)$ and $A_{2\delta}^- := \omega_{2\delta}^- \times (-h/3, h/3)$. The function $u_{\varepsilon,\delta}$ coincides with u_ε^{AB} outside A_δ^- and with z_ε inside $A_{2\delta}^-$. A similar construction is used in Step 2, with u_ε^{AB} replaced by u_ε^{AA}, and z_ε replaced by u.

Proposition 5.7 (Lateral matching) *Assume that* $\Omega = \omega \times (-h, h)$, *where* $\omega \subset \mathbb{R}^{N-1}$ *is a bounded, open connected set with* $\mathcal{H}^{N-1}(\partial\omega) = 0$, *and let* W *satisfy* (H_1), $(H_2)'$ *and* (H_3). *Suppose, in addition, that* W *is differentiable at* A *and* B. *Let* $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, *with* $\nabla u \in BV(\Omega; \{A, B\})$. *Let* $\{\varepsilon_n\} \subset \mathbb{R}_+$ *be a sequence converging to zero. Then there exists* $\{u_n\} \subset W^{2,2}(\Omega; \mathbb{R}^d)$ *such that* $u_n \to u$ *in* $W^{1,p}(\Omega; \mathbb{R}^d)$ *for all* $p \in [1, +\infty)$, $u_n = u$ *nearby* $x_N = \pm h$,

$$u_n(x) = u_{\varepsilon_n}^{FG}(x) + u(x', 0) \text{ nearby } \partial\omega \times (-h, h) \text{ where } \nabla u = \begin{cases} F & \text{if } x_N > 0, \\ G & \text{if } x_N < 0, \end{cases} \quad F, G \in \{A, B\},$$

and

$$\lim_{n \to +\infty} \int_\Omega \frac{1}{\varepsilon_n} W(\nabla u_n) + \varepsilon_n \left|\nabla^2 u_n\right|^2 dx = K^* \mathcal{H}^{N-1}(S(\nabla u) \cap \Omega).$$

Proof. In light of the argument used in Remark 5.4, we claim that given an arbitrary sequence $\{\varepsilon_n\}$ converging to 0^+, it suffices to construct a double indexed sequence $\{u_{n,k}\}$ satisfying the prescribed boundary conditions, such that

$$\lim_{k \to +\infty} \lim_{n \to +\infty} \|u_{k,n} - u\|_{L^1(\Omega; \mathbb{R}^d)} = 0,$$

and

$$\limsup_{k \to +\infty} \limsup_{n \to +\infty} \int_\Omega \frac{1}{\varepsilon_n} W(\nabla u_{k,n}) + \varepsilon \left|\nabla^2 u_{k,n}\right|^2 dx \leq K^* \mathcal{H}^{N-1}(\omega).$$

We fix one such sequence $\{\varepsilon_n\}$, abbreviated $\varepsilon := \varepsilon_n$, and we divide the proof leading to the construction of the double indexed sequence $u_{k,n}$ into two steps corresponding to the cases where ∇u has either no interfaces or one interface in the cylinder Ω.

Step 1: Assume that $\nabla u \equiv \operatorname{sgn}(x_N)\, a \otimes e_N$ in Ω. Without loss of generality we may take $u(x) = |x_N|\, a$. We proceed as in Step 1 of the proof of Theorem 5.6 until (5.16) which should now be replaced by

$$u_{\varepsilon,\delta}(x) := \psi_\delta(x)\, z_\varepsilon(x) + (1 - \psi_\delta(x))\, u_\varepsilon^{AB}, \tag{5.34}$$

where δ stands for the elements δ_m of a sequence converging to 0^+ as $m \to +\infty$, $z_\varepsilon(x) := \int_0^{x_N} g\left(\frac{s}{\varepsilon}\right) ds$ as before, $\psi_\delta \in C_c^\infty(\mathbb{R}^N; [0, 1])$ is a a smooth cut-off function such that $\psi_\delta = 1$ in $\omega_{2\delta}^- \times \left(-\frac{h}{3}, \frac{h}{3}\right)$ and $\psi_\delta = 0$

outside $\omega_\delta^- \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, where for $s > 0$ we denote $\omega_s^- := \{x' \in \omega : \operatorname{dist}(x', \partial\omega) > s\}$ (see Figure 6). In turn (5.17) becomes

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx = &\int_{\{\psi_\delta = 1\}} \frac{1}{\varepsilon} W(\nabla z_\varepsilon) + \varepsilon \left|\nabla^2 z_\varepsilon\right|^2 dx \\ &+ \int_{\{\psi_\delta = 0\}} \frac{1}{\varepsilon} W\left(\nabla u_\varepsilon^{AB}\right) + \varepsilon \left|\nabla^2 u_\varepsilon^{AB}\right|^2 dx \\ &+ \int_{\{0<\psi_\delta<1\}} \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx.\end{aligned} \tag{5.35}$$

Then (5.18) should be replaced by

$$\int_{\{\psi_\delta = 1\}} \frac{1}{\varepsilon} W(\nabla z_\varepsilon) + \varepsilon \left|\nabla^2 z_\varepsilon\right|^2 dx \le \mathcal{H}^{N-1}\left(\omega_{2\delta}^-\right) \int_{-L}^{L} W(0, g(s)) + |g'(s)|^2 ds, \tag{5.36}$$

(5.21) continues to hold, while

$$\begin{aligned}\int_{\{\psi_\delta = 0\}} \frac{1}{\varepsilon} W\left(\nabla u_\varepsilon^{AB}\right) + \varepsilon \left|\nabla^2 u_\varepsilon^{AB}\right|^2 dx &\le \mathcal{H}^{N-1}\left(\omega \backslash \omega_\delta^-\right) \int_{-1}^{1} W(0, \tau'(|t|) \operatorname{sgn}(t)\, a) + |\tau''(|t|)|^2 dt \\ &\le C\mathcal{H}^{N-1}\left(\omega \backslash \omega_\delta^-\right).\end{aligned} \tag{5.37}$$

To estimate the right hand side of (5.21) we replace (5.22) with

$$\begin{aligned}|\nabla u_{\varepsilon,\delta} - \nabla u| &\le |\nabla\psi_\delta| \left|z_\varepsilon - u_\varepsilon^{AB}\right| + \psi_\delta |\nabla z_\varepsilon - \nabla u| + (1-\psi_\delta)\left|\nabla u_\varepsilon^{AB} - \nabla u\right|, \\ \left|\nabla^2 u_{\varepsilon,\delta}\right| &\le \left|\nabla^2 \psi_\delta\right| \left|z_\varepsilon - u_\varepsilon^{AB}\right| + 2|\nabla\psi_\delta| \left|\nabla z_\varepsilon - \nabla u_\varepsilon^{AB}\right| + \psi_\delta \left|\nabla^2 z_\varepsilon\right| + (1-\psi_\delta)\left|\nabla^2 u_\varepsilon^{AB}\right|.\end{aligned} \tag{5.38}$$

Let $L_1 := \max\{1, L\}$. Then for $|x_N| \ge \varepsilon L_1$ we have $u_\varepsilon^{AB} = |x_N|\, a$ and so from (5.23) and (5.38) the bound (5.24) continues to hold for $|x_N| \ge \varepsilon L_1$. Moreover

$$\left|\nabla\left(u_\varepsilon^{AB} - u\right)\right| \le \left|\tau'\left(\pm\frac{x_N}{\varepsilon}\right) a\right| + \|\nabla u\|_\infty \le C, \quad \left|\nabla^2 u_\varepsilon^{AB}\right| \le \frac{1}{\varepsilon}\left|\tau''\left(\pm\frac{x_N}{\varepsilon}\right)\right| \le \frac{C}{\varepsilon} \tag{5.39}$$

for $0 < |x_N| \le \varepsilon L_1$, and thus

$$\left|z_\varepsilon(x) - u_\varepsilon^{AB}(x)\right| \le \left|z_\varepsilon(x', \varepsilon L_1) - u_\varepsilon^{AB}(x, \varepsilon L_1)\right| + \int_{-\varepsilon L_1}^{\varepsilon L_1} \left|\nabla\left(z_\varepsilon - u_\varepsilon^{AB}\right)\right| dx_N \le \varepsilon C. \tag{5.40}$$

Hence, from (5.25), (5.38), (5.39) and (5.40) the estimate (5.26) is still valid for $|x_N| \le \varepsilon L_1$, while (5.27) becomes

$$\begin{aligned}\int_{\{0<\psi_\delta<1\}} \frac{1}{\varepsilon}\eta(|\nabla u_{\varepsilon,\delta} - \nabla u|) + \varepsilon\left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx \le &\left(\frac{1}{\varepsilon}\eta(\varepsilon C_\delta) + C_\delta\varepsilon^3\right)\mathcal{H}^{N-1}(\omega)\, 2h \\ &+ \mathcal{H}^{N-1}\left(\omega_\delta^- \backslash \omega_{2\delta}^-\right) 2L_1\varepsilon\left(\frac{1}{\varepsilon}\eta(\varepsilon C_\delta + C) + \varepsilon\left(C_\delta + \frac{C}{\varepsilon^2}\right)\right).\end{aligned} \tag{5.41}$$

In view of (5.35), (5.37), (5.36), (5.21) and (5.41), we now have

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon\left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx \le &C\mathcal{H}^{N-1}\left(\omega \backslash \omega_{2\delta}^-\right) + \left(\int_{-L}^{L} W(0, g(s)) + |g'(s)|^2 ds\right)\mathcal{H}^{N-1}(\omega) \\ &+ C\left(\frac{1}{\varepsilon}\eta(\varepsilon C_\delta) + C_\delta\varepsilon^3\right) \\ &+ C\left(\eta(\varepsilon C_\delta + C) + \varepsilon^2 C_\delta + C\right)\mathcal{H}^{N-1}\left(\omega_\delta^- \backslash \omega_{2\delta}^-\right),\end{aligned}$$

and letting $\varepsilon \to 0^+$ and then $\delta \to 0^+$ yields, by (5.19),

$$\limsup_{\delta\to 0^+} \limsup_{\varepsilon\to 0^+} \int_\Omega \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon\left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx \le \mathcal{H}^{N-1}(\omega) \int_{-L}^{L} W(0, g(s)) + |g'(s)|^2 ds.$$

We can now continue the argument of Step 1 in the proof of Theorem 5.6 from (5.29) onwards.
Step 2: Assume that $\nabla u \equiv A = a \otimes e_N$ in Ω. Without loss of generality we may assume that $u(x) = ax_N$. Fix $\delta > 0$ and let ψ_δ be defined as in Step 1. Define

$$u_{\varepsilon,\delta}(x) := \psi_\delta(x) u(x) + (1 - \psi_\delta(x)) u_\varepsilon^{AA}(x).$$

For $|x_N| \geq \varepsilon$ we have $u_\varepsilon^{AA}(x) = u(x)$ and so $u_{\varepsilon,\delta}(x) = u(x)$. Since $\psi_\delta = 1$ in $\omega_{2\delta}^- \times \left(-\frac{h}{3}, \frac{h}{3}\right)$ and $\psi_\delta = 0$ outside $\omega_\delta^- \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, for $0 < \varepsilon < h$ we have

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx &\leq \int_{(\omega \backslash \omega_\delta^-) \times (-\varepsilon,\varepsilon)} \frac{1}{\varepsilon} W\left(\nabla u_\varepsilon^{AA}\right) + \varepsilon \left|\nabla^2 u_\varepsilon^{AA}\right|^2 dx \\ &\quad + \int_{(\omega_\delta^- \backslash \omega_{2\delta}^-) \times (-\varepsilon,\varepsilon)} \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx \\ &\leq C\mathcal{H}^{N-1}\left(\omega \backslash \omega_\delta^-\right) + \int_{(\omega_\delta^- \backslash \omega_{2\delta}^-) \times (-\varepsilon,\varepsilon)} \frac{1}{\varepsilon} \eta\left(\left|\nabla u_{\varepsilon,\delta} - a \otimes e_N\right|\right) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx,\end{aligned} \tag{5.42}$$

where we used (5.37) (which continues to hold, provided we replace the derivatives of $\tau(|\cdot|)$ with those of $\tau(\cdot)$) and (5.20).

To estimate the last integral on the right hand side of (5.42) note that

$$\begin{aligned}|\nabla u_{\varepsilon,\delta}(x) - \nabla u(x)| &\leq |\nabla \psi_\delta(x)| \left|u_\varepsilon^{AA}(x) - u(x)\right| + \psi_\delta(x) \left|\nabla u_\varepsilon^{AA}(x) - \nabla u(x)\right|, \\ \left|\nabla^2 u_{\varepsilon,\delta}(x)\right| &\leq \left|\nabla^2 \psi_\delta(x)\right| \left|u_\varepsilon^{AA}(x) - u(x)\right| + 2 |\nabla \psi_\delta(x)| \left|\nabla u_\varepsilon^{AA}(x) - \nabla u(x)\right| + \psi_\delta(x) \left|\nabla^2 u_\varepsilon^{AA}(x)\right|.\end{aligned} \tag{5.43}$$

The bounds (5.39) and (5.40) are still valid for $|x_N| \leq \varepsilon$, with $L_1 := 1$ and u_ε^{AB} replaced by u_ε^{AA}. Hence from (5.43) we deduce that (5.26) holds for $|x_N| \leq \varepsilon$, and thus by (5.42)

$$\int_\Omega \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx \leq C\mathcal{H}^{N-1}\left(\omega \backslash \omega_\delta^-\right) + \mathcal{H}^{N-1}\left(\omega_\delta^- \backslash \omega_{2\delta}^-\right) 2 \left(\eta(\varepsilon C_\delta + C) + \varepsilon^2 C_\delta + C\right),$$

and by (5.19), letting $\varepsilon \to 0^+$ and then $\delta \to 0^+$ yields

$$\limsup_{\delta \to 0^+} \limsup_{\varepsilon \to 0^+} \int_\Omega \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 u_{\varepsilon,\delta}\right|^2 dx = 0.$$

The argument of Remark 5.4 brings the proof of this case to a closure. The remaining two cases where $\nabla u \equiv -a \otimes e_N$ in Ω and $\nabla u \equiv -\operatorname{sgn}(x_N)\, a \otimes e_N$ in Ω are treated in a way similar to Steps 1 and 2. We omit the details. ■

In preparation for the main result of this section, Theorem 5.10, we establish the following inequality for level sets.

Lemma 5.8 *For each $t \in \mathbb{R}$ let $\Omega_t := \{(x', x_N) \in \Omega : x_N = t\}$ denote a horizontal interface of Ω. Then*

$$\limsup_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathcal{H}^{N-1}(\Omega_{s+t})\, ds \leq \mathcal{H}^{N-1}\left(\overline{\Omega}_t\right).$$

Proof. Write $\overline{\Omega}_t = \overline{\omega} \times \{t\}$, fix $\delta > 0$, and consider an open set $\Sigma \supset\supset \overline{\omega}$ such that

$$\mathcal{H}^{N-1}(\Sigma \times \{t\}) \leq \mathcal{H}^{N-1}\left(\overline{\Omega}_t\right) + \delta.$$

We claim that if ε is sufficiently small then

$$\{x \in \Omega : |x_N - t| < \varepsilon\} \subset \Sigma \times (t - \varepsilon, t + \varepsilon).$$

Indeed, if this was not the case then there would exist a sequence $\{x_n\} \subset \Omega$, with $x_n = (x_n', (x_n)_N)$, such that

$$(x_n)_N \to t \text{ and } x_n' \notin \Sigma. \tag{5.44}$$

By extracting a subsequence, if necessary, we may assume that $x_n \to (x', t)$. But then $(x', t) \in \overline{\Omega}_t \subset \Sigma \times \{t\}$, which is in contradiction with (5.44) since Σ is open. Hence the claim holds, and in turn by Fubini's Theorem

$$\int_{-\varepsilon}^{\varepsilon} \mathcal{H}^{N-1}(\Omega_{s+t})\, ds = |\{x \in \Omega : |x_N - t| < \varepsilon\}| \leq |\Sigma \times (t-\varepsilon, t+\varepsilon)| = 2\varepsilon \mathcal{H}^{N-1}(\Sigma \times \{t\}).$$

Hence

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \mathcal{H}^{N-1}(\Omega_{s+t})\, ds \leq \mathcal{H}^{N-1}(\Sigma \times \{t\}) \leq \mathcal{H}^{N-1}\left(\overline{\Omega}_t\right) + \delta,$$

and by letting first $\varepsilon \to 0^+$ and then $\delta \to 0^+$ we conclude the proof. ■

The next result is a generalization of the Isoperimetric Inequality (see [4, 13]).

Theorem 5.9 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded connected domain with Lipschitz boundary. Then there exists a constant C_{iso}, depending on N and on Ω, such that for every set $E \subset \Omega$ of finite perimeter there holds*

$$\min\{|E|, |\Omega \backslash E|\}^{1-\frac{1}{N}} \leq C_{\text{iso}} \operatorname{Per}_\Omega(E). \tag{5.45}$$

Proof. By Poincaré's inequality there exists a constant $C(N, \Omega)$ such that

$$\int_\Omega |v - v_\Omega|^{N/(N-1)}\, dx \leq C\, |Dv|(\Omega)$$

for all $v \in BV(\Omega; \mathbb{R})$, where

$$v_\Omega := \frac{1}{|\Omega|}\int_\Omega v\, dx$$

and $|Dv|(\Omega)$ is the total variation of Dv. If we take $v := \chi_E$ then $v_\Omega := \frac{|E|}{|\Omega|}$ and so

$$\begin{aligned} C \operatorname{Per}_\Omega(E) = C\,|Dv|(\Omega) &\geq \int_\Omega |v - v_\Omega|^{N/(N-1)}\, dx = \int_\Omega \left|\chi_E - \frac{|E|}{|\Omega|}\right|^{N/(N-1)} dx \\ &= |\Omega|^{-N/(N-1)}(|\Omega| - |E|)^{N/(N-1)}|E| + |\Omega|^{-N/(N-1)}|E|^{N/(N-1)}|\Omega \backslash E| \\ &\geq |\Omega|^{-N/(N-1)} \min\{|E|, |\Omega \backslash E|\}^{1-\frac{1}{N}}(|E| + |\Omega \backslash E|) \\ &= |\Omega|^{-1/(N-1)} \min\{|E|, |\Omega \backslash E|\}^{1-\frac{1}{N}}. \end{aligned}$$

Setting $C_{\text{iso}} := C(N, \Omega)\,|\Omega|^{1/(N-1)}$ we conclude the proof. ■

Theorem 5.10 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume that W satisfies the conditions (H_1), $(H_2)'$ and (H_3). Suppose, in addition, that W is differentiable at A and B. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, with $\nabla u \in BV(\Omega; \{A, B\})$. Then*

$$\Gamma - \lim_{\varepsilon \to 0^+} I_\varepsilon(u; \Omega) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = (1 - \chi_E(x))\, A + \chi_E(x)\, B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Proof. Just as in the proofs of the previous Γ-limit results, in view of Theorem 4.1 and of Remark 5.4 it suffices to show that given an arbitrary sequence $\{\varepsilon_n\}$ converging to 0^+ we may construct a double indexed sequence $\{u_{n,k}\}$ such that

$$\lim_{k \to +\infty} \lim_{n \to +\infty} \|u_{k,n} - u\|_{L^1(\Omega; \mathbb{R}^d)} = 0,$$

and

$$\limsup_{k \to +\infty} \limsup_{n \to +\infty} \int_\Omega \frac{1}{\varepsilon_n} W(\nabla u_{k,n}) + \varepsilon \left|\nabla^2 u_{k,n}\right|^2 dx \leq K^* \mathcal{H}^{N-1}(\omega).$$

We fix one such sequence $\{\varepsilon_n\}$, abbreviated $\varepsilon := \varepsilon_n$, and we divide the proof leading to the construction of the double indexed sequence $u_{k,n}$ into four steps.

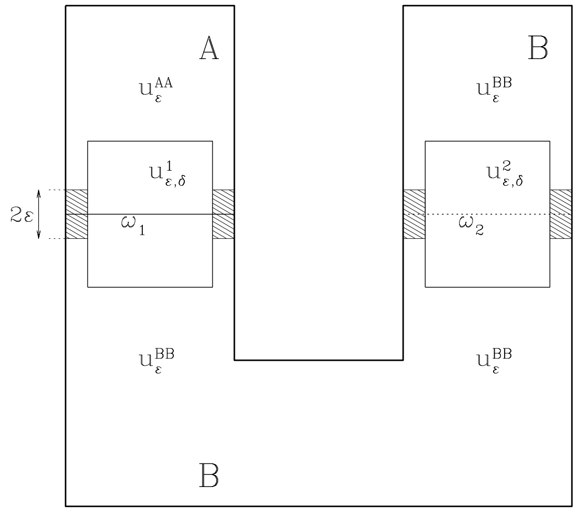

Figure 7: Construction for the case of one hyperplane, with continuously varying interface area, in Step 1 of Theorem 5.10. The functions $u^1_{\varepsilon,\delta}$ and $u^2_{\varepsilon,\delta}$ used in the two cylinders have been constructed in Proposition 5.7, and on the cylinder boundaries agree with u_ε^{AB} and u_ε^{BB}, respectively. The shaded region represents the set $E_{\varepsilon,\delta}$.

Step 1: *One hyperplane– case 1.* We assume first that

$$S(\nabla u) \cap \Omega \subset \Omega_0 := \{x = (x', x_N) \in \Omega : \ x_N = 0\}$$

and that (see Figure 1)

$$t \mapsto \mathcal{H}^{N-1}(\Omega_t) \ \text{ is continuous at } t = 0. \tag{5.46}$$

Then we may write

$$\Omega_0 = \bigcup_{i=1}^{\infty} \omega_i \times \{0\},$$

where the open sets $\omega_i \subset \mathbb{R}^{N-1}$ are pairwise disjoint and connected. Since Ω is bounded we know that

$$\sum_{i=1}^{\infty} \mathcal{H}^{N-1}(\omega_i \times \{0\}) < \infty.$$

For fixed $\delta > 0$, standing for an arbitrary element of a sequence $\{\delta_k\}$ converging to 0^+, choose $M > 1$ so large that

$$\sum_{i=M+1}^{\infty} \mathcal{H}^{N-1}(\omega_i \times \{0\}) < \delta, \tag{5.47}$$

and for each $i = 1, \cdots, M$, let $\omega_{i,\delta} \subset\subset \omega_i$ be such that

$$\mathcal{H}^{N-1}((\omega_i \backslash \omega_{i,\delta}) \times \{0\}) < \frac{\delta}{M}. \tag{5.48}$$

Since $\omega_{i,\delta} \times \{0\} \subset\subset \Omega$ there exists $h > 0$ such that the cylinders $\omega_{i,\delta} \times (-h, h) \subset\subset \Omega$ for each $i = 1, \cdots, M$ (see Figure 7). In each cylinder $\omega_{i,\delta} \times (-h, h)$ we may apply Proposition 5.7 to obtain sequences $\left\{u^i_{\varepsilon,\delta}\right\} \subset W^{2,2}\left(\omega_{i,\delta} \times (-h, h); \mathbb{R}^d\right)$ such that $u^i_{\varepsilon,\delta} \to u$ in $W^{1,2}\left(\omega_{i,\delta} \times (-h, h); \mathbb{R}^d\right)$, $u^i_{\varepsilon,\delta} = u$ nearby $x_N = \pm h$,

$$u^i_{\varepsilon,\delta}(x) = u_\varepsilon^{FG}(x) + u(x', 0) \ \text{ nearby } \partial\omega_{i,\delta} \times (-h, h), \ \text{ where } \nabla u = \begin{cases} F & \text{in } \omega_{i,\delta} \times (0, h), \\ G & \text{in } \omega_{i,\delta} \times (-h, 0), \end{cases} \ F, G \in \{A, B\},$$

and

$$\lim_{\varepsilon\to 0^+}\int_{\omega_{i,\delta}\times(-h,h)}\frac{1}{\varepsilon}W\left(\nabla u^i_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 u^i_{\varepsilon,\delta}\right|^2 dx = K^*\mathcal{H}^{N-1}\left(S\left(\nabla u\right)\cap\left(\omega_{i,\delta}\times\{0\}\right)\right). \tag{5.49}$$

Define

$$v_{\varepsilon,\delta}(x):=\begin{cases} u^i_{\varepsilon,\delta}(x) & \text{if } x\in\omega_{i,\delta}\times(0,h),\ i=1,\cdots,M,\\ u^{AA}_\varepsilon(x)+u(x',0) & \text{if } x\in\Omega\backslash\left(\bigcup_{i=1}^M\omega_{i,\delta}\times(-h,h)\right) \text{ and } \nabla u(x)=A,\\ u^{BB}_\varepsilon(x)+u(x',0) & \text{if } x\in\Omega\backslash\left(\bigcup_{i=1}^M\omega_{i,\delta}\times(-h,h)\right) \text{ and } \nabla u(x)=B.\end{cases} \tag{5.50}$$

We claim that $\{v_{\varepsilon,\delta}\}\subset W^{2,\infty}\left(\Omega;\mathbb{R}^d\right)$, $v_{\varepsilon,\delta}\to u$ in $L^1\left(\Omega;\mathbb{R}^d\right)$ and

$$\limsup_{\delta\to 0^+}\limsup_{\varepsilon\to 0^+}\int_\Omega\frac{1}{\varepsilon}W\left(\nabla v_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 v_{\varepsilon,\delta}\right|^2\,dx\le K^*\mathcal{H}^{N-1}\left(S\left(\nabla u\right)\cap\Omega\right).$$

Define

$$E_{\varepsilon,\delta}:=\left\{x\in\Omega\backslash\left(\bigcup_{i=1}^M\omega_{i,\delta}\times(-h,h)\right):\ |x_N|<\varepsilon\right\}.$$

Since $u^{FG}_\varepsilon(x)+u(x',0)=u(x)$ for $|x_N|\ge\varepsilon$ we have

$$\begin{aligned}\int_\Omega\frac{1}{\varepsilon}W\left(\nabla v_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 v_{\varepsilon,\delta}\right|^2\,dx =&\sum_{i=1}^M\int_{\omega_{i,\delta}\times(-h,h)}\frac{1}{\varepsilon}W\left(\nabla u^i_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 u^i_{\varepsilon,\delta}\right|^2\,dx\\ &+\int_{E_{\varepsilon,\delta}}\frac{1}{\varepsilon}W\left(\nabla v_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 v_{\varepsilon,\delta}\right|^2\,dx\\ =&\sum_{i=1}^M\int_{\omega_{i,\delta}\times(-h,h)}\frac{1}{\varepsilon}W\left(\nabla u^i_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 u^i_{\varepsilon,\delta}\right|^2\,dx\\ &+\frac{1}{\varepsilon}\int_{E_{\varepsilon,\delta}}W\left(0,\pm\tau'\left(\pm\frac{x_N}{\varepsilon}\right)a\right)+\left|\tau''\left(\pm\frac{x_N}{\varepsilon}\right)\right|^2\,dx.\end{aligned} \tag{5.51}$$

By Fubini's Theorem we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_{E_{\varepsilon,\delta}}W\left(0,\pm\tau'\left(\pm\frac{x_N}{\varepsilon}\right)a\right)+\left|\tau''\left(\pm\frac{x_N}{\varepsilon}\right)\right|^2 dx&\le C\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(\mathcal{H}^{N-1}\left(\Omega_s\right)-\sum_{i=1}^M\mathcal{L}^{N-1}\left(\omega_{i,\delta}\right)\right)ds\\ &=C\left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\mathcal{H}^{N-1}\left(\Omega_s\right)ds-2\sum_{i=1}^M\mathcal{L}^{N-1}\left(\omega_{i,\delta}\right)\right)\\ &\to 2C\left(\mathcal{H}^{N-1}\left(\Omega_0\right)-\sum_{i=1}^M\mathcal{H}^{N-1}\left(\omega_{i,\delta}\times\{0\}\right)\right)\end{aligned}$$

as $\varepsilon\to 0^+$, and where we have used (5.46). By (5.47), (5.48), (5.49) and (5.51) we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}\int_\Omega\frac{1}{\varepsilon}W\left(\nabla v_{\varepsilon,\delta}\right)+\varepsilon\left|\nabla^2 v_{\varepsilon,\delta}\right|^2\,dx&\le K^*\sum_{i=1}^M\mathcal{H}^{N-1}\left(S\left(\nabla u\right)\cap\left(\omega_{i,\delta}\times\{0\}\right)\right)\\ &\quad+C\sum_{i=M+1}^\infty\mathcal{H}^{N-1}\left(\omega_i\times\{0\}\right)+C\delta\\ &\le K^*\mathcal{H}^{N-1}\left(S\left(\nabla u\right)\cap\Omega\right)+C\delta.\end{aligned}$$

It now suffices to let $\delta\to 0^+$.

Step 2: *One hyperplane– case 2.* Next we remove condition (5.46), and thus we only assume that (see Figure 2)

$$S\left(\nabla u\right)\cap\Omega\subset\Omega_0:=\left\{x=\left(x',x_N\right)\in\Omega:\ x_N=0\right\}.$$

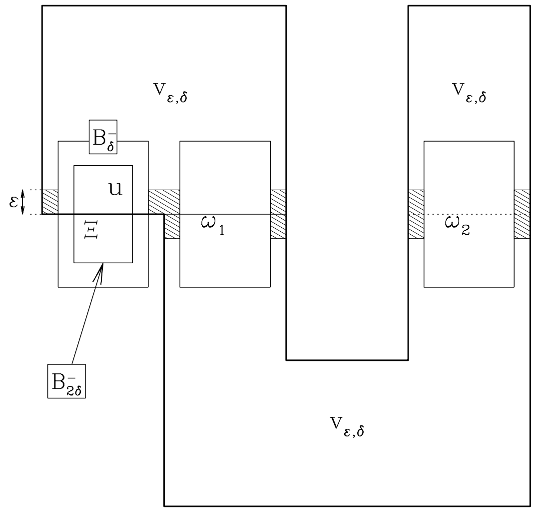

Figure 8: Construction for one hyperplane in Step 2 of Theorem 5.10. The function $v_{\varepsilon,\delta}$, which is used everywhere except for the set $B^-_\delta := \Xi^-_\delta \times (-h/2, h/2)$, is the one constructed in Step 1. Inside $B^-_{2\delta} := \Xi^-_{2\delta} \times (-h/3, h/3)$, the limit u is used directly. The set $F_{\varepsilon,\delta}$ is the shaded region. The two boxes $\omega^-_{1,\delta} \times (-h/2, h/2)$ and $\omega^-_{2,\delta} \times (-h/2, h/2)$ are also shown.

Write

$$\partial\Omega \cap \left\{x = (x', x_N) \in \mathbb{R}^N : x_N = 0\right\} =: (\Xi \times \{0\}) \cup E,$$

where Ξ is an open subset of $\mathbb{R}^{N-1}$ and $\mathcal{H}^{N-1}(E) = 0$. Fix $\delta > 0$ and consider a smooth cut-off function ψ_δ such that $\psi_\delta = 1$ in $\Xi^-_{2\delta} \times \left(-\frac{h}{3}, \frac{h}{3}\right)$ and $\psi_\delta = 0$ outside $\Xi^-_\delta \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, where for $s > 0$ we denote $\Xi^-_s := \{x' \in \Xi : \operatorname{dist}(x', \partial\Xi) > s\}$ (see Figure 8). For $x \in \Omega$ define

$$w_{\varepsilon,\delta}(x) := \psi_\delta(x)\, u(x) + (1 - \psi_\delta(x))\, v_{\varepsilon,\delta},$$

where $\{v_{\varepsilon,\delta}\}$ is the sequence defined in (5.50). Note that $u \in W^{2,2}\left(\Omega \cap \Xi^-_\delta \times \left(-\frac{h}{2}, \frac{h}{2}\right); \mathbb{R}^d\right)$. Set

$$F_{\varepsilon,\delta} := \left\{x \in \Omega \backslash \left(\left(\Xi^-_\delta \times (-h, h)\right) \cup \bigcup_{i=1}^{M} (\omega_{i,\delta} \times (-h, h)) \right) : |x_N| < \varepsilon \right\}.$$

Using (5.51) we have

$$\begin{aligned}
\int_\Omega \frac{1}{\varepsilon} W(\nabla w_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 w_{\varepsilon,\delta}\right|^2 dx &= \int_{\{\psi_\delta = 0\}} \frac{1}{\varepsilon} W(\nabla v_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 v_{\varepsilon,\delta}\right|^2 dx \\
&+ \int_{\{0<\psi_\delta<1\}} \frac{1}{\varepsilon} W(\nabla w_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 w_{\varepsilon,\delta}\right|^2 dx \\
&\leq \sum_{i=1}^{M} \int_{\omega_{i,\delta}\times(-h,h)} \frac{1}{\varepsilon} W\left(\nabla u^i_{\varepsilon,\delta}\right) + \varepsilon \left|\nabla^2 u^i_{\varepsilon,\delta}\right|^2 dx + \frac{1}{\varepsilon} \int_{F_{\varepsilon,\delta}} W\left(0, \pm\tau'\left(\pm\frac{x_N}{\varepsilon}\right) a\right) + \left|\tau''\left(\pm\frac{x_N}{\varepsilon}\right)\right|^2 dx \qquad (5.52) \\
&+ \int_{\{0<\psi_\delta<1\}} \frac{1}{\varepsilon} W(\nabla w_{\varepsilon,\delta}) + \varepsilon \left|\nabla^2 w_{\varepsilon,\delta}\right|^2 dx.
\end{aligned}$$

The second integral on the right hand side of (5.52) may be estimated as before to obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{F_{\varepsilon,\delta}} W\left(0,\pm\tau'\left(\pm\frac{x_N}{\varepsilon}\right)a\right)+\left|\tau''\left(\pm\frac{x_N}{\varepsilon}\right)\right|^2 dx \qquad (5.53)\\
&\leq \limsup_{\varepsilon\to0^+} C\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(\mathcal{H}^{N-1}(\Omega_s)-\mathcal{L}^{N-1}\left(\Xi_\delta^-\right)-\sum_{i=1}^{M}\mathcal{L}^{N-1}(\omega_{i,\delta})\right)ds\\
&= C\limsup_{\varepsilon\to0^+}\left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\mathcal{H}^{N-1}(\Omega_s)\,ds-2\mathcal{L}^{N-1}\left(\Xi_\delta^-\right)-2\sum_{i=1}^{M}\mathcal{L}^{N-1}(\omega_{i,\delta})\right)\\
&\leq 2C\left(\mathcal{H}^{N-1}\left(\overline{\Omega}_0\right)-\mathcal{L}^{N-1}\left(\Xi_\delta^-\right)-\sum_{i=1}^{M}\mathcal{H}^{N-1}(\omega_{i,\delta}\times\{0\})\right)\\
&\leq 2C\left(\mathcal{H}^{N-1}\left(\left(\Xi\backslash\Xi_\delta^-\right)\times\{0\}\right)+\sum_{i=M+1}^{\infty}\mathcal{H}^{N-1}(\omega_i\times\{0\})+\delta\right),
\end{aligned}$$

where we have used Lemma 5.8.

The estimate of the third integral on the right hand side of (5.52) is very similar to the proof of Step 2 of Proposition 5.7. Indeed, for $|x_N|\geq\varepsilon$ we have $u_\varepsilon^{FG}(x)+u(x',0)=u(x)$ and so $w_{\varepsilon,\delta}=u$ in the set $\{x\in\Omega:0<\psi_\delta(x)<1,\ |x_N|\geq\varepsilon\}$. Hence

$$\int_{\{0<\psi_\delta<1\}}\frac{1}{\varepsilon}W(\nabla w_{\varepsilon,\delta})+\varepsilon\left|\nabla^2 w_{\varepsilon,\delta}\right|^2 dx\leq\int_{\left(\Xi_\delta^-\backslash\Xi_{2\delta}^-\right)\times(-\varepsilon,\varepsilon)}\frac{1}{\varepsilon}\eta\left(|\nabla w_{\varepsilon,\delta}-\nabla u|\right)+\varepsilon\left|\nabla_{\varepsilon,\delta}^2 w\right|^2 dx, \qquad (5.54)$$

where we used (5.20) and we have extended u to all of $\mathbb{R}^N$ as an affine function. The estimates (5.38), (5.39) and (5.40) continue to hold for $x\in\left(\Xi_\delta^-\backslash\Xi_{2\delta}^-\right)\times(-\varepsilon,\varepsilon)$, with u_ε^{AB} replaced by u_ε^{FG}. Hence from (5.38) we deduce that (5.26) is still valid for $x\in\left(\Xi_\delta^-\backslash\Xi_{2\delta}^-\right)\times(-\varepsilon,\varepsilon)$, and thus by (5.52)-(5.54) we obtain

$$\limsup_{\varepsilon\to0^+}\int_\Omega\frac{1}{\varepsilon}W(\nabla w_{\varepsilon,\delta})+\varepsilon\left|\nabla^2 w_{\varepsilon,\delta}\right|^2 dx\leq K^*\mathcal{H}^{N-1}(S(\nabla u)\cap\Omega)+C\delta+\mathcal{H}^{N-1}\left(\Xi_\delta^-\backslash\Xi_{2\delta}^-\right)2(\eta(C)+C).$$

Letting $\delta\to0^+$ concludes the proof.

Step 3: *Finitely many hyperplanes.* Assume that

$$S(\nabla u)\cap\Omega\subset\bigcup_{i=1}^{M}\Omega_{l_i},$$

where $\Omega_{l_i}:=\{x=(x',x_N)\in\Omega:\ x_N=l_i\}=\omega_i\times\{l_i\}$ for some finite family $l_1<\cdots<l_M$.

Fix $0<h<\frac{1}{2}\min\{l_{i+1}-l_i:\ i=1,\cdots,M-1\}$ and let $\{\varepsilon\}\subset\mathbb{R}_+$ be a sequence converging to zero. Define

$$\Omega^i:=\{x\in\Omega:\ |x_N-l_i|<2h\}.$$

By Step 2 applied to Ω^i we may find sequences $\{u_\varepsilon^i\}\subset W^{2,2}\left(\Omega^i;\mathbb{R}^d\right)$ such that $u_\varepsilon^i\to u$ in $W^{1,2}\left(\Omega^i;\mathbb{R}^d\right)$, $u_\varepsilon^i=u$ nearby $x_N=l_i\pm h$, and

$$\lim_{\varepsilon\to0^+}\int_{\Omega^i}\frac{1}{\varepsilon}W\left(\nabla u_\varepsilon^i\right)+\varepsilon\left|\nabla^2 u_\varepsilon^i\right|^2 dx=K^*\mathcal{H}^{N-1}\left(S(\nabla u)\cap\Omega^i\right).$$

It suffices to define

$$u_\varepsilon(x):=\begin{cases}u_\varepsilon^i(x) & \text{if } x\in\Omega^i,\ i=1,\cdots,M,\\ u(x) & \text{otherwise.}\end{cases}$$

Then $\{u_\varepsilon\}\subset W^{2,2}\left(\Omega;\mathbb{R}^d\right)$, $u_\varepsilon\to u$ in $W^{1,2}\left(\Omega;\mathbb{R}^d\right)$, and

$$\lim_{\varepsilon\to0^+}\int_\Omega\frac{1}{\varepsilon}W(\nabla u_\varepsilon)+\varepsilon\left|\nabla^2 u_\varepsilon\right|^2 dx=K^*\mathcal{H}^{N-1}(S(\nabla u)\cap\Omega).$$

Step 4: *Countably many hyperplanes.* Assume that

$$S(\nabla u)\cap\Omega=\bigcup_{i=1}^{\infty}\omega_i\times\{l_i\},$$

where $\omega_i\subset\mathbb{R}^{N-1}$ are connected open sets with $\partial(\omega_i\times\{l_i\})\subset\partial\Omega$ and $\{l_i\}$ is a sequence of real numbers (not necessarily distinct). Fix $0<\delta<\min\left\{1,\frac{1}{2}|\Omega|\right\}$. Since

$$\mathcal{H}^{N-1}(S(\nabla u)\cap\Omega)=\sum_{i=1}^{\infty}\mathcal{H}^{N-1}(\omega_i\times\{l_i\})<\infty, \tag{5.55}$$

there exists an integer $M=M(\delta)$ such that

$$\sum_{i=M+1}^{\infty}\mathcal{H}^{N-1}(\omega_i\times\{l_i\})<\frac{1}{c_0}\delta<1,$$

where $c_0:=\max\left\{1,C_{\text{iso}}^{\frac{N}{N-1}}\right\}$ and C_{iso} is the isoperimetric constant introduced in (5.45). As Ω is simply connected, for each $i=M+1,\cdots$, the set $\Omega\backslash(\omega_i\times\{l_i\})$ may be written as the union of two open connected disjoint sets E_i and $\Omega\backslash E_i$, where $|E_i|\geq\frac{1}{2}|\Omega|$. By Theorem 5.9

$$\min\{|E_i|,|\Omega\backslash E_i|\}\leq c_0\mathcal{H}^{N-1}(\partial E_i\cap\Omega)^{\frac{N}{N-1}}=c_0\mathcal{H}^{N-1}(\omega_i\times\{l_i\})^{\frac{N}{N-1}}<\delta<\frac{1}{2}|\Omega|,$$

and so $|\Omega\backslash E_i|\leq c_0\mathcal{H}^{N-1}(\omega_i\times\{l_i\})^{\frac{N}{N-1}}$. Set $u^M:=u$, and for each $i=M+1,\cdots$, define u^i as u^{i-1} in the set E_i, while we extend u^i as an affine function outside E_i. Thus ∇u^i is continuous across $\omega_i\times\{l_i\}$ and

$$S\left(\nabla u^i\right)\cap\Omega\subset\left(\bigcup_{j=1}^{M}\omega_j\times\{l_j\}\right)\cup\left(\bigcup_{j=i+1}^{\infty}\omega_j\times\{l_j\}\right). \tag{5.56}$$

Clearly $\nabla u^i\in BV(\Omega;\{A,B\})$ and by (5.55) and (5.56)

$$\sup_i\mathcal{H}^{N-1}\left(S\left(\nabla u^i\right)\cap\Omega\right)<\infty.$$

In addition, $u^i=u$ on $\Omega\backslash\left(\bigcup_{j=M+1}^{\infty}(\Omega\backslash E_j)\right)=\bigcap_{j=M+1}^{\infty}E_j$, with

$$\left|\bigcap_{j=M+1}^{\infty}E_j\right|\geq|\Omega|-c_0\sum_{j=M+1}^{\infty}\mathcal{H}^{N-1}(\omega_j\times\{l_j\})^{\frac{N}{N-1}}\geq|\Omega|-c_0\sum_{j=M+1}^{\infty}\mathcal{H}^{N-1}(\omega_j\times\{l_j\})\geq\frac{|\Omega|}{2}. \tag{5.57}$$

By Poincaré's inequality we can extract a subsequence (not relabelled) converging in $W^{1,1}\left(\Omega;\mathbb{R}^d\right)$ to a function v^M, with $\nabla v^M\in BV(\Omega;\{A,B\})$,

$$v^M=u\text{ on }\bigcap_{i=M+1}^{\infty}E_i, \tag{5.58}$$

and

$$S\left(\nabla v^M\right)\cap\Omega\subset\bigcup_{j=1}^{M}\omega_j\times\{l_j\}.$$

In order to assert the latter inclusion, consider a point $x_0\notin\bigcup_{j=1}^{M}\omega_j\times\{l_j\}$, and find $r>0$ such that

$$Q(x_0,r)\subset\Omega\backslash\left(\bigcup_{j=1}^{M}\omega_j\times\{l_j\}\right).$$

If i is large enough so that $\mathcal{H}^{N-1}(\omega_i \times \{l_i\}) < r^{N-1}$ then, clearly, and in light of (5.56),

$$Q(x_0, r) \cap S(\nabla u^i) = \emptyset.$$

We deduce, therefore, that $\nabla u^i \in W^{1,\infty}(Q(x_0, r); \mathbb{R}^{d\times N})$ and thus $\nabla v^M \in W^{1,\infty}(Q(x_0, r); \mathbb{R}^{d\times N})$. In particular, $x_0 \notin S(\nabla v^M)$.

Let $\{\varepsilon\} \subset \mathbb{R}_+$ be a sequence converging to zero. By Step 3 we may find sequences $\{v_\varepsilon^M\} \subset W^{2,2}(\Omega; \mathbb{R}^d)$ such that $v_\varepsilon^M \to v^M$ in $W^{1,2}(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0^+$, and

$$\lim_{\varepsilon\to 0^+} \int_\Omega \frac{1}{\varepsilon} W(\nabla v_\varepsilon^M) + \varepsilon |\nabla^2 v_\varepsilon^M|^2 \, dx = K^* \mathcal{H}^{N-1}(S(\nabla v^M) \cap \Omega) \le K^* \sum_{i=1}^{M} \mathcal{H}^{N-1}(\omega_i \times \{l_i\}).$$

In turn

$$\begin{aligned}\limsup_{M\to\infty} \lim_{\varepsilon\to 0^+} \int_\Omega \frac{1}{\varepsilon} W(\nabla v_\varepsilon^M) + \varepsilon |\nabla^2 v_\varepsilon^M|^2 \, dx &\le K^* \sum_{i=1}^{\infty} \mathcal{H}^{N-1}(\omega_i \times \{l_i\}) \\ &= K^* \mathcal{H}^{N-1}(S(\nabla u) \cap \Omega).\end{aligned}$$

Since $v_\varepsilon^M \to v^M$ in $W^{1,2}(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0^+$, by means of a standard diagonalization process it suffices to prove that $v^M \to u$ in $L^1(\Omega; \mathbb{R}^d)$ as $M \to \infty$. By construction we have

$$\begin{aligned}\int_\Omega |\nabla v^M - \nabla u| \, dx &\le \int_{\bigcup_{i=M+1}^\infty \Omega\setminus E_i} |\nabla v^M - \nabla u| \, dx \\ &\le 2|a \otimes e_N| \left| \bigcup_{i=M+1}^{\infty} \Omega\setminus E_i \right| \le C \sum_{i=M+1}^{\infty} |\Omega\setminus E_i| \\ &\le C \sum_{i=M+1}^{\infty} |\Omega\setminus E_i|^{1-\frac{1}{N}} \le C \sum_{i=M+1}^{\infty} \mathcal{H}^{N-1}(\omega_i \times \{l_i\}) \le C\delta,\end{aligned}$$

and by Poincaré's inequality and (5.58) we obtain

$$\int_\Omega |v^M - u| \, dx \le \tilde{C} \int_\Omega |\nabla v^M - \nabla u| \, dx \le \tilde{C}\delta,$$

where the Poincaré constant $\tilde{C}$ may be taken independently of δ in view of (5.57). It now suffices to let $\delta \to 0^+$. ■

6 Γ−limsup: the upper bound. The symmetry hypotheses.

We introduce the notation

$$\xi = (\xi_1, \cdots, \xi_N) \in \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N \text{ times}}, \quad \xi' = (\xi_1, \cdots, \xi_N - 1) \in \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N-1 \text{ times}},$$

so that $\xi = (\xi', \xi_N) \in \mathbb{R}^{d\times(N-1)} \times \mathbb{R}^d$. Throughout this section we assume that

(H_1) W is continuous, $W(\xi) = 0$ if and only if $\xi \in \{A, B\}$, where $A = -B = a \otimes e_N$, for some $a \in \mathbb{R}^d \setminus \{0\}$;

$(H_2)''$ there exist an exponent $p \ge 2$ and a constant $C > 1$ such that

$$\frac{1}{C}|\xi|^p - C \le W(\xi) \le C(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{d\times N}$;

(H_4) there exist constants ρ, $\gamma > 0$ such that

$$\frac{1}{\gamma}|\xi - A|^p \le W(\xi) \le \gamma |\xi - A|^p \quad \text{if } |\xi - A| \le \rho,$$
$$\frac{1}{\gamma}|\xi - B|^p \le W(\xi) \le \gamma |\xi - B|^p \quad \text{if } |\xi - B| \le \rho.$$

(H_5) W is even in each variable ξ_i, $i = 1, \cdots, N-1$, that is $W(\xi_1, \cdots, -\xi_i, \cdots, \xi_N) = W(\xi_1, \cdots, \xi_i, \cdots, \xi_N)$ for each $i = 1, \cdots, N-1$.

Hypothesis (H_4) may be improved as

$(H_4)'$ there exist a constant $\rho > 0$ and a convex function $g : [0, \infty) \to [0, \infty)$, with $g(s) = 0$ if and only if $s = 0$, such that g is derivable in $s = 0$,

$$g(2t) \le cg(t), \tag{6.1}$$

for all $0 \le t \le \rho$,

$$g(|\xi - A|) \le W(\xi) \le cg(|\xi - A|) \text{ if } |\xi - A| \le \rho,$$

and

$$g(|\xi - B|) \le W(\xi) \le cg(|\xi - B|) \text{ if } |\xi - B| \le \rho,$$

for some constant $c > 0$.

Condition (6.1) is called the *doubling condition* – it prevents g to be too degenerate near $t = 0$, precisely, it is satisfied if $g(t) \sim$const. t^p as $t \to 0^+$, for some $p \ge 1$, while it does not hold if g grows exponentially near the origin, i.e., $g(t) \sim$const. e^{-1/t^2} as $t \to 0^+$.

Remark 6.1 In what follows, and without loss of generality, we will consider the model case where $A = -B = a \otimes e_N$. It is easy to check that $(H_2)''$ and (H_4) yield

$$W(\xi) \ge C_1|\xi'|^p$$

for all $\xi \in \mathbb{R}^{d\times N}$ and for some constant $C_1 > 0$. Moreover, we claim that

$$W(\xi) \le C_2 (W(\eta) + |\xi - \eta|^p), \tag{6.2}$$

for all $\xi, \eta \in \mathbb{R}^{d\times N}$, and for some constant $C_2 > 0$. Indeed, assume by contradiction that (6.2) does not hold. Then there exist two sequences $\{\xi_n\}$, $\{\eta_n\} \subset \mathbb{R}^{d\times N}$ such that

$$W(\xi_n) > n(W(\eta_n) + |\xi_n - \eta_n|^p). \tag{6.3}$$

We have by $(H_2)''$

$$\begin{aligned} C(|\xi_n|^p + 1) \ge W(\xi_n) > n(W(\eta_n) + |\xi_n - \eta_n|^p) &\ge n\left(\frac{1}{C}|\eta_n|^p - C + \frac{1}{C}|\xi_n - \eta_n|^p\right) \\ &\ge n\left(\frac{1}{2^{p-1}C}|\xi_n|^p - C\right), \end{aligned}$$

where we have used the inequality $|a|^p + |b|^p \ge \frac{1}{2^{p-1}}|a - b|^p$. This clearly implies that the sequence $\{\xi_n\}$ is bounded, and by (6.3) it follows that $W(\eta_n) + |\xi_n - \eta_n|^p \to 0$. In view of (H_1) we may assume, without loss of generality, that $\eta_n \to A$. For n sufficiently large the quantities $|\xi_n - A|, |\eta_n - A|$ are so small that (H_2) may be applied, and (6.3) yields

$$\begin{aligned} \gamma|\xi_n - A|^p \ge W(\xi_n) &> n(W(\eta_n) + |\xi_n - \eta_n|^p) \\ &\ge n\left(\frac{1}{\gamma}|\eta_n - A|^p + |\xi_n - \eta_n|^p\right) \\ &\ge \frac{1}{2^{p-1}} n \min\left\{1, \frac{1}{\gamma}\right\}|\xi_n - A|^p, \end{aligned}$$

which is clearly a contradiction for n large.

For simplicity we will present the proof of the analog of the results of Section 5 first under hypothesis (H_4), and then on Section 7 we move on to the general case where $(H_4)'$ holds.

6.1 Characterization of K^*

In this subsection we prove that under conditions (H_1), $(H_2)''$, (H_4) and (H_5)

$$K^* = \inf\left\{\int_Q L\,W(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2 dx : L > 0, v \in W^{2,\infty}\left(Q;\mathbb{R}^d\right),\right.$$
$$\left.\nabla v = \pm a\otimes e_N \text{ nearby } x_N = \pm\frac{1}{2},\ v \text{ periodic of period one in } x'\right\}.$$

The next two propositions will establish that, for cubes, realizing sequences may be taken periodic in the transversal directions, and that there is a matching of vertical boundary conditions, precisely:

Proposition 6.2 (Vertical matching) *Assume that W satisfies (H_1), $(H_2)''$ and (H_4). Then there exist sequences $\{\varepsilon_n\}\subset\mathbb{R}_+$, $\{c_n\}\subset\mathbb{R}$ and $\{z_n\}\subset W^{2,2}\left(Q;\mathbb{R}^d\right)$, such that $\varepsilon_n\to 0^+$, $c_n\to 0$, $z_n\to |x_N|\,a$ in $W^{1,p}\left(Q;\mathbb{R}^d\right)$,*

$$z_n(x) = -x_N\,a \text{ nearby } x_N = -\frac{1}{2},\quad z_n(x) = x_N\,a + c_n \text{ nearby } x_N = \frac{1}{2}, \tag{6.4}$$

and

$$\lim_{n\to+\infty}\int_Q \frac{1}{\varepsilon_n}W(\nabla z_n) + \varepsilon_n\left|\nabla^2 z_n\right|^2 dx = K^*.$$

Proof. By definition of K^* there exist sequences $\{\varepsilon_n\}\subset\mathbb{R}_+$, $\{u_n\}\subset W^{2,2}\left(Q;\mathbb{R}^d\right)$, such that $\varepsilon_n\to 0^+$, $u_n\to u_0 := |x_N|\,a$ in $L^1\left(Q;\mathbb{R}^d\right)$ and

$$\lim_{n\to+\infty}\int_Q \frac{1}{\varepsilon_n}W(\nabla u_n) + \varepsilon_n\left|\nabla^2 u_n\right|^2 dx = K^*.$$

We abbreviate $\varepsilon := \varepsilon_n$ and $u_\varepsilon := u_n$. Due to Theorem 3.1 and Remark 3.2 (ii), we may assume, up to extraction of a subsequence, that $u_\varepsilon\to u$ in $W^{1,p}\left(Q;\mathbb{R}^d\right)$. Partition $Q'\times\left(\frac{1}{6},\frac{1}{3}\right)$ into $\left[\frac{1}{\varepsilon}\right]$ horizontal layers of height $\left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{6}$. In view of Remark 4.4, choose one such layer, $L_\varepsilon = Q'\times\left(\theta_\varepsilon - \left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{6},\theta_\varepsilon\right)$, such that

$$\begin{aligned}&\left[\frac{1}{\varepsilon}\right]\int_{L_\varepsilon}\frac{1}{\varepsilon}W(\nabla u_\varepsilon) + \varepsilon\left|\nabla^2 u_\varepsilon\right|^2 + |\nabla u_\varepsilon - a\otimes e_N|^p + |u_\varepsilon - u_0|^p\,dx \\ &\le \int_{Q'\times\left(\frac{1}{6},\frac{1}{3}\right)}\frac{1}{\varepsilon}W(\nabla u_\varepsilon) + \varepsilon\left|\nabla^2 u_\varepsilon\right|^2 + |\nabla u_\varepsilon - a\otimes e_N|^p + |u_\varepsilon - u_0|^p\,dx =: \alpha_\varepsilon\to 0.\end{aligned} \tag{6.5}$$

In L_ε select a height $z_\varepsilon\in\left(\theta_\varepsilon - \left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{6},\theta_\varepsilon\right)$ such that

$$\begin{aligned}\int_{Q'}\frac{1}{\varepsilon}W(\nabla u_\varepsilon(x',z_\varepsilon)) + \varepsilon\left|\nabla^2 u_\varepsilon(x',z_\varepsilon)\right|^2 + |\nabla u_\varepsilon(x',z_\varepsilon) - a\otimes e_N|^p \\ + |u_\varepsilon(x',z_\varepsilon) - u_0(x',z_\varepsilon)|^p\,dx' \le 6\alpha_\varepsilon.\end{aligned} \tag{6.6}$$

First matching: Set

$$v_\varepsilon(x) := u_0(x) + \bar{u}_\varepsilon(x) + \varphi_\varepsilon(x_N)\left(u_\varepsilon(x) - u_0(x) - \bar{u}_\varepsilon(x)\right),$$

where $\bar{u}_\varepsilon(x) := u_\varepsilon(x',z_\varepsilon) - \tilde{u}(z_\varepsilon)$, $u_0(x) =: \tilde{u}(x_N)$, and let φ_ε be a smooth cut-off function such that $\{0<\varphi_\varepsilon<1\}\subset L_\varepsilon$, $\varphi_\varepsilon = 1$ if $x_N < \theta_\varepsilon - \left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{6}$, $\varphi_\varepsilon = 0$ if $x_N > \theta_\varepsilon$, and

$$\|\varphi_\varepsilon'\|_\infty \le \frac{c}{\varepsilon},\quad \|\varphi_\varepsilon''\|_\infty \le \frac{c}{\varepsilon^2}.$$

We claim that

(i) $\int_{L_\varepsilon} |v_\varepsilon - u_0|^p \, dx \to 0$;

(ii) $\frac{1}{\varepsilon}\int_{L_\varepsilon} |\nabla v_\varepsilon - a \otimes e_N|^p \, dx \to 0$;

(iii) $\int_{L_\varepsilon} \frac{1}{\varepsilon} W(\nabla v_\varepsilon) \, dx \to 0$;

(iv) $\int_{L_\varepsilon} \varepsilon \left|\nabla^2 v_\varepsilon\right|^2 dx \to 0$.

It is easy to deduce (i) from (6.5) and (6.6). Now

$$\begin{aligned}\frac{1}{\varepsilon}\int_{L_\varepsilon} |\nabla v_\varepsilon - a \otimes e_N|^p \, dx &\le C\int_{L_\varepsilon} \frac{1}{\varepsilon}|\nabla_{x'} u_\varepsilon(x', z_\varepsilon)|^p + \frac{1}{\varepsilon}|\nabla u_\varepsilon - a\otimes e_N|^p + \frac{1}{\varepsilon^{p+1}}|u_\varepsilon - u_0 - \bar{u}_\varepsilon|^p \, dx \\ &\le C\left\{\int_{Q'} |\nabla_{x'} u_\varepsilon(x', z_\varepsilon)|^p \, dx' + \frac{1}{\varepsilon}\int_{L_\varepsilon} |\nabla u_\varepsilon - a\otimes e_N|^p \, dx\right\} \\ &\le C\left\{\int_{Q'} |\nabla u_\varepsilon(x', z_\varepsilon) - a\otimes e_N|^p \, dx' + \frac{1}{\varepsilon}\int_{L_\varepsilon} |\nabla u_\varepsilon - a\otimes e_N|^p \, dx\right\} \to 0\end{aligned} \tag{6.7}$$

by (6.5) and (6.6), where we have invoked the Poincaré's inequality

$$\int_{L_\varepsilon} |u_\varepsilon - u_0 - \bar{u}_\varepsilon|^p \, dx \le C\varepsilon^p \int_{L_\varepsilon} |\nabla(u_\varepsilon - a\otimes e_N)|^p \, dx \tag{6.8}$$

due to the fact that $(u_\varepsilon - u_0 - \bar{u}_\varepsilon)(x', z_\varepsilon) \equiv 0$, and using the identity $|\xi - \nabla u_0|^p = \left(|\xi'|^2 + |\xi_N \pm a|^2\right)^{p/2}$. In addition,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{L_\varepsilon} W(\nabla v_\varepsilon) \, dx &\le \frac{1}{\varepsilon}\int_{L_\varepsilon \cap \{|\nabla v_\varepsilon - \nabla u_0| < \rho\}} C|\nabla v_\varepsilon - a\otimes e_N|^p \, dx + \frac{1}{\varepsilon}\int_{L_\varepsilon\cap\{|\nabla v_\varepsilon - \nabla u_0| \ge \rho\}} C(1 + |\nabla v_\varepsilon|^p) \, dx \\ &\le \frac{C}{\varepsilon}\int_{L_\varepsilon} |\nabla v_\varepsilon - a\otimes e_N|^p \, dx,\end{aligned}$$

where we have used $(H_2)''$, (H_4), and the fact that

$$\mathcal{L}^N(L_\varepsilon \cap \{|\nabla v_\varepsilon - a\otimes e_N| \ge \rho\}) \le \frac{1}{\rho^p}\int_{L_\varepsilon} |\nabla v_\varepsilon - a\otimes e_N|^p \, dx.$$

By (ii) we easily deduce (iii).

Finally,

$$\begin{aligned}\int_{L_\varepsilon} \varepsilon\left|\nabla^2 v_\varepsilon\right|^2 dx \le& C\int_{L_\varepsilon} \varepsilon\left|\nabla^2_{x'} u_\varepsilon(x', z_\varepsilon)\right|^2 + \varepsilon\left|\nabla^2 u_\varepsilon\right|^2 + \frac{1}{\varepsilon}|\nabla u_\varepsilon - a\otimes e_N|^2 \\ &+ \frac{1}{\varepsilon}|\nabla_{x'} u_\varepsilon(x', z_\varepsilon)|^2 + \frac{1}{\varepsilon^3}|u_\varepsilon - u_0 - \bar{u}_\varepsilon|^2 \, dx \\ \le& C\left\{\varepsilon^2\int_{Q'} \left|\nabla^2_{x'} u_\varepsilon(x', z_\varepsilon)\right|^2 dx' + \varepsilon\int_{L_\varepsilon} \left|\nabla^2 u_\varepsilon\right|^2 dx + \int_{L_\varepsilon} \frac{1}{\varepsilon}|\nabla u_\varepsilon - a\otimes e_N|^2 \, dx\right. \\ &\left.+ \int_{Q'} |\nabla_{x'} u_\varepsilon(x', z_\varepsilon)|^2 \, dx' + \frac{1}{\varepsilon^3}\varepsilon^{(p-2)/p}\left(\int_{L_\varepsilon} |u_\varepsilon - u_0 - \bar{u}_\varepsilon|^p \, dx\right)^{2/p}\right\} \\ \le& C\left\{\varepsilon^2\int_{Q'} \left|\nabla^2_{x'} u_\varepsilon(x', z_\varepsilon)\right|^2 dx' + \varepsilon\int_{L_\varepsilon} \left|\nabla^2 u_\varepsilon\right|^2 dx + \left(\frac{1}{\varepsilon}\int_{L_\varepsilon} |\nabla u_\varepsilon - a\otimes e_N|^p \, dx\right)^{2/p}\right. \\ &\left.+ \left(\int_{Q'} |\nabla u_\varepsilon(x', z_\varepsilon) - a\otimes e_N|^p \, dx'\right)^{2/p} dx'\right\} \to 0\end{aligned}$$

by (6.5), (6.6), Hölder's inequality, and (6.7), (6.8).

Second matching: Set

$$w_\varepsilon(x) := u_0 + c_\varepsilon + \psi(x_N)(\bar{u}_\varepsilon - c_\varepsilon),$$

where, by (6.6),

$$c_\varepsilon := \int_{Q'} \bar{u}_\varepsilon(x', z_\varepsilon)\, dx' \to 0,$$

and ψ is a smooth cut-off function such that $\psi = 1$ nearby $x_N = \theta_\varepsilon\ \left(< \frac{1}{3}\right)$, $\psi = 0$ nearby $x_N = \frac{1}{2}$, and

$$\|\psi'\|_\infty \le c, \quad \|\psi''\|_\infty \le c.$$

We claim that

(i) $\int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} |w_\varepsilon - u_0|^p\, dx \to 0$;

(ii) $\int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} \frac{1}{\varepsilon} W(\nabla w_\varepsilon)\, dx \to 0$ or, as seen before, $\frac{1}{\varepsilon} \int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} |\nabla w_\varepsilon - a \otimes e_N|^p\, dx \to 0$;

(iii) $\int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} \varepsilon \left|\nabla^2 w_\varepsilon\right|^2 dx \to 0$.

It is clear that (i) is a consequence of (6.6). To prove (ii) and (iii) we notice that

$$\begin{aligned}\frac{1}{\varepsilon} \int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} |\nabla w_\varepsilon - a \otimes e_N|^p\, dx &\le C \frac{1}{\varepsilon} \int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} |\nabla_{x'} u_\varepsilon(x', z_\varepsilon)|^p + |\bar{u}_\varepsilon - c_\varepsilon|^p\, dx \\ &\le C \frac{1}{\varepsilon} \int_{Q'} |\nabla_{x'} u_\varepsilon(x', z_\varepsilon)|^p\, dx' \\ &\le C \int_{Q'} \frac{1}{\varepsilon} W(\nabla u_\varepsilon(x', z_\varepsilon))\, dx' \to 0\end{aligned}$$

by (6.6), and where we have used Poincaré-Friedrichs's inequality and Remark 6.1. Furthermore, also by (6.6), and using Hölder's inequality

$$\begin{aligned}\int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} \varepsilon \left|\nabla^2 w_\varepsilon\right|^2 dx &\le C \int_{Q' \times \left(\theta_\varepsilon, \frac{1}{2}\right)} \varepsilon \left|\nabla^2_{x'} \bar{u}_\varepsilon(x', z_\varepsilon)\right|^2 + \varepsilon \left|\nabla_{x'} u_\varepsilon(x', z_\varepsilon)\right|^2 dx \\ &\le C\varepsilon \left\{ \int_{Q'} \left|\nabla^2_{x'} u_\varepsilon(x', z_\varepsilon)\right|^2 dx' + \left(\int_{Q'} |\nabla u_\varepsilon(x', z_\varepsilon) - a \otimes e_N|^p\, dx' \right)^{2/p} \right\} \to 0.\end{aligned}$$

To conclude the proof, note that the sequence

$$U_\varepsilon := \begin{cases} u_\varepsilon & \text{if } x_N < \theta_\varepsilon - \left[\frac{1}{\varepsilon}\right]^{-1} \frac{1}{6}, \\ v_\varepsilon & \text{if } \theta_\varepsilon - \left[\frac{1}{\varepsilon}\right]^{-1} \frac{1}{6} \le x_N \le \theta_\varepsilon, \\ w_\varepsilon & \text{if } x_N > \theta_\varepsilon, \end{cases}$$

satisfies condition (6.4) with

$$\limsup_{\varepsilon \to 0^+} \int_Q \frac{1}{\varepsilon} W(\nabla U_\varepsilon) + \varepsilon \left|\nabla^2 U_\varepsilon\right|^2 dx \le \lim_{\varepsilon \to 0^+} \int_Q \frac{1}{\varepsilon} W(\nabla u_\varepsilon) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx = K^*.$$

This procedure pins down the boundary conditions at $x_N = 0$, and nearby $x_N = \frac{1}{2}$ we have $U_\varepsilon = u_0 + c_\varepsilon$. Now we repeat the argument in $Q' \times \left(-\frac{1}{2}, 0\right)$ with the obvious adaptations, in order to change U_ε on the bottom half of the cylinder so that the new field V_ε is equal to $u_0 + c_\varepsilon^+$ on top and to $u_0 + c_\varepsilon^-$ on bottom. It suffices to set $z_\varepsilon := V_\varepsilon - C_\varepsilon$, with $C_\varepsilon := c_\varepsilon^+ - c_\varepsilon^-$ and to invoke Theorem 4.1. ■

Proposition 6.3 (Transversal Periodicity) *Assume that* W *satisfies conditions* (H_1), $(H_2)''$, (H_4) *and* (H_5). *Then there exist sequences* $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\{u_n\} \subset W^{2,\infty}\left(Q; \mathbb{R}^d\right)$, *such that* $\varepsilon_n \to 0^+$, $u_n \to |x_N|\, a$ *in* $L^1\left(Q; \mathbb{R}^d\right)$, $\nabla u_n = \pm a \otimes e_N$ *nearby* $x_N = \pm\frac{1}{2}$ *(resp.),* u_n *is periodic of period one in* x', *and*

$$\liminf_{n \to +\infty} \int_Q \frac{1}{\varepsilon_n} W(\nabla u_n) + \varepsilon_n \left|\nabla^2 u_n\right|^2 dx = K^*.$$

Proof. We claim that we may find sequences $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\{v_n\} \subset W^{2,\infty}_{\mathrm{loc}}\left(\mathbb{R}^N;\mathbb{R}^d\right)$, such that $\varepsilon_n \to 0^+$, $v_n(\cdot, x_N)$ is $2Q'$-periodic for all $x_N \in \mathbb{R}$, $\nabla v_n = \pm a \otimes e_N$ nearby $x_N = \pm\frac{1}{2}$ (resp.), and

$$\lim_{n\to\infty} \int_{2Q'\times\left(-\frac{1}{2},\frac{1}{2}\right)} \frac{1}{\varepsilon_n} W(\nabla v_n) + \varepsilon_n \left|\nabla^2 v_n\right|^2 dx = 2^{N-1} K^*,$$

$$\lim_{n\to\infty} \int_{2Q'\times\left(-\frac{1}{2},\frac{1}{2}\right)} |v_n(x) - |x_N| a|\, dx = 0.$$

If the claim holds, then extend v_n linearly to $2Q$ and define $u_n(x) := \frac{1}{2} v_n(2x)$ for $x \in 2Q$. Then $\{u_n\} \subset W^{2,\infty}_{\mathrm{loc}}\left(\mathbb{R}^N;\mathbb{R}^d\right)$, $u_n(\cdot, x_N)$ is Q'-periodic for all $x_N \in \mathbb{R}$, $\nabla u_n = \pm a \otimes e_N$ nearby $x_N = \pm\frac{1}{2}$ (resp.), and

$$\lim_{n\to\infty} \int_Q \frac{2}{\varepsilon_n} W(\nabla u_n) + \frac{\varepsilon_n}{2} \left|\nabla^2 u_n\right|^2 dx = K^*,$$

$$\lim_{n\to\infty} \int_Q |u_n(x) - |x_N| a|\, dx = 0,$$

thus completing the proof.

We divide the proof of the claim in two steps, where, as before, for simplicity of notation we write $\varepsilon := \varepsilon_n$.

Step 1: *The two dimensional case* $N = 2$. In view of Lemma 4.2, consider sequences $\{\varepsilon\} \subset \mathbb{R}_+$, $\{u_\varepsilon\} \subset W^{2,2}\left(Q;\mathbb{R}^d\right)$, such that $\varepsilon \to 0^+$, $u_\varepsilon \to u_0 := |x_2|\, a$ in $L^1\left(Q;\mathbb{R}^d\right)$, and

$$\lim_{\varepsilon\to 0^+} \int_Q \frac{1}{\varepsilon} W(\nabla u_\varepsilon) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx = K^*.$$

By $(H_2)''$ we may assume, without loss of generality, that $\{u_\varepsilon\} \subset W^{2,2}\left(Q;\mathbb{R}^d\right) \cap C^2\left(Q;\mathbb{R}^d\right)$, and by Proposition 6.2 that $\nabla u_\varepsilon(x) = \pm\, a \otimes e_N$ nearby $x_2 = \pm\frac{1}{2}$ (resp.). By Theorem 4.1 we have

$$\begin{aligned} K^* &= \lim_{\varepsilon\to 0^+} \int_Q \frac{1}{\varepsilon} W(\nabla u_\varepsilon) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx \geq \liminf_{\varepsilon\to 0^+} \int_{Q\setminus I_m} \frac{1}{\varepsilon} W(\nabla u_\varepsilon) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx \\ &\geq K^* \mathcal{H}^1\left(\left(-\frac{1}{2} + \frac{1}{m}, \frac{1}{2} - \frac{1}{m}\right) \times \{0\}\right) = K^*\left(1 - \frac{2}{m}\right), \end{aligned}$$

where $I_m := \left(\left(-\frac{1}{2}, -\frac{1}{2} + \frac{1}{m}\right) \cup \left(\frac{1}{2} - \frac{1}{m}, \frac{1}{2}\right)\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, and so

$$\limsup_{\varepsilon\to 0^+} \int_{I_m} \frac{1}{\varepsilon} W(\nabla u_\varepsilon) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx \leq K^* \frac{2}{m}.$$

Divide $\left(-\frac{1}{2}, -\frac{1}{2} + \frac{1}{m}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ into $\left[\frac{1}{\varepsilon}\right]$ vertical strips of horizontal width $\frac{1}{m}\left[\frac{1}{\varepsilon}\right]^{-1}$, and proceed symmetrically in $\left(\frac{1}{2} - \frac{1}{m}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Order these strips in pairs $\left(R^-_{\varepsilon,m,i}, R^+_{\varepsilon,m,i}\right)$ with $R^+_{\varepsilon,m,i} \subset \left(\frac{1}{2} - \frac{1}{m}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, $R^-_{\varepsilon,m,i} \subset \left(-\frac{1}{2}, -\frac{1}{2} + \frac{1}{m}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then for all $\varepsilon > 0$ sufficiently small we have

$$\sum_{i=1}^{\left[\frac{1}{\varepsilon}\right]} \int_{R^+_{\varepsilon,m,i}\cup R^-_{\varepsilon,m,i}} \frac{1}{\varepsilon} W(\nabla u_\varepsilon) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 + m^p \left|\nabla u_\varepsilon - \nabla u_0\right|^p + |u_\varepsilon - u_0|\, dx \leq K^* \frac{3}{m}, \tag{6.9}$$

where we have used the fact that $u_\varepsilon \to u_0$ in $W^{1,p}\left(Q;\mathbb{R}^d\right)$ (see Theorem 3.1 and Remark 3.2 (ii)). Choose one pair $\left(R^-_{\varepsilon,m,i}, R^+_{\varepsilon,m,i}\right)$, with $i = i(\varepsilon, m)$, such that

$$\int_{R^+_{\varepsilon,m,i}\cup R^-_{\varepsilon,m,i}} \frac{1}{\varepsilon} W(\nabla u_\varepsilon) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 + m^p \left|\nabla u_\varepsilon - \nabla u_0\right|^p + |u_\varepsilon - u_0|\, dx \leq K^* \frac{3}{m}\left[\frac{1}{\varepsilon}\right]^{-1}. \tag{6.10}$$

For simplicity, from now on we denote $R^+_{\varepsilon,m,i} =: R^+_{\varepsilon,m} = (b_{\varepsilon,m}, c_{\varepsilon,m}) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $R^-_{\varepsilon,m,i} =: R^-_{\varepsilon,m} =$

$\left(-c_{\varepsilon,m},-b_{\varepsilon,m}\right)\times\left(-\frac{1}{2},\frac{1}{2}\right)$. Since

$$\int_{\frac{b_{\varepsilon,m}+c_{\varepsilon,m}}{2}}^{c_{\varepsilon,m}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon}\left(x\right)\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon}\left(x\right)\right|^{2}+m^{p}\left|\nabla\left(u_{\varepsilon}-u_{0}\right)\right|^{p}\left(x\right)+\left|\left(u_{\varepsilon}-u_{0}\right)\left(x\right)\right|\right.$$
$$+\frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon}\left(-x_{1},x_{2}\right)\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon}\left(-x_{1},x_{2}\right)\right|^{2}+m^{p}\left|\nabla\left(u_{\varepsilon}-u_{0}\right)\left(-x_{1},x_{2}\right)\right|^{p}$$
$$\left.+\left|\left(u_{\varepsilon}-u_{0}\right)\left(-x_{1},x_{2}\right)\right|\right]dx\leq K^{*}\frac{3}{m}\left[\frac{1}{\varepsilon}\right]^{-1},$$

with $c_{\varepsilon,m}-\frac{b_{\varepsilon,m}+c_{\varepsilon,m}}{2}=\frac{1}{2}\frac{1}{m}\left[\frac{1}{\varepsilon}\right]^{-1}$, there exists $a_{\varepsilon,m}\in\left(\frac{b_{\varepsilon,m}+c_{\varepsilon,m}}{2},c_{\varepsilon,m}\right)$ such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon}\left(a_{\varepsilon,m},x_{2}\right)\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon}\left(a_{\varepsilon,m},x_{2}\right)\right|^{2}+\left|\left(u_{\varepsilon}-u_{0}\right)\left(a_{\varepsilon,m},x_{2}\right)\right|\right.$$
$$+m^{p}\left|\nabla\left(u_{\varepsilon}-u_{0}\right)\left(a_{\varepsilon,m},x_{2}\right)\right|^{p}+m^{p}\left|\nabla\left(u_{\varepsilon}-u_{0}\right)\left(-a_{\varepsilon,m},x_{2}\right)\right|^{p}$$
$$\left.+\frac{1}{\varepsilon}W\left(\nabla u_{\varepsilon}\left(-a_{\varepsilon,m},x_{2}\right)\right)+\varepsilon\left|\nabla^{2}u_{\varepsilon}\left(-a_{\varepsilon,m},x_{2}\right)\right|^{2}+\left|\left(u_{\varepsilon}-u_{0}\right)\left(-a_{\varepsilon,m},x_{2}\right)\right|\right]dx_{2}\leq 6K^{*}. \tag{6.11}$$

Now $\left(-a_{\varepsilon,m},-a_{\varepsilon,m}+\frac{\varepsilon}{2m}\right)\subset\left(-c_{\varepsilon,m},-b_{\varepsilon,m}\right)$ because $a_{\varepsilon,m}\in\left(\frac{b_{\varepsilon,m}+c_{\varepsilon,m}}{2},c_{\varepsilon,m}\right)$ and $\frac{b_{\varepsilon,m}+c_{\varepsilon,m}}{2}-b_{\varepsilon,m}=\frac{1}{2}\frac{1}{m}\left[\frac{1}{\varepsilon}\right]^{-1}$. We will now modify u_{ε} on $\left(-a_{\varepsilon,m},-b_{\varepsilon,m}\right)\times\left(-\frac{1}{2},\frac{1}{2}\right)$ so that the new sequence will match u_{ε} near $x_{1}=-a_{\varepsilon,m}+\frac{\varepsilon}{2m}$, hence near $-b_{\varepsilon,m}$, and will coincide with $u_{\varepsilon}\left(-a_{\varepsilon,m},\cdot\right)$ near $x_{1}=-a_{\varepsilon,m}$. Let $\varphi_{\varepsilon,m}$ be a smooth cut-off function such that $\varphi_{\varepsilon,m}=1$ if $x_{1}>-a_{\varepsilon,m}+\frac{\varepsilon}{2m}$, $\varphi_{\varepsilon,m}=0$ if $x_{1}<-a_{\varepsilon,m}$, and

$$\left\|\varphi_{\varepsilon,m}^{\prime}\right\|_{\infty}\leq\frac{cm}{\varepsilon},\quad\left\|\varphi_{\varepsilon,m}^{\prime\prime}\right\|_{\infty}\leq\frac{cm^{2}}{\varepsilon^{2}}.$$

Define

$$w_{\varepsilon,m}\left(x\right):=\varphi_{\varepsilon,m}\left(x_{1}\right)u_{\varepsilon}\left(x\right)+\left(1-\varphi_{\varepsilon,m}\left(x_{1}\right)\right)u_{\varepsilon}\left(-a_{\varepsilon,m},x_{2}\right).$$

Then $\left\{w_{\varepsilon,m}\right\}\subset W^{2,\infty}\left(Q;\mathbb{R}^{d}\right)$, and $\nabla w_{\varepsilon,m}\left(x\right)=\pm\,a\otimes e_{N}$ nearby $x_{2}=\pm\frac{1}{2}$ (resp.). We show that

$$\limsup_{m\to\infty}\limsup_{\varepsilon\to0^{+}}\int_{Q}\frac{1}{\varepsilon}W\left(\nabla w_{\varepsilon,m}\right)+\varepsilon\left|\nabla^{2}w_{\varepsilon,m}\right|^{2}dx\leq K^{*},\quad\limsup_{m\to\infty}\limsup_{\varepsilon\to0^{+}}\int_{Q}\left|w_{\varepsilon,m}-u_{0}\right|dx=0. \tag{6.12}$$

If (6.12) holds, and repeating the argument now nearby $x_{1}=a_{\varepsilon,m}$, then after a diagonalization procedure and invoking Theorem 4.1, we can find a subsequence $\varepsilon_{m}\to0^{+}$ and a sequence $\left\{w_{m}\right\}\subset W^{2,\infty}\left(Q;\mathbb{R}^{d}\right)$ such that $w_{m}=w_{m}\left(\pm a_{m},x_{2}\right)$ nearby $x_{1}=\pm\frac{1}{2}$ resp, $\nabla w_{m}\left(x\right)=\pm\,a\otimes e_{N}$ nearby $x_{2}=\pm\frac{1}{2}$ (resp.), and

$$\lim_{m\to\infty}\int_{Q}\frac{1}{\varepsilon_{m}}W\left(\nabla w_{m}\right)+\varepsilon_{m}\left|\nabla^{2}w_{m}\right|^{2}dx=K^{*},\quad\lim_{m\to\infty}\int_{Q}\left|w_{m}-u_{0}\right|dx=0.$$

Construct by reflection about $x_{1}=\frac{1}{2}$ a new function, still denoted w_{m}, x_{1}-periodic with period 2. Precisely, for $x_{1}\in\left(\frac{1}{2},\frac{3}{2}\right)$, $x_{2}\in\left(-\frac{1}{2},\frac{1}{2}\right)$, set $w_{m}(x_{1},x_{2}):=w_{m}(1-x_{1},x_{2})$. Since the problem is translation invariant, for simplicity of notation in what follows we identify w with its translation $(x_{1},x_{2})\mapsto w(x_{1}-1/2,x_{2})$, and in this way we work with periodic functions with period $(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)$, such that

$$\lim_{m\to\infty}\int_{(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)}\frac{1}{\varepsilon_{m}}W\left(\nabla w_{m}\right)+\varepsilon_{m}\left|\nabla^{2}w_{m}\right|^{2}dx=2K^{*},\quad\lim_{m\to\infty}\int_{(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)}\left|w_{m}-u_{0}\right|dx=0.$$

Note that here we have used condition (H_{5}), and note also that the new function w_{m} extended to $\mathbb{R}\times\left(-\frac{1}{2},\frac{1}{2}\right)$ is still in $W_{\mathrm{loc}}^{2,\infty}\left(\mathbb{R}\times\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^{d}\right)$ because w_{m} does not depend on x_{1} nearby the axis of reflection $x_{1}=\frac{1}{2}$.

The remainder of Step 1 is devoted to the proof of (6.12), where we use the notation $I_m^- := (-1/2, -b_m) \times (-1/2, 1/2)$. We have

$$\begin{aligned}\int_Q \frac{1}{\varepsilon} W\left(\nabla w_{\varepsilon,m}\right) + \varepsilon \left|\nabla^2 w_{\varepsilon,m}\right|^2 dx &= \int_{Q\setminus I_m^-} \frac{1}{\varepsilon} W\left(\nabla u_\varepsilon\right) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx + \int_{R_{\varepsilon,m}^-} \frac{1}{\varepsilon} W\left(\nabla w_{\varepsilon,m}\right) + \varepsilon \left|\nabla^2 w_{\varepsilon,m}\right|^2 dx \\ &\quad + \int_{I_m^-\setminus R_{\varepsilon,m}^-} \frac{1}{\varepsilon} W\left(0, \frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m}, x_2\right)\right) + \varepsilon \left|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}\left(-a_{\varepsilon,m}, x_2\right)\right|^2 dx \\ &\le \int_Q \frac{1}{\varepsilon} W\left(\nabla u_\varepsilon\right) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx + \int_{R_{\varepsilon,m}^-} \frac{1}{\varepsilon} W\left(\nabla w_{\varepsilon,m}\right) + \varepsilon \left|\nabla^2 w_{\varepsilon,m}\right|^2 dx \\ &\quad + \frac{1}{m} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\varepsilon} W\left(0, \frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m}, x_2\right)\right) + \varepsilon \left|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}\left(-a_{\varepsilon,m}, x_2\right)\right|^2 dx.\end{aligned}$$

By Remark 6.1 we have

$$W\left(0, \frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m}, x_2\right)\right) \le CW\left(\nabla u_\varepsilon\left(-a_{\varepsilon,m}, x_2\right)\right) + C\left|\frac{\partial u_\varepsilon}{\partial x_1}\left(-a_{\varepsilon,m}, x_2\right)\right|^p \le CW\left(\nabla u_\varepsilon\left(-a_{\varepsilon,m}, x_2\right)\right), \quad (6.13)$$

hence, by (6.11)

$$\begin{aligned}&\frac{1}{m} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\varepsilon} W\left(0, \frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m}, x_2\right)\right) + \varepsilon \left|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}\left(-a_{\varepsilon,m}, x_2\right)\right|^2 dx \\ &\le \frac{C}{m} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\varepsilon} W\left(\nabla u_\varepsilon\left(-a_{\varepsilon,m}, x_2\right)\right) + \varepsilon \left|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}\left(-a_{\varepsilon,m}, x_2\right)\right|^2 dx \le \frac{C}{m},\end{aligned}$$

and, in turn,

$$\begin{aligned}\int_Q \frac{1}{\varepsilon} W\left(\nabla w_{\varepsilon,m}\right) + \varepsilon \left|\nabla^2 w_{\varepsilon,m}\right|^2 dx &\le \int_Q \frac{1}{\varepsilon} W\left(\nabla u_\varepsilon\right) + \varepsilon \left|\nabla^2 u_\varepsilon\right|^2 dx \\ &\quad + \int_{R_{\varepsilon,m}^-} \frac{1}{\varepsilon} W\left(\nabla w_{\varepsilon,m}\right) + \varepsilon \left|\nabla^2 w_{\varepsilon,m}\right|^2 dx + \frac{C}{m}.\end{aligned} \quad (6.14)$$

Similarly, again by (6.11),

$$\begin{aligned}\int_Q \left|w_{\varepsilon,m} - u_0\right| dx &\le \int_{Q\setminus I_m^-} \left|u_\varepsilon - u_0\right| dx + \int_{R_{\varepsilon,m}^-} \left|w_{\varepsilon,m} - u_0\right| dx \\ &\quad + \frac{1}{m} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|u_\varepsilon\left(-a_{\varepsilon,m}, x_2\right) - \left|x_2\right| a\right| dx \\ &\le \int_{Q\setminus I_m^-} \left|u_\varepsilon - u_0\right| dx + \int_{R_{\varepsilon,m}^-} \left|w_{\varepsilon,m} - u_0\right| dx + \frac{C}{m},\end{aligned}$$

and thus, also by (6.14), to prove (6.12) it is sufficient to show that:

(i) $\lim_{m\to\infty} \lim_{\varepsilon\to 0^+} \int_{R_{\varepsilon,m}^-} \frac{1}{\varepsilon} W\left(\nabla w_{\varepsilon,m}\right) dx = 0$;

(ii) $\lim_{m\to\infty} \lim_{\varepsilon\to 0^+} \int_{R_{\varepsilon,m}^-} \varepsilon \left|\nabla^2 w_{\varepsilon,m}\right|^2 dx = 0$;

(iii) $\lim_{m\to\infty} \lim_{\varepsilon\to 0^+} \int_{R_{\varepsilon,m}^-} \left|w_{\varepsilon,m} - u_\varepsilon\right| dx = 0$.

Now

$$\begin{aligned}\nabla w_{\varepsilon,m}(x) &= \varphi_{\varepsilon,m}\left(x_1\right) \nabla u_\varepsilon(x) + \left(1 - \varphi_{\varepsilon,m}\left(x_1\right)\right)\left(0 \left| \frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m}, x_2\right)\right.\right) \\ &\quad + \left(u_\varepsilon\left(x_1, x_2\right) - u_\varepsilon\left(-a_{\varepsilon,m}, x_2\right)\right) \otimes \varphi_{\varepsilon,m}'\left(x_1\right) e_1.\end{aligned} \quad (6.15)$$

By $(H_2)''$ we have

$$W(\xi) \le C(1+|\xi|^p) \le C\left(1+2^{p-1}|\nabla u_0|^p + 2^{p-1}|\xi - \nabla u_0|^p\right),$$

and so, using (6.15),

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{R^-_{\varepsilon,m}} W(\nabla w_{\varepsilon,m})\,dx &\le C\frac{1}{\varepsilon}\int_{R^-_{\varepsilon,m}} (1+|\nabla u_0|^p + |\nabla w_{\varepsilon,m}-\nabla u_0|^p)\,dx \\
&\le C\frac{1}{\varepsilon m}\left[\frac{1}{\varepsilon}\right]^{-1}(1+|a\otimes e_2|^p) \\
&\quad + C\frac{1}{\varepsilon}\int_{R^-_{\varepsilon,m}} |\nabla u_\varepsilon - \nabla u_0|^p + \left|\nabla u_0 - \left(0, \frac{\partial u_\varepsilon}{\partial x_2}(-a_{\varepsilon,m}, x_2)\right)\right|^p dx \\
&\quad + C\frac{1}{\varepsilon}\int_{R^-_{\varepsilon,m}} \left(\frac{m}{\varepsilon}\right)^p |u_\varepsilon(x) - u_\varepsilon(-a_{\varepsilon,m}, x_2)|^p\,dx \\
&\le C\left(\frac{1}{m} + \frac{1}{\varepsilon}\int_{R^-_{\varepsilon,m}} |\nabla u_\varepsilon - \nabla u_0|^p\,dx + \frac{1}{m}\left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{\varepsilon}\int_{-\frac{1}{2}}^{\frac{1}{2}} |\nabla(u_0-u_\varepsilon)(-a_{\varepsilon,m}, x_2)|^p\,dx_2\right) \\
&\quad + C\frac{m^p}{\varepsilon^{p+1}}\int_{R^-_{\varepsilon,m}} |u_\varepsilon(x) - u_\varepsilon(-a_{\varepsilon,m}, x_2)|^p\,dx \\
&\le C\left(\frac{1}{m} + \frac{1}{\varepsilon}\frac{3K^*}{m^{p+1}}\left[\frac{1}{\varepsilon}\right]^{-1} + \frac{6K^*}{m^{p+1}}\left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{\varepsilon} + \frac{m^p}{\varepsilon^{p+1}}\int_{R^-_{\varepsilon,m}} |u_\varepsilon(x) - u_\varepsilon(-a_{\varepsilon,m}, x_2)|^p\,dx\right),
\end{aligned}$$

where we have used (6.9), (6.10), (6.11). Thus to prove (i) it remains to show that

$$\lim_{m\to\infty}\lim_{\varepsilon\to 0^+}\frac{m^p}{\varepsilon^{p+1}}\int_{R^-_{\varepsilon,m}} |u_\varepsilon(x) - u_\varepsilon(-a_{\varepsilon,m}, x_2)|^p\,dx = 0.$$

Indeed, by Hölder's inequality

$$\begin{aligned}
|u_\varepsilon(x_1,x_2) - u_\varepsilon(-a_{\varepsilon,m}, x_2)|^p &\le \left(\int_{-b_{\varepsilon,m}}^{-c_{\varepsilon,m}} \left|\frac{\partial u_\varepsilon}{\partial x_1}(s,x_2)\right| ds\right)^p \\
&\le C\left(\frac{\varepsilon}{m}\right)^{p/p'}\int_{-b_{\varepsilon,m}}^{-c_{\varepsilon,m}} \left|\frac{\partial u_\varepsilon}{\partial x_1}(s,x_2)\right|^p ds \\
&\le C\left(\frac{\varepsilon}{m}\right)^{p-1}\int_{-b_{\varepsilon,m}}^{-c_{\varepsilon,m}} |\nabla(u_\varepsilon - u_0)(s,x_2)|^p\,ds,
\end{aligned}$$

and thus by (6.10)

$$\begin{aligned}
\int_{R^-_{\varepsilon,m}} |u_\varepsilon(x_1,x_2) - u_\varepsilon(-a_{\varepsilon,m}, x_2)|^p\,dx &\le C\int_{R^-_{\varepsilon,m}} \left(\frac{\varepsilon}{m}\right)^{p-1}\int_{-b_{\varepsilon,m}}^{-c_{\varepsilon,m}} |\nabla(u_\varepsilon - u_0)(s,x_2)|^p\,ds dx \qquad (6.16) \\
&\le C\left[\frac{1}{\varepsilon}\right]^{-1}\frac{\varepsilon^{p-1}}{m^p}\int_{R^-_{\varepsilon,m}} |\nabla u_\varepsilon - \nabla u_0|^p\,dx \le C\frac{\varepsilon^{p+1}}{m^{2p+1}}
\end{aligned}$$

and the (6.12) holds.

To show (ii) note that for $x_1 \in (-b_{\varepsilon,m}, -c_{\varepsilon,m})$

$$\begin{aligned}
\left|\nabla^2 w_{\varepsilon,m}(x)\right| &\le \varphi_{\varepsilon,m}(x_1)\left|\nabla^2 u_\varepsilon(x)\right| + (1-\varphi_{\varepsilon,m}(x_1))\left|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}(-a_{\varepsilon,m}, x_2)\right| \\
&\quad + |u_\varepsilon(x_1,x_2) - u_\varepsilon(-a_{\varepsilon,m}, x_2)|\left|\varphi''_{\varepsilon,m}(x_1)\right| + 2\left|\varphi'_{\varepsilon,m}(x_1)\right|\left|\nabla u_\varepsilon(x) - \left(0\left|\frac{\partial u_\varepsilon}{\partial x_2}(-a_{\varepsilon,m}, x_2)\right.\right)\right|,
\end{aligned}$$

and so

$$\left|\nabla^2 w_{\varepsilon,m}\right|^2 \leq C\left(\left|\nabla^2 u_\varepsilon\right|^2+\left|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}\left(-a_{\varepsilon,m},x_2\right)\right|^2\right. \\ \left.+\left(\frac{m}{\varepsilon}\right)^4\left|u_\varepsilon-u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)\right|^2+\left(\frac{m}{\varepsilon}\right)^2\left|\nabla u_\varepsilon-\left(0\left|\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)\right.\right)\right|^2\right). \tag{6.17}$$

We now estimate the two terms on the right hand side of (6.17). By (6.16) and Hölder's inequality

$$\int_{R^-_{\varepsilon,m}}\left|u_\varepsilon\left(x_1,x_2\right)-u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)\right|^2 dx \leq \left|R^-_{\varepsilon,m}\right|^{(p-2)/p}\left(\int_{R^-_{\varepsilon,m}}\left|u_\varepsilon\left(x_1,x_2\right)-u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)\right|^p dx\right)^{2/p} \\ \leq\left(\frac{1}{m}\left[\frac{1}{\varepsilon}\right]^{-1}\right)^{(p-2)/p}\left(C\frac{\varepsilon^{p+1}}{m^{2p+1}}\right)^{2/p}\leq C\frac{\varepsilon^3}{m^5}, \tag{6.18}$$

while

$$\left|\nabla u_\varepsilon(x)-\left(0\left|\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)\right.\right)\right|^2 \leq 2\left|\frac{\partial u_\varepsilon}{\partial x_1}(x)\right|^2+2\left|\frac{\partial u_\varepsilon}{\partial x_2}(x)-\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)\right|^2 \\ \leq 2\left|\nabla u_\varepsilon-\nabla u_0\right|^2+2\left(\int_{-c_{\varepsilon,m}}^{-b_{\varepsilon,m}}\left|\frac{\partial^2 u_\varepsilon}{\partial x_2\partial x_1}\left(s,x_2\right)\right|ds\right)^2 \\ \leq 2\left|\nabla u_\varepsilon-\nabla u_0\right|^2+\frac{3\varepsilon}{m}\int_{-c_{\varepsilon,m}}^{-b_{\varepsilon,m}}\left|\nabla^2 u_\varepsilon\left(s,x_2\right)\right|^2 ds.$$

Hence, by (6.10) and Hölder's inequality,

$$\int_{R^-_{\varepsilon,m}}\left|\nabla u_\varepsilon(x)-\left(0\left|\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)\right.\right)\right|^2 dx \leq 2\int_{R^-_{\varepsilon,m}}\left|\nabla u_\varepsilon-\nabla u_0\right|^2 dx \\ +\frac{3\varepsilon}{m}\int_{R^-_{\varepsilon,m}}\int_{-c_{\varepsilon,m}}^{-b_{\varepsilon,m}}\left|\nabla^2 u_\varepsilon\left(s,x_2\right)\right|^2 ds dx \\ \leq 2\left|R^-_{\varepsilon,m}\right|^{(p-2)/p}\left(\int_{R^-_{\varepsilon,m}}\left|\nabla u_\varepsilon-\nabla u_0\right|^p dx\right)^{2/p} \\ +\frac{3}{m^2}\left[\frac{1}{\varepsilon}\right]^{-1}\varepsilon\int_{R^-_{\varepsilon,m}}\left|\nabla^2 u_\varepsilon\right|^2 dx \\ \leq C\frac{\varepsilon}{m^3}+C\frac{\varepsilon^2}{m^3}. \tag{6.19}$$

By (6.17), (6.18) and (6.19) we conclude that

$$\varepsilon\int_{R^-_{\varepsilon,m}}\left|\nabla^2 w_{\varepsilon,m}\right|^2 dx \leq C\varepsilon\int_{R^-_{\varepsilon,m}}\left|\nabla^2 u_\varepsilon\right|^2 dx+C\frac{\varepsilon^2}{m}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\nabla^2 u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)\right|^2 dx_2 \\ +C\frac{m^2}{\varepsilon}\int_{R^-_{\varepsilon,m}}\left|\nabla u_\varepsilon-\left(0\left|\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)\right.\right)\right|^2 dx \\ +C\frac{m^4}{\varepsilon^3}\int_{R^-_{\varepsilon,m}}\left|u_\varepsilon-u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)\right|^2 dx \\ \leq C\left(K^*\frac{3}{m}\left[\frac{1}{\varepsilon}\right]^{-1}+6K^*\frac{\varepsilon}{m}+\frac{1}{m}+\frac{\varepsilon}{m}+\frac{1}{m}\right)\to 0$$

as $\varepsilon \to 0^+$ and $m \to \infty$.

Finally, note that by (6.18)

$$\int_{R^-_{\varepsilon,m}} |w_{\varepsilon,m} - u_\varepsilon|\, dx \le C \left(\frac{\varepsilon}{m} \int_{R^-_{\varepsilon,m}} |u_\varepsilon - u_\varepsilon(-a_{\varepsilon,m}, x_2)|^2\, dx \right)^{1/2} \to 0$$

as $\varepsilon \to 0^+$.

Step 2: *The N-dimensional case.* Take sequences $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\{u_n\} \subset W^{2,2}(Q;\mathbb{R}^d)$, such that $\varepsilon_n \to 0^+$, $u_n \to u_0 := |x_N|\, a$ in $L^1(Q;\mathbb{R}^d)$ and

$$\lim_{\varepsilon \to 0^+} \int_Q \frac{1}{\varepsilon_n} W(\nabla u_n) + \varepsilon_n \left|\nabla^2 u_n\right|^2 dx = K^*.$$

By Step 1 there exist a subsequence $\{\varepsilon_m\} \subset \{\varepsilon_n\}$ such that the corresponding fields u_m may be modified so as to obtain a new sequence $\left\{w^{(1)}_m\right\} \subset W^{2,\infty}_{\text{loc}}\left(\mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}; \mathbb{R}^d\right)$, x_1-periodic with period 2, such that $\nabla w^{(1)}_m(x) = \pm\, a \otimes e_N$ nearby $x_N = \pm\frac{1}{2}$ (resp.) and

$$\lim_{m\to\infty} \int_{(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)^{N-1}} \frac{1}{\varepsilon_m} W\left(\nabla w^{(1)}_m\right) + \varepsilon_m \left|\nabla^2 w^{(1)}_m\right|^2 dx = 2K^*,$$
$$\lim_{m\to\infty} \int_{(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)^{N-1}} \left|w^{(1)}_m - u_0\right| dx = 0.$$

We treat x_2 just as above. Starting from the $w^{(1)}_m$ above we construct

$$w^{(2)}_k(x) := \varphi_k(x_2)\, w^{(1)}_{m_k}(x) + (1 - \varphi_k(x_2))\, w^{(1)}_{m_k}(x_1, \pm\, b_k, x_3, \cdots, x_N).$$

with $b_k \to \frac{1}{2}^-$, such that

$$\lim_{k\to\infty} \int_{(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)^{N-1}} \frac{1}{\varepsilon_{m_k}} W\left(\nabla w^{(2)}_k\right) + \varepsilon_{m_k} \left|\nabla^2 w^{(2)}_k\right|^2 dx = 2K^*,$$
$$\lim_{k\to\infty} \int_{(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)^{N-1}} \left|w^{(2)}_k - u_0\right| dx = 0.$$

Note that $w^{(2)}_k(x)$ is still periodic in x_1 with period 2 and $\nabla w^{(2)}_k(x) = \pm\, a \otimes e_N$ nearby $x_N = \pm\frac{1}{2}$ (resp.). After reflection about $x_2 = \frac{1}{2}$ we obtain a sequence $w^{(2)}_k(x)$ periodic in x_1 and in x_2, of period 2, such that $\nabla w^{(2)}_k(x) = \pm\, a \otimes e_N$ nearby $x_N = \pm\frac{1}{2}$ (resp.), and

$$\lim_{k\to\infty} \int_{(-1,1)\times(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)^{N-2}} \frac{1}{\varepsilon_{m_k}} W\left(\nabla w^{(2)}_k\right) + \varepsilon_{m_k} \left|\nabla^2 w^{(2)}_k\right|^2 dx = 4K^*,$$
$$\lim_{k\to\infty} \int_{(-1,1)\times(-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)^{N-2}} \left|w^{(2)}_k - u_0\right| dx = 0.$$

By repeating this process for all remaining variables $x_3, \cdots, x_{N-1}$, we obtain sequences fulfilling the claim.
■

Define

$$K_{\text{per}} := \inf\left\{ \int_Q L\, W(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2\, dx : L > 0, v \in W^{2,\infty}(Q;\mathbb{R}^d), \right.$$
$$\left. \nabla v = \pm a \otimes e_N \text{ nearby } x_N = \pm\frac{1}{2},\ v \text{ periodic of period one in } x' \right\}.$$

Proposition 6.4 *Assume that W satisfies conditions (H_1), $(H_2)''$, (H_4) and (H_5). Then*

$$K^* = K_{\text{per}}.$$

Proof. By Proposition 6.3 $K^* \geq K_{\text{per}}$. To prove the opposite inequality, fix $\delta > 0$ and let $L > 0$, $v \in W^{2,\infty}_{\text{loc}}(Q;\mathbb{R}^d)$, $v(\cdot, x_N)$ Q'-periodic for all $x_N \in \mathbb{R}$, such that $\nabla v = \pm a \otimes e_N$ nearby $x_N = \pm\frac{1}{2}$ (resp.) and

$$\int_Q L\,W(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2\, dx \leq K_{\text{per}} + \delta.$$

Let $\{\varepsilon_n\}$ be a sequence converging to 0^+, and writing $\varepsilon := \varepsilon_n$ define

$$z_\varepsilon(x) := \begin{cases} \varepsilon L v\left(\dfrac{x'}{\varepsilon L}, \frac{1}{2}\right) + a\left(x_N - \frac{\varepsilon L}{2}\right) & \text{if } x_N > \frac{\varepsilon L}{2}, \\ \varepsilon L v\left(\dfrac{x}{\varepsilon L}\right) & \text{if } |x_N| \leq \frac{\varepsilon L}{2}, \\ \varepsilon L v\left(\dfrac{x'}{\varepsilon L}, -\frac{1}{2}\right) - a\left(x_N + \frac{\varepsilon L}{2}\right) & \text{if } x_N < -\frac{\varepsilon L}{2}, \end{cases} \tag{6.20}$$

so that

$$\nabla z_\varepsilon(x) = \begin{cases} a \otimes e_N & \text{if } x_N > \frac{\varepsilon L}{2}, \\ \nabla v\left(\dfrac{x}{\varepsilon L}\right) & \text{if } |x_N| \leq \frac{\varepsilon L}{2}, \\ -a \otimes e_N & \text{if } x_N < -\frac{\varepsilon L}{2}. \end{cases} \tag{6.21}$$

Then

$$\begin{aligned} \int_{Q^+} |\nabla z_\varepsilon - \nabla u_0|^p\, dx &= \int_{Q'}\int_0^{\frac{\varepsilon L}{2}} \left|\nabla v\left(\frac{x}{\varepsilon L}\right) - a \otimes e_N\right|^p dx'dx_N \\ &= \varepsilon L \int_{Q'}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\nabla v\left(\frac{x'}{\varepsilon L}, t\right) - a \otimes e_N\right|^p dx'dx_N \to 0 \end{aligned}$$

since $v \in W^{2,\infty}(Q;\mathbb{R}^d)$, and where $u_0 := |x_N|\, a$, $Q^+ := Q' \times \left(0, \frac{1}{2}\right)$. A similar conclusion holds in $Q^- := Q' \times \left(-\frac{1}{2}, 0\right)$ and so $\nabla z_\varepsilon \to \nabla u_0$ in $L^p(Q;\mathbb{R}^d)$. Moreover, by the Riemann-Lebesgue Lemma

$$\begin{aligned} \int_Q \frac{1}{\varepsilon} W(\nabla z_\varepsilon) + \varepsilon\left|\nabla^2 z_\varepsilon\right|^2 dx &= \int_{-\frac{\varepsilon L}{2}}^{\frac{\varepsilon L}{2}}\int_{Q'} \frac{1}{\varepsilon} W\left(\nabla v\left(\frac{x}{\varepsilon L}\right)\right) + \frac{1}{\varepsilon L^2}\left|\nabla^2 v\left(\frac{x}{\varepsilon L}\right)\right|^2 dx'dx_N \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{Q'} LW\left(\nabla v\left(\frac{x'}{\varepsilon L}, t\right)\right) + \frac{1}{L}\left|\nabla^2 v\left(\frac{x'}{\varepsilon L}, t\right)\right|^2 dx'dt \to \int_Q LW(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2 dx, \end{aligned}$$

and so

$$K^* \leq \int_Q LW(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2 dx \leq K_{\text{per}} + \delta.$$

It now suffices to let $\delta \to 0^+$. ∎

Remark 6.5 If $\Omega = \omega \times (-h, h)$, where $\omega \subset \mathbb{R}^{N-1}$ is a bounded open set, and if we consider the sequence defined in (6.20), then we have by the Riemann-Lebesgue Lemma

$$\begin{aligned} \int_{\omega\times(-h,h)} \frac{1}{\varepsilon} W(\nabla z_\varepsilon) + \varepsilon\left|\nabla^2 z_\varepsilon\right|^2 dx &= \int_{-\frac{\varepsilon L}{2}}^{\frac{\varepsilon L}{2}}\int_\omega \frac{1}{\varepsilon} W\left(\nabla v\left(\frac{x}{\varepsilon L}\right)\right) + \frac{1}{\varepsilon L^2}\left|\nabla^2 v\left(\frac{x}{\varepsilon L}\right)\right|^2 dx'dx_N \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\omega LW\left(\nabla v\left(\frac{x'}{\varepsilon L}, t\right)\right) + \frac{1}{L}\left|\nabla^2 v\left(\frac{x'}{\varepsilon L}, t\right)\right|^2 dx'dt \to \mathcal{H}^{N-1}(\omega)\int_Q LW(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2 dx. \end{aligned}$$

6.2 x'-connected domains

Theorem 6.6 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume* (5.14) *and let W satisfy the conditions (H_1), $(H_2)''$, (H_4) and (H_5). Let $u \in W^{1,1}(\Omega;\mathbb{R}^d)$, with $\nabla u \in BV(\Omega;\{A,B\})$. Then*

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon(u;\Omega) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = (1-\chi_E(x))A + \chi_E(x)\, B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Proof. The proof is very similar to that of Theorem 5.6.
Step 1: *One interface.* We assume first that u has the form

$$u(x) = |x_N|\, a \quad \text{a.e. in } \Omega.$$

We proceed as in Step 1 of Theorem 5.6 and in place of the function g we consider a function $v \in W^{2,\infty}(Q;\mathbb{R}^d)$ admissible for K_{per}, and, consequently, we define

$$u_{\varepsilon,\delta}(x) := \psi_\delta(x)\, z_\varepsilon(x) + (1-\psi_\delta(x))\, u(x)$$

where z_ε is now defined as in (6.20). The estimate (5.18) should be replaced by

$$\begin{aligned}\int_{\{\psi_\delta=1\}} \frac{1}{\varepsilon} W(\nabla z_\varepsilon) + \varepsilon\left|\nabla^2 z_\varepsilon\right|^2 dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\omega_\delta^+} LW\left(\nabla v\left(\frac{x'}{\varepsilon L}, t\right)\right) + \frac{1}{L}\left|\nabla^2 v\left(\frac{x'}{\varepsilon L}, t\right)\right|^2 dx' dt \\ &\to \mathcal{H}^{N-1}(\omega_\delta^+)\left(\int_Q LW(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2 dx\right)\end{aligned} \tag{6.22}$$

as $\varepsilon \to 0^+$, by Remark 6.5. By (6.20) and (6.21) the bound (5.24) continues to hold, while (5.25) should be replaced by

$$|\nabla(z_\varepsilon - u)| \le \left|\nabla v\left(\frac{x}{\varepsilon L}\right)\right| + \|\nabla u\|_\infty \le C, \quad \left|\nabla^2 z_\varepsilon\right| \le \frac{1}{\varepsilon}\left|\nabla v\left(\frac{x}{\varepsilon L}\right)\right| \le \frac{C}{\varepsilon}, \tag{6.23}$$

for $|x_N| \le \varepsilon L$. We can continue essentially as before, using (6.22) in the right hand side of the new formula corresponding to (5.28). We omit the details.
Steps 2 and 3: In the cases of finitely many and countably many interfaces we may proceed, respectively, as in Steps 2 and 3 of Theorem 5.5. We omit the details. ∎

6.3 General domains

In this section we remove the condition (5.14).

Theorem 6.7 *Assume that W satisfies the conditions (H_1), $(H_2)''$, (H_4) and (H_5). Let $u \in W^{1,1}(\Omega;\mathbb{R}^d)$, with $\nabla u \in BV(\Omega;\{A,B\})$. Then*

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon(u;\Omega) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = (1-\chi_E(x))A + \chi_E(x)\, B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Proof. The proof follows closely that of Subsection 5.4, with the only differences that in (5.34) of Proposition 5.7 the function z_ε is now defined as in (6.20) and, in turn, the estimate (5.25) should be replaced by (6.23). ∎

7 Condition $(H_4)'$

In this section we weaken the condition (H_4) on the bounds of W near the wells.

Theorem 7.1 *All the results of the previous section continue to hold if condition (H_4) is replaced by $(H_4)'$.*

The next lemma ensures that the function g introduced in $(H_4)'$ to control the behavior of W near the wells may be extended to a function G still satisfying the doubling condition, and such that $G(|\cdot - A|)$ and $G(|\cdot - B|)$ may be compared with W in the whole $\mathbb{R}^{d\times N}$. As before, in what follows we are assuming without loss of generality that A and B satisfy (2.1).

Lemma 7.2 *Let $g:[0,\infty)\to[0,\infty)$ be a convex function, with $g(s)=0$ if and only if $s=0$, such that*

$$g(2t) \leq Cg(t) \tag{7.1}$$

for all $0 \leq t \leq \rho$,

$$g(|\xi - A|) \leq W(\xi) \leq Cg(|\xi - A|) \tag{7.2}$$

for all $\xi \in \mathbb{R}^{d\times N}$ with $|\xi - A| \leq \rho$, and

$$g(|\xi - B|) \leq W(\xi) \leq Cg(|\xi - B|) \tag{7.3}$$

for all $\xi \in \mathbb{R}^{d\times N}$ with $|\xi - B| \leq \rho$, for some constant $C = C(\rho) > 0$. Then there exists a convex function $G:[0,\infty)\to[0,\infty)$ such that $G(t) = g(t)$ for all $t \in [0,\rho]$,

$$G(s+t) \leq C_1 (G(s) + G(t)) \tag{7.4}$$

for all $s,t \geq 0$ and for some constant $C_1 > 0$,

$$\lim_{t\to\infty} \frac{G(t)}{t^p} = 1, \tag{7.5}$$

$$\frac{1}{C_2} G(|\xi'|) \leq \frac{1}{C_2} \min\{G(|\xi - A|), G(|\xi - B|)\} \leq W(\xi) \leq C_2 \min\{G(|\xi - A|), G(|\xi - B|)\} \tag{7.6}$$

for all $\xi \in \mathbb{R}^{d\times N}$ and for some constant $C_2 > 0$,

$$W(\xi) \leq C_3 (W(\eta) + G(|\xi - \eta|)) \tag{7.7}$$

for all $\xi, \eta \in \mathbb{R}^{d\times N}$ and for some constant $C_3 > 0$,

$$C_4 G(|\xi - A|) \leq W(\xi) \tag{7.8}$$

for all $\xi \in \mathbb{R}^{d\times N}$ such that $|\xi - A|, |\xi - B| \geq \rho$ and for some constant $C_4 > 0$.

Proof. Let $a > \rho$ be any Lebesgue point for g' (recall that, since g is convex, g' is a function of bounded variation, precisely $g' \in BV_{\text{loc}}([0,\infty))$), and define

$$G(t) = \begin{cases} g(t) & \text{for } 0 \leq t \leq a, \\ t^p + \left(g'(a) - pa^{p-1}\right)t + g(a) - g'(a)a + (p-1)a^p & \text{for } t > a. \end{cases} \tag{7.9}$$

We claim that G is convex. Assume first that $g \in C^2([0,\infty))$. Then

$$G'(t) = \begin{cases} g'(t) & \text{for } 0 \leq t < a, \\ pt^{p-1} + \left(g'(a) - pa^{p-1}\right) & \text{for } t > a, \end{cases}$$

and

$$G''(t) = \begin{cases} g''(t) & \text{for } 0 \leq t < a, \\ p(p-1)t^{p-2} & \text{for } t > a. \end{cases}$$

Hence G' is continuous and nondecreasing, since $G''(t) \geq 0$ for all $t \neq a$, and, as G is continuous, this implies that G is convex. In the general case, consider $g_\varepsilon := \eta_\varepsilon * g$ and let G_ε be the corresponding convex functions defined as in (7.9). Since $g_\varepsilon \to g$ pointwise and $g'_\varepsilon(t) \to g'(t)$ for every Lebesgue point t of g', we obtain that $G_\varepsilon \to G$ pointwise, and thus G is convex. Condition (7.5) is now immediate. To prove (7.4) we first show that

$$G(2t) \leq C_1 G(t) \tag{7.10}$$

for all $t \geq 0$ and for some constant $C_1 > 0$. It $t \leq \rho$ this follows from (7.1). Let $\rho_1 > a$ be so large that

$$2t^p > -\left(2^{p+1}-2\right)\left(g'(a)-2a\right)t-\left(2^{p+1}-1\right)\left(g(a)-g'(a)a+(p-1)a^p\right)$$

for all $t \geq \rho_1$. Then

$$\begin{aligned} G(2t) &= 2^p t^p + \left(g'(a) - pa^{p-1}\right)2t + g(a) - g'(a)a + (p-1)a^p \\ &= 2^{p+1}G(t) - 2t^p - \left(2^{p+1}-2\right)\left(g'(a)-2a\right)t - \left(2^{p+1}-1\right)\left(g(a)-g'(a)a+(p-1)a^p\right) \\ &< 2^{p+1}G(t) \end{aligned}$$

for all $t \geq \rho_1$. Thus (7.10) holds for $t \leq \rho$ and $t \geq \rho_1$ taking as a constant $\max\left\{C, 2^{p+1}\right\}$. For $t \in [\rho, \rho_1]$ we have

$$G(2t) \leq \frac{\max_{[2\rho,2\rho_1]} G}{\min_{[\rho,\rho_1]} G} G(t)$$

and thus (7.10) holds for all $t \geq 0$ with

$$C_1 := \max\left\{C, 2^{p+1}, \frac{\max_{[2\rho,2\rho_1]} G}{\min_{[\rho,\rho_1]} G}\right\}.$$

To prove that (7.10) implies (7.4) is standard, note that by convexity and (7.10)

$$G(s+t) = G\left(\frac{2(s+t)}{2}\right) \leq \frac{1}{2}\left(G(2s)+G(2t)\right) \leq \frac{1}{2}C_1\left(G(s)+G(t)\right).$$

By (7.2) and (7.3) condition (7.6) holds if either $|\xi - A| \leq \rho$ or $|\xi - B| \leq \rho$. For $k > 1$ set $E_k := \left\{\xi \in \mathbb{R}^{d\times N} : |\xi - A|, |\xi - B| \geq \rho, |\xi| \leq k\right\}$. For $\xi \in E_k$

$$\frac{\min_{E_k} W}{\max_{E_k} G(|\cdot - A|)} G(|\xi - A|) \leq W(\xi) \leq \frac{\max_{E_k} W}{\min_{E_k} G(|\cdot - A|)} G(|\xi - A|),$$

and a similar inequality holds when the $G(|\xi - A|)$ is replaced by $G(|\xi - B|)$. Thus it is sufficient to prove (7.6) and (7.8) for $|\xi| \geq k$ where $k > 1$ remains to be chosen. This is an obvious consequence of $(H_2)''$ and (7.5). ■

Remark 7.3 In light of Lemma 7.2, and in spite of the fact that the qualitative properties of g are only given nearby zero, in the remaining of this section, and without loss of generality, we will assume that g satisfies (7.4)-(7.6) and (7.7)-(7.8) with g in place of G.

The next result has been proved by Bhattacharya and Leonetti [12] in the case where Ω is convex and $S = \Omega$, and a generalized version for for open, bounded domains with the cone property may be found in the Appendix.

Proposition 7.4 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, starshaped with respect to a set $S \subset \Omega$, with $|S| > 0$. Let $g : [0,\infty) \to [0,\infty)$ be a convex function, with $g(0) = 0$. Let $u \in W^{1,1}\left(\Omega; \mathbb{R}^d\right)$ be such that $g(|\nabla u|) \in L^1(\Omega)$. Then*

$$\int_\Omega g\left(\frac{|u(x) - u_S|}{d}\right) dx \leq \left(\frac{\alpha_N d^N}{|\Omega|}\right)^{1-\frac{1}{N}} \frac{|\Omega|}{|S|} \int_\Omega g(|\nabla u|)\, dx,$$

where $u_S := \frac{1}{|S|}\int_S u\, dx$, d is any number greater or equal than the diameter of Ω, and α_N is the volume of the unit ball in $\mathbb{R}^N$.

Proof of Theorem 7.1. Condition (H_4) was used only in the proof of Propositions 6.2 and 6.3. Thus it remains to show that these propositions continue to work under the weaker hypothesis $(H_4)'$. We begin with Proposition 6.2.

Vertical matching– first matching: The proof of the first matching continues to work up to (6.8). By (7.6) and since g is increasing

$$\begin{aligned}\frac{1}{\varepsilon}\int_{L_\varepsilon} W\left(\nabla v_\varepsilon\right)dx &\le \frac{1}{\varepsilon}\int_{L_\varepsilon} Cg\left(\left|\nabla v_\varepsilon - a\otimes e_N\right|\right)dx \\ &\le \frac{1}{\varepsilon}\int_{L_\varepsilon} g\left(C\left(\left|\nabla_{x'}u_\varepsilon\left(x',z_\varepsilon\right)\right| + \left|\nabla u_\varepsilon - a\otimes e_N\right| + \frac{1}{\varepsilon}\left|u_\varepsilon - u_0 - \bar{u}_\varepsilon\right|\right)\right)dx \\ &\le \frac{C}{\varepsilon}\int_{L_\varepsilon} g\left(\left|\nabla_{x'}u_\varepsilon\left(x',z_\varepsilon\right)\right|\right) + g\left(\left|\nabla u_\varepsilon - a\otimes e_N\right|\right) + g\left(\frac{1}{\varepsilon}\left|u_\varepsilon - u_0 - \bar{u}_\varepsilon\right|\right)dx\end{aligned} \tag{7.11}$$

where we have used (7.4). We now estimate the three terms on the right hand side of (7.11). By (7.6)

$$\begin{aligned}\frac{1}{\varepsilon}\int_{L_\varepsilon} g\left(\left|\nabla_{x'}u_\varepsilon\left(x',z_\varepsilon\right)\right|\right)dx &= \left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{6\varepsilon}\int_{Q'} g\left(\left|\nabla_{x'}u_\varepsilon\left(x',z_\varepsilon\right)\right|\right)dx' \\ &\le C\int_{Q'} W\left(\nabla u_\varepsilon\left(x',z_\varepsilon\right)\right)dx' \to 0\end{aligned} \tag{7.12}$$

as $\varepsilon \to 0^+$ by (6.6). If $\left|\nabla u_\varepsilon - a\otimes e_N\right| \le \rho$ or $\left|\nabla u_\varepsilon - a\otimes e_N\right|, \left|\nabla u_\varepsilon + a\otimes e_N\right| \ge \rho$ then by (7.8)

$$g\left(\left|\nabla u_\varepsilon - a\otimes e_N\right|\right) \le CW\left(\nabla u_\varepsilon\right),$$

while if $\left|\nabla u_\varepsilon - a\otimes e_N\right| \ge \rho$ and $\left|\nabla u_\varepsilon + a\otimes e_N\right| \le \rho$ then

$$g\left(\left|\nabla u_\varepsilon - a\otimes e_N\right|\right) \le g\left(\rho + 2\left|a\otimes e_N\right|\right) \le g\left(\rho + 2\left|a\otimes e_N\right|\right)\frac{\left|\nabla u_\varepsilon - a\otimes e_N\right|^p}{\rho^p}.$$

Hence

$$\frac{C}{\varepsilon}\int_{L_\varepsilon} g\left(\left|\nabla u_\varepsilon - a\otimes e_N\right|\right)dx \le \frac{C}{\varepsilon}\int_{L_\varepsilon} W\left(\nabla u_\varepsilon\right) + \left|\nabla u_\varepsilon - a\otimes e_N\right|^p dx \to 0 \tag{7.13}$$

as $\varepsilon \to 0^+$ by (6.5). Finally, by Jensen's inequality

$$\begin{aligned}\frac{C}{\varepsilon}\int_{L_\varepsilon} g\left(\frac{1}{\varepsilon}\left|u_\varepsilon - u_0 - \bar{u}_\varepsilon\right|\right)dx &\le \frac{C}{\varepsilon}\int_{L_\varepsilon} g\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon - \left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{6}}^{\theta_\varepsilon}\left|\nabla u_\varepsilon\left(x',t\right) - a\otimes e_N\right|dt\right)dx \\ &\le \frac{C}{\varepsilon}\int_{L_\varepsilon}\frac{1}{\varepsilon}\int_{\theta_\varepsilon - \left[\frac{1}{\varepsilon}\right]^{-1}\frac{1}{6}}^{\theta_\varepsilon} g\left(\left|\nabla u_\varepsilon\left(x',t\right) - a\otimes e_N\right|\right)dtdx \\ &= \frac{C}{\varepsilon}\int_{L_\varepsilon} g\left(\left|\nabla u_\varepsilon - a\otimes e_N\right|\right)dx \to 0\end{aligned} \tag{7.14}$$

as $\varepsilon \to 0^+$ by (6.5). Thus, by (7.11)-(7.14) the claim (iii) holds as before.
Vertical matching–second matching: To prove (ii) in the second matching, note that by (7.6) and since g is increasing,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{Q'\times\left(\theta_\varepsilon,\frac{1}{2}\right)} W\left(\nabla v_\varepsilon\right)dx &\le \frac{1}{\varepsilon}\int_{Q'\times\left(\theta_\varepsilon,\frac{1}{2}\right)} Cg\left(\left|\nabla v_\varepsilon - a\otimes e_N\right|\right)dx \\ &\le \frac{C}{\varepsilon}\int_{Q'\times\left(\theta_\varepsilon,\frac{1}{2}\right)} g\left(\left|\nabla_{x'}u_\varepsilon\left(x',z_\varepsilon\right)\right|\right) + g\left(\left|\bar{u}_\varepsilon - c_\varepsilon\right|\right)dx \\ &\le \frac{C}{\varepsilon}\int_{Q'\times\left(\theta_\varepsilon,\frac{1}{2}\right)} W\left(\nabla u_\varepsilon\right) + g\left(\left|\bar{u}_\varepsilon - c_\varepsilon\right|\right)dx,\end{aligned}$$

where we have used (7.4). Thus by (6.5), it suffices to prove that

$$\frac{C}{\varepsilon}\int_{Q'\times\left(\theta_\varepsilon,\frac{1}{2}\right)} g\left(\left|\bar{u}_\varepsilon - c_\varepsilon\right|\right)dx \to 0.$$

By Proposition 7.4

$$\begin{aligned}\frac{C}{\varepsilon}\int_{Q'\times\left(\theta_\varepsilon,\frac{1}{2}\right)} g\left(\left|\bar{u}_\varepsilon-c_\varepsilon\right|\right)dx &= \frac{C}{\varepsilon}\int_{\theta_\varepsilon}^{\frac{1}{2}}\int_{Q'} g\left(\left|\bar{u}_\varepsilon-\frac{1}{|Q'|}\int_{Q'}\bar{u}_\varepsilon\left(x',z_\varepsilon\right)dx'\right|\right)dx'dx_N\\ &\le \frac{C}{\varepsilon}\int_{\theta_\varepsilon}^{\frac{1}{2}}\int_{Q'} g\left(\left|\nabla_{x'}\bar{u}_\varepsilon\right|\right)dx'dx_N\\ &\le \frac{C}{\varepsilon}\int_{Q'} g\left(\left|\nabla_{x'}u_\varepsilon\left(x',z_\varepsilon\right)\right|\right)dx'\\ &\le \frac{C}{\varepsilon}\int_{Q'} W\left(\nabla u_\varepsilon\left(x',z_\varepsilon\right)\right)dx'\to 0\end{aligned}$$

as $\varepsilon\to 0^+$ by (6.6), and where we have used (7.6).

It remains to ensure that Proposition 6.3 still holds.

Transversal periodicity: The proof of Proposition 6.3 continues to work. The only difference is on the estimate (6.13) which continues to hold since, by (7.6) and the fact that g is increasing, we have

$$\begin{aligned}W\left(0,\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)\right) &\le C_2\min\left\{g\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)-a\right|\right),g\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\left(-a_{\varepsilon,m},x_2\right)+a\right|\right)\right\}\\ &\le C_2\min\left\{g\left(\left|\nabla u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)-a\otimes e_2\right|\right),g\left(\left|\nabla u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)+a\otimes e_2\right|\right)\right\}\\ &\le C_2^2 W\left(\nabla u_\varepsilon\left(-a_{\varepsilon,m},x_2\right)\right),\end{aligned}$$

and we can now proceed as before. ■

8 Example of a non one-dimensional interface

In this section we show that when (H_5) holds but (H_3) fails, the asymptotic limiting problem may not have a one dimensional character, namely,

$$K_{\text{per}}<K.$$

Consider the case where $N=2$, $d=1$, so that, with $x=(x_1,x_2)$, we have

$$\begin{aligned}K_{\text{per}}:=\inf\Bigg\{\int_Q L\,W\left(\nabla v\right)+\frac{1}{L}\left|\nabla^2 v\right|^2\,dx_1dx_2 : L>0,\ v\in W^{2,\infty}\left(Q;\mathbb{R}\right),\\ \nabla v=\pm e_2 \text{ nearby } x_2=\pm\frac{1}{2},\ v\text{ is periodic of period one in } x_1\Bigg\}.\end{aligned}$$

In what follows we say that $v\in W^{2,\infty}_{\text{loc}}\left(\left(-\frac{1}{2},\frac{1}{2}\right)\times\mathbb{R};\mathbb{R}\right)$ is such that $\nabla v=\pm e_2$ nearby $x_2=\pm\infty$ if there exists a constant $M>0$ such that $\nabla v=\pm e_2$ for all $x\in\left(-\frac{1}{2},\frac{1}{2}\right)\times\mathbb{R}$ (resp.) with $x_2\ge M$ (resp. $x_2\le -M$).

Proposition 8.1

$$\begin{aligned}K_{\text{per}}=\inf\Bigg\{\int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} L\,W\left(\nabla v\right)+\frac{1}{L}\left|\nabla^2 v\right|^2\,dx_1dx_2 : L>0,\ v\in W^{2,\infty}_{loc}\left(\left(-\frac{1}{2},\frac{1}{2}\right)\times\mathbb{R};\mathbb{R}\right),\\ \nabla v=\pm e_2\ \textit{nearby}\ x_2=\pm\infty,\ v\ \textit{is periodic of period one in}\ x_1\Bigg\}=:K_\infty.\end{aligned}$$

Proof. By linear continuation it is easy to see that $K_\infty\le K_{\text{per}}$. To prove the converse inequality, fix $\delta>0$ and let $v\in W^{2,\infty}\left(\left(-\frac{1}{2},\frac{1}{2}\right)\times\mathbb{R};\mathbb{R}\right)$ be an admissible function for K_∞ such that, for some $L>0$,

$$\int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} L\,W\left(\nabla v\right)+\frac{1}{L}\left|\nabla^2 v\right|^2\,dx_1dx_2\le K_\infty+\delta.$$

Since $\nabla v = \pm e_2$ nearby $x_2 = \pm\infty$ we may find a positive integer m such that

$$K_\infty + \delta \geq \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} L\,W(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2\,dx_1dx_2 = \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} L\,W(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2\,dx_1dx_2$$

and $\nabla v = \pm e_2$ for $\pm x_2 \geq m$. Due to the periodicity of v with respect to x_1, we have

$$\begin{aligned}
K_\infty + \delta &\geq \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} L\,W(\nabla v(x)) + \frac{1}{L}\left|\nabla^2 v(x)\right|^2\,dx_1dx_2\\
&= \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}}\frac{1}{2m+1}\sum_{k=-m}^{m}\int_{-\frac{1}{2}}^{\frac{1}{2}} L\,W(\nabla v(x_1+k,x_2)) + \frac{1}{L}\left|\nabla^2 v(x_1+k,x_2)\right|^2\,dx_1dx_2\\
&= \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}}\frac{1}{2m+1}\sum_{k=-m}^{m}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} L\,W(\nabla v(x_1,x_2)) + \frac{1}{L}\left|\nabla^2 v(x_1,x_2)\right|^2\,dx_1dx_2\\
&= \frac{1}{2m+1}\int_{-m-\frac{1}{2}}^{m+\frac{1}{2}}\int_{-m-\frac{1}{2}}^{m+\frac{1}{2}} L\,W(\nabla v(x_1,x_2)) + \frac{1}{L}\left|\nabla^2 v(x_1,x_2)\right|^2\,dx_1dx_2\\
&= \frac{1}{2m+1}\int_{(2m+1)Q} L\,W(\nabla v(x)) + \frac{1}{L}\left|\nabla^2 v(x)\right|^2\,dx.
\end{aligned}$$

Via the change of variables $x := (2m+1)\,y$ we have

$$\begin{aligned}
K_\infty + \delta &\geq (2m+1)\int_Q LW(\nabla v((2m+1)\,y)) + \frac{1}{L}\left|\nabla^2 v((2m+1)\,y)\right|^2\,dy\\
&= \int_Q (2m+1)\,L\,W(\nabla z(y)) + \frac{1}{(2m+1)\,L}\left|\nabla^2 z(y)\right|^2\,dy,
\end{aligned}$$

where we have set $z(y) := \frac{1}{2m+1}v((2m+1)\,y)$. Note that

$$\begin{aligned}
z(y_1+1,y_2) &= \frac{1}{2m+1}v((2m+1)\,y_1 + (2m+1),(2m+1)\,y_2)\\
&= \frac{1}{2m+1}v((2m+1)\,y_1,(2m+1)\,y_2)\\
&= z(y_1,y_2),
\end{aligned}$$

since v periodic of period one in x_1, and $\nabla z(y) = \pm e_2$ if y_2 is nearby $\pm\frac{1}{2}$, resp. Hence z is admissible for K_{per}, and so

$$\begin{aligned}
K_\infty + \delta &\geq \frac{1}{2m+1}\int_{(2m+1)Q} L\,W(\nabla v(x)) + \frac{1}{L}\left|\nabla^2 v(x)\right|^2\,dx\\
&= (2m+1)\int_Q L\,W(\nabla v((2m+1)\,y)) + \frac{1}{L}\left|\nabla^2 v((2m+1)\,y)\right|^2\,dy\\
&= \int_Q (2m+1)\,L\,W(\nabla z(y)) + \frac{1}{(2m+1)\,L}\left|\nabla^2 z(y)\right|^2\,dy \geq K_{\text{per}}.
\end{aligned}$$

It now suffices to let $\delta \to 0^+$. ■

Next we exhibit an example of an energy density satisfying (H_1), $(H_2)''$, $(H_4)'$ and (H_5) for which $K_{\text{per}} < K$. Define

$$W(\xi) = W(\xi_1,\xi_2) := \left(1-\alpha\xi_1^2-\xi_2^2\right)^2 + \xi_1^2,$$

where $\alpha > 0$. Then $W(\xi) = 0$ if and only if $\xi \in \{(0,1),(0,-1)\}$. By Lemma 3.5 in [23] we have that

$$\begin{aligned}
K &= \inf\left\{\int_{-\infty}^{\infty} W(0,g(s)) + \left|g'(s)\right|^2 ds :\ g \text{ piecewise } C^1,\ g(-\infty) = -1,\ g(\infty) = 1\right\} \qquad (8.1)\\
&= \inf\left\{\int_{-\infty}^{\infty} \left(1-g^2(s)\right)^2 + \left|g'(s)\right|^2 ds :\ g \text{ piecewise } C^1,\ g(-\infty) = -1,\ g(\infty) = 1\right\}.
\end{aligned}$$

It is not difficult to see that K is realized by the unique solution of the boundary value problem

$$\begin{cases} g'' + 2g - 2g^3 = 0 \\ g(-\infty) = -1,\ g(\infty) = 1, \end{cases}$$

which is given by $g(s) := \tanh s$. Define $\bar{u}(t) := \int_0^t g(s)\, ds = \ln\cosh t$.

Proposition 8.2 *If α is sufficiently large then we have $K_{\mathrm{per}} < K$.*

Proof. Set

$$v(x) = v(x_1, x_2) := \bar{u}(x_2) + \lambda\psi(x_1, x_2),$$

where $\psi(x_1, x_2) := \sin(2\pi x_1) f(x_2)$ and f is a smooth nonnegative function with compact support. With $L = 1$ we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} W(\nabla v) + \left|\nabla^2 v\right|^2 dx_1 dx_2 = \int_{-\infty}^{\infty} \left(1 - g^2(s)\right)^2 + |g'(s)|^2 ds \\
&+ \lambda^2 \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[\left|\nabla^2\psi\right|^2 + 2\left(3g^2(x_2) - 1\right)\sin^2(2\pi x_1)\left(f'(x_2)\right)^2 + 4\pi^2\cos^2(2\pi x_1) f^2(x_2) \\
&\qquad -8\pi^2\alpha\left(1 - g^2(x_2)\right)\cos^2(2\pi x_1) f^2(x_2)\Big]\, dx_1 dx_2 \\
&+ \lambda^4 \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[\sin^2(2\pi x_1)\left(f'(x_2)\right)^2 + 4\pi^2\alpha\cos^2(2\pi x_1) f^2(x_2)\Big]^2 dx_1 dx_2 \\
&=: I_1 + \lambda^2 I_2(\alpha) + \lambda^4 I_3(\alpha).
\end{aligned}$$

We now choose $\alpha > 0$ so large that $I_2(\alpha) < 0$, and then λ so small that $\lambda^2 I_2(\alpha) + \lambda^4 I_3(\alpha) < 0$. In view of (8.1)

$$\int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} W(\nabla v) + \left|\nabla^2 v\right|^2 dx_1 dx_2 < \int_{-\infty}^{\infty} \left(1 - g^2(s)\right)^2 + |g'(s)|^2 ds = K.$$

Let $\{u_n\}$ be a sequence of smooth functions converging to $\bar{u}$ strongly in $W^{2,p}_{\mathrm{loc}}(\mathbb{R})$ for all $p \geq 1$, and such that $u_n'(x_2) = \pm e_2$ (resp.) for all $x_2 \geq n$ (resp. $x_2 \leq -n$). Define

$$v_n(x) := u_n(x_2) + \lambda\psi(x_1, x_2).$$

Then v_n are admissible for K_∞, and so

$$\begin{aligned}
K_\infty &\leq \lim_{n\to\infty} \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} W(\nabla v_n) + \left|\nabla^2 v_n\right|^2 dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} W(\nabla v) + \left|\nabla^2 v\right|^2 dx_1 dx_2 < K.
\end{aligned}$$

This, together with Proposition 8.1, concludes the proof. ■

9 Appendix

In the Appendix we generalize Poincaré's inequality to Orlicz-Sobolev spaces (see Proposition 9.2). Although it is probably known to experts, we have not been able to find it in the literature. The proof follows that of Maz'ja [29] for the case $g(s) = |s|^p$. A first version has been proved by Bhattacharya and Leonetti [12] in the case where Ω is convex and $S = \Omega$.

We recall that an open set $\Omega \subset \mathbb{R}^N$ is *starshaped with respect to a set* $S \subset \Omega$ if Ω is starshaped with respect to each point of S, i.e. if $x \in \Omega$ and $s \in S$ then $\theta x + (1 - \theta) s \in \Omega$ for all $\theta \in (0, 1)$.

Proposition 9.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, starshaped with respect to a set $S \subset \Omega$, with $|S| > 0$. Let $g : [0, \infty) \to [0, \infty)$ be a convex function, with $g(0) = 0$. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$ be such that $g(|\nabla u|) \in L^1(\Omega)$. Then*

$$\int_\Omega g\left(\frac{|u(x) - u_S|}{d}\right) dx \le \left(\frac{\alpha_N d^N}{|\Omega|}\right)^{1-\frac{1}{N}} \frac{|\Omega|}{|S|} \int_\Omega g(|\nabla u|)\, dx,$$

where $u_S := \frac{1}{|S|}\int_S u\, dx$, d is any number greater or equal than the diameter of Ω, and α_N is the volume of the unit ball in $\mathbb{R}^N$.

Proof. We follow Lemma 7.16 in Gilbarg and Trudinger [24]. Assume first that $u \in W^{1,1}(\Omega; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d)$. Since Ω is starshaped with respect to $S \subset \Omega$, for $x \in \Omega$ and $y \in S$ we have

$$u(x) - u(y) = -\int_0^{|x-y|} D_r u(x + r\omega)\, dr, \quad \omega = \frac{y-x}{|y-x|}.$$

Averaging with respect to y over S yields

$$u(x) - u_S = -\frac{1}{|S|}\int_S dy \int_0^{|x-y|} D_r u(x + r\omega)\, dr.$$

Since $|x - y| \le d$ we have

$$\frac{|u(x) - u_S|}{d} \le \frac{1}{|S|}\int_S \frac{1}{|x-y|}\int_0^{|x-y|} |D_r u(x + r\omega)|\, drdy.$$

As g is convex, it now follows from applying twice Jensen's inequality that

$$g\left(\frac{|u(x) - u_S|}{d}\right) \le \frac{1}{|S|}\int_S \frac{1}{|x-y|}\int_0^{|x-y|} g(|D_r u(x + r\omega)|)\, drdy.$$

Defining

$$V(x) = \begin{cases} |\nabla u(x)| & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases}$$

and, as g is increasing, we have

$$\begin{aligned} g\left(\frac{|u(x) - u_S|}{d}\right) &\le \frac{1}{|S|}\int_{\{y:|x-y|<d\}} \frac{1}{|x-y|}\int_0^\infty g(V(x + r\omega))\, drdy \\ &= \frac{1}{|S|}\int_0^\infty \int_{|\omega|=1} \int_0^d g(V(x + r\omega))\, \rho^{N-2} d\rho d\omega dr \\ &= \frac{d^{N-1}}{(N-1)|S|}\int_0^\infty \int_{|\omega|=1} g(V(x + r\omega))\, d\omega dr \\ &= \frac{d^{N-1}}{(N-1)|S|}\int_\Omega |x-y|^{1-N} g(|\nabla u(y)|)\, dy, \end{aligned}$$

where we have used the fact that $g(0) = 0$. The theory of Riesz potentials (Lemma 7.12 in Gilbarg and Trudinger [24]) now yields

$$\int_\Omega g\left(\frac{|u(x) - u_S|}{d}\right) dx \le \frac{1}{N}(\alpha_N)^{1-\frac{1}{N}} |\Omega|^{\frac{1}{N}} \frac{d^{N-1}}{(N-1)|S|}\int_\Omega g(|\nabla u(x)|)\, dx$$

and the proof is complete. ■

Proposition 9.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain having the cone property, let $g : [0, \infty) \to [0, \infty)$ be a convex function satisfying the doubling condition, with $g(0) = 0$. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$ be such that $g(|\nabla u|) \in L^1(\Omega)$. Then*

$$\int_\Omega g(|u(x) - u_B|)\, dx \leq C \int_\Omega g(|\nabla u|)\, dx,$$

where

$$u_B := \frac{1}{|B|} \int_B u(y)\, dy,$$

B is any fixed ball whose closure is contained in Ω, and C is a positive constant depending only on Ω and on the ball B.

Proof. Since Ω has the cone property, it is the union of a finite number of domains starshaped with respect to a ball. Let d be a number greater than the diameter of all these domains, and let A be any of these subdomains with D being the corresponding ball. Construct a finite family of balls $B_0, \cdots, B_M$ contained in Ω and such that $B_0 = D$, $B_i \cap B_{i+1} \neq \emptyset$, $B_M = B$. Since A is starshaped with respect to any fixed ball $\tilde{B}$ contained in $B_0 \cap B_1$, by Proposition 7.4 we obtain

$$\int_A g\left(\frac{|u(x) - u_{\tilde{B}}|}{d}\right) dx \leq \left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx.$$

By Remark 7.3 and (7.4)

$$\begin{aligned}\int_A g\left(\frac{|u(x)|}{d}\right) dx &\leq C\,|A|\, g\left(\frac{|u_{\tilde{B}}|}{d}\right) + C\left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx \\ &\leq C\frac{|A|}{\left|\tilde{B}\right|} \int_{\tilde{B}} g\left(\frac{|u(x)|}{d}\right) dx + C\left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx,\end{aligned}$$

where we have used Jensen's inequality. Hence

$$\int_A g\left(\frac{|u(x)|}{d}\right) dx \leq C\frac{|A|}{\left|\tilde{B}\right|} \int_{B_0 \cap B_1} g\left(\frac{|u(x)|}{d}\right) dx + C\left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx.$$

Similarly, since for $i = 1, \cdots, M-1$ the ball B_i is starshaped with respect to any fixed ball $\tilde{B}_i$ contained in $B_i \cap B_{i+1} \neq \emptyset$, we obtain

$$\int_{B_i} g\left(\frac{|u(x)|}{d}\right) dx \leq C\frac{|B_i|}{\left|\tilde{B}_i\right|} \int_{B_i \cap B_{i+1}} g\left(\frac{|u(x)|}{d}\right) dy + C\left(\frac{\alpha_N\, d^N}{|B_i|}\right)^{1-\frac{1}{N}} \frac{|B_i|}{\left|\tilde{B}_i\right|} \int_{B_i} g(|\nabla u|)\, dx.$$

Therefore

$$\int_A g\left(\frac{|u(x)|}{d}\right) dx \leq C\left(\int_B g\left(\frac{|u(x)|}{d}\right) dx + \int_\Omega g(|\nabla u|)\, dx\right).$$

Summing over all A gives

$$\int_\Omega g\left(\frac{|u(x)|}{d}\right) dx \leq C\left(\int_B g\left(\frac{|u(x)|}{d}\right) dx + \int_\Omega g(|\nabla u|)\, dx\right). \tag{9.1}$$

Since B is convex, by Proposition 7.4

$$\int_B g\left(\frac{|u(x) - u_B|}{d}\right) dx \leq \left(\frac{\alpha_N\, d^N}{|B|}\right)^{1-\frac{1}{N}} \int_B g(|\nabla u|)\, dx,$$

where $u_B := \frac{1}{|B|}\int_B u\,dx$. Replacing u by $u - u_B$ in (9.1) we obtain

$$\int_\Omega g\left(\frac{|u(x) - u_B|}{d}\right) dx \le C\left(\int_B g\left(\frac{|u(x) - u_B|}{d}\right) + \int_\Omega g(|\nabla u|)\,dx\right)$$
$$\le C\int_\Omega g(|\nabla u|)\,dx.$$

Applying the latter inequality to du in place of u yields

$$\int_\Omega g(|u(x) - u_B|)\,dx \le C\int_\Omega g(d\,|\nabla u|)\,dx \le C_1\int_\Omega g(|\nabla u|)\,dx,$$

where we have used the fact that $g(dz) \le$const. $g(z)$ for all $z \ge 0$ (see Remark 7.3 and (7.4)). This concludes the proof. ■

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