MATERIAL VOIDS IN ELASTIC SOLIDS WITH ANISOTROPIC SURFACE ENERGIES

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Abstract. This work discusses the role of highly anisotropic interfacial energy for problems involving a material void in a linearly elastic solid. Using the calculus of variations it is shown that important qualitative features of the equilibrium shape of the void may be deduced from smoothness and convexity properties of the interfacial energy.

1. Introduction

Understanding surface roughening of materials plays a central role in many fields of physics, chemistry, and metallurgy. Since the pioneer work of Asaro & Tiller [3] (see also [28,37], and the references therein), it has been recognized that in continuous models of crystals surface instability is driven by the competition between elastic energy and surface energy.

The stress, acting parallel to a flat surface of an elastic solid, causes atoms to diffuse on the surface and the surface to undulate. In turn such a migration of atoms has an energetic prize in terms of surface tension. This phenomenon may lead to the formation of isolated islands on the substrate surface (see, e.g., [30,31], and [32]), or of cracks running into the bulk of the solid. Island formation in systems such as In-GaAs/GaAs or SiGe/Si turns out to be useful in the fabrication of modern semiconductor electronic and optoelectronic devices such as quantum dots laser.

Similarly, a void in a grain can collapse into a crack by surface diffusion when the applied stress exceeds a critical value (see [9,19,20,33,35,36]). Note that, since the lattice diffusion is much slower as compared to the surface diffusion, the evolving void in a grain can be assumed to conserve its volume, only changes its shape.

In [36], Suo & Wang have conducted numerical experiments on the shape change of a pore in an infinite solid. Assuming that the surface tension is isotropic and that the solid is under a uniaxial stress σ_1, they observed that the pore changes shape as the atoms diffuse on the surface driven by surface and elastic energy variation, expressed in term of the dimensionless number

$$\Lambda = \frac{\sigma_1^2 R_0}{Y\gamma},$$

where Y is the Young's modulus, R_0 the initial circular pre radius, and γ the surface tension. Their experiments showed that under no stress, the pore has a rounded shape maintained by surface tension. On the other hand, if the applied stress is small (Λ small), the pore reaches an equilibrium shape close to an ellipse (thus compromising the stress and the surface tension), while if the applied stress Λ is large, the pore does not reach equilibrium and noses emerge, which sharpen into crack tips. Similar results were also obtained for anisotropic surface tension.

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The purpose of this paper is to formulate a simple variational model describing the competition between elastic energy and highly anisotropic surface energy for problems involving a material void in a linearly elastic solid. Following the fundamental work of Herring [23] (see also [34]), we take the surface free energy of a body to be an integral of the form

$$\int \varphi(\nu)\, dS \tag{1.1}$$

extended over the surface of the body, where the surface energy density φ is, for anisotropic bodies, a function of the orientation of the outer unit normal ν at each surface point. The shape that minimizes (1.1) for fixed volume is known as the *Wulff shape* (see [14,17,21] and the references therein). Under no stress, Herring [23] argued that if a given macroscopic surface of a crystal does not coincide in orientation with some portion of the boundary of the Wulff shape, then there exists a hill-and-valley structure that has a lower free energy than a flat surface.

On the other hand, the minimum energy configuration of the bulk material occurs at the stress-free state for each solid. Thus, at the interface between the void and the elastic solid these two opposing mechanisms compete to determine the resulting structure.

We now describe the model considered in this paper. Our formulation follows Siegel, Miksis, and Voorhees [28]. Consider a *starshaped* void, which occupies a closed region $F \subset \mathbb{R}^2$, embedded in an elastic solid. The solid region is assumed to obey the usual laws of linear elasticity, so that the bulk energy takes the form $\int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\, dz$, where B_0 is a large ball, $\mathcal{W}(\mathbf{E}) = \frac{1}{2}\mathbb{C}(\mathbf{E}) \cdot \mathbf{E}$ is the elastic energy density, with $\mathbb{C}$ a constant positive definite fourth order tensor, and $\mathbf{E}(u)$ is the symmetrized gradient, i.e.,

$$\mathbf{E}(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)\,.$$

We assume that far from the void $u = u_0$ a.e. in $\mathbb{R}^2 \setminus B_0$[a]. Thus, we are led to minimize the functional

$$\mathcal{F}(F,u) := \int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\, dz + \int_{\partial F} \varphi(\nu_F^i)\, d\mathcal{H}^1 \tag{1.2}$$

over all pairs (F,u) for which $u = u_0$ a.e. in $\mathbb{R}^2 \setminus B_0$ and for which the void F has a fixed area. Notice that, since the inner normal ν_F^i is equal to the outer normal to the elastic body, the surface integral in (1.2) coincides with (1.1).

The paper is divided into two parts. In the first part we prove an integral representation result for the *relaxed* or effective energy of (1.2) (see Theorem 3.2). This result is closely related to recent work of Braides, Chambolle, and Solci [4] (see also [6,8], and [15]), who proved a similar relaxation result in the N-dimensional case but with Hausdorff convergence of sets replaced by L^1-convergence of their characteristics functions.

In the second part of this work we study the regularity of minimizers (F,u) of the relaxed functional $\overline{\mathcal{F}}$ (see (3.8)), under volume constraint. The strategy of the proof is similar to the one in [15], where the case of isotropic surface energy was considered. As in that paper we are able to show that volume constrained minimizers of the limiting energy $\overline{\mathcal{F}}$ are also unconstrained minimizers if we add to $\overline{\mathcal{F}}$ a suitable volume penalization. This allows us to consider a larger class of variations of F and to prove, adapting an argument contained in [7], an exterior Wulff shape condition. It is at this point that our analysis significantly departs from previous work [15] in the isotropic case (in which the Wulff shape was a ball), see also [18].

[a]Since our approach is variational, here we depart sligthly from the work of [28], where the solid is assumed to occupy the infinite region $\mathbb{R}^2 \setminus F$ and far from the void a state of biaxial stress is imposed, precisely,

$$\mathbf{T}\left(\mathbf{E}\left(u\right)\right) \to \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

as $\sqrt{x^2+y^2} \to \infty$. Note that this condition would force the energy $\int_{\mathbb{R}^2\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\, dz$ to be infinite.

We first study polygonal Wulff shapes. This is the appropriate setting to address physical crystals (see [34]). Surface integrands φ for which the Wulff shape is a polygon are called *crystalline* and it can be shown that if $W \subset \mathbb{R}^2$ is a convex, bounded, closed set, then it is the Wulff shape of its *support function* (see Proposition 3.5 in [14])

$$\varphi(z) := \sup\{y \cdot z : \, y \in W\}\,, \qquad z \in \mathbb{R}^2\,.$$

Under the assumption that the internal angles of the Wulff shape are strictly greater than $\frac{\pi}{2}$, we can prove that if (F, u) is a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$, then ∂F is the union of finitely many Lipschitz graphs. To the best of our knowledge these are the first regularity results in this context. In the absence of the elastic energy but without the restriction that F is starshaped, we refer to the recent work of Ambrosio, Novaga, and Paolini [2], and of Novaga and Paolini [27] as well as to the references contained therein.

We then study the case in which the anisotropy is weak, that is, the surface energy density φ in (1.2) (extended to be 1-homogeneous) is strictly convex. For example, for helium the surface energy is almost isotropic and its Wulff shape is nearly spherical (see [26]). For this type of surface energies the Wulff shape is of class C^1 and thus many of the arguments obtained in [15] can be adapted, although the proofs are significantly more involved.

2. Preliminaries

2.1. Sets of finite perimeter, functions of finite pointwise variation, and polar coordinates.

First, we recall some basic properties of sets of finite perimeter. If $E \subset \mathbb{R}^N$ is a measurable set, then E^0 and E^1 denote the *set of points of density* 0 with respect to E and the *set of points of density* 1, respectively. Recall that the density of $z \in \mathbb{R}^N$ with respect to E is defined as

$$\lim_{r \to 0^+} \frac{|E \cap B_r(z)|}{|B_r|}$$

whenever this limit exists, where $B_r(z)$ denotes the ball of center z and radius r.

A set $E \subset \mathbb{R}^N$ is said to be of *finite perimeter* if the distributional derivative of the characteristic function χ_E is a Radon measure with finite total variation. Then, the *reduced boundary* $\partial^* E$ is defined as the set of points $z \in \operatorname{spt}|D\chi_E|$ such that the limit

$$\nu_E(z) := -\lim_{r \to 0^+} \frac{D\chi_E(B_r(z))}{|D\chi_E|(B_r(z))}$$

exists and satisfies $|\nu_E(z)| = 1$. It may be verified that $\partial^* E$ is a Borel set and that $\nu_E : \partial^* E \to \mathbb{S}^1$ is a Borel map (see e.g. [1]). We call ν_E the (generalized) *outer normal*, and

$$\nu_E^i := -\nu_E$$

is the (generalized) *inner normal*.

We shall need the following lemma which is a consequence of Proposition 3.38, Example 3.68, and Example 3.97 in [1], and [13, Lemma 2.2].

Lemma 2.1. *Let A and B be sets of finite perimeter in $\mathbb{R}^N$. Then $A \cap B$, $A \setminus B$ and $A \cup B$ are sets of finite perimeter. Moreover,*

$$\partial^*(A \cap B) = \big(\partial^* A \cap B^1\big) \cup \big(\partial^* B \cap A^1\big) \cup \big(\partial^* A \cap \partial^* B \cap \{\nu_A = \nu_B\}\big) \quad (\textit{mod. } \mathcal{H}^{N-1})\,, \tag{2.1}$$

and for $\mathcal{H}^{N-1}$-a.e. $z \in \partial^*(A \cap B)$,

$$\nu_{A \cap B}(z) = \begin{cases} \nu_A(z) & \textit{if } z \in \partial^* A \cap B^1\,, \\ \nu_B(z) & \textit{if } z \in \partial^* B \cap A^1\,, \\ \nu_A(z) & \textit{if } z \in \partial^* A \cap \partial^* B \cap \{\nu_A = \nu_B\}\,. \end{cases}$$

In addition, if $|A \cap B| = 0$, then

$$\partial^*(A \cup B) = \big(\partial^* A \setminus \partial^* B\big) \cup \big(\partial^* B \setminus \partial^* A\big) \quad (\textit{mod. } \mathcal{H}^{N-1}), \tag{2.2}$$

and for $\mathcal{H}^{N-1}$-a.e. $z \in \partial^*(A \cup B)$,

$$\nu_{A\cup B}(z) = \begin{cases} \nu_A(z) & \textit{if } z \in \partial^* A \setminus \partial^* B\,, \\ \nu_B(z) & \textit{if } z \in \partial^* B \setminus \partial^* A\,. \end{cases} \tag{2.3}$$

In this paper $\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$ centered at the origin and oriented counterclockwise. If $\sigma = \big(\sigma^1, \sigma^2\big) \in \mathbb{S}^1$, then $\sigma^\perp$ is obtained rotating σ counterclockwise by $\pi/2$, i.e.,

$$\sigma^\perp := \big(-\sigma^2, \sigma^1\big)\ .$$

Given $\sigma_1, \sigma_2 \in \mathbb{S}^1$, we set

$$(\sigma_1, \sigma_2) := \big\{\sigma \in \mathbb{S}^1 : \sigma_1 < \sigma < \sigma_2\big\}\ , \qquad [\sigma_1, \sigma_2] := \big\{\sigma \in \mathbb{S}^1 : \sigma_1 \le \sigma \le \sigma_2\big\}\ ,$$

and

$$A(\sigma_1, \sigma_2) := \{r\sigma : \sigma \in (\sigma_1, \sigma_2),\, r > 0\}\ , \qquad A[\sigma_1, \sigma_2] := \{r\sigma : \sigma \in [\sigma_1, \sigma_2]\,,\, r \ge 0\}\ ,$$

where the order relation $\le$ between unit vectors is inherited from the orientation.

Similarly, the notions of *left* and *right limits* of sequences and functions defined on $\mathbb{S}^1$ are to be understood according to orientation, precisely, *right convergence* means clockwise convergence, and *left convergence* means counterclockwise.

If $\rho : \mathbb{S}^1 \to [0, \infty)$ is a given function, then for $\sigma \in \mathbb{S}^1$ we define

$$\rho^+(\sigma) := \sup\left\{\limsup_{n\to\infty} \rho(\sigma_n) : \sigma_n \to \sigma\,,\ \sigma_n \ne \sigma\right\}\ ,\quad \rho^-(\sigma) := \inf\left\{\liminf_{n\to\infty} \rho(\sigma_n) : \sigma_n \to \sigma\,,\ \sigma_n \ne \sigma\right\}\ .$$

Note that ρ^+ and ρ^- are upper and lower semicontinuous, respectively.

The *pointwise total variation* of ρ is defined by

$$\mathrm{pV}(\rho, \mathbb{S}^1) := \sup\left\{\sum_{i=0}^{n-1} |\rho(\sigma_{i+1}) - \rho(\sigma_i)| : \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1} < \sigma_n = \sigma_0\,,\ \sigma_i \in \mathbb{S}^1 \text{ for } i = 1, \ldots, n\right\},$$

and we say that the function ρ has *finite pointwise variation* if $\mathrm{pV}(\rho, \mathbb{S}^1)$ is finite.

If ρ has finite pointwise variation, then ρ has left and right limits at every $\sigma \in \mathbb{S}^1$, that we write $\rho(\sigma-)$ and $\rho(\sigma+)$ respectively, and $\rho^+(\sigma) = \max\{\rho(\sigma-), \rho(\sigma+)\}$, $\rho^-(\sigma) = \min\{\rho(\sigma-), \rho(\sigma+)\}$. In addition, the 2π-periodic function

$$\rho^*(\theta) := \rho(\sigma(\theta)) \tag{2.4}$$

then belongs to $BV_{\mathrm{loc}}(\mathbb{R})$, where

$$\sigma(\theta) := (\cos\theta, \sin\theta)\ , \tag{2.5}$$

and the functions $\rho^\pm(\sigma(\cdot)) : \mathbb{R} \to \mathbb{R}$ coincide with the the approximate upper and lower limits of ρ^* in the sense of Federer that we denote by $(\rho^*)^\pm$, respectively.

In the sequel it will be useful to consider *polar coordinates*, and to this purpose we introduce the map $\Psi : \mathbb{R} \times [0, \infty) \to \mathbb{R}^2$ given by

$$\Psi(\theta, r) := r\sigma(\theta) = r(\cos\theta, \sin\theta)\ .$$

If $S \subset \mathbb{R} \times [0, \infty)$ is a countably $\mathcal{H}^1$-rectifiable set, since Ψ is locally Lipschitz, then $\Psi(S)$ is also a countably $\mathcal{H}^1$-rectifiable set. Moreover, if $\Psi_{|S}$ is one-to-one and $f : \mathbb{R}^2 \to [0, \infty]$ is a Borel function, we have

$$\begin{aligned}\int_{\Psi(S)} f(z)\, d\mathcal{H}^1(z) &= \int_S f(\Psi(\theta,r))\, |\nabla_\tau \Psi(\theta,r)|\, d\mathcal{H}^1(\theta,r) \\ &= \int_S f(\Psi(\theta,r)) \sqrt{r^2\tau_1^2(\theta,r)+\tau_2^2(\theta,r)}\, d\mathcal{H}^1(\theta,r)\,,\end{aligned} \tag{2.6}$$

where $\tau = (\tau_1(\theta,r), \tau_2(\theta,r))$ is the approximate tangent unit vector to S for $\mathcal{H}^1$-a.e. $(\theta,r) \in S$. Indeed, the first equality follows from the area formula proved in [1, Theorem 2.91], and then we observe that the Jacobian of $\Psi_{|S}$ is given by $|\nabla_\tau \Psi|$ where $\nabla_\tau \Psi$ denotes the tangential gradient of Ψ along S, i.e.,

$$\nabla_\tau \Psi(\theta,r) = \tau\cdot\nabla\Psi(\theta,r) = (-r\tau_1\sin\theta+\tau_2\cos\theta, r\tau_1\cos\theta+\tau_2\sin\theta) = \tau_2\sigma(\theta) + r\tau_1(\sigma(\theta))^{\perp}\,. \tag{2.7}$$

2.2. Starshaped sets and radial functions.

Throughout the paper we consider $R_0 > 0$ fixed, and we set $B_0 := B_{R_0}(0) \subset \mathbb{R}^2$. We are interested in closed sets $F \subset \overline{B}_0$ starshaped with respect to the origin. For such a set, we can write

$$F = \left\{r\sigma \in \mathbb{R}^2 :\, \sigma \in \mathbb{S}^1\,,\ 0 \le r \le \rho_F(\sigma)\right\}\,,$$

where $\rho_F : \mathbb{S}^1 \to [0, R_0]$ is the *radial function* of F, that is,

$$\rho_F(\sigma) := \sup\left\{r \ge 0 :\ r\sigma \in F\right\}.$$

It may be shown that ρ_F is upper semicontinuous, and that the supremum in the definition of ρ_F is attained. Moreover, since ρ_F^+ is upper semicontinuous, the set

$$F^+ := \left\{r\sigma \in \mathbb{R}^2 :\, \sigma \in \mathbb{S}^1\,,\ 0 \le r \le \rho_F^+(\sigma)\right\} \tag{2.8}$$

is closed and starshaped with respect to the origin. In addition, $\rho_{F^+} = \rho_F^+$.

Given a closed set $F \subset \overline{B}_0$ starshaped with respect to the origin, in place of ρ_F we will often use the 2π-periodic function

$$\rho_F^*(\theta) := \rho_F(\sigma(\theta))\,, \tag{2.9}$$

where $\sigma(\theta)$ is defined in (2.5).

Lemma 2.2. *Let $F \subset \overline{B}_0$ be a closed set starshaped with respect to the origin. Set*

$$\Gamma := \left\{r\sigma :\, \sigma \in \mathbb{S}^1\,,\ \rho_F^-(\sigma) \le r \le \rho_F(\sigma)\right\}\,.$$

Then $\partial F = \Gamma$ is a connected set. In particular, ∂F is pathwise connected whenever $\mathcal{H}^1(\partial F) < \infty$.

Proof. We first prove that $\partial F \subset \Gamma$. Let $z \in \partial F$. If $z = 0$, then we claim that there exists σ such that $\rho_F^-(\sigma) = 0$, which implies that $0 \in \Gamma$. To prove the claim, assume by contradiction that $\rho_F^-(\sigma) > 0$ for all $\sigma \in \mathbb{S}^1$. Since ρ_F^- is lower semicontinuous, we have that $r_0 := \inf_{\sigma\in\mathbb{S}^1}\rho_F^- > 0$, and thus $B_{r_0}(0) \subset F$, which is a contradiction. If $z \ne 0$, we may write $z = r\sigma$ with $r > 0$ and $\sigma \in \mathbb{S}^1$. Let $r_n\sigma_n \notin F$ be such that $r_n \to r$ and $\sigma_n \to \sigma$, with $\sigma_n \ne \sigma$. We have

$$\rho_F^-(\sigma) \le \liminf_{n\to\infty}\rho_F(\sigma_n) \le \lim_{n\to\infty} r_n = r \le \rho_F(\sigma)\,.$$

Hence, $z \in \Gamma$, and we conclude that $\partial F \subset \Gamma$.

Since $\Gamma \subset F$, to show that $\Gamma \subset \partial F$, it is enough to prove that for every $r\sigma \in \Gamma$ there exists a sequence $r_n\sigma_n \notin F$ converging to $r\sigma$. Let $\sigma_n \to \sigma$, $\sigma_n \ne \sigma$, such that $\rho_F(\sigma_n) \to \rho_F^-(\sigma)$. Then the points $\sigma_n\left[\rho_F(\sigma_n) + \left(r - \rho_F^-(\sigma)\right) + \frac{1}{n}\right]$ do not belong to F and converge to $r\sigma$. Thus, $\Gamma = \partial F$.

To prove that Γ is connected, assume that U and V are two disjoint open sets such that $\Gamma \subset U \cup V$ and $\Gamma \cap U \ne \emptyset$. Without loss of generality, we may assume that $\rho_F(\sigma_0)\sigma_0 \in \Gamma\cap U$, where $\sigma_0 = (1,0)$. Set

$$\bar\theta := \sup\left\{\theta \in [0,2\pi) :\ \Gamma \cap A[\sigma_0, \sigma(\theta)] \subset U\right\}\,.$$

We claim that $\bar\theta = 2\pi$. Indeed, if $\bar\theta < 2\pi$, consider the segment

$$S_{\bar\theta} := \left\{ r\sigma(\bar\theta) : \rho_F^- \left(\sigma(\bar\theta)\right) \le r \le \rho_F \left(\sigma(\bar\theta)\right) \right\} .$$

Since $S_{\bar\theta}$ is connected and contained in Γ, we have that either $S_{\bar\theta} \subset U$ or $S_{\bar\theta} \subset V$. Assume first that $S_{\bar\theta} \subset U$ and let $\sigma_n = \sigma(\theta_n)$ be such that $\theta_n \to \bar\theta^+$ and $r_n \ge 0$, with $r_n\sigma_n \in \Gamma \cap V$. Since $\Gamma = \partial F$ is closed, up to a subsequence, we may assume that $r_n\sigma_n \to r\sigma(\bar\theta) \in S_{\bar\theta} \subset U$. Therefore for n sufficiently large we would get that $r_n\sigma_n \in U \cap V$, which is a contradiction. Taking into account the fact that $S_0 \subset U$, a similar argument ensures that $\bar\theta > 0$.

Finally, if $S_{\bar\theta} \subset V$, since $\bar\theta > 0$, there exist $\sigma_n = \sigma(\theta_n)$ such that $\theta_n \to \bar\theta^-$ and $r_n \ge 0$, with $r_n\sigma_n \in \Gamma \cap U$. As before, $r_n\sigma_n \to r\sigma(\bar\theta) \in S_{\bar\theta} \subset V$, and so $r_n\sigma_n \in V$ for all n large, which is again a contradiction. This shows that $\bar\theta = 2\pi$, so that $\Gamma \cap V = \emptyset$, thus proving that Γ is connected.

If $\mathcal{H}^1(\partial F) < \infty$, then the connectedness of ∂F implies that ∂F is pathwise connected by Theorem 4.46 in [24]. $\square$

Remark 2.3. Arguing as in the proof above, if $\mathcal{H}^1(\partial F) < \infty$, we also obtain that $\partial F \cap A[\sigma_1, \sigma_2]$ is pathwise connected for every $\sigma_1, \sigma_2 \in \mathbb{S}^1$.

Let us now define the class

$$\mathcal{A} := \left\{ F \subset \overline{B}_0 \text{ closed, starshaped with respect to the origin, and } \mathcal{H}^1(\partial F) < \infty \right\} .$$

We endow $\mathcal{A}$ with the topology induced by the Hausdorff distance $d_{\mathcal{H}}$. We recall that given two sets $A, B \subset \mathbb{R}^2$, the *Hausdorff distance* between A and B is defined by

$$d_{\mathcal{H}}(A, B) := \inf \left\{ \varepsilon > 0 : A \subset \mathscr{N}_\varepsilon(B) \text{ and } B \subset \mathscr{N}_\varepsilon(A) \right\} ,$$

where $\mathscr{N}_\varepsilon(C)$ denotes the ε–neighborhood of a set $C \subset \mathbb{R}^2$, i.e.,

$$\mathscr{N}_\varepsilon(C) := \left\{ z \in \mathbb{R}^2 : \operatorname{dist}(z, C) < \varepsilon \right\} .$$

In the sequel, we also consider the subfamily

$$\mathcal{A}_{\mathrm{Lip}} := \left\{ F \in \mathcal{A} : \rho_F \in \mathrm{Lip}(\mathbb{S}^1) \right\} . \tag{2.10}$$

Consider now a closed set $F \subset \overline{B}_0$ starshaped with respect to the origin. In Lemma 2.4 below we will prove that ρ_F has finite pointwise variation if and only if $\mathcal{H}^1(\partial F) < \infty$ (i.e., $F \in \mathcal{A}$). In this case, ρ_F has a left and right limit at every point $\sigma \in \mathbb{S}^1$ and, as mentioned in Subsection 2.1, the 2π-periodic function $\rho_F^*(\theta)$ defined in (2.9) belongs to $BV_{\mathrm{loc}}(\mathbb{R})$. Therefore, its distributional derivative $D\rho_F^*$ can be decomposed into three mutually singular measures,

$$D\rho_F^* = D^a\rho_F^* + D^c\rho_F^* + D^j\rho_F^* ,$$

where $D^a\rho_F^* =: (\rho_F^*)'\, d\theta$ stands for the absolutely continuous part of $D\rho_F^*$ with respect to the 1-dimensional Lebesgue measure on $\mathbb{R}$, $D^j\rho_F^*$ is the *jump part* or purely atomic part of $D\rho_F^*$, and $D^c\rho_F^*$ is the remaining part or *Cantor part* of $D\rho_F^*$. We denote by $D^s\rho_F^*$ the singular part of $D\rho_F^*$, i.e., $D^j\rho_F^* + D^c\rho_F^*$. In addition, it is well known that there is a $\mathcal{L}^1$-negligible (Borel) set $\Sigma_F^* \subset \mathbb{R}$ such that $D^s\rho_F^* = D\rho_F^* \lfloor \Sigma_F^*$.

Since ρ_F is upper semicontinuous, if σ is a point of discontinuity of ρ_F, then $\sigma \in J_F \cup S_F$, where

$$J_F := \left\{ \sigma \in \mathbb{S}^1 : \rho_F^-(\sigma) < \rho_F^+(\sigma) \right\} \quad \text{and} \quad S_F := \left\{ \sigma \in \mathbb{S}^1 : \rho_F^+(\sigma) < \rho_F(\sigma) \right\} . \tag{2.11}$$

Note that the sets J_F and S_F may not be disjoint and, in view of Lemma 2.2, ∂F can be decomposed as

$$\partial F = \Gamma_{\mathrm{cut}} \cup \Gamma_{\mathrm{jump}} \cup \Gamma_{\mathrm{reg}} , \tag{2.12}$$

with

$$\Gamma_{\rm cut} := \{ r\sigma : \sigma \in S_F ,\ \rho_F^+(\sigma) < r \le \rho_F(\sigma) \} ,$$

$$\Gamma_{\rm jump} := \{ r\sigma : \sigma \in J_F ,\ \rho_F^-(\sigma) < r < \rho_F^+(\sigma) \} ,$$

$$\Gamma_{\rm reg} := \partial F \setminus \big(\Gamma_{\rm jump} \cup \Gamma_{\rm cut}\big) .$$

In view of (2.9), if ρ_F has finite pointwise variation, then the sets J_F, S_F are countable. Also

$$J_F = \{ \sigma(\theta) : |D\rho_F^*|(\{\theta\}) > 0 \} ,$$

and we define

$$M_F := \{ \sigma(\theta) : \theta \in \Sigma_F^* \} \setminus J_F .$$

Finally, we denote by G_F^- the *subgraph* of ρ_F^*, i.e.,

$$G_F^- := \{ (\theta, r) \in \mathbb{R}^2 : r \le \rho_F^*(\theta) \} .$$

Recall that $\rho_F^* \in BV_{\rm loc}(\mathbb{R})$ if and only if G_F^- has locally finite perimeter in $\mathbb{R}^2$. The *extended graph* of ρ_F^* will be the set

$$G_F := \{ (\theta, r) \in \mathbb{R}^2 : (\rho_F^*)^-(\theta) \le r \le (\rho_F^*)^+(\theta) \} . \tag{2.13}$$

Lemma 2.4. *Let $F \subset \overline{B}_0$ be a closed set starshaped with respect to the origin. Then $\mathcal{H}^1(\partial F) < \infty$ if and only if ρ_F has finite pointwise variation. Moreover, in this case,*

$$\partial^* F = \partial^* F^+ \quad \textit{and} \quad \mathcal{H}^1(\partial F^+ \Delta \partial^* F^+) = 0 . \tag{2.14}$$

Proof. We start by proving that $\mathcal{H}^1(\partial F) < \infty$ implies that ρ_F has finite pointwise variation. To this purpose, it suffices to prove that for any distinct points $\sigma_1, \sigma_2 \in \mathbb{S}^1$, we have

$$|\rho_F(\sigma_1) - \rho_F(\sigma_2)| \le \mathcal{H}^1(\partial F \cap A) \tag{2.15}$$

where $A := A[\sigma_1, \sigma_2]$. The estimate above then yields

$$\mathrm{pV}\left(\rho_F, \mathbb{S}^1\right) \le 2\mathcal{H}^1(\partial F) .$$

To prove (2.15), denote by $P : \mathbb{R}^2 \to \mathbb{R}_+$ the function $P(z) = |z|$. Since P is Lipschitz continuous with Lipschitz constant equal to 1, we have

$$\mathcal{H}^1\big(P(\partial F \cap A)\big) \le \mathcal{H}^1(\partial F \cap A) .$$

Hence it suffices to prove that the interval $[\rho_F(\sigma_1), \rho_F(\sigma_2)] \subset P(\partial F \cap A)$, assuming, without loss of generality, that $\rho_F(\sigma_1) < \rho_F(\sigma_2)$. Indeed, given $\rho_F(\sigma_1) < r < \rho_F(\sigma_2)$, let $z := r\sigma_3$, where

$$\sigma_3 := \sup \{ \sigma : \sigma_1 \le \sigma \le \sigma_2 ,\ r\sigma \notin F \} .$$

Then z lies on $\partial F \cap A$.

Conversely, assume that ρ_F has finite pointwise variation. Then $\rho_F^* \in BV_{\rm loc}(\mathbb{R})$ and the extended graph G_F of ρ_F^* defined in (2.13) has locally finite $\mathcal{H}^1$-measure (see [11]). On the other hand, it can be checked that $\mathcal{H}^1(\Gamma_{\rm cut}) \le \mathrm{pV}\left(\rho_F, \mathbb{S}^1\right)$. Observe that Lemma 2.2 yields

$$\partial F^+ = \{ r\sigma \in \mathbb{R}^2 : \sigma \in \mathbb{S}^1 ,\ \rho_F^-(\sigma) \le r \le \rho_F^+(\sigma) \} = \Psi\big(G_F\big) . \tag{2.16}$$

In view of (2.12), we have

$$\partial F = \Gamma_{\rm cut} \cup \Psi\big(G_F\big) .$$

Since $G_F \subset \mathbb{R} \times [0, R_0]$ and Ψ is globally Lipschitz in $\mathbb{R} \times [0, R_0]$, we conclude that $\mathcal{H}^1(\partial F) < \infty$.

Finally, we prove (2.14) assuming that $\mathcal{H}^1(\partial F) < \infty$. Since $F\Delta F^+ = \Gamma_{\mathrm{cut}} \subset \partial F$ we have $\mathcal{H}^1(\Gamma_{\mathrm{cut}}) < \infty$, and then $|F\Delta F^+| = 0$. Hence $\partial^* F = \partial^* F^+$. Next, denote by $\partial^M F$ *the measure-theoretic boundary* of F, i.e., $\partial^M F := \mathbb{R}^2 \setminus \left(F^0 \cup F^1\right)$, and notice that $\partial^M F = \partial^M F^+$. By Theorem 3.61 in [1], $\partial^* F \subset \partial^M F$ and $\mathcal{H}^1(\partial^M F \setminus \partial^* F) = 0$. Therefore, to prove the assertion, it is enough to show

$$\mathcal{H}^1(\partial F^+ \Delta \partial^M F) = 0 . \tag{2.17}$$

We claim that

$$\partial^M F \setminus \{0\} = \Psi\left(\partial^M G_F^- \cap (\mathbb{R}\times(0,\infty))\right) . \tag{2.18}$$

Let us assume that the claim holds, and complete the proof of (2.17). Since $\partial^M G_F^- \subset \mathbb{R}\times[0,R_0]$, Ψ is Lipschitz in $\mathbb{R}\times[0,R_0]$, and $\Psi(\mathbb{R}\times\{0\}) = \{0\}$, we infer from (2.16),

$$\mathcal{H}^1(\partial F^+ \Delta \partial^M F) = \mathcal{H}^1(\Psi(G_F)\,\Delta\Psi(\partial^M G_F^-)) \le C\mathcal{H}^1(G_F \Delta \partial^M G_F^-) = C\mathcal{H}^1(G_F\Delta\partial^* G_F^-) ,$$

where the last equality follows from the fact that $\mathcal{H}^1(\partial^M G_F^- \setminus \partial^* G_F^-) = 0$ (see [1]). On the other hand, it follows from in [12, Theorem 4.5.9 (5)] that

$$\mathcal{H}^1\left(G_F \Delta \partial^* G_F^-\right) = 0 , \tag{2.19}$$

which would give (2.17).

It remains to prove the claim. Fix a point $r_0(\cos\theta_0, \sin\theta_0) = r_0\sigma_0 \in F^0\setminus\{0\}$. Since the map Ψ is a local diffeomorphism in $\mathbb{R}\times(0,\infty)$, and G_F^- is the subgraph of ρ_F^*, the area formula yields

$$\lim_{\varepsilon\to0^+}\frac{\left|G_F^-\cap B_\varepsilon((\theta_0,r_0))\right|}{\varepsilon^2} = \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^2}\int_{F^+\cap\Psi(B_\varepsilon((\theta_0,r_0)))}\left|J\Psi^{-1}(z)\right|\,dz = 0 ,$$

where the last equality follows from the assumption $r_0\sigma_0 \in F^0\setminus\{0\}$ (note that there exists $c > 0$ such that for all $\varepsilon > 0$ small enough, $\Psi(B_\varepsilon((\theta_0,r_0))) \subset B_{c\varepsilon}(r_0\sigma_0)$). This proves the inclusion $\Psi^{-1}\left(F^0\setminus\{0\}\right) \subset \left(G_F^-\right)^0 \cap (\mathbb{R}\times(0,\infty))$. The opposite one is proved in a similar way. The same argument yields $\Psi^{-1}\left(F^1\setminus\{0\}\right) = (G_F)^1\cap(\mathbb{R}\times(0,\infty))$, and (2.18) is proved. □

In the next two lemmas we relate the inner normal to $\partial^* F$ and the length of $\partial^* F$ to the derivative of ρ_F^*, extending well known formulas in the case of a smooth radial function.

Lemma 2.5. *Let $F \in \mathcal{A}$. Then for $\mathcal{H}^1$-a.e. $z = r\sigma(\theta) \in \partial^* F$, we have*

$$\nu_F^i(z) = \begin{cases} \dfrac{1}{\sqrt{(\rho_F^*)^2(\theta) + \left((\rho_F^*)'\right)^2(\theta)}}\left((\rho_F^*)'(\theta)\,(\sigma(\theta))^\perp - \rho_F^*(\theta)\sigma(\theta)\right) & \text{if } \sigma(\theta)\in\mathbb{S}^1\setminus(J_F\cup M_F) , \\[2ex] \dfrac{dD^s\rho_F^*}{d|D^s\rho_F^*|}(\theta)\;(\sigma(\theta))^\perp & \text{if } \sigma(\theta)\in J_F\cup M_F . \end{cases}$$

Proof. Since G_F^- is the subgraph of the BV_{loc} function ρ_F^*, using Theorems 3 and 4 in Section 1.5 of Chapter 4 in [22], we have that for $\mathcal{L}^1$-a.e. $\theta\in\mathbb{R}\setminus\Sigma_F^*$,

$$\nu^i_{G_F^-}(\theta,\rho_F^*(\theta)) = \frac{1}{\sqrt{\left((\rho_F^*)'\right)^2(\theta)+1}}\left((\rho_F^*)'(\theta), -1\right) , \tag{2.20}$$

while for $|D^c\rho_F^*|$-a.e. $\theta\in\mathbb{R}$ with $\sigma(\theta)\notin J_F$,

$$\nu^i_{G_F^-}(\theta,\rho_F^*(\theta)) = \left(\frac{dD^s\rho_F^*}{d|D^s\rho_F^*|}(\theta), 0\right) . \tag{2.21}$$

Finally for every $\theta\in\mathbb{R}$ such that $|D^j\rho_F^*|(\{\theta\}) > 0$, and every $r \in \left]\left(\rho_F^*\right)^-(\theta), \left(\rho_F^*\right)^+(\theta)\right[$, we have

$$\nu^i_{G_F^-}(\theta,r) = \left(\frac{dD^s\rho_F^*}{d|D^s\rho_F^*|}(\theta), 0\right) . \tag{2.22}$$

Since $\partial^* G_F^- \subset \partial^M G_F^- \subset \mathbb{R} \times [0, R_0]$, $\mathcal{H}^1(\partial^M F \setminus \partial^* F) = 0$, and $\mathcal{H}^1(\partial^M G_F^- \setminus \partial^* G_F^-) = 0$, we infer from (2.18) and the Lipschitz continuity of Ψ in $\mathbb{R} \times [0, R_0]$ that

$$\partial^* F \Delta \Psi \left(\partial^* G_F^- \cap (\mathbb{R} \times (0, \infty))\right) = E \,, \tag{2.23}$$

with $\mathcal{H}^1(E) = 0$.

From the proof of Theorem 2.90 in [1] it follows that for $\mathcal{H}^1$-a.e. $z = r\sigma(\theta) \in \partial^* F$, a counterclockwise oriented tangent vector to $\partial^* F$ at z is given by $\nabla_\tau \Psi(\theta, r)$, where $\tau = (\tau_1, \tau_2)$ is the unit tangent vector to $\partial^* G_F^-$ at (θ, r) given by $\left(\nu^i_{G_F^-}(\theta, r)\right)^\perp =: (-\nu_2, \nu_1)$. By (2.7),

$$\nabla_\tau \Psi(\theta, r) = \tau_2 \sigma(\theta) + \tau_1 r (\sigma(\theta))^\perp \,,$$

and so

$$\nu^i_F(z) = \frac{(\nabla_\tau \Psi(\theta, r))^\perp}{|\nabla_\tau \Psi(\theta, r)|} = \frac{\tau_2 (\sigma(\theta))^\perp - \tau_1 r \sigma(\theta)}{\sqrt{\tau_2^2 + (\tau_1 r)^2}} = \frac{\nu_1 (\sigma(\theta))^\perp + \nu_2 r \sigma(\theta)}{\sqrt{\nu_1^2 + (\nu_2 r)^2}} \,. \tag{2.24}$$

Set $\Pi : (\theta, r) \in \mathbb{R}^2 \mapsto \theta$ to be the projection on the θ-axis. Since the periodic function ρ_F^* belongs to $BV_{\mathrm{loc}}(\mathbb{R})$, we have (see e.g. [22], Chapter 4, Section 1.5, Theorem 1)

$$\Pi_\sharp \left(\mathcal{H}^1 \lfloor \partial^* G_F^-\right) = \sqrt{1 + ((\rho_F^*)')^2}\, d\theta + |D^s \rho_F^*| =: \mu \,, \tag{2.25}$$

i.e., $\mu(A) = \mathcal{H}^1 \lfloor \partial^* G_F^- (A \times \mathbb{R})$ for any Borel set $A \subset \mathbb{R}$. It follows that if $E \subset \mathbb{R}$ is such that $\mathcal{L}^1(E) = 0$ and $|D^s \rho_F^*|(E) = 0$, then

$$\mathcal{H}^1(\{(\theta, \rho_F^*(\theta)) : \theta \in E\}) = 0 \,. \tag{2.26}$$

Therefore the result follows from (2.20)-(2.22), (2.24), and (2.26). □

Remark 2.6. Note that in view of (2.7), for $\mathcal{L}^1$-a.e. $\theta \in \mathbb{R} \setminus \Sigma_F^*$,

$$\nabla_\tau \Psi(\theta, \rho_F^*(\theta)) = \frac{1}{\sqrt{((\rho_F^*)')^2(\theta) + 1}} \left((\rho_F^*)'(\theta)\sigma(\theta) + \rho_F^*(\theta)(\sigma(\theta))^\perp\right) \,, \tag{2.27}$$

while for $|D^c \rho_F^*|$-a.e. $\theta \in \mathbb{R}$ with $\sigma(\theta) \notin J_F$, then

$$\nabla_\tau \Psi(\theta, \rho_F^*(\theta)) = \frac{dD^s \rho_F^*}{d|D^s \rho_F^*|}(\theta)\, \sigma(\theta) \,. \tag{2.28}$$

Finally for any $\theta \in \mathbb{R}$ such that $|D\rho_F^*|(\{\theta\}) > 0$ and any $r \in \left]\, (\rho_F^*)^-(\theta), (\rho_F^*)^+(\theta) \right[$,

$$\nabla_\tau \Psi(\theta, r) = \frac{dD^s \rho_F^*}{d|D^s \rho_F^*|}(\theta)\, \sigma(\theta) \,. \tag{2.29}$$

Lemma 2.7. *For every $F \in \mathcal{A}$, we have*

$$\mathcal{H}^1(\partial^* F) = \int_0^{2\pi} \sqrt{(\rho_F^*)^2(\theta) + ((\rho_F^*)')^2(\theta)}\, d\theta + |D^s \rho_F^*|([0, 2\pi)) \,. \tag{2.30}$$

Proof. Set

$$\begin{aligned} S^a &:= \partial^* G_F^- \cap \left(([0, 2\pi) \setminus \Sigma_F^*) \times \mathbb{R}\right) , \\ S^s &:= \partial^* G_F^- \cap \left(([0, 2\pi) \cap \Sigma_F^*) \times \mathbb{R}\right) . \end{aligned} \tag{2.31}$$

In view of (2.23) the area formula (2.6) yields

$$\begin{aligned} \mathcal{H}^1(\partial^* F) &= \int_{\partial^* G_F^- \cap \{r > 0\}} |\nabla_\tau \Psi(\theta, r)|\, d\mathcal{H}^1(\theta, r) \\ &= \int_{S^a \cap \{r > 0\}} \frac{\sqrt{(\rho_F^*)^2(\theta) + ((\rho_F^*)')^2(\theta)}}{\sqrt{((\rho_F^*)')^2(\theta) + 1}}\, d\mathcal{H}^1(\theta, r) + \mathcal{H}^1\left(S^s \cap \{r > 0\}\right) , \end{aligned}$$

where we have used (2.27), (2.28), (2.29) in the last equality. Since $\mathcal{H}^1(S^s \cap \{r=0\}) = 0$, $(\rho_F^*)' = 0$ $\mathcal{L}^1$-a.e. in $\{\rho_F^* = 0\}$, and ρ_F^* is nonnegative, we infer that

$$\mathcal{H}^1(\partial^* F) = \int_{S^a} \frac{\sqrt{(\rho_F^*)^2(\theta) + \left((\rho_F^*)'\right)^2(\theta)}}{\sqrt{\left((\rho_F^*)'\right)^2(\theta) + 1}} \, d\mathcal{H}^1(\theta, r) + \mathcal{H}^1(S^s)\,,$$

which combined with (2.25) yields (2.30). □

We conclude this section with a compactness result for sequences of sets in $\mathcal{A}$ (note that any sequence $\{F_n\} \subset \mathcal{A}$ is relatively compact for the Hausdorff distance between compact sets by Blaschke's theorem, see Theorem 6.1 in [1]).

Lemma 2.8. *Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be such that $F_n \to F$ as $n \to \infty$ in the Hausdorff metric for some $F \subset \overline{B}_0$. Then F is closed and starshaped with respect to the origin. Moreover, if $\sup_n \mathcal{H}^1(\partial F_n) < \infty$, then $\mathcal{H}^1(\partial F) < \infty$ and*

(i) $\rho_F(\sigma) = \sup\left\{\limsup_{n\to\infty} \rho_{F_n}(\sigma_n) :\, \sigma_n \to \sigma\right\}$,

(ii) $\rho_{F_n}^* \to \rho_F^*$ *in* $L^1((0,2\pi))$, $|F_n \Delta F| \to 0$ *and* $D\chi_{F_n} \overset{*}{\rightharpoonup} D\chi_F$ *weakly* in the sense of measures.*

Proof. *Step 1.* The closedness of F is a consequence of Blaschke's theorem. To prove that F is starshaped with respect to the origin, we assume by contradiction that there exists $\sigma_0 \in \mathbb{S}^1$ and $r_0 \in (0, \rho_F(\sigma_0))$ such that $r_0\sigma_0$ does not belong to F. Since F is closed, there exists $B_\varepsilon(r_0\sigma_0) \subset \mathbb{R}^2 \backslash F$, and so, by Hausdorff convergence, $B_\varepsilon(r_0\sigma_0) \subset \mathbb{R}^2 \setminus F_n$ for all n sufficiently large.

Consider the smallest infinite cone C with vertex at the origin containing $B_\varepsilon(r_0\sigma_0)$. Note that the axis of the cone is the half-line $\{t\sigma_0 :\, t \geq 0\}$. By the definition of $\rho_F(\sigma_0)$ there exists $r > r_0 + \varepsilon$ such that $r\sigma_0 \in F$. Let $\delta > 0$ be such that $B_\delta(r\sigma_0) \subset C$. Let $z_n \in F_n$ be such that $z_n \to r\sigma_0$, and consider n so large that $z_n \in B_\delta(r\sigma_0)$. Since F_n is starshaped with respect to the origin, the segment joining z_n to the origin must be contained in F_n. However, this segment must intersect $B_\varepsilon(r_0\sigma_0)$ in a segment of positive length and this contradicts the fact that $B_\varepsilon(r_0\sigma_0) \subset \mathbb{R}^2 \setminus F_n$.

Step 2. We prove *(i)*. Let $\sigma_n \to \sigma$. Since $\rho_{F_n}(\sigma_n)\sigma_n \in F_n$ and $F_n \to F$ in the Hausdorff metric, we have that $(\limsup_{n\to\infty} \rho_{F_n}(\sigma_n))\,\sigma \in F$. This proves that

$$\rho_F(\sigma) \geq \sup\left\{\limsup_{n\to\infty} \rho_{F_n}(\sigma_n) :\, \sigma_n \to \sigma\right\}.$$

To show the opposite inequality, it is enough to consider the case in which $\rho_F(\sigma) > 0$. In this case, there exist $r_n\sigma_n \in F_n$ such that $r_n \to \rho_F(\sigma)$ and $\sigma_n \to \sigma$. Thus $\rho_F(\sigma) = \lim_{n\to\infty} r_n \leq \limsup_{n\to\infty} \rho_{F_n}(\sigma_n)$.

Step 3. Since $F_n \in \mathcal{A}$, we infer from (2.30) that

$$\left|D\rho_{F_n}^*\right|(0,2\pi) \leq \int_0^{2\pi} \sqrt{(\rho_{F_n}^*)^2 + \left((\rho_{F_n}^*)'\right)^2}\, d\theta + |D^s\rho_{F_n}^*|(0,2\pi) \leq \mathcal{H}^1(\partial F_n)\,,$$

so that the sequence $\left\{\rho_{F_n}^*\right\}$ is bounded in $BV((0,2\pi))$. Therefore, up to a subsequence (not relabeled), we may assume that $\rho_{F_n}^* \to \rho^*$ in $L^1((0,2\pi))$ and $\mathcal{L}^1$-a.e. in $(0,2\pi)$.

We claim that $\rho^* = \rho_F^*$ $\mathcal{L}^1$-a.e. in $(0,2\pi)$. Let $N_0 \subset (0,2\pi)$ be such that $\mathcal{L}^1(N_0) = 0$ and $\rho_{F_n}^*(\theta) \to \rho^*(\theta)$ for all $\theta \in (0,2\pi)\setminus N_0$. From *(i)* it follows that $\rho^*(\theta) \leq \rho_F^*(\theta)$ for all $\theta \in (0,2\pi)\setminus N_0$. Next we prove the opposite inequality. Up to a subsequence (not relabeled), there exists a compact set K such that $\partial F_n \to K$ in the Hausdorff metric. Since ∂F_n is connected, by Golab's theorem it follows that K is connected and

$$\mathcal{H}^1(K) \leq \liminf_{n\to\infty} \mathcal{H}^1(\partial F_n) < \infty\,. \tag{2.32}$$

We claim that $\partial F \subset K$. Indeed, assume that there exists $z \in \partial F \setminus K$. Then, for n large enough $B_\varepsilon(z) \cap \partial F_n = \emptyset$ for some $\varepsilon > 0$ independent of n. In other words, $B_\varepsilon(z) \subset \operatorname{int} F_n$ or $B_\varepsilon(z) \subset \mathbb{R}^2 \setminus F_n$

for n large. Since $F_n \to F$ in the Hausdorff metric, we deduce that $B_\varepsilon(z) \subset F$ or $B_\varepsilon(z) \subset \mathbb{R}^2 \setminus F$, which is impossible. Therefore (2.32) yields $\mathcal{H}^1(\partial F) < \infty$.

Fix $\sigma \in \mathbb{S}^1$ and set $K_\sigma := K \cap \{r\sigma : r > 0\}$. We claim that K_σ is connected. Indeed, if $r_1\sigma$, $r_2\sigma \in K_\sigma$, with $0 < r_1 < r_2$, and $r \in (r_1, r_2)$, then there exist two sequences $r_{i,n}\sigma_{i,n} \in \partial F_n$, $i = 1, 2$, such that $r_{i,n}\sigma_{i,n} \to r_i\sigma$ as $n \to \infty$. Up to a subsequence, we may assume that $\mathcal{H}^1((\sigma_{1,n}, \sigma_{2,n})) \to 0$ (the opposite case $\mathcal{H}^1((\sigma_{2,n}, \sigma_{1,n})) \to 0$ is analogous) and $r_{1,n} < r < r_{2,n}$ for all n. By Remark 2.3 $\partial F_n \cap A\,[\sigma_{1,n}, \sigma_{2,n}]$ is pathwise connected, and thus for every n there exists $\sigma_n \in [\sigma_{1,n}, \sigma_{2,n}]$ such that $r\sigma_n \in \partial F_n$. Using the fact that $r\sigma_n \to r\sigma$, we deduce that $r\sigma \in K$, thus proving that K_σ is connected.

Denote by $\tilde{N}_1$ the set of points $\sigma \in \mathbb{S}^1$ such that $\mathcal{H}^1(K_\sigma) > 0$. Then N_1 is at most countable since $\mathcal{H}^1(K) < \infty$. Moreover, since K_σ is connected and $\rho_F(\sigma)\sigma \in \partial F \subset K$ for every σ, we infer that $K_\sigma = \{\rho_F(\sigma)\,\sigma\}$ for all $\sigma \in \mathbb{S}^1 \setminus \tilde{N}_1$ such that $\rho_F(\sigma) > 0$. Consider $N_1 := \{\theta \in (0, 2\pi) : \sigma(\theta) \in \tilde{N}_1\}$. Then the set N_1 is at most countable.

Take $\theta \in (0, 2\pi) \setminus (N_0 \cup N_1)$. We claim that $\rho^*(\theta) \geq \rho^*_F(\theta)$. Indeed, assume that $\rho^*(\theta) < \rho^*_F(\theta)$. By *(i)* there exists $\theta_{n_k} \to \theta$ such that $\rho^*_{F_{n_k}}(\theta_{n_k}) \to \rho^*_F(\theta)$. Fix $r \in (\rho^*(\theta), \rho^*_F(\theta))$. Since $\theta \notin N_0$, $\rho^*_{F_n}(\theta) \to \rho^*(\theta)$. Hence, for all k large enough, $\rho^*_{F_{n_k}}(\theta) < r < \rho^*_{F_{n_k}}(\theta_{n_k})$. Note that $r\sigma(\theta_{n_k}) \in \partial F_{n_k}$ for finitely many k's. Indeed, if the opposite case were true we would conclude that $r\sigma(\theta) \in K$, which contradicts our assumption since $K_{\sigma(\theta)} = \{\rho^*_F(\theta)\,\sigma(\theta)\}$. Thus we may assume that for all k large enough, $r\sigma(\theta_{n_k}) \notin \partial F_{n_k}$. Since $r < \rho^*_{F_{n_k}}(\theta_{n_k})$, we deduce that $r\sigma(\theta_{n_k}) \in \operatorname{int} F_{n_k}$. On the other hand, since $r > \rho^*_{F_{n_k}}(\theta)$, we have $r\sigma(\theta) \notin F_{n_k}$. Using the fact that ∂F_{n_k} is connected, we conclude that there exist $\theta'_{n_k} \to \theta$ such that $r\sigma(\theta'_{n_k}) \in \partial F_{n_k}$, but this would imply that $r\sigma(\theta) \in K$, which again contradicts the fact that $K_{\sigma(\theta)} = \{\rho^*_F(\theta)\,\sigma(\theta)\}$. Hence we have shown that $\rho^*(\theta) = \rho^*_F(\theta)$ for all $\theta \in (0, 2\pi) \setminus (N_0 \cup N_1)$.

To prove that $|F_n \Delta F| \to 0$, it suffices to observe that

$$|F_n \Delta F| = |F_n \setminus F| + |F \setminus F_n| \leq R_0 \int_0^{2\pi} \left|\rho^*_{F_n}(\theta) - \rho^*_F(\theta)\right| d\theta \to 0\,.$$

Consequently, $\chi_{F_n} \to \chi_F$ in $L^1(\mathbb{R}^2)$. Since $\sup_n \mathcal{H}^1(\partial F_n) < \infty$, it follows from (2.14) that χ_{F_n} is bounded in $BV(\mathbb{R}^2)$, and thus $D\chi_{F_n} \overset{*}{\rightharpoonup} D\chi_F$ weakly* in the sense of measures. □

3. The minimization problem

Let us fix a Lipschitz map $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$. For every $F \in \mathcal{A}$, we set

$$\mathcal{C}(F) := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus F; \mathbb{R}^2) : \; u = u_0 \text{ a.e. in } \mathbb{R}^2 \setminus B_0 \right\}.$$

We define a class of admissible pairs *set-function* as

$$X := \left\{ (F, u) : \; F \in \mathcal{A}\,, \; u \in \mathcal{C}(F) \right\}$$

and its subspace (see (2.10))

$$X_{\mathrm{Lip}} := \left\{ (F, u) \in X : \; F \in \mathcal{A}_{\mathrm{Lip}} \right\}. \tag{3.1}$$

On the class X we shall consider the following notion of convergence motivated by Lemma 2.8.

Definition 3.1. *A sequence of pairs $\{(F_n, u_n)\}_{n\in\mathbb{N}} \subset X$ is said to converge to $(F, u) \in X$ as $n \to \infty$, and we write $(F_n, u_n) \xrightarrow{X} (F, u)$, if the following conditions hold:*

(i) $\sup_n \mathcal{H}^1(\partial F_n) < \infty$;

(ii) $F_n \to F$ *for the Hausdorff metric;*

(iii) $u_n \rightharpoonup u$ *weakly in $H^1(\omega; \mathbb{R}^2)$ for any bounded open set ω compactly contained in $\mathbb{R}^2 \setminus F$.*

Let us now consider a functional $\mathcal{F} : X_{\mathrm{Lip}} \to [0,\infty)$ defined by

$$\mathcal{F}(F,u) := \int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\,dz + \int_{\partial F} \varphi(\nu_F^i)\,d\mathcal{H}^1\,, \tag{3.2}$$

where $\mathbf{E}(u)$ is the symmetrized gradient, i.e.,

$$\mathbf{E}(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)\,.$$

Throughout the paper, we assume that

(H1) $\mathcal{W}(\mathbf{E}) = \mathbb{C}(\mathbf{E})\cdot\mathbf{E}$ for some constant positive definite fourth order tensor $\mathbb{C}$;

(H2) $\varphi : \mathbb{R}^2 \to [0,\infty)$ is Lipschitz continuous and positively 1-homogeneous.

Note that, by homogeneity, φ satisfies

$$m|z| \le \varphi(z) \le M|z| \tag{3.3}$$

for all $z \in \mathbb{R}^2$ and some positive constants m and M.

We are interested in minimizing the functional $\mathcal{F}$ over the class X_{Lip} under a volume constraint on the admissible sets. But we note that such minimization problem might be ill-posed since an arbitrary sequence in X_{Lip} with uniformly bounded energy is not precompact in X_{Lip}. However such sequences always admit a converging subsequence in X in the sense of Definition 3.1, thanks to Lemma 2.8 (see the proof of Theorem 3.3). To effectively address the minimization problem, we introduce the relaxed energy $\overline{\mathcal{F}} : X \to [0,\infty]$ defined by

$$\overline{\mathcal{F}}(F,u) := \inf\left\{ \liminf_{n\to\infty} \mathcal{F}(F_n,u_n) :\ (F_n,u_n) \in X_{\mathrm{Lip}}\,,\ (F_n,u_n) \xrightarrow{X} (F,u) \right\}.$$

The first main result of this paper is an integral representation of $\overline{\mathcal{F}}$ (see Theorem 3.2 below). Define the function $\Phi : \mathbb{S}^1\times\mathbb{R}\times\mathbb{R} \to (0,\infty)$ by

$$\Phi(\sigma,p,q) := \varphi\big(q\sigma^\perp - p\sigma\big)\,. \tag{3.4}$$

Note that if $\nu \in \mathbb{S}^1$ then

$$\varphi(\nu) = \Phi(\sigma, -\nu\cdot\sigma, \nu\cdot\sigma^\perp)$$

for *all* $\sigma \in \mathbb{S}^1$. We denote by $\overline{\Phi}$ the convexification of Φ with respect to the q–variable, i.e.,

$$\overline{\Phi}(\sigma,p,q) = \inf\left\{ \sum_{i=1}^{2} \eta_i \Phi(\sigma,p,q_i) :\ \eta_1,\eta_2 \in \mathbb{R}_+\,,\ \eta_1+\eta_2 = 1\,,\ q_1,q_2\in\mathbb{R}\,,\ \eta_1 q_1 + \eta_2 q_2 = q \right\}, \tag{3.5}$$

and if $(z,\nu) \in (\mathbb{R}^2\setminus\{0\})\times\mathbb{S}^1$, then we set

$$\mathcal{K}(z,\nu) := \overline{\Phi}\left(\frac{z}{|z|}, -\nu\cdot\frac{z}{|z|}, \nu\cdot\frac{z^\perp}{|z|}\right). \tag{3.6}$$

Observe that

$$\mathcal{K}(z,\nu) \le \Phi\left(\frac{z}{|z|}, -\nu\cdot\frac{z}{|z|}, \nu\cdot\frac{z^\perp}{|z|}\right) = \varphi(\nu) \tag{3.7}$$

for all $(z,\nu) \in (\mathbb{R}^2\setminus\{0\})\times\mathbb{S}^1$.

By (3.3),

$$0 \le \overline{\Phi}(\sigma,p,q) \le M\,(1+|q|)$$

for all $\sigma \in \mathbb{S}^1$, $p \in [-1,1]$, and every $q \in \mathbb{R}$. Hence, by Proposition 4.64 in [16],

$$\big|\overline{\Phi}(\sigma,p,q_1) - \overline{\Phi}(\sigma,p,q_2)\big| \le M\,|q_1 - q_2|$$

for all $\sigma \in \mathbb{S}^1$, $p \in [-1,1]$, and every q_1, $q_2 \in \mathbb{R}$. This, together with (9.1) and Lemma 9.1 in the Appendix, implies that $\mathcal{K}$ is continuous in $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}^1$.

The next two sections will be devoted to the proof of the following theorem.

Theorem 3.2. *Assume (H1)-(H2). Then*

$$\overline{\mathcal{F}}(F,u) = \int_{B_0 \setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\,dz + \int_{\partial^* F} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}} \tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1 \tag{3.8}$$

for every $(F,u) \in X$, where ν_F^i denotes a normal unit vector on $\Gamma_{\rm cut}$, and

$$\tilde{\mathcal{K}}(z,\nu_F^i) := \mathcal{K}(z,\nu_F^i) + \mathcal{K}(z,-\nu_F^i)\,. \tag{3.9}$$

A straightforward argument based on Theorem 3.2 and Lemma 2.8 yields the following existence result.

Theorem 3.3. *Assume (H1)-(H2). Then $\overline{\mathcal{F}}$ is lower semicontinuous with respect to the convergence introduced in Definition 3.1 and, given $0 < d < \pi R_0^2$, the constrained minimization problem*

$$\min\left\{\overline{\mathcal{F}}(F,u) :\ (F,u) \in X\,,\ |F| = d\right\} \tag{3.10}$$

admits at least one solution.

Proof. Let $\{(F_n,u_n)\} \subset X$ be such that $(F_n,u_n) \xrightarrow{X} (F,u)$. Without loss of generality, we may assume that

$$C := \liminf_{n\to\infty} \overline{\mathcal{F}}(F_n,u_n) = \lim_{n\to\infty} \overline{\mathcal{F}}(F_n,u_n) < \infty\,. \tag{3.11}$$

For every $n \in \mathbb{N}$ find $(F_m^{(n)}, u_m^{(n)}) \in X_{\rm Lip}$ such that $(F_m^{(n)}, u_m^{(n)}) \xrightarrow{X} (F_n,u_n)$ as $m \to \infty$, and

$$\sup_m \mathcal{F}(F_m^{(n)}, u_m^{(n)}) \le \overline{\mathcal{F}}(F_n,u_n) + \frac{1}{n}\,, \tag{3.12}$$

By (H2), (3.11), and (3.12), we have that

$$\sup_{n,m} \mathcal{H}^1(\partial F_m^{(n)}) < \infty\,, \qquad \sup_n \int_{B_0 \setminus F_m^{(n)}} |\mathbf{E}(u_{n,m})|^2\,dz < \infty\,. \tag{3.13}$$

Let $\{\omega_i\}$ be an increasing sequence of open sets compactly contained in $\mathbb{R}^2 \setminus F$ and such that

$$\mathbb{R}^2 \setminus F = \bigcup_{i=1}^{\infty} \omega_i\,. \tag{3.14}$$

Since

$$\lim_{n\to\infty} \lim_{m\to\infty} d_{\mathcal{H}}(F_m^{(n)}, F) = 0\,,$$

for every fixed $i \in \mathbb{N}$, we have that ω_i is compactly contained in $\mathbb{R}^2 \setminus F^{(n)}$ for all $n \ge \bar{n}_i$ and in turn for every $n \ge \bar{n}_i$, ω_i is compactly contained in $\mathbb{R}^2 \setminus F_m^{(n)}$ for all $m \ge \bar{m}_{i,n}$. Hence, we have that

$$\lim_{\substack{n\to\infty\\ n\ge \bar{n}_i}} \lim_{\substack{m\to\infty\\ m\ge \bar{m}_{i,n}}} \left\| u_m^{(n)} - u \right\|_{L^2(\omega_i;\mathbb{R}^2)} = 0\,.$$

Recursively, we construct two increasing sequences $\{n_i\}_i$ and $\{m_i\}_i$ with $n_i \ge \bar{n}_i$ and $m_i \ge \bar{m}_{i,n_i}$ such that

$$d_{\mathcal{H}}(F_{m_i}^{(n_i)}, F) + \left\| u_{m_i}^{(n_i)} - u \right\|_{L^2(\omega_i;\mathbb{R}^2)} \le \frac{1}{i}\,. \tag{3.15}$$

Set $v_i := u_{m_i}^{(n_i)}$ and $G_i := F_{m_i}^{(n_i)}$. We claim that $(G_i,v_i) \xrightarrow{X} (F,u)$. Indeed, properties (i) and (ii) in Definition 3.1 follow from (3.13) and (3.15). In order to establish (iii), let ω be a open set compactly contained in $\mathbb{R}^2 \setminus F$. Let $\omega \subset \widetilde{\omega} \subset\subset \mathbb{R}^2 \setminus F$, with $\widetilde{\omega}$ an open set with Lipschitz boundary and choose i_1 so large that $\widetilde{\omega} \subset \omega_i \cap \big(\mathbb{R}^2 \setminus G_i\big)$ for all $i \geq i_1$. Hence, for all $i \geq i_1$,

$$\|v_i - u\|_{L^2(\widetilde{\omega};\mathbb{R}^2)} \leq \|v_i - u\|_{L^2(\omega_i;\mathbb{R}^2)} \leq \frac{1}{i}$$

and by Korn's inequality and (3.13),

$$\sup_{i\geq i_1} \|v_i\|_{H^1(\widetilde{\omega};\mathbb{R}^2)} \leq C\,(\widetilde{\omega}, u_0) \sup_i \left(1 + \int_{B_0\setminus G_i} |\mathbf{E}(v_i)|^2\, dz\right) < \infty\,.$$

This proves the claim. Therefore,

$$\overline{\mathcal{F}}(F, u) \leq \liminf_{i\to\infty} \overline{\mathcal{F}}(G_i, v_i) \leq \lim_{i\to\infty} \overline{\mathcal{F}}(F_{n_i}, u_{n_i}) = C\,,$$

where we have used (3.11) and (3.12).

To prove the second part of the statement, let $\{(F_n, u_n)\} \subset X$ be a minimizing sequence. Since $F_n \subset \overline{B}_0$, by Blaschke's Theorem (see Theorem 6.1 in [1]), up to a subsequence, not relabeled, $F_n \to F$ in the Hausdorff metric for some set F. By Lemma 2.8, F is closed and starshaped with respect to the origin. Since $\sup_n \mathcal{H}^1(\partial F_n) < \infty$ by (H2), Lemma 2.8 yields $F \in \mathcal{A}$ and $|F| = d$.

Let $\{\omega_i\}$ be as in (3.14), with ω_i Lipschitz. Since

$$\lim_{n\to\infty} d_{\mathcal{H}}(F_n, F) = 0\,,$$

for every fixed $i \in \mathbb{N}$, we have that ω_i is compactly contained in $\mathbb{R}^2 \setminus F_n$ for all $n \geq n_i$, where $\{n_i\}_i$ is increasing. Recalling that $u_n = u_0$ in $\mathbb{R}^2 \setminus B_0$, since by (H1),

$$\sup_{n\geq n_i} \int_{B_0\cap\omega_i} |\mathbf{E}(u_n)|^2\, dz < \infty\,,$$

an application of Korn's inequality implies that $\{u_n\}_{n\geq n_i}$ is bounded in $H^1(\omega_i;\mathbb{R}^2)$. Hence, there exists a subsequence converging to some function $v_i \in H^1(\omega_i;\mathbb{R}^2)$. A standard diagonalization argument and the fact that $\{\omega_i\}$ is increasing yield the existence of a subsequence, not relabeled, of $\{u_n\}$ and of a function $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus F;\mathbb{R}^2)$ such that $u = v_i$ a.e. in ω_i for every i, and $u_n \rightharpoonup u$ weakly in $H^1(\omega;\mathbb{R}^2)$ for every bounded open set ω compactly included in $\mathbb{R}^2 \setminus F$. The conclusion follows the first part of the theorem. $\square$

Remark 3.4. Note that the formula (3.2) defining $\mathcal{F}$ actually makes sense for starshaped sets F with smooth (Lipschitz) boundary and for which ρ_F is not necessarily Lipschitz continuous. In other words, we could have defined (in a more natural way)

$$\overline{\mathcal{G}}(F, u) := \inf\left\{ \liminf_{n\to\infty} \mathcal{F}(F_n, u_n) :\ (F_n, u_n) \in X\,,\ \partial F_n \text{ Lipschitz}\,,\ (F_n, u_n) \xrightarrow{X} (F, u) \right\},$$

for $(F, u) \in X$, in place of $\overline{\mathcal{F}}$. It turns out that

$$\overline{\mathcal{F}} = \overline{\mathcal{G}}\,.$$

Indeed, it follows from the definitions of $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ that $\overline{\mathcal{G}}(F, u) \leq \overline{\mathcal{F}}(F, u)$ for every $(F, u) \in X$. To prove the opposite inequality, let $(F_n, u_n) \in X$ be such that ∂F_n is Lipschitz and $(F_n, u_n) \xrightarrow{X} (F, u)$. Since $\mathcal{K}(z, \nu) \leq \varphi(\nu)$, we have that $\overline{\mathcal{F}}(F_n, u_n) \leq \mathcal{F}(F_n, u_n)$, and using the lower semicontinuity of $\overline{\mathcal{F}}$ (see Theorem 3.3), we infer that

$$\overline{\mathcal{F}}(F, u) \leq \liminf_{n\to\infty} \overline{\mathcal{F}}(F_n, u_n) \leq \liminf_{n\to\infty} \mathcal{F}(F_n, u_n)\,.$$

Given the arbitrariness of $\{(F_n, u_n)\}$, we conclude that $\overline{\mathcal{F}}(F, u) \leq \overline{\mathcal{G}}(F, u)$.

4. Lower bound of the relaxed energy

The purpose of this section is to prove the lower bound in Theorem 3.2, precisely,

Theorem 4.1. *Assume (H1)-(H2). Then*

$$\overline{\mathcal{F}}(F,u) \geq \int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\,dz + \int_{\partial^* F} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\mathrm{cut}}} \tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1$$

for every $(F,u)\in X$*, where the functions* $\mathcal{K}$ *and* $\tilde{\mathcal{K}}$ *are given in* (3.6) *and* (3.9)*.*

To prove Theorem 4.1, we begin by studying the lower semicontinuous envelope of the surface energy with respect to the Hausdorff convergence of sets. More precisely, for $F\in\mathcal{A}$, we consider

$$\mathcal{J}(F) := \inf\left\{ \liminf_{n\to\infty} \int_{\partial F_n} \varphi(\nu_{F_n}^i)\,d\mathcal{H}^1 :\ F_n\in\mathcal{A}_{\mathrm{Lip}}\,,\ d_{\mathcal{H}}(F_n,F) \underset{n\to\infty}{\longrightarrow} 0 \right\}. \tag{4.1}$$

The key point for proving Theorem 4.1 is the following lower inequality on $\mathcal{J}(F)$.

Proposition 4.2. *Assume (H2). Then for every* $F\in\mathcal{A}$*,*

$$\mathcal{J}(F) \geq \int_{\partial^* F} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\mathrm{cut}}} \tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1\,. \tag{4.2}$$

We start with some preliminary results.

Lemma 4.3. *Assume (H2). Then for every* $F\in\mathcal{A}$*,*

$$\int_{\partial^* F} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 = \int_0^{2\pi} \overline{\Phi}\left(\sigma(\theta),\rho_F^*,(\rho_F^*)'\right)\,d\theta + \int_{[0,2\pi)} \overline{\Phi}\left(\sigma(\theta),0,\frac{dD^s\rho_F^*}{d|D^s\rho_F^*|}\right)\,d|D^s\rho_F^*|\,.$$

Proof. Consider the sets S^a and S^s given by (2.31). Arguing as in the proof of Lemma 2.7, the area formula yields

$$\int_{\partial^* F} \mathcal{K}(z,\nu_F^i(z))\,d\mathcal{H}^1(z) = \int_{(S^a\cup S^s)\cap\{r>0\}} \mathcal{K}\left(\Psi(\theta,r),\nu_F^i(\Psi(\theta,r))\right)|\nabla_\tau\Psi(\theta,r)|\,d\mathcal{H}^1(\theta,r)\,. \tag{4.3}$$

We split the integral on the right-hand side in two parts. Arguing again as in the proof of Lemma 2.7 and using Lemma 2.5, (2.27) and (2.25), we get

$$\begin{aligned}
&\int_{S^a\cap\{r>0\}} \mathcal{K}\left(\Psi(\theta,r),\nu_F^i(\Psi(\theta,r))\right)|\nabla_\tau\Psi(\theta,r)|\,d\mathcal{H}^1(\theta,r)\\
&= \int_{S^a\cap\{r>0\}} \overline{\Phi}\left(\sigma(\theta),-\sigma(\theta)\cdot\nu_F^i(\Psi(\theta,r)),\sigma^\perp(\theta)\cdot\nu_F^i(\Psi(\theta,r))\right)\frac{\sqrt{(\rho_F^*)^2(\theta)+\left((\rho_F^*)'\right)^2(\theta)}}{\sqrt{1+\left((\rho_F^*)'\right)^2(\theta)}}\,d\mathcal{H}^1(\theta,r)\\
&= \int_{S^a\cap\{r>0\}} \overline{\Phi}\left(\sigma(\theta),\frac{\rho_F^*}{\sqrt{(\rho_F^*)^2+\left((\rho_F^*)'\right)^2}},\frac{(\rho_F^*)'}{\sqrt{(\rho_F^*)^2+\left((\rho_F^*)'\right)^2}}\right)\frac{\sqrt{(\rho_F^*)^2+\left((\rho_F^*)'\right)^2}}{\sqrt{1+\left((\rho_F^*)'\right)^2}}\,d\mathcal{H}^1(\theta,r)\\
&= \int_0^{2\pi} \overline{\Phi}\left(\sigma(\theta),\rho_F^*,(\rho_F^*)'\right)\,d\theta\,,
\end{aligned}$$

where we have used the fact that $\overline{\Phi}(\sigma,\cdot,\cdot)$ is positively homogeneous of degree one.

Similarly, we infer from (2.28), (2.29) and (2.25) that

$$\begin{aligned}
\int_{S^s\cap\{r>0\}} \mathcal{K}\left(\Psi(\theta,r),\nu_F^i(\Psi(\theta,r))\right)|\nabla_\tau\Psi(\theta,r)|\,d\mathcal{H}^1(\theta,r) &= \int_{S^s} \overline{\Phi}\left(\sigma(\theta),0,\frac{dD^s\rho_F^*}{d|D^s\rho_F^*|}(\theta)\right)\,d\mathcal{H}^1(\theta,r)\\
&= \int_{[0,2\pi)} \overline{\Phi}\left(\sigma(\theta),0,\frac{dD^s\rho_F^*}{d|D^s\rho_F^*|}\right)\,d|D^s\rho_F^*|\,,
\end{aligned}$$

and the proof is complete. ◻

We shall also need the following (local) lower semicontinuity result for the surface energy.

Proposition 4.4. *Assume (H2). Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $F \in \mathcal{A}$ be such that $\rho^*_{F_n} \to \rho^*_F$ in $L^1((0,2\pi))$ as $n \to \infty$ and $\sup_n \mathcal{H}^1(\partial F_n) < \infty$. Then for every $\zeta \in C_c(\mathbb{R}^2)$ with $\zeta \geq 0$,*

$$\liminf_{n\to\infty} \int_{\partial^* F_n} \zeta(z)\mathcal{K}(z, \nu^i_{F_n}(z))\, d\mathcal{H}^1(z) \geq \int_{\partial^* F} \zeta(z)\mathcal{K}(z, \nu^i_F(z))\, d\mathcal{H}^1(z)\,.$$

In particular,

$$\liminf_{n\to\infty} \int_{\partial^* F_n} \mathcal{K}(z, \nu^i_{F_n}(z))\, d\mathcal{H}^1(z) \geq \int_{\partial^* F} \mathcal{K}(z, \nu^i_F(z))\, d\mathcal{H}^1(z)\,.$$

Proof. *Step 1.* Fix $\delta > 0$ and $\zeta \in C_c(\mathbb{R}^2)$ with $\zeta \geq 0$. Given $\rho \in BV_{\rm loc}(\mathbb{R})$ and a bounded interval $I \subset \mathbb{R}$, we define

$$\mathcal{G}(\rho, I) := \int_I g(\theta, \rho, \rho')\, d\theta + \int_I g^\infty\left(\theta, \rho, \frac{dD^c\rho}{d|D^c\rho|}\right) d|D^c\rho| + \int_I \left(\fint_{\rho^-(\theta)}^{\rho^+(\theta)} g^\infty\left(\theta, r, \frac{dD^j\rho}{d|D^j\rho|}\right) dr\right) d|D^j\rho|\,,$$

where (see (3.5))

$$g(\theta, p, q) := \big(\zeta(p\sigma(\theta)) + \delta\big)\overline{\Phi}(\sigma(\theta), p, q)\,,$$

and

$$\begin{aligned} g^\infty(\sigma(\theta), p, q) &:= \lim_{t\to+\infty} \frac{g(\theta, p, tq)}{t} \\ &= \lim_{t\to+\infty} \big(\zeta(p\sigma(\theta)) + \delta\big)\overline{\Phi}(\sigma(\theta), p/t, q) = \big(\zeta(p\sigma(\theta)) + \delta\big)\overline{\Phi}(\sigma(\theta), 0, q)\,, \end{aligned} \quad (4.4)$$

since $\overline{\Phi}(\sigma(\theta), \cdot, \cdot)$ is positively homogeneous of degree one.

We claim that for every $F \in \mathcal{A}$,

$$\int_{\partial^* F} (\zeta(z) + \delta)\mathcal{K}(z, \nu^i_F(z))\, d\mathcal{H}^1(z) = \mathcal{G}\big(\rho^*_F, [0, 2\pi)\big)\,. \quad (4.5)$$

Indeed, consider the sets S^a and S^s given by (2.31), and write $S^s = S^c \cup S^j$ with

$$S^c := S^s \cap (\{\theta \in [0, 2\pi) : \sigma(\theta) \notin J_F\} \times \mathbb{R})\,, \quad S^j := S^s \cap (\{\theta \in [0, 2\pi) : \sigma(\theta) \in J_F\} \times \mathbb{R})\,.$$

As in (4.3), we have

$$\begin{aligned} &\int_{\partial^* F} (\zeta(z) + \delta)\mathcal{K}(z, \nu^i_F(z))\, d\mathcal{H}^1(z) \\ &\qquad = \int_{S\cap\{r>0\}} \big(\zeta(\Psi(\theta, r)) + \delta\big)\mathcal{K}\left(\Psi(\theta, r), \nu^i_F(\Psi(\theta, r))\right) |\nabla_\tau \Psi(\theta, r)|\, d\mathcal{H}^1(\theta, r)\,, \end{aligned}$$

where $S = S^a \cup S^s$, and we split the integral in the right hand-side in two parts. Arguing exactly as in the proof of Lemma 4.3 and using (4.4), we first obtain

$$\begin{aligned} &\int_{(S^a\cup S^c)\cap\{r>0\}} \big(\zeta(\Psi(\theta, r)) + \delta\big)\mathcal{K}\left(\Psi(\theta, r), \nu^i_F(\Psi(\theta, r))\right) |\nabla_\tau \Psi(\theta, r)|\, d\mathcal{H}^1(\theta, r) \\ &\qquad = \int_0^{2\pi} g(\theta, \rho^*_F, (\rho^*_F)')\, d\theta + \int_{[0,2\pi)} g^\infty\left(\theta, \rho^*_F, \frac{dD^c\rho^*_F}{d|D^c\rho^*_F|}\right) d|D^c\rho^*_F|\,. \end{aligned}$$

On the other hand, we have by (2.19),

$$\int_{S^j\cap\{r>0\}} \big(\zeta(\Psi(\theta,r))+\delta\big)\mathcal{K}\left(\Psi\left(\theta,r\right),\nu_F^i\left(\Psi\left(\theta,r\right)\right)\right)\left|\nabla_\tau\Psi\left(\theta,r\right)\right|\,d\mathcal{H}^1(\theta,r)$$

$$= \int_{S^j\cap\{r>0\}} \big(\zeta(r\sigma(\theta))+\delta\big)\overline{\Phi}\left(\sigma\left(\theta\right),0,\frac{dD^j\rho_F^*}{d|D^j\rho_F^*|}\left(\theta\right)\right)\,d\mathcal{H}^1(\theta,r)$$

$$= \sum_{\{\theta\in[0,2\pi):\,\sigma(\theta)\in J_F\}}\int_{G_F\cap(\{\theta\}\times\mathbb{R})} \big(\zeta(r\sigma(\theta))+\delta\big)\overline{\Phi}\left(\sigma\left(\theta\right),0,\frac{dD^j\rho_F^*}{d|D^j\rho_F^*|}\left(\theta\right)\right)\,d\mathcal{H}^1(\theta,r)$$

$$= \sum_{\{\theta\in[0,2\pi):\,\sigma(\theta)\in J_F\}}\int_{(\rho_F^*)^-(\theta)}^{(\rho_F^*)^+(\theta)} g^\infty\left(\theta,r,\frac{dD^j\rho_F^*}{d|D^j\rho_F^*|}\right)\,dr$$

$$= \int_{[0,2\pi)}\left(\fint_{(\rho_F^*)^-(\theta)}^{(\rho_F^*)^+(\theta)} g^\infty\left(\theta,r,\frac{dD^j\rho_F^*}{d|D^j\rho_F^*|}\right)\,dr\right)\,d|D^j\rho_F^*|\,,$$

and (4.5) follows.

Step 2. Without loss of generality, we may assume that $\mathcal{H}^1(\partial^*F\cap\{(x,0):\,x\geq 0\})=0$. By (4.5) we have that

$$\int_{\partial^*F}(\zeta(z)+\delta)\mathcal{K}(z,\nu_F^i(z))\,d\mathcal{H}^1(z)=\mathcal{G}\big(\rho_F^*,(0,2\pi)\big)\,,$$

and for all n,

$$\int_{\partial^*F_n}(\zeta(z)+\delta)\mathcal{K}(z,\nu_{F_n}^i)\,d\mathcal{H}^1(z)\geq\mathcal{G}\big(\rho_{F_n}^*,(0,2\pi)\big)\,.$$

In view of Lemma 9.1 in the Appendix, and the fact that $g(\theta,p,q)\geq\delta m\,|q|$, it follows that g satisfies the hypotheses of Theorem 3.1 in [10], and thus $\mathcal{G}(\cdot,(0,2\pi))$ is lower semicontinuous with respect to convergence in $L^1\left((0,2\pi)\right)$. Therefore,

$$\liminf_{n\to\infty}\int_{\partial^*F_n}(\zeta(z)+\delta)\mathcal{K}(z,\nu_{F_n}^i(z))\,d\mathcal{H}^1(z)\geq\int_{\partial^*F}\zeta(z)\mathcal{K}(z,\nu_F^i(z))\,d\mathcal{H}^1(z)\,. \tag{4.6}$$

Since $\sup_n\mathcal{H}^1(\partial F_n)<\infty$, we have

$$\sup_n\int_{\partial^*F_n}\mathcal{K}(z,\nu_{F_n}^i(z))\,d\mathcal{H}^1(z)\leq C<\infty\,.$$

Hence (4.6) yields

$$\int_{\partial^*F}\zeta(z)\mathcal{K}(z,\nu_F^i(z))\,d\mathcal{H}^1(z)\leq\liminf_{n\to\infty}\int_{\partial^*F_n}\zeta(z)\mathcal{K}(z,\nu_{F_n}^i(z))\,d\mathcal{H}^1(z)+C\delta\,,$$

and the conclusion follows from the arbitrariness of δ. $\square$

Proof of Proposition 4.2. *Step 1.* Let $\{F_n\}_{n\in\mathbb{N}}\subset\mathcal{A}_{\mathrm{Lip}}$ be such that $F_n\to F$ as $n\to\infty$ in the Hausdorff metric. Without loss of generality, we may assume that

$$\liminf_{n\to\infty}\int_{\partial F_n}\varphi(\nu_n^i)\,d\mathcal{H}^1=\lim_{n\to\infty}\int_{\partial F_n}\varphi(\nu_n^i)\,d\mathcal{H}^1<\infty\,,$$

where $\nu_n^i:=\nu_{F_n}^i$ for all n. Since $\mathcal{K}(z,\nu)\leq\varphi(\nu)$ by (3.7) and $\mathcal{K}(z,\nu)\geq m>0$ for every $(z,\nu)\in\mathbb{R}^2\setminus\{0\}\times\mathbb{S}^1$, we have

$$\sup_{n\in\mathbb{N}}\int_{\partial F_n}\mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1<\infty\quad\text{and}\quad\sup_{n\in\mathbb{N}}\mathcal{H}^1(\partial F_n)<\infty\,.$$

Extracting a subsequence (not relabeled), we find a nonnegative Radon measure μ such that

$$\mu_n:=\mathcal{K}(z,\nu_n^i(z))\mathcal{H}^1\lfloor\partial F_n\overset{*}{\rightharpoonup}\mu\quad\text{as }n\to\infty\,,$$

weakly* in the sense of measures. Since $\partial F^+ \cap \Gamma_{\rm cut} = \emptyset$ by (2.8), by Lemma 2.4 we have that $\partial^* F \cap \Gamma_{\rm cut} = \emptyset$. Hence, the measures $\mathcal{H}^1 \lfloor \Gamma_{\rm cut}$ and $\mathcal{H}^1 \lfloor \partial^* F$ are mutually singular and to prove (4.2), it suffices to show that

$$\frac{d\mu}{d\mathcal{H}^1 \lfloor \Gamma_{\rm cut}}(z_0) \geq \tilde{\mathcal{K}}(z_0, \nu_F^i(z_0)) \quad \text{for } \mathcal{H}^1\text{-a.e. } z_0 \in \Gamma_{\rm cut}\,, \tag{4.7}$$

and

$$\frac{d\mu}{d\mathcal{H}^1 \lfloor \partial^* F}(z_0) \geq \mathcal{K}(z_0, \nu_F^i(z_0)) \quad \text{for } \mathcal{H}^1\text{-a.e. } z_0 \in \partial^* F\,. \tag{4.8}$$

Step 2. By the Besicovitch derivation theorem (see, e.g. Theorem 1.153 in [16]), we have

$$\frac{d\mu}{d\mathcal{H}^1 \lfloor \Gamma_{\rm cut}}(z_0) = \lim_{\varepsilon \to 0^+} \frac{\mu\big(Q_{\nu_0}(z_0, \varepsilon)\big)}{2\varepsilon} \quad \text{for } \mathcal{H}^1\text{-a.e. } z_0 \in \Gamma_{\rm cut}\,, \tag{4.9}$$

where $\nu_0 := (z_0/|z_0|)^\perp$ and $Q_{\nu_0}(z_0,\varepsilon)$ is the square of side length 2ε, centered at z_0 with two sides parallel to ν_0. Observe that (2.12) implies

$$\rho_F^+(\sigma_0) < |z_0| < \rho_F(\sigma_0) \quad \text{for } \mathcal{H}^1\text{-a.e. } z_0 \in \Gamma_{\rm cut}\,, \tag{4.10}$$

where $\sigma_0 := z_0/|z_0|$, because S_F is at most countable (see (2.11)). Now fix $z_0 \in \Gamma_{\rm cut}$ such that (4.9) and (4.10) hold. We may assume without loss of generality that $\sigma_0 = (0,1)$ and $\nu_0 = (-1,0)$. Then we write $z_0 = r_0\sigma_0$ and $Q_{\nu_0}(z_0,\varepsilon) = Q(z_0,\varepsilon)$. We claim that that there exists $\varepsilon_0 > 0$ such that

$$F \cap Q(z_0,\varepsilon) = \{0\} \times (r_0 - \varepsilon, r_0 + \varepsilon) \quad \text{for every } 0 < \varepsilon \leq \varepsilon_0\,. \tag{4.11}$$

Indeed, consider the function $\tilde\rho_F : \mathbb{S}^1 \to \mathbb{R}_+$ defined by $\tilde\rho_F(\sigma) = \rho_F(\sigma)$ if $\sigma \neq \sigma_0$ and $\tilde\rho_F(\sigma_0) = \rho_F^+(\sigma_0)$. Then $\tilde\rho_F$ is upper semicontinuous. Hence, the set $\tilde F = \big\{r\sigma \,:\, \sigma \in \mathbb{S}^1\,,\, 0 \leq r \leq \tilde\rho_F(\sigma)\big\}$ is closed in $\mathbb{R}^2$. Since $z_0 \notin \tilde F$ by (4.10), there exists $\varepsilon_0 > 0$ such that $\tilde F \cap Q(z_0,\varepsilon_0) = \emptyset$, and so (4.11) follows because $F = \tilde F \cup \big(\{0\} \times (\rho_F^+(\sigma_0), \rho_F(\sigma_0)]\big)$.

Next we choose a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0^+$, $\varepsilon_k \ll \varepsilon_0$, and $\mu\big(\partial Q(z_0,\varepsilon_k)\big) = 0$ for every $k \in \mathbb{N}$. Then

$$\frac{d\mu}{d\mathcal{H}^1 \lfloor \Gamma_{\rm cut}}(z_0) = \lim_{k\to\infty} \frac{\mu\big(Q(z_0,\varepsilon_k)\big)}{2\varepsilon_k} = \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{2\varepsilon_k}\int_{\partial F_n \cap \overline{Q}(z_0,\varepsilon_k)} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1\,. \tag{4.12}$$

Since $F_n \to F$ in the Hausdorff sense, there exists $n_k \in \mathbb{N}$ such that $F_n \subset \mathscr{N}_{\varepsilon_k/2}(F)$ for every $n \geq n_k$. By (4.11) we have

$$\begin{aligned}\mathscr{N}_{\varepsilon_k/2}(F) \cap Q(z_0,\varepsilon_k) &= \mathscr{N}_{\varepsilon_k/2}(F \cap Q(z_0,\varepsilon_0)) \cap Q(z_0,\varepsilon_k)\\ &= (-\varepsilon_k/2, \varepsilon_k/2) \times (r_0 - \varepsilon_k, r_0 + \varepsilon_k)\end{aligned}$$

for ε_k small enough (see Figure 1). Therefore

$$F_n \cap Q(z_0,\varepsilon_k) \subset (-\varepsilon_k/2, \varepsilon_k/2) \times (r_0 - \varepsilon_k, r_0 + \varepsilon_k) \tag{4.13}$$

for ε_k small enough and $n \geq n_k$. Set

$$p_k := (\varepsilon_k/2, r_0 - \varepsilon_k)\,, \quad q_k := (-\varepsilon_k/2, r_0 - \varepsilon_k)\,, \quad \sigma_k^- := \frac{p_k}{|p_k|}\,, \quad \sigma_k^+ := \frac{q_k}{|q_k|}\,,$$

and note that, in view of (4.13), $\rho_n(\sigma_k^-) \leq |p_k|$ and $\rho_n(\sigma_k^+) \leq |q_k|$ with $\rho_n := \rho_{F_n}$. Denoting by Π_2 the projection $z = (x,y) \mapsto y$, we deduce that

$$\Pi_2\big(\rho_n(\sigma_k^-)\sigma_k^-\big) \leq r_0 - \varepsilon_k \quad \text{and} \quad \Pi_2\big(\rho_n(\sigma_k^+)\sigma_k^+\big) \leq r_0 - \varepsilon_k\,. \tag{4.14}$$

Now we fix some $0 < \delta \ll 1/2$ and we consider $z_k = (0, r_0 + (1-\delta)\varepsilon_k) \in \Gamma_{\rm cut} \cap Q(z_0,\varepsilon_k)$. Since $F_n \to F$, for n large enough, we may find $z_{n,k} \in F_n \cap B_{\delta\varepsilon_k}(z_k)$. Setting $\sigma_{n,k} := z_{n,k}/|z_{n,k}|$, we have

$$\sigma_k^- < \sigma_{n,k} < \sigma_k^+ \quad \text{and} \quad \Pi_2\big(\rho_n(\sigma_{n,k})\sigma_{n,k}\big) \geq r_0 + (1-2\delta)\varepsilon_k\,. \tag{4.15}$$

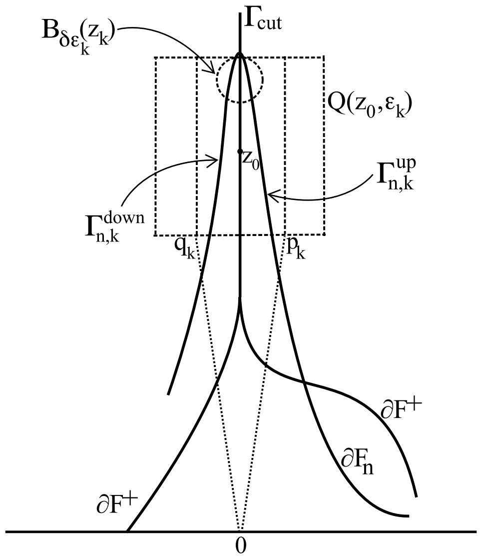

Fig. 1. The construction described in the proof of Proposition 4.2.

Consider the Lipschitz continuous scalar function H_n defined on $\mathbb{S}^1$ by $H_n(\sigma) := \Pi_2\big(\rho_n(\sigma)\sigma\big)$. By (4.14) and (4.15), we have that $[r_0-\varepsilon_k, r_0+(1-2\delta)\varepsilon_k] \subset H_n([\sigma_k^-, \sigma_{n,k}])$.

Therefore, there exists at least one arc $[\sigma^1_{n,k}, \sigma^2_{n,k}] \subset [\sigma_k^-, \sigma_{n,k}]$ with $\sigma^1_{n,k} < \sigma^2_{n,k}$ such that $H_n([\sigma^1_{n,k}, \sigma^2_{n,k}]) = [r_0-\varepsilon_k, r_0+(1-2\delta)\varepsilon_k]$, $H_n(\sigma^1_{n,k}) = r_0-\varepsilon_k$, and $H_n(\sigma^2_{n,k}) = r_0+(1-2\delta)\varepsilon_k$. By construction, it follows that

$$\Gamma^{up}_{n,k} := \big\{\rho_n(\sigma)\sigma :\ \sigma^1_{n,k} \le \sigma \le \sigma^2_{n,k}\big\} \subset \partial F_n \cap \overline{Q}(z_0,\varepsilon_k)\,.$$

Arguing in the same way, we find an arc $[\sigma^3_{n,k}, \sigma^4_{n,k}] \subset [\sigma_{n,k}, \sigma_k^+]$ with $\sigma^3_{n,k} < \sigma^4_{n,k}$ such that $H_n(\sigma^3_{n,k}) = r_0+(1-2\delta)\varepsilon_k$, $H_n(\sigma^4_{n,k}) = r_0-\varepsilon_k$, and

$$\Gamma^{down}_{n,k} := \big\{\rho_n(\sigma)\sigma :\ \sigma^3_{n,k} \le \sigma \le \sigma^4_{n,k}\big\} \subset \partial F_n \cap \overline{Q}(z_0,\varepsilon_k)\,.$$

From the construction of $\Gamma^{up}_{n,k}$ and $\Gamma^{down}_{n,k}$, we infer that

$$\int_{\partial F_n \cap \overline{Q}(z_0,\varepsilon_k)} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 \ge \int_{\Gamma^{up}_{n,k}} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 + \int_{\Gamma^{down}_{n,k}} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1\,,$$

and consequently,

$$\lim_{n\to\infty} \int_{\partial F_n \cap \overline{Q}(x_0,\varepsilon_k)} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 \ge \liminf_{n\to\infty} \int_{\Gamma^{up}_{n,k}} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 + \liminf_{n\to\infty} \int_{\Gamma^{down}_{n,k}} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1\,. \tag{4.16}$$

Now we claim that

$$\liminf_{n\to\infty} \int_{\Gamma^{up}_{n,k}} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 \ge \int_{\Gamma_{\star,k}} \mathcal{K}(z,\nu_0)\,d\mathcal{H}^1\,, \tag{4.17}$$

and

$$\liminf_{n\to\infty} \int_{\Gamma^{down}_{n,k}} \mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 \ge \int_{\Gamma_{\star,k}} \mathcal{K}(z,-\nu_0)\,d\mathcal{H}^1\,, \tag{4.18}$$

where $\Gamma_{\star,k} := \{r\sigma_0 \in \mathbb{R}^2\,,\ r_0-\varepsilon_k \le r \le r_0+(1-2\delta)\varepsilon_k\}$. Before proving (4.17) and (4.18), we complete the proof of (4.7). Since K is 0-homogeneous with respect to the z-variable, we have

$$\int_{\Gamma_{\star,k}} \mathcal{K}(z,\nu_0)\,d\mathcal{H}^1 = \mathcal{K}(z_0,\nu_0)\mathcal{H}^1(\Gamma_{\star,k}) = 2\varepsilon_k(1-\delta)\mathcal{K}(z_0,\nu_0)\,,$$

and

$$\int_{\Gamma_{\star,k}} \mathcal{K}(z,-\nu_0)\,d\mathcal{H}^1 = \mathcal{K}(z_0,-\nu_0)\mathcal{H}^1(\Gamma_{\star,k}) = 2\varepsilon_k(1-\delta)\mathcal{K}(z_0,-\nu_0)\,,$$

so that (4.12) , (4.16) , (4.17), and (4.18) lead to

$$\frac{d\mu}{d\mathcal{H}^1\lfloor\Gamma_{\rm cut}}(z_0) \geq (1-\delta)\tilde{\mathcal{K}}(z_0,\nu_0)\,.$$

Then the conclusion follows from the arbitrariness of δ.

Proof of (4.17)-(4.18). We only present the proof of (4.17) since the proof of (4.18) is similar. Observe first that, by construction and by the convergence of F_n to F in the Hausdorff metric, we have

$$\Gamma^{\rm up}_{n,k} \to \Gamma_{\star,k} \quad \text{in the Hausdorff metric as } n\to\infty\,,$$

and

$$\sigma^i_{n,k}\to\sigma_0 \text{ for } i=1,2\,,\;\; \rho_n(\sigma^1_{n,k})\to r_0-\varepsilon_k \text{ and } \rho_n(\sigma^2_{n,k})\to r_0+(1-2\delta)\varepsilon_k \text{ as } n\to\infty\,. \tag{4.19}$$

Next we construct a test function $\hat\rho_n\in \mathrm{Lip}(\mathbb{S}^1)$ in the following way. Write $\sigma^i_{n,k}=\sigma(\theta^i_{n,k})$ for $i=1,2$ with $\theta^1_{n,k}\in(0,\pi)$ and $\theta^2_{n,k}\in(\theta^1_{n,k},2\pi)$. Note that in view of (4.19),

$$\theta^1_{n,k}\to\pi/2 \quad\text{and}\quad \theta^2_{n,k}\to\pi/2 \quad \text{as } n\to\infty\,. \tag{4.20}$$

Set (see (2.4))

$$\hat\rho^*_n(\theta) := \begin{cases} \rho^*_n(\theta^1_{n,k})\dfrac{\theta}{\theta^1_{n,k}} + \dfrac{R_0}{2}\dfrac{\theta^1_{n,k}-\theta}{\theta^1_{n,k}} & \text{if } \theta\in[0,\theta^1_{n,k})\,,\\ \rho^*_n(\theta) & \text{if } \theta\in[\theta^1_{n,k},\theta^2_{n,k}]\,,\\ \rho^*_n(\theta^2_{n,k})\dfrac{2\pi-\theta}{2\pi-\theta^2_{n,k}} + \dfrac{R_0}{2}\dfrac{\theta-\theta^2_{n,k}}{2\pi-\theta^2_{n,k}} & \text{if } \theta\in(\theta^2_{n,k},2\pi]\,. \end{cases}$$

By (4.19) and (4.20), we have that $\hat\rho^*_n\to\hat\rho^*$ in $L^1((0,2\pi))$, where

$$\hat\rho^*(\theta) := \begin{cases} (r_0-\varepsilon_k)\dfrac{2\theta}{\pi} + \dfrac{R_0}{2}\dfrac{\pi-2\theta}{\pi} & \text{if } \theta\in[0,\pi/2)\,,\\ (r_0+(1-2\delta)\varepsilon_k)\dfrac{4\pi-2\theta}{3\pi} + \dfrac{R_0}{2}\dfrac{2\theta-\pi}{3\pi} & \text{if } \theta\in[\pi/2,2\pi]\,. \end{cases}$$

Setting $\hat F_n\in\mathcal{A}_{\rm Lip}$ and $\hat F\in\mathcal{A}$ to be the closed set generated by $\hat\rho_n$ and $\hat\rho$, respectively, (note that $\hat F$ has a Lipschitz boundary), we deduce from Proposition 4.4 that

$$\liminf_{n\to\infty}\int_{\partial\hat F_n}\mathcal{K}(z,\nu^i_{\hat F_n})\,d\mathcal{H}^1 \geq \int_{\partial\hat F}\mathcal{K}(z,\nu^i_{\hat F})\,d\mathcal{H}^1\,. \tag{4.21}$$

Then we observe that we can split $\partial\hat F_n$ and $\partial\hat F$ as

$$\partial\hat F_n = \Gamma^{\rm up}_{n,k}\cup\hat\Gamma_n\,,\quad \partial\hat F = \Gamma_{\star,k}\cup\hat\Gamma \tag{4.22}$$

with disjoint unions, $\hat\Gamma_n$ and $\hat\Gamma$ are smooth and ν_0 is the inner normal to $\hat F$ along $\Gamma_{\star,k}$. Now straightforward computations using polar coordinates yield

$$\begin{aligned}\int_{\hat\Gamma_n}\mathcal{K}(z,\nu^i_{\hat F_n})\,d\mathcal{H}^1 = \int_0^{\theta^1_{n,k}}\overline{\Phi}\Big(\sigma(\theta),\hat\rho^*_n(\theta),\frac{\rho^*_n(\theta^1_{n,k})-R_0/2}{\theta^1_{n,k}}\Big)d\theta\\ +\int_{\theta^2_{n,k}}^{2\pi}\overline{\Phi}\Big(\sigma(\theta),\hat\rho^*_n(\theta),\frac{-\rho^*_n(\theta^2_{n,k})+R_0/2}{2\pi-\theta^2_{n,k}}\Big)d\theta\,,\end{aligned} \tag{4.23}$$

and

$$\int_{\hat\Gamma} \mathcal{K}(z,\nu^i_{\hat F})\,d\mathcal{H}^1 = \int_0^{\frac{\pi}{2}} \overline{\Phi}\Big(\sigma(\theta),\hat\rho^*(\theta),(r_0-\varepsilon_k)\frac{2}{\pi}-\frac{R_0}{\pi}\Big)d\theta + \int_{\frac{\pi}{2}}^{2\pi} \overline{\Phi}\Big(\sigma(\theta),\hat\rho^*(\theta),-(r_0+(1-2\delta)\varepsilon_k)\frac{2}{3\pi}+\frac{R_0}{3\pi}\Big)d\theta\,. \tag{4.24}$$

Using (4.23) and (4.24), by Lebesgue's dominated convergence theorem, we derive that

$$\int_{\hat\Gamma_n} \mathcal{K}(z,\nu^i_{\hat F_n})\,d\mathcal{H}^1 \underset{n\to\infty}{\longrightarrow} \int_{\hat\Gamma} \mathcal{K}(z,\nu^i_{\hat F})\,d\mathcal{H}^1\,. \tag{4.25}$$

Then (4.17) follows from (4.21), (4.22) and (4.25).

Step 3: Proof of (4.8). Proving (4.8) is equivalent to show that

$$\mu \geq \mathcal{K}(\cdot,\nu^i_F)\,d\mathcal{H}^1\lfloor\partial^*F\,. \tag{4.26}$$

Fix $\zeta \in C_c(\mathbb{R}^2;\mathbb{R})$ such that $\zeta \geq 0$. From the weak* convergence of $\mathcal{K}(\cdot,\nu^i_n)\,d\mathcal{H}^1\lfloor\partial F_n$ to μ together with Proposition 4.4 and Lemma 2.8 we obtain that

$$\int_{\mathbb{R}^2}\zeta\,d\mu = \lim_{n\to\infty}\int_{\partial F_n}\zeta(z)\mathcal{K}(z,\nu^i_n)\,d\mathcal{H}^1 \geq \int_{\partial^*F}\zeta(z)\mathcal{K}(z,\nu^i_F)\,d\mathcal{H}^1\,,$$

which yields (4.26) since ζ is arbitrary. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Fix $(F,u)\in X$ and let $(F_n,u_n)\in X_{\rm Lip}$ be such that $(F_n,u_n)\xrightarrow{X}(F,u)$. Let $\{\omega_i\}$ be an increasing sequence of open sets compactly contained in $B_0\setminus F$ and such that

$$B_0\setminus F = \bigcup_{i=1}^{\infty}\omega_i\,.$$

Since $\lim_{n\to\infty} d_{\mathcal{H}}(F_n,F)=0$, for every fixed $i\in\mathbb{N}$, we have that ω_i is compactly contained in $\mathbb{R}^2\setminus F_n$ for all $n\geq n_i$ for some $n_i\in\mathbb{N}$. Since $\mathbf{E}(u_n)\rightharpoonup\mathbf{E}(u)$ in $L^2\big(\omega_i;\mathbb{R}^{2\times2}\big)$ and $\mathcal{W}$ is convex and nonnegative by (H1),

$$\liminf_{n\to\infty}\int_{B_0\setminus F_n}\mathcal{W}\big(\mathbf{E}(u_n)\big)\,dz \geq \liminf_{n\to\infty}\int_{\omega_i}\mathcal{W}\big(\mathbf{E}(u_n)\big)\,dz \geq \int_{\omega_i}\mathcal{W}\big(\mathbf{E}(u)\big)\,dz\,.$$

Using Lebesgue's monotone convergence theorem, we conclude that

$$\liminf_{n\to\infty}\int_{B_0\setminus F_n}\mathcal{W}\big(\mathbf{E}(u_n)\big)\,dz \geq \int_{B_0\setminus F}\mathcal{W}\big(\mathbf{E}(u)\big)\,dz\,.$$

In turn, by Proposition 4.2,

$$\liminf_{n\to\infty}\int_{\partial F_n}\varphi(\nu^i_{F_n})\,d\mathcal{H}^1 \geq \int_{\partial^*F}\mathcal{K}(z,\nu^i_F)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}}\tilde{\mathcal{K}}(z,\nu^i_F)\,d\mathcal{H}^1\,.$$

Therefore

$$\liminf_{n\to\infty}\mathcal{F}(F_n,u_n) \geq \int_{B_0\setminus F}\mathcal{W}\big(\mathbf{E}(u)\big)\,dz + \int_{\partial^*F}\mathcal{K}(z,\nu^i_F)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}}\tilde{\mathcal{K}}(z,\nu^i_F)\,d\mathcal{H}^1\,,$$

and the proof is complete. □

5. Upper bound of the relaxed energy

In this section we establish the upper bound in Theorem 3.2, precisely,

Theorem 5.1. *Assume (H1)-(H2). Then*

$$\overline{\mathcal{F}}(F,u) \le \int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\,dz + \int_{\partial^* F} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}} \tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1 \tag{5.1}$$

for every $(F,u)\in X$, where the functions $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are given in (3.6) *and* (3.9).

The proof relies on the following proposition.

Proposition 5.2. *Let $F\in\mathcal{A}$ be such that $F\subset B_0$. Then there exists a sequence $\{F_n\}_{n\in\mathbb{N}}\subset\mathcal{A}_{\rm Lip}$ such that $F\subset F_n$ for every n, $F_n\to F$ as $n\to\infty$ in the Hausdorff metric and*

$$\limsup_{n\to\infty}\int_{\partial F_n}\varphi(\nu_{F_n}^i)\,d\mathcal{H}^1 \le \int_{\partial^* F} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}} \tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1\,. \tag{5.2}$$

In particular,

$$\mathcal{J}(F) = \int_{\partial^* F} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}} \tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1\,,$$

where $\mathcal{J}$ is defined in (4.1).

To prove Proposition 5.2, we begin with two auxiliary lemmas.

Lemma 5.3. *For every $F\in\mathcal{A}$ such that $\rho_F=\rho_F^+<R_0$, there exists a sequence $\{F_n\}_{n\in\mathbb{N}}\subset\mathcal{A}_{\rm Lip}$ such that $F\subset F_n\subset B_0$ for every n, $F_n\to F$ in the Hausdorff metric, and $\mathcal{H}^1(\partial F_n)\to\mathcal{H}^1(\partial F)$ as $n\to\infty$.*

Proof. Without loss of generality, we can assume that $\mathcal{H}^1(G_F\cap(\{0\}\times[0,R_0)))=0$. Then, from the proof of Lemma 1 in [6] (given in Subsections 5.1 and 5.2 of [6]) it follows that there exists a sequence of 2π-periodic Lipschitz functions $\rho_n^*:\mathbb{R}\to[0,\infty)$, $\rho_n^*\ge\rho_F^*$, converging in $L^1_{\rm loc}(\mathbb{R})$ to ρ_F^* and such that

$$d_{\mathcal{H}}(G^-_{F_n},G^-_F)\to 0 \quad\text{and}\quad \mathcal{H}^1(G_{F_n}\cap((0,2\pi)\times\mathbb{R}))\to\mathcal{H}^1(G_F\cap((0,2\pi)\times\mathbb{R}))\,, \tag{5.3}$$

where $F_n:=\{r\sigma(\theta):\,0\le r\le\rho_n^*(\theta)\}$. In particular from the Hausdorff convergence of $G^-_{F_n}$ to G^-_F it follows that $F_n\to F$ in the Hausdorff metric and that $F_n\subset B_0$ for all n sufficiently large.

Moreover, since $\rho_n^*\to\rho_F^*$ in $L^1((0,2\pi))$, from (5.3) and (2.19), we deduce that $D\chi_{G^-_{F_n}}\overset{*}{\rightharpoonup}D\chi_{G^-_F}$ in the sense of measures in $(0,2\pi)\times(-\infty,R_0)$, and that

$$\begin{aligned}\mathcal{H}^1(G_{F_n}\cap((0,2\pi)\times\mathbb{R})) &= |D\chi_{G^-_{F_n}}|((0,2\pi)\times(-\infty,R_0))\\ &\to |D\chi_{G^-_F}|((0,2\pi)\times(-\infty,R_0)) = \mathcal{H}^1(G_F\cap((0,2\pi)\times\mathbb{R}))\,.\end{aligned}$$

Consider the function $g:(0,2\pi)\times(-\infty,R_0)\times\mathbb{S}^1\to\mathbb{R}$ defined by

$$g\,(\theta,r,\nu):=\begin{cases}\left|\nabla\Psi\,(\theta,r)\,\nu^\perp\right| & \text{if } 0<r<R_0\\ |\nu_1| & \text{if } r\le 0\,.\end{cases}$$

Since g is a continuous bounded function, by Reshetnyak continuity theorem (see Theorem 2.39 in [1] or [29]) we have

$$\int_{G_{F_n}\cap((0,2\pi)\times(-\infty,R_0))} g(\theta,r,\nu_{G^-_{F_n}})\,d\mathcal{H}^1(\theta,r)\to\int_{G_F\cap((0,2\pi)\times(-\infty,R_0))} g(\theta,r,\nu_{G^-_F})\,d\mathcal{H}^1(\theta,r) \tag{5.4}$$

as $n\to\infty$. Arguing as in the proof Lemma 2.7, and using the fact that $\mathcal{H}^1\left(G_F\cap(\{0\}\times(-\infty,R_0))\right)=0$, we obtain

$$\mathcal{H}^1\left(\partial^* F\right)=\int_{G_F\cap((0,2\pi)\times(-\infty,R_0))}|\nabla_\tau\Psi\,(\theta,r)|\,d\mathcal{H}^1(\theta,r)=\int_{G_F\cap((0,2\pi)\times(-\infty,R_0))} g(\theta,r,\nu_{G^-_F})\,d\mathcal{H}^1(\theta,r)\,,$$

and similarly for F_n. In view of (5.4) we deduce that

$$\mathcal{H}^1(\partial F_n) \to \mathcal{H}^1(\partial^* F) = \mathcal{H}^1(\partial F) \,,$$

and the proof is complete. $\square$

Lemma 5.4. *For every $F \in \mathcal{A}_{\mathrm{Lip}}$ such that $F \subset B_0$, there exists a sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_{\mathrm{Lip}}$ such that $F \subset F_n$ for every n, $F_n \to F$ in the Hausdorff metric, and*

$$\int_{\partial F_n} \varphi(\nu^i_{F_n})\, d\mathcal{H}^1 \to \int_{\partial F} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1 \tag{5.5}$$

as $n \to \infty$.

Proof. By Lemma 4.3, (9.1) and (9.4), there exists a sequence of closed sets F_n starshaped with respect to the origin such that $\rho_{F_n} \overset{*}{\rightharpoonup} \rho_F$ in $W^{1,\infty}(\mathbb{S}^1)$ and such that (5.5) holds. Since $\rho_{F_n} \to \rho_F$ uniformly, and $\rho_F < R_0$, we may replace ρ_{F_n} by $\rho_{F_n} + \|\rho_{F_n} - \rho_F\|_\infty$ and since $\rho_{F_n} < R_0$ for n sufficiently large, the conclusion follows. $\square$

We now turn to the proof of Proposition 5.2.

Proof of Proposition 5.2. *Step 1.* First, we prove (5.2) for $F \in \mathcal{A}$ such that $\rho_F = \rho_F^+ < R_0$. We consider the sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_{\mathrm{Lip}}$ given by Lemma 5.3 and the associated ρ_n's, so that

$$\lim_{n\to\infty} |D\chi_{F_n}|(\mathbb{R}^2) = \lim_{n\to\infty} \mathcal{H}^1(\partial F_n) = \mathcal{H}^1(\partial F)\,.$$

On the other hand, since $\rho_F = \rho_F^+ = \rho_{F^+}$, we have

$$\mathcal{H}^1 \lfloor \partial F = \mathcal{H}^1 \lfloor \partial F^+ = \mathcal{H}^1 \lfloor \partial^* F$$

where we used Lemma 2.4 in the second equality. Hence,

$$\lim_{n\to\infty} |D\chi_{F_n}|(\mathbb{R}^2) = \mathcal{H}^1(\partial^* F) = |D\chi_F|(\mathbb{R}^2)\,. \tag{5.6}$$

Since

$$|D\chi_{F_n}|\left(\mathbb{R}^2 \setminus \{0\}\right) = \mathcal{H}^1(\partial F_n \setminus \{0\}) = \mathcal{H}^1(\partial F_n) \quad \text{and} \quad |D\chi_F|\left(\mathbb{R}^2 \setminus \{0\}\right) = \mathcal{H}^1(\partial^* F \setminus \{0\}) = \mathcal{H}^1(\partial^* F)\,,$$

by (5.6), we have that

$$\lim_{n\to\infty} |D\chi_{F_n}|\left(\mathbb{R}^2 \setminus \{0\}\right) = |D\chi_F|\left(\mathbb{R}^2 \setminus \{0\}\right)\,.$$

Moreover, by Lemma 2.8, $D\chi_{F_n} \overset{*}{\rightharpoonup} D\chi_F$ weakly* in the sense of measures. Thus, by applying Reshetnyak continuity theorem to the measures $|D\chi_{F_n}|$ and $|D\chi_F|$ in $\mathbb{R}^2 \setminus \{0\}$ (see Theorem 2.39 in [1] or [29]), and recalling that $\mathcal{K}$ is continuous on $\mathbb{R}^2 \setminus \{0\} \times \mathbb{S}^1$, we derive

$$\lim_{n\to\infty} \int_{\partial F_n} \mathcal{K}(z, \nu^i_n)\, d\mathcal{H}^1 = \int_{\partial^* F} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1\,,$$

where we have set $\nu^i_n = \nu^i_{F_n}$.

Since $F_n \subset B_0$ for all n sufficiently large, we may use Lemma 5.4 to construct sequences $\{F_{n,k}\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\mathrm{Lip}}$ such that $F \subset F_n \subset F_{n,k}$, with

$$\lim_{n\to\infty} \lim_{k\to\infty} d_{\mathcal{H}}(F, F_{n,k}) = \lim_{n\to\infty} d_{\mathcal{H}}(F, F_n) = 0$$

and

$$\lim_{n\to\infty} \lim_{k\to\infty} \int_{\partial F_{n,k}} \varphi(\nu^i_{n,k})\, d\mathcal{H}^1 = \lim_{n\to\infty} \int_{\partial F_n} \mathcal{K}(z, \nu^i_n)\, d\mathcal{H}^1 = \int_{\partial^* F} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1\,,$$

where $\nu^i_{n,k} = \nu^i_{F_{n,k}}$. By diagonalizing, we obtain (5.2).

Step 2. Next we consider $F \in \mathcal{A}$, $F \subset B_0$, such that the set S_F is finite (see 2.11), i.e., $S_F = \{\sigma_1, \dots, \sigma_N\}$ with

$$\sigma_1 < \sigma_2 < \cdots < \sigma_N .$$

We claim that there exists a sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\rho^+_{F_n} = \rho_{F_n}$, $F_n \supset F$, $F_n \to F$ as $n \to \infty$ in the Hausdorff metric, and

$$\lim_{n\to\infty} \int_{\partial^* F_n} \mathcal{K}(z, \nu^i_n)\, d\mathcal{H}^1 = \int_{\partial^* F} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}} \tilde{\mathcal{K}}(z, \nu^i_F)\, d\mathcal{H}^1 . \tag{5.7}$$

Let $\varepsilon_0 := 1/2 \min\big\{d_{\mathbb{S}^1}(\sigma_i, \sigma_j) : i, j = 1, \dots N,\, i \neq j\big\}$, where $d_{\mathbb{S}^1}$ denotes the geodesic distance on $\mathbb{S}^1$, and select a decreasing sequence $\varepsilon_n \to 0^+$ as $n \to \infty$, $\varepsilon_n \leq \varepsilon_0$, such that the points $\sigma^-_{n,i}$ and $\sigma^+_{n,i}$ defined by $[\sigma^-_{n,i}, \sigma^+_{n,i}] = \overline{B}_{\mathbb{S}^1}(\sigma_i, \varepsilon_n)$, belong to $\mathbb{S}^1 \setminus (J_F \cup S_F)$. Note that here we are using the fact that J_F is countable. Define

$$c_{n,i} := \max_{\sigma\in[\sigma^-_{n,i}, \sigma^+_{n,i}]} \rho_F(\sigma)$$

and

$$\rho_n(\sigma) := \begin{cases} c_{n,i} & \text{if } \sigma \in [\sigma^-_{n,i}, \sigma^+_{n,i}] \text{ for some } i \in \{1, \dots, N\}\,,\\ \rho_F(\sigma) & \text{otherwise}\,, \end{cases}$$

and $F_n := \big\{r\sigma : \sigma \in \mathbb{S}^1\,,\, 0 \leq r \leq \rho_n(\sigma)\big\}$. Since ρ_F is upper semicontinuous, ρ_n is upper semicontinuous, ρ_n converges pointwise to ρ_F, and $R_0 > \rho_n \geq \rho_F$ for all n. Hence, $F_n \in \mathcal{A}$ and $B_0 \supset F_n \supset F$. Moreover, from the construction of ρ_n it follows that $F_n \to F$ in the Hausdorff metric. Setting $\Xi_n := \big\{r\sigma \in \mathbb{R}^2 : r \in \mathbb{R}_+\,,\, \sigma \in \mathbb{S}^1 \setminus \bigcup_{i=1}^N [\sigma^-_{n,i}, \sigma^+_{n,i}]\big\}$, a straightforward computation using the 0-homogeneity of $\mathcal{K}$ with respect to the z variable, yields

$$\begin{aligned}\int_{\partial^* F_n} \mathcal{K}(z, \nu^i_n)\, d\mathcal{H}^1 &= \int_{\partial^* F\cap\Xi_n} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1 + \sum_{i=1}^N \big(c_{n,i} - \rho_F(\sigma^-_{n,i})\big)\mathcal{K}\big(\sigma^-_{n,i}, (\sigma^-_{n,i})^\perp\big)\\ &\quad + \sum_{i=1}^N \big(c_{n,i} - \rho_F(\sigma^+_{n,i})\big)\mathcal{K}\big(\sigma^+_{n,i}, -(\sigma^+_{n,i})^\perp\big) + \sum_{i=1}^N c_{n,i} \int_{[\sigma^-_{n,i}, \sigma^+_{n,i}]} \mathcal{K}(\sigma, -\sigma)\, d\mathcal{H}^1 .\end{aligned}$$

Observe that $\Xi_n \nearrow \Xi_\star := \big\{r\sigma \in \mathbb{R}^2 : r \in \mathbb{R}_+\,,\, \sigma \in \mathbb{S}^1 \setminus S_F\big\}$ as $n \to \infty$ so that, by the Lebesgue monotone convergence theorem,

$$\lim_{n\to\infty} \int_{\partial^* F\cap\Xi_n} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1 = \int_{\partial^* F\cap\Xi_\star} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1 .$$

Since $\rho_F(\sigma^-_{n,i}) \to \rho_F(\sigma_i-)$, $\rho_F(\sigma^+_{n,i}) \to \rho_F(\sigma_i+)$, and $c_{n,i} \to \rho_F(\sigma_i)$ as $n \to \infty$, we derive that for every $i = 1, \dots, N$,

$$\begin{aligned}&\big(c_{n,i} - \rho_F(\sigma^-_{n,i})\big)\mathcal{K}\big(\sigma^-_{n,i}, (\sigma^-_{n,i})^\perp\big) + \big(c_{n,i} - \rho_F(\sigma^+_{n,i})\big)\mathcal{K}\big(\sigma^+_{n,i}, -(\sigma^+_{n,i})^\perp\big)\\ &\qquad \underset{n\to\infty}{\longrightarrow} \big(\rho_F(\sigma_i) - \rho_F(\sigma_i-)\big)\mathcal{K}\big(\sigma_i, \sigma_i^\perp\big) + \big(\rho_F(\sigma_i) - \rho_F(\sigma_i+)\big)\mathcal{K}\big(\sigma_i, -\sigma_i^\perp\big)\\ &= \big(\rho_F(\sigma_i) - \rho^+_F(\sigma_i)\big)\tilde{\mathcal{K}}\big(\sigma_i, \sigma_i^\perp\big) + \big(\rho^+_F(\sigma_i) - \rho_F(\sigma_i-)\big)\mathcal{K}\big(\sigma_i, \sigma_i^\perp\big) + \big(\rho^+_F(\sigma_i) - \rho_F(\sigma_i+)\big)\mathcal{K}\big(\sigma_i, -\sigma_i^\perp\big) .\end{aligned}$$

In addition, we have

$$\sum_{i=1}^N c_{n,i} \int_{[\sigma^-_{n,i}, \sigma^+_{n,i}]} \mathcal{K}(\sigma, -\sigma)\, d\mathcal{H}^1 \leq C N R_0 \varepsilon_n ,$$

and consequently

$$\begin{aligned}\lim_{n\to\infty} \int_{\partial^* F_n} \mathcal{K}(z, \nu^i_n)\, d\mathcal{H}^1 &= \int_{\partial^* F\cap\Xi_\star} \mathcal{K}(z, \nu^i_F)\, d\mathcal{H}^1 + \sum_{i=1}^N \big(\rho_F(\sigma_i) - \rho^+_F(\sigma_i)\big)\tilde{\mathcal{K}}\big(\sigma_i, \sigma_i^\perp\big)\\ &\quad + \sum_{i=1}^N \big(\rho^+_F(\sigma_i) - \rho_F(\sigma_i-)\big)\mathcal{K}\big(\sigma_i, \sigma_i^\perp\big) + \sum_{i=1}^N \big(\rho^+_F(\sigma_i) - \rho_F(\sigma_i+)\big)\mathcal{K}\big(\sigma_i, -\sigma_i^\perp\big) .\end{aligned} \tag{5.8}$$

Note that $\Gamma_{\rm cut} = \bigcup_{i=1}^N \{r\sigma_i : \rho_F^+(\sigma_i) < r \le \rho_F(\sigma_i)\}$ and this union is disjoint. Hence, using the 0-homogeneity of $\mathcal{K}$ with respect to the z variable, we derive

$$\int_{\Gamma_{\rm cut}} \tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1 = \sum_{i=1}^N \int_{\rho_F^+(\sigma_i)}^{\rho_F(\sigma_i)} \tilde{\mathcal{K}}(r\sigma_i,\sigma_i^\perp)\,dr = \sum_{i=1}^N \big(\rho_F(\sigma_i)-\rho_F^+(\sigma_i)\big)\tilde{\mathcal{K}}\big(\sigma_i,\sigma_i^\perp\big)\,. \tag{5.9}$$

Hence, in view of (5.8) and (5.9), (5.7) follows, provided we show that

$$\int_{\partial^*F\setminus\Xi_\star} \mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 = \sum_{i=1}^N \big(\rho_F^+(\sigma_i)-\rho_F(\sigma_i-)\big)\mathcal{K}\big(\sigma_i,\sigma_i^\perp\big) + \sum_{i=1}^N \big(\rho_F^+(\sigma_i)-\rho_F(\sigma_i+)\big)\mathcal{K}\big(\sigma_i,-\sigma_i^\perp\big)\,.$$

To see this, observe that by Lemma 2.4, $\partial^*F = \partial F^+$ except for a set of null $\mathcal{H}^1$-measure so that $\partial^*F\setminus\Xi_\star = \partial F^+\setminus\Xi_\star$ except for a set of null $\mathcal{H}^1$-measure. In view of Lemma 2.4,

$$\partial F^+\setminus\Xi_\star = \big\{r\sigma\in\mathbb{R}^2 :\ \sigma\in S_F\,,\ \rho_F^-(\sigma)\le r\le\rho_F^+(\sigma)\big\} = \bigcup_{i=1}^N \{r\sigma_i\in\mathbb{R}^2 :\ \rho_F^-(\sigma_i)\le r\le\rho_F^+(\sigma_i)\}\,,$$

where the sets in the union are disjoint except possibly at the origin. By Lemma 2.5 on each segment $\Gamma_i := \{r\sigma_i\in\mathbb{R}^2 :\ \rho_F^-(\sigma_i)\le r\le\rho_F^+(\sigma_i)\}$, $i=1,\dots,N$, we have $\nu_F^i = \sigma_i^\perp$ if $\rho_F^+(\sigma_i) = \rho_F(\sigma_i+)$ and $\nu_F^i = -\sigma_i^\perp$ otherwise. This concludes the proof of this step.

Step 3. Finally, if the set S_F is countable, then we claim that there exists a decreasing sequence $\{F_n\}_{n\in\mathbb{N}}\subset\mathcal{A}$ such that for every $n\in\mathbb{N}$, S_{F_n} is finite, $F\subset F_n\subset B_0$, $F_n\to F$ in the Hausdorff metric, and

$$\limsup_{n\to\infty}\left(\int_{\partial^*F_n}\mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 + \int_{\Gamma_{{\rm cut},n}}\tilde{\mathcal{K}}(z,\nu_n^i)\,d\mathcal{H}^1\right) \le \int_{\partial^*F}\mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}}\tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1\,.$$

Since ρ_F is upper semicontinuous and less than R_0, for all n sufficiently large, we have that

$$\rho_n(\sigma) := \max\big\{\rho_F^+(\sigma)+1/n,\ \rho_F(\sigma)\big\} < R_0$$

for all $\sigma\in\mathbb{S}^1$. Note that ρ_n is upper semicontinuous and has finite pointwise variation, so that the closed set F_n generated by ρ_n belongs to $\mathcal{A}$. From the construction we have that $F_n\to F$ in the Hausdorff metric. We also observe that $\rho_n^+ = \rho_F^+ + 1/n$ so that

$$S_{F_n} = \big\{\sigma\in\mathbb{S}^1 :\ \rho_F(\sigma) > \rho_F^+(\sigma)+1/n\big\}\subset S_F \tag{5.10}$$

and S_{F_n} is finite because ρ_F has finite pointwise variation. Moreover, $\chi_{F_n}\to\chi_F$ in $L^1(\mathbb{R}^2)$ and $\mathcal{H}^1(\partial F_n^+)\to\mathcal{H}^1(\partial F^+)$. Hence, arguing as in Step 1, we obtain

$$\lim_{n\to\infty}\int_{\partial F_n}\mathcal{K}(z,\nu_n^i)\,d\mathcal{H}^1 = \int_{\partial^*F}\mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1\,.$$

Moreover, from (5.10), it follows that $\Gamma_{{\rm cut},n}\subset\Gamma_{\rm cut}$ and $\tilde{\mathcal{K}}(z,\nu_n^i) = \mathcal{K}(z,\nu_F^i)$ on $\Gamma_{{\rm cut},n}$. Hence,

$$\limsup_{n\to\infty}\int_{\Gamma_{{\rm cut},n}}\tilde{\mathcal{K}}(z,\nu_n^i)\,d\mathcal{H}^1 \le \int_{\Gamma_{\rm cut}}\tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1\,,$$

and this completes the proof of the claim.

End of the proof. Combining Step 1, Step 2, and Step 3 and applying a standard diagonalization argument, we obtain the required sequence. □

Finally, we prove Theorem 5.1.

Proof of Theorem 5.1. To prove (5.1), given $(F,u)\in X$, we have to construct a sequence $\{(F_n,u_n)\}_{n\in\mathbb{N}}\subset X_{\rm Lip}$ such that $(F_n,u_n)\xrightarrow{X}(F,u)$ and

$$\limsup_{n\to\infty}\mathcal{F}(F_n,u_n) \le \int_{B_0\setminus F}\mathcal{W}\big(\mathbf{E}(u)\big)\,dz + \int_{\partial^*F}\mathcal{K}(z,\nu_F^i)\,d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}}\tilde{\mathcal{K}}(z,\nu_F^i)\,d\mathcal{H}^1\,. \tag{5.11}$$

Assume first that $F \subset B_0$ and let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence as in Proposition 5.2. By (H3),

$$\mathcal{H}^1(\partial F_n) \leq \frac{1}{m}\int_{\partial F_n} \varphi(\nu^i_{F_n})\, d\mathcal{H}^1\,,$$

therefore $\{\mathcal{H}^1(\partial F_n)\}$ is bounded in view of (5.2). Then, since $F \subset F_n$, we have $(F_n, u) \in X_{\rm Lip}$ (see (3.1)) and consequently $(F_n, u) \overset{X}{\longrightarrow} (F, u)$. By Lebesgue's dominated convergence theorem,

$$\int_{B_0\setminus F_n} \mathcal{W}\big(\mathbf{E}(u)\big) dz \underset{n\to\infty}{\longrightarrow} \int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big) dz\,,$$

and so (5.11) holds for the sequence $\{(F_n, u)\}$.

Suppose now that $\partial F \cap \partial B_0 \neq \emptyset$ and let $\alpha_k > 0$ be such that $\alpha_k \nearrow 1$. Set $F_k := \alpha_k F \subset B_0$ and define

$$u_k\,(z) := \begin{cases} u\left(\dfrac{z}{\alpha_k}\right) & \text{if } z \in \alpha_k B_0 \setminus F_k\,, \\ u_0\left(\dfrac{zR_0}{|z|}\right) & \text{if } z \in B_0 \setminus \alpha_k B_0\,. \end{cases}$$

Then, from the first part of the prood for every fixed k there exists a sequence $\{F_{k,n}\}_{n\in\mathbb{N}} \subset \mathcal{A}_{\rm Lip}$ such that $(F_{k,n}, u_k) \underset{n\to\infty}{\longrightarrow} (F_k, u_k)$ and

$$\limsup_{n\to\infty} \mathcal{F}(F_{k,n}, u_k) \leq \int_{B_0\setminus F_k} \mathcal{W}\big(\mathbf{E}(u_k)\big)\, dz + \int_{\partial^* F_k} \mathcal{K}(z, \nu^i_{F_k})\, d\mathcal{H}^1 + \int_{(\Gamma_k)_{\rm cut}} \tilde{\mathcal{K}}(z, \nu^i_{F_k})\, d\mathcal{H}^1\,.$$

Letting $k \to \infty$, we obtain

$$\limsup_{k\to\infty}\, \limsup_{n\to\infty} \mathcal{F}(F_{k,n}, u_k) \leq \int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\, dz + \int_{\partial^* F} \mathcal{K}(z, \nu^i_{F})\, d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}} \tilde{\mathcal{K}}(z, \nu^i_{F})\, d\mathcal{H}^1\,,$$

and so (5.11) follows by a standard diagonalization argument. ◻

6. The exterior Wulff condition

We now start to investigate the regularity issue for solutions of (3.10). In the remaining of the paper we assume that

(H3) $\varphi : \mathbb{R}^2 \to [0, \infty)$ is convex.

The convexity of φ is justified by the fact that the Wulff set of φ is also the Wulff set of φ^{**} (see Proposition 3.5 in [14]).
Note that, (H3) implies that $K(z, \nu) = \varphi(\nu)$ (see (3.5) and (3.6)) and thus by Theorem 3.2,

$$\overline{\mathcal{F}}\,(F, u) = \int_{B_0\setminus F} \mathcal{W}\,(\mathbf{E}\,(u))\; dz + \int_{\partial^* F} \varphi\left(\nu^i_F\right)\; d\mathcal{H}^1 + \int_{\Gamma_{\rm cut}} \left(\varphi\left(\nu^i_F\right) + \varphi\left(-\nu^i_F\right)\right)\; d\mathcal{H}^1\,.$$

Given $0 < d < \pi R_0^2$ and $\ell > 0$, we set

$$\mathcal{F}_\ell\,(F, u) := \overline{\mathcal{F}}\,(F, u) + \ell\, ||F| - d|\,.$$

As in [15], we shall prove that if ℓ is sufficiently large the constrained minimization problem for $\overline{\mathcal{F}}$ is equivalent to the unconstrained minimization problem for the penalized energy $\mathcal{F}_\ell$. The advantage of working with $\mathcal{F}_\ell$ is that we are allowed more freedom in admissible variations.

Proposition 6.1. *Assume that (H1)-(H3) hold. There exists $\ell_0 > 0$ such that for all $\ell \geq \ell_0$, $(F, u) \in X$ is a minimizer of the constrained problem* (3.10) *if and only if it is a minimizer in X of $\mathcal{F}_\ell$.*

We start with a minimality property of line segments. To fix ideas in what follows a Lipschitz function $\gamma : [a, b] \to \mathbb{R}^2$ is a *parametrization* of a curve if γ is injective, $\gamma'\,(t) \neq 0$ for a.e. $t \in [a, b]$, and $|\gamma'|$ is constant. With a slight abuse of notations we shall identify a parametrization γ with its image $\gamma([a,b])$.

Lemma 6.2. *Let $\psi : \mathbb{R}^2 \to [0,+\infty)$ be a positively 1-homogenous convex function. Let z_1, $z_2 \in \mathbb{R}^2$ be two distinct points, and let γ, $\chi : [0,1] \to \mathbb{R}^2$ be parametrizations of curves such that $\gamma(0) = \chi(0) = z_1$ and $\gamma(1) = \chi(1) = z_2$, with $\gamma([0,1]) = [z_1, z_2]$. Then*

$$\int_\chi \psi(\nu_\chi)\, d\mathcal{H}^1 \geq \int_\gamma \psi(\nu_\gamma)\, d\mathcal{H}^1 ,$$

where $\nu_\gamma := \dfrac{(\gamma')^\perp}{|\gamma'|}$ and $\nu_\chi := \dfrac{(\chi')^\perp}{|\chi'|}$.

Proof. Let $z_i = (x_i, y_i)$, $i = 1,2$, and $\chi(t) = (\chi_1(t), \chi_2(t))$. From Jensen's inequality and the homogeneity of ψ, we get

$$\begin{aligned}\int_\chi \psi(\nu_\chi)\, d\mathcal{H}^1 &= \int_0^1 \psi(-\chi_2'(t), \chi_1'(t))\, dt \\ &\geq \psi\left(\int_0^1 (-\chi_2'(t), \chi_1'(t))\, dt\right) = \psi(y_1 - y_2, x_2 - x_1) = \int_\gamma \psi(\nu_\gamma)\, d\mathcal{H}^1 ,\end{aligned}$$

which completes the proof. □

Proof of Proposition 6.1. Let (F_ℓ, u_ℓ) be a minimizer of $\mathcal{F}_\ell$. The existence of minimizers is guaranteed via an argument similar to the one used in the proof of Theorem 3.3. Then for every $\ell > 0$,

$$\mathcal{F}_\ell(F_\ell, u_\ell) \leq \mathcal{F}_\ell(F, u) = \overline{\mathcal{F}}(F, u) \leq \overline{\mathcal{F}}\left(\overline{B}_{R_d}, u_0\right) =: \Lambda ,$$

where $\pi R_d^2 = d$, and so by (H2),

$$\ell\,||F_\ell| - d| \leq \Lambda , \quad \mathcal{H}^1(\partial F_\ell) \leq \frac{\Lambda}{m} . \tag{6.1}$$

Thus, there exist $\ell_1 > 0$ depending only on d and Λ, such that

$$|F_\ell| > \frac{d}{2} \quad \text{and} \quad \left|\frac{d}{|F_\ell|} - 1\right| < 1 \tag{6.2}$$

for all $\ell \geq \ell_1$.

We claim that $|F_\ell| = d$ for ℓ large enough. Note that this being the case, then

$$\overline{\mathcal{F}}(F, u) \leq \overline{\mathcal{F}}(F_\ell, u_\ell) = \mathcal{F}_\ell(F_\ell, u_\ell) \leq \mathcal{F}_\ell(F, u) = \overline{\mathcal{F}}(F, u) .$$

Step 1. For $\ell > \ell_1$, assume first that $|F_\ell| > d$. Set

$$\alpha := \left(\frac{d}{|F_\ell|}\right)^{\frac{1}{2}} < 1 , \quad \widetilde{F}_\ell := \alpha F_\ell \in \mathcal{A} ,$$

so that $|\widetilde{F}_\ell| = d$, and consider

$$\widetilde{u}_\ell(w) := \begin{cases} u_\ell\left(\dfrac{w}{\alpha}\right) & \text{if } w \in \alpha B_0 \setminus \widetilde{F}_\ell , \\ u_0\left(\dfrac{w R_0}{|w|}\right) & \text{if } w \in B_0 \setminus \alpha B_0 . \end{cases}$$

Since $\mathcal{J}(\widetilde{F}_\ell) = \alpha \mathcal{J}(F_\ell) < \mathcal{J}(F_\ell)$, we infer that

$$\begin{aligned}\mathcal{F}_\ell(\widetilde{F}_\ell, \widetilde{u}_\ell) - \mathcal{F}_\ell(F_\ell, u_\ell) \leq{}& \int_{B_0 \setminus \alpha B_0} \mathcal{W}(\mathbf{E}(\widetilde{u}_\ell))\, dw + \int_{\alpha B_0 \setminus \widetilde{F}_\ell} \mathcal{W}(\mathbf{E}(\widetilde{u}_\ell))\, dw \\ & - \int_{B_0 \setminus F_\ell} \mathcal{W}(\mathbf{E}(u_\ell))\, dz - \ell(|F_\ell| - d) .\end{aligned}$$

Since the second and third integrals on the right-hand side are equal, recalling (6.2), we have, for some constant $c_0 > 0$ depending only on d, u_0, R_0, $\mathbb{C}$ and φ,

$$\mathcal{F}_\ell(\widetilde{F}_\ell, \widetilde{u}_\ell) - \mathcal{F}_\ell(F_\ell, u_\ell) \leq c_0 \left(1 - \frac{d}{|F_\ell|}\right) - \ell\,(|F_\ell| - d) \leq (|F_\ell| - d)\left(\frac{c_0}{d} - \ell\right) < 0$$

provided $\ell > \ell_2 := \max\{\ell_1, c_0/d\}$. This contradicts the minimality of (F_ℓ, u_ℓ), and thus $|F_\ell| \leq d$ for all $\ell > \ell_2$.

Step 2. To conclude the proof assume by contradiction that there exist a sequence $\ell_k > \ell_2$, $k = 3, 4, \dots$ and a sequence of minimizers $\{(F_{\ell_k}, u_{\ell_k})\}$ of $\mathcal{F}_{\ell_k}$ such that $\ell_k \to \infty$ and $|F_{\ell_k}| < d$ for all $k \geq 3$. By Blaschke's Theorem (see Theorem 6.1 in [1]), (6.1) and Lemma 2.8, we may assume without loss of generality that the sets F_{ℓ_k} converge for the Hausdorff metric to some $F \in \mathcal{A}$, with $0 < |F| = d < \pi R_0^2$ and that $\rho_{F_{\ell_k}} \to \rho_F$ in $L^1(\mathbb{S}^1)$.

We now distinguish two different cases.

Case 1: Assume that there exists a point σ_0 such that ρ_F is continuous at σ_0 and $0 < \rho_F(\sigma_0) < R_0$. Fix $0 < \varepsilon < R_0 - \rho_F(\sigma_0)$ and let $\delta > 0$ be such that $0 < \rho_F(\sigma) < R_0 - \varepsilon$ whenever $|\sigma - \sigma_0| < \delta$. By taking $\varepsilon > 0$ smaller if necessary, we can assume that $\varepsilon < \delta$. By the Hausdorff convergence of $\{F_{\ell_k}\}$ to F there exists k_0 such that $\rho_{F_{\ell_k}}(\sigma) < R_0 - \varepsilon$ for all $k \geq k_0$ and for all $|\sigma - \sigma_0| < \varepsilon$.
For such k's, define $\widetilde{F}_{\ell_k}$ by taking $\rho_{\widetilde{F}_{\ell_k}}(\sigma) := \rho_{F_{\ell_k}}(\sigma)$ if $|\sigma - \sigma_0| > \varepsilon$ and $\rho_{\widetilde{F}_{\ell_k}}(\sigma) := \rho_{F_{\ell_k}}(\sigma) + \eta_k$ if $|\sigma - \sigma_0| \leq \varepsilon$, where $\eta_k > 0$ is chosen such that $|\widetilde{F}_{\ell_k}| \leq d$ and $\eta_k \to 0$.

Since we are adding two segments at $\sigma_0 \pm \varepsilon$, we have that

$$\begin{aligned}\mathcal{F}_{\ell_k}(\widetilde{F}_{\ell_k}, u_{\ell_k}) - \mathcal{F}_{\ell_k}(F_{\ell_k}, u_{\ell_k}) &\leq c_0\eta_k - \ell_k|\widetilde{F}_{\ell_k} \setminus F_{\ell_k}| \\ &= c_0\eta_k - \frac{\ell_k}{2}\int_{\sigma_0-\varepsilon}^{\sigma_0+\varepsilon}\left(2\eta_k\rho_{F_{\ell_k}}(\sigma) + \eta_k^2\right)d\sigma \leq \eta_k\left(c_0 - \ell_k\int_{\sigma_0-\varepsilon}^{\sigma_0+\varepsilon}\rho_{F_{\ell_k}}(\sigma)\,d\sigma\right),\end{aligned}$$

for a constant $c_0 > 0$ independent of k. Since

$$\int_{\sigma_0-\varepsilon}^{\sigma_0+\varepsilon}\rho_{F_{\ell_k}}(\sigma)\,d\sigma \underset{k\to\infty}{\longrightarrow} \int_{\sigma_0-\varepsilon}^{\sigma_0+\varepsilon}\rho_F(\sigma)\,d\sigma > 0\,,$$

we conclude that $\mathcal{F}_{\ell_k}(\widetilde{F}_{\ell_k}, u_{\ell_k}) - \mathcal{F}_{\ell_k}(F_{\ell_k}, u_{\ell_k}) < 0$ for k sufficiently large, which contradicts again the minimality of (F_{ℓ_k}, u_{ℓ_k}).
Case 2: Assume that ρ_F only takes a.e. the two values 0 and R_0. Since $\mathcal{H}^1(\partial F) < \infty$, by Lemma 2.4, ρ_F has finite pointwise variation, and thus it is piecewice constant with finitely many jump points in $\mathbb{S}^1$. We claim that the sets F_{ℓ_k}, and hence also F, are convex (note that this fact immediately rules out that $d > \pi R_0^2/2$). In particular, F_{ℓ_k} has a Lipschitz boundary.

We argue by contradiction, i.e., we assume that F_{ℓ_k} is not convex. Then there exist two distinct points $z_0, z_1 \in \partial F_{\ell_k}$ such that the segment $[z_0, z_1]$ is not contained in F_{ℓ_k} (observe that neither z_0 nor z_1 can be origin and that z_0 and z_1 cannot be on the same ray from the origin). Moreover, using the upper semicontinuity of $\rho_{F_{\ell_k}}$, we can choose the points z_0 and z_1 in such a way that the open segment (z_0, z_1) is contained in $\mathbb{R}^2 \setminus F_{\ell_k}$. Then, the new domain $\widetilde{F}_{\ell_k}$ obtained by the union of F_{ℓ_k} and the closed triangle T of vertices $\{0, z_0, z_1\}$ belongs to $\mathcal{A}$ and $|\widetilde{F}_{\ell_k}| > |F_{\ell_k}|$. In addition, moving the points z_0 and z_1 on ∂F_{ℓ_k} if necessary, we may always construct the set $\widetilde{F}_{\ell_k}$ in such a way that $|\widetilde{F}_{\ell_k}| \leq d$. As in Remark 2.3, it can be shown that $\partial F_{\ell_k} \cap T$ is a connected set. Hence (see Theorem 4.46 in [24]), there exists a curve $\gamma \subset \partial F_{\ell_k} \cap T$ connecting z_0 and z_1. By Lemma 6.2 we have that the resulting surface energy decreases, i.e., $\mathcal{J}(\widetilde{F}_{\ell_k}) \leq \mathcal{J}(F_{\ell_k})$. Therefore $\mathcal{F}_{\ell_k}(\widetilde{F}_{\ell_k}, u_{\ell_k}) < \mathcal{F}_{\ell_k}(F_{\ell_k}, u_{\ell_k})$, which contradicts the minimality of (F_{ℓ_k}, u_{ℓ_k}), and thus proves the convexity of each F_{ℓ_k}.

Since F is convex, as observed before we have necessarily that $|F| = d \leq \pi R_0^2/2$. Therefore, without loss of generality, we may assume that $\rho_F(\sigma) = R_0$ if $\sigma \in [\sigma_0, \sigma_1]$ and $\rho_F(\sigma) \equiv 0$ elsewhere, for some $\sigma_0 = \sigma(\theta_0), \sigma_1 = \sigma(\pi - \theta_0)$, with $0 \leq \theta_0 < \pi/2$. Then, setting $z_0 = (0, y_0)$ for some $y_0 > 0$, by the Hausdorff convergence of F_{ℓ_k} to F, there exists a ball $B_{r_0}(z_0) \subset F \cap F_{\ell_k}$, for all k large enough. By

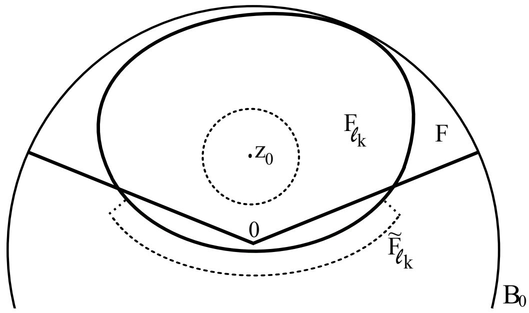

Fig. 2. The set $\widetilde{F}_{\ell_k}$ is obtained by adding to F_{ℓ_k} the region enclosed by the dotted curves .

the convexity of F and F_{ℓ_k}, we can consider the radial functions of the sets F and F_{ℓ_k} with respect to z_0, respectively denoted by $\rho_{F,z_0}(\tau)$ and $\rho_{F_{\ell_k},z_0}(\tau)$ where $\tau \in \mathbb{S}^1$. As before, we shall write for $\theta \in \mathbb{R}$, $\tau(\theta) := (\cos\theta, \sin\theta)$. We construct the sets $\widetilde{F}_{\ell_k}$ as follows.

First observe that there exists $\overline{\theta} \in (-\pi/2, \pi/2)$ such that, setting $\tau_1 := \tau(\overline{\theta})$ and $\tau_2 := \tau(\pi - \overline{\theta})$, $z_0 + \rho_{F,z_0}(\tau)\tau \in \partial B_0$ if and only if $\tau \in [\tau_1, \tau_2]$. Then, by the Hausdorff convergence of F_{ℓ_k} to F, given $\hat{\theta} \in (-\pi/2, \overline{\theta})$ and setting $\tau_1' := \tau(\hat{\theta})$, $\tau_2' := \tau(\pi - \hat{\theta})$, there exists $\delta > 0$ such that for k large enough $z_0 + \rho_{F_{\ell_k},z_0}(\tau)(1+\delta)\tau \in B_0$ whenever $\tau \notin (\tau_1', \tau_2')$. Then, for k large, choose $0 < \delta_k < \delta$ such that

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \rho^2_{F_{\ell_k},z_0}(\tau(\theta))(1+\delta_k)^2\, d\theta = d$$

(i.e., $\delta_k \approx c(d - |F_{\ell_k}|)$). Next denote by ζ the function defined on $\mathbb{S}^1$ satisfying $\zeta(\tau) := 0$ in (τ_1', τ_2'), and $\zeta(\tau) := 1$ if $\tau \notin (\tau_1', \tau_2')$. Since the sets $\{z = z_0 + r\tau : 0 \le r \le \rho_{F_{\ell_k},z_0}(\tau)(1+\delta_k),\ \tau \in \mathbb{S}^1\}$ are all convex, and thus starshaped with respect to the origin, it follows that the sets

$$\widetilde{F}_{\ell_k} := \left\{ z = z_0 + r\tau : 0 \le r \le \rho_{\widetilde{F}_{\ell_k},z_0}(\tau) := \rho_{F_{\ell_k},z_0}(\tau)(1 + \delta_k\zeta(\tau)),\ \tau \in \mathbb{S}^1 \right\},$$

which are not convex, are still starshaped with respect to the origin (see Figure 2). Moreover, from the definition and the choice of δ_k and ζ, it is clear that $\widetilde{F}_{\ell_k} \in \mathcal{A}$, $|\widetilde{F}_{\ell_k}| \le d$, $F_{\ell_k} \subset \widetilde{F}_{\ell_k} \subset \overline{B}_0$ for all k large enough, and that $\widetilde{F}_{\ell_k}$ has a Lipschitz boundary. Then straightforward computations yield

$$\begin{aligned}
\mathcal{F}_{\ell_k}(\widetilde{F}_{\ell_k}, u_{\ell_k}) - \mathcal{F}_{\ell_k}(F_{\ell_k}, u_{\ell_k}) &\le \int_{\partial \widetilde{F}_{\ell_k}} \varphi(\nu^i_{\widetilde{F}_{\ell_k}})\, d\mathcal{H}^1 - \int_{\partial F_{\ell_k}} \varphi(\nu^i_{F_{\ell_k}})\, d\mathcal{H}^1 - \ell_k \big|\widetilde{F}_{\ell_k} \setminus F_{\ell_k}\big| \\
&\le \delta_k \left(\varphi(-(\tau_1')^\perp) + \varphi((\tau_2')^\perp)\right) + \delta_k \int_{\partial F_{\ell_k} \cap \{z_0 + r\tau : \tau \notin (\tau_1', \tau_2')\}} \varphi(\nu^i_{F_{\ell_k}})\, d\mathcal{H}^1 \\
&\quad - \frac{\ell_k}{2}\int_{(-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (\hat{\theta}, \pi - \hat{\theta})} \rho^2_{F_{\ell_k},z_0}(\tau(\theta))(2\delta_k + \delta_k^2)\, d\theta\,.
\end{aligned}$$

Recalling that $\rho_{F_{\ell_k},z_0}(\tau) \ge r_0$ for all $\tau \in \mathbb{S}^1$, from (6.1) we deduce that

$$\mathcal{F}_{\ell_k}(\widetilde{F}_{\ell_k}, u_{\ell_k}) - \mathcal{F}_{\ell_k}(F_{\ell_k}, u_{\ell_k}) \le \delta_k \left[\left(2M + \Lambda \frac{M}{m} \right) - \ell_k r_0^2 (\pi + 2\hat{\theta}) \right] < 0$$

whenever k is large enough. This contradicts again the minimality of (F_{ℓ_k}, u_{ℓ_k}) and concludes the proof. □

Next we prove that if (F, u) is a minimum for the penalized problem, then it satisfies an exterior Wulff shape condition, i.e., there exists $\varrho_0 > 0$ such that for every $z \in \partial F$ there exists a translation of $\varrho_0 W$ contained in $\mathbb{R}^2 \setminus F$ such that its boundary either touches ∂F only at z or it coincides with ∂F near z. We recall that, given a function $\varphi : \mathbb{S}^1 \to (0, \infty)$, the (open) *Wulff set* is defined by

$$W := \left\{ w \in \mathbb{R}^2 : \varphi^\circ(w) < 1 \right\}, \tag{6.3}$$

where φ° is the polar function of φ, i.e.,

$$\varphi^\circ(w) := \max_{|z|=1} \frac{z \cdot w}{\varphi(z)}, \quad w \in \mathbb{R}^2 .$$

It can be shown (see [14,17,34]) that up to translations, the Wulff set is the unique solution of the minimization problem

$$\min \left\{ \int_{\partial E} \varphi(\nu_E)\, d\mathcal{H}^1 : E \subset \mathbb{R}^2 \text{ has finite perimeter, } |E| = |W| \right\} =: c_W\, |W|^{\frac{1}{2}} . \tag{6.4}$$

We begin with an auxiliary result, which is of interest in itself.

Proposition 6.3. *There exists a constant $c_0 > 0$, depending only on W, such that the following holds. Let $F \in \mathcal{A}$ and let $C := z_0 + \varrho_0 W$ with $z_0 \in \mathbb{R}^2$ and $\varrho_0 > 0$, be such that $0 \notin \partial C$, $C \subset \mathbb{R}^2 \setminus F$, and $\partial C \cap \partial F$ contains at least two points $P_1 = r_1\sigma_1$, $P_2 = r_2\sigma_2$, with $r_1 > 0$, $r_2 > 0$, and $\sigma_1 \neq \sigma_2$. Let G be the bounded component of $A(\sigma_1,\sigma_2) \cap (\mathbb{R}^2 \setminus \overline{C})$ and let $D := G \setminus F$. Then,*

$$\int_{\partial^* D \setminus \partial C} \varphi(\nu_D)\, d\mathcal{H}^1 - \int_{\partial C \cap \partial^* D} \varphi(\nu_C)\, d\mathcal{H}^1 \geq \frac{c_0}{\varrho_0} |D| , \tag{6.5}$$

where ν_D and ν_C denote the exterior normals to D and C, respectively.

Proof. By rescaling we may assume, without loss of generality, that $\varrho_0 = 1$, so that $|C| = |W|$. Consider a function $\rho_0 : (\sigma_1,\sigma_2) \subset \mathbb{S}^1 \to \mathbb{R}_+$ such that $\sigma \in (\sigma_1,\sigma_2) \mapsto \rho_0(\sigma)\sigma$ is a parametrization of $\partial G \cap A(\sigma_1,\sigma_2)$.

Since $D = G \cap (\mathbb{R}^2 \setminus F)$ and by Lemma 2.4, $\partial F^+ = \partial^* F = \partial^*(\mathbb{R}^2 \setminus F)$ (mod. $\mathcal{H}^1$), using (2.1) we infer that

$$\partial^* D = \big(\partial G \cap F^0\big) \cup \big(\partial F^+ \cap G^1\big) \cup \big(\partial G \cap \partial F^+ \cap \{\nu_G = \nu_F^i\}\big) \quad (\text{mod. } \mathcal{H}^1) . \tag{6.6}$$

In addition, setting $r_i' := \min\{r\sigma_i : r\sigma_i \in \partial C\}$ for $i = 1,2$, we have that, up to a set of vanishing $\mathcal{H}^1$-measure,

$$\begin{aligned} \partial^* D \cap \{r\sigma_1 : r \geq 0\} &= \{r\sigma_1 : \rho_F(\sigma_1+) \leq r \leq r_1'\} , \\ \partial^* D \cap \{r\sigma_2 : r \geq 0\} &= \{r\sigma_2 : \rho_F(\sigma_2-) \leq r \leq r_2'\} . \end{aligned} \tag{6.7}$$

Step 1. We assume, as in Figure 3, that $\partial C \cap \partial F^+ \cap A(\sigma_1,\sigma_2) = \emptyset$, i.e., $\rho_F^+(\sigma) < \rho_0(\sigma)$ for all $\sigma \in (\sigma_1,\sigma_2)$.

Assume first that

$$\int_{\partial^* D \setminus \partial C} \varphi(\nu_D)\, d\mathcal{H}^1 \leq 2 \int_{\partial W} \varphi(\nu_W)\, d\mathcal{H}^1 . \tag{6.8}$$

Then we have

$$\begin{aligned} c_W\, |C \cup D|^{\frac{1}{2}} \leq \int_{\partial^*(C \cup D)} \varphi(\nu_{C\cup D})\, d\mathcal{H}^1 &= \int_{\partial^* D \setminus \partial C} \varphi(\nu_D)\, d\mathcal{H}^1 + \int_{\partial C \setminus \partial^* D} \varphi(\nu_C)\, d\mathcal{H}^1 \\ &\leq 3 \int_{\partial W} \varphi(\nu_W)\, d\mathcal{H}^1 = 3c_W\, |W|^{\frac{1}{2}} = 3c_W\, |C|^{\frac{1}{2}} , \end{aligned}$$

where in the first inequality and in the last equality we used (6.4), while in the first equality we applied (2.2) and (2.3) to the (disjoint) union of C and D, and the second inequality is a consequence

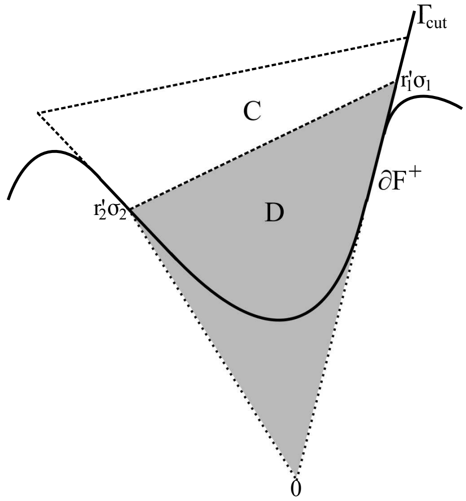

Fig. 3. In the above picture $\rho_F^+ < \rho_0$ in (σ_1, σ_2). D is enclosed by ∂C and ∂F^+ and G is the shaded region.

of (6.8). In turn, by (6.4),

$$\begin{aligned}\int_{\partial^* D\setminus\partial C}&\varphi(\nu_D)\,d\mathcal{H}^1-\int_{\partial C\cap\partial^* D}\varphi(\nu_C)\,d\mathcal{H}^1\\ &=\int_{\partial^*(C\cup D)}\varphi(\nu_{C\cup D})\,d\mathcal{H}^1-\int_{\partial C\setminus\partial^* D}\varphi(\nu_C)\,d\mathcal{H}^1-\int_{\partial C\cap\partial^* D}\varphi(\nu_C)\,d\mathcal{H}^1\\ &=\int_{\partial^*(C\cup D)}\varphi(\nu_{C\cup D})\,d\mathcal{H}^1-\int_{\partial C}\varphi(\nu_C)\,d\mathcal{H}^1\\ &\geq c_W|C\cup D|^{\frac12}-c_W|C|^{\frac12}\geq\frac{c_W|D|}{|C\cup D|^{\frac12}+|C|^{\frac12}}\geq\frac{c_W}{4|C|^{\frac12}}|D|\,.\end{aligned}$$

This concludes the proof in this case.

If the opposite inequality to (6.8) holds, then

$$\int_{\partial^* D\setminus\partial C}\varphi(\nu_D)\,d\mathcal{H}^1-\int_{\partial C\cap\partial^* D}\varphi(\nu_C)\,d\mathcal{H}^1\geq\frac12\int_{\partial^* D\setminus\partial C}\varphi(\nu_D)\,d\mathcal{H}^1\geq\frac{m}{2}\mathcal{H}^1(\partial^* D\setminus\partial C)\,. \tag{6.9}$$

From (6.6), (6.7), and the assumption $\rho_F^+ < \rho_0$ in (σ_1, σ_2), we deduce that up to a set of $\mathcal{H}^1$-measure zero,

$$\partial^* D\setminus\partial C=\big(\partial F^+\cap A(\sigma_1,\sigma_2)\big)\cup\{r\sigma_1:\,\rho_F(\sigma_1+)\leq r\leq r_1'\}\cup\{r\sigma_2:\,\rho_F(\sigma_2-)\leq r\leq r_2'\}\,.$$

Setting $\bar r := \max\{r_1', r_2'\}$ and $r_0 := \mathrm{dist}(\partial F^+\cap A[\sigma_1,\sigma_2], 0)$, it follows that

$$\mathcal{H}^1(\partial^* D\setminus\partial C)\geq\bar r-r_0\,. \tag{6.10}$$

Note that D is contained in the region inside $A[\sigma_1, \sigma_2]$ bounded from above by the segment with endpoints $r_1'\sigma_1 \in \overline{C}$ and $r_2'\sigma_2 \in \overline{C}$ whose length is smaller than $\mathrm{diam}\, W$, and from below by the open disc of radius r_0. Therefore, from (6.10) we get that

$$|D|\leq c(\bar r-r_0)\leq c\mathcal{H}^1(\partial^* D\setminus\partial C)\,,$$

where the constant $c > 0$ only depends on W, and in view of (6.9), we conclude (6.5).

Step 2. We now consider the general case. Since $\rho_0 - \rho_F^+$ is a lower semicontinuous function, the set $\{\rho_0 - \rho_F^+ > 0\} \cap (\sigma_1, \sigma_2)$ is open, therefore it can be written as the union of countably many open intervals (σ_i', σ_i''), $i \in J \subset \mathbb{N}$. For each $i \in J$, the set $D_i := D \cap A(\sigma_i', \sigma_i'')$ satisfies the hypotheses of Step 1, and (6.5) follows observing that $|D| = \sum_i |D_i|$ and that, by (6.6), $\partial^* D$ coincides with the essentially disjoint union of the $\partial^* D_i$'s, up to a set of $\mathcal{H}^1$-measure zero. $\square$

Proposition 6.4. *Let $(F,u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$ and let $0 < \varrho_0 < \frac{c_0}{\ell_0}$, where c_0 and ℓ_0 are the constants given in Propositions 6.3 and 6.1, respectively. If $C := z_0 + \varrho_0 W$ is contained in $\mathbb{R}^2 \setminus F$, then $\partial F \cap \partial C$ is a connected closed arc (possibly empty).*

Proof. If $\partial F \cap \partial C$ is empty or contains just one point there is nothing to prove. Otherwise assume that $\partial F \cap \partial C$ contains two distinct points P_1 and P_2. We want to show that one of the two arcs on ∂C connecting P_1 to P_2 is contained in $\partial F \cap \partial C$. If one of the two points coincides with the origin, since C is convex and contained in $\mathbb{R}^2 \setminus F$ and since F is starshaped with respect to 0, then the segment $[P_1, P_2]$ is contained in $\partial F \cap \partial C$. A similar argument applies if $P_1 = r_1\sigma_1$, $P_2 = r_2\sigma_2$ with $\sigma_1 = \sigma_2$. Therefore, we may assume that $r_1, r_2 > 0$ and $\sigma_1 \neq \sigma_2$. If $0 \in \partial C$, the union of the segments $[P_1, 0]$ and $[0, P_2]$ is contained in $\partial F \cap \partial C$ so that we may also assume that $0 \notin \partial C$.

Let D be as in Proposition 6.3. The proof will be concluded provided we show that the open set D is empty. Assume by contradiction that $D \neq \emptyset$, and set $\widetilde{F} := F \cup \overline{D}$. Then $\widetilde{F} \in \mathcal{A}$ and

$$\partial \widetilde{F} \cap A(\sigma_1, \sigma_2) = \gamma\,,$$

where $\gamma := \{r\sigma : r = \rho_0(\sigma),\ \sigma_1 < \sigma < \sigma_2\}$ with ρ_0 is as in the proof of Proposition 6.3. By Lemma 2.4, Lemma 2.1 and (6.6), we obtain (see Figure 3 again)

$$\partial^* F \cap A(\sigma_1,\sigma_2) = \partial F^+ \cap A(\sigma_1,\sigma_2) = \big(\big(\partial^* D \setminus \partial C\big) \cap A(\sigma_1,\sigma_2)\big) \cup (\partial^* F \cap \gamma) \qquad (\text{mod. } \mathcal{H}^1)$$

with a disjoint union on the right-hand side, and

$$\gamma = ((\partial^* D \cap \partial C) \cap A(\sigma_1,\sigma_2)) \cup (\partial^* F \cap \gamma) \qquad (\text{mod. } \mathcal{H}^1)$$

with an $\mathcal{H}^1$-essentially disjoint union. Consequently,

$$\begin{aligned}\int_{\partial^* F \cap A(\sigma_1,\sigma_2)} \varphi(\nu_F^i)\, d\mathcal{H}^1 &- \int_{\partial^* \widetilde{F} \cap A(\sigma_1,\sigma_2)} \varphi(\nu_{\widetilde{F}}^i)\, d\mathcal{H}^1 \\ &= \int_{(\partial^* D \setminus \partial C) \cap A(\sigma_1,\sigma_2)} \varphi(\nu_D)\, d\mathcal{H}^1 - \int_{\partial^* D \cap \partial C} \varphi(\nu_C)\, d\mathcal{H}^1\,, \qquad (6.11)\end{aligned}$$

where we have used the fact that $\partial C \cap \partial^* D = \partial C \cap \partial^* D \cap A(\sigma_1,\sigma_2)$ (mod. $\mathcal{H}^1$), which is a consequence of (6.7).

Using (6.7) again and denoting by $\widetilde{\Gamma}_{\text{cut}}$ the "cut part" of $\partial \widetilde{F}$ (see (2.12)), we have for $i = 1, 2$,

$$\begin{aligned}\int_{\partial^* F \cap \{r\sigma_i :\, r \geq 0\}} \varphi(\nu_F^i)\, d\mathcal{H}^1 &+ \int_{\Gamma_{\text{cut}} \cap \{r\sigma_i :\, r \geq 0\}} \big(\varphi(\nu_F^i) + \varphi(-\nu_F^i)\big)\, d\mathcal{H}^1 \\ = \int_{\partial^* \widetilde{F} \cap \{r\sigma_i :\, r \geq 0\}} \varphi(\nu_{\widetilde{F}}^i)\, d\mathcal{H}^1 &+ \int_{\widetilde{\Gamma}_{\text{cut}} \cap \{r\sigma_i :\, r \geq 0\}} \big(\varphi(\nu_{\widetilde{F}}^i) + \varphi(-\nu_{\widetilde{F}}^i)\big)\, d\mathcal{H}^1 \\ &+ \int_{\partial^* D \cap \{r\sigma_i :\, r \geq 0\}} \varphi(\nu_D)\, d\mathcal{H}^1\,. \qquad (6.12)\end{aligned}$$

Combining (6.11) and (6.12) we obtain

$$\mathcal{J}(F) - \mathcal{J}(\widetilde{F}) \geq \int_{\partial^* D \setminus \partial C} \varphi(\nu_D)\, d\mathcal{H}^1 - \int_{\partial^* D \cap \partial C} \varphi(\nu_C)\, d\mathcal{H}^1$$

with equality if $\Gamma_{\text{cut}} \cap A(\sigma_1,\sigma_2) = \emptyset$. In view of Proposition 6.3 we conclude that

$$\begin{aligned}\mathcal{F}_{\ell_0}(F,u) - \mathcal{F}_{\ell_0}(\widetilde{F},u) &\geq \mathcal{J}(F) - \mathcal{J}(\widetilde{F}) - \ell_0|\widetilde{F} \setminus F| \\ &\geq \int_{\partial^* D \setminus \partial C} \varphi(\nu_D)\, d\mathcal{H}^1 - \int_{\partial^* D \cap \partial C} \varphi(\nu_C)\, d\mathcal{H}^1 - \ell_0|D| \geq \left(\frac{c_0}{\varrho_0} - \ell_0\right)|D| > 0\,,\end{aligned}$$

which contradicts the minimality of (F,u). Therefore $D = \emptyset$ and the proof is complete. □

Theorem 6.5 (Uniform Exterior Wulff Condition). *Let $(F,u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$. Then for all $z \in \partial F$ there exists $w \in \mathbb{R}^2$ such that $w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ and $z \in \partial\,(w + \varrho_0 W)$, where ϱ_0 is as in Proposition 6.4.*

Proof. Set

$$U := \bigcup \left\{ w + \varrho_0 W : w + \varrho_0 W \subset \mathbb{R}^2 \setminus F \right\} . \tag{6.13}$$

Then U is an open set. To prove the statement, it suffices to show that

$$U = \mathbb{R}^2 \setminus F . \tag{6.14}$$

Indeed, in this case $\partial F = \partial U$, and so if $z \in \partial F$, there exist sequences $\{a_n\}$, $\{w_n\} \subset \mathbb{R}^2$ such that $a_n \in w_n + \varrho_0 W \subset U$ and $a_n \to z$. Then the sequence $\{w_n\}$ is bounded, and so, up to a subsequence, $w_n \to w$ for some $w \in \mathbb{R}^2$. Note that $z \in w + \varrho_0 \overline{W}$. We claim that $w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$. To see this, assume that there exists $\bar{w} \in F \cap (w + \varrho_0 W)$ and let $r > 0$ be such that $B_r(\bar{w}) \subset w + \varrho_0 W$. Let n be so large that $|w_n - w| < r/2$. Then if $\bar{z} \in B_{r/2}(\bar{w})$, we have that $\bar{z} - w_n + w \in B_r(\bar{w}) \subset w + \varrho_0 W$, therefore $\bar{z} - w_n \in \varrho_0 W$, i.e., $\bar{z} \in w_n + \varrho_0 W$, which shows that $B_{r/2}(\bar{w}) \subset w_n + \varrho_0 W \subset \mathbb{R}^2 \setminus F$. This contradicts the fact that $\bar{w} \in F$. Hence, the claim holds. Finally, using the facts that $z \in w + \varrho_0 \overline{W} = (w + \varrho_0 W) \cup \partial (w + \varrho_0 W)$, $z \in \partial U$, and $w + \varrho_0 W \subset U$, we conclude that $z \in \partial (w + \varrho_0 W)$.

The remaining of the proof is dedicated to prove that $U = \mathbb{R}^2 \setminus F$. Observe that, since $\mathbb{R}^2 \setminus F$ is pathwise connected, this is equivalent to having $\partial U \cap \left(\mathbb{R}^2 \setminus F\right) = \emptyset$. We argue by contradiction and assume that there exists $a \in \partial U \cap \left(\mathbb{R}^2 \setminus F\right)$. Since $a \in \partial U$ we may find two sequences $\{w_n\}$ and $\{a_n\}$ in $\mathbb{R}^2$ such that $a_n \in w_n + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ and $a_n \to a$. Arguing as above, there exists w_0 such that $C := w_0 + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ and $a \in \partial C$. Observe that $\partial C \cap \partial F$ is nonempty, since otherwise we could slightly translate C in such a way that the resulting set C' would still be contained in $\mathbb{R}^2 \setminus F$ and would contain a. By the definition of U, this would imply that a belongs to (the interior of) C' and C' is contained in U, and in turn that $a \in U$, which contradicts the fact that $a \in \partial U$. Hence, by Proposition 6.4, $\partial C \cap \partial F$ is either a point or a connected arc.

Up to a rotation, we may assume that the projection of C on the (horizontal) x-axis is the interval (α, β) with $\alpha < 0 < \beta$, and that C is contained in the (vertical) half line $\{y > 0\}$. This is obvious if $0 \notin \partial C$, but it can be easily shown to be true also when $0 \in \partial C$, by the convexity of C. Then there exist two functions f, $g : [\alpha, \beta] \to [0, \infty)$, with f convex and g concave such that

$$C = \left\{ (x, y) \in \mathbb{R}^2 : x \in (\alpha, \beta) \ , \ f(x) < y < g(x) \right\} .$$

Since F is starshaped with respect to the origin, $\partial F \cap \partial C$ is contained in the graph of f. Denote by $z_0 = (x_0, f(x_0))$ and $z_1 = (x_1, f(x_1))$ the left and right endpoints of $\partial C \cap \partial F$, respectively, and set

$$\gamma := \{(x, f(x)) : x_0 \le x \le x_1\} . \tag{6.15}$$

We now consider several cases:

Case 1: $a = (x, g(x))$ for some $x \in (\alpha, \beta)$. In this case and as before, by slightly translating C upwards we would obtain a set $C' := w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ containing a. This would contradict the fact that $a \in \partial U$.

Case 2: $a = (\beta, y)$ for some $y \in (f(\beta), g(\beta)]$, assuming that this interval is nonempty (the case $a = (\alpha, y)$ with $y \in (f(\alpha), g(\alpha)]$ is analogous). In this case, to get a contradiction we first translate C slightly upwards thus obtaining a set C' with positive distance from ∂F and such that $a \in \partial C'$. Then we translate C' to the right to obtain a set C'' that includes a in its interior and is contained in $\mathbb{R}^2 \setminus F$. This again contradicts the fact that $a \in \partial U$.

We are now left with the situation in which $a = (\overline{x}, f(\overline{x}))$ for some $\overline{x} \in [\alpha, \beta]$. Since $a \notin F$, by (6.15), without loss of generality we may assume that $\overline{x} < x_0$ (the case $\overline{x} > x_1$ is analogous).

Case 3: Consider first the case in which f is not affine in the interval $(\overline{x}, x_0)$ (see Figure 4). Then there exists $\widetilde{x} \in (\overline{x}, x_0)$ such that f is not affine in the interval $(\overline{x}, \widetilde{x})$. Note that the arc

$$\gamma' := \{(x, f(x)) : \alpha \le x \le \widetilde{x}\}$$

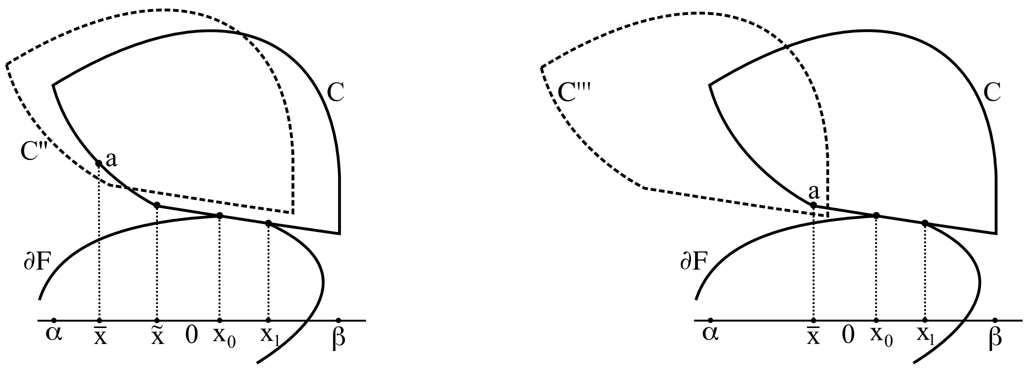

Fig. 4. The situation analyzed in Cases 3 and 4 of the proof of Theorem 6.5. Notice that in the left picture, the quantity s defined by (6.16) is negative, hence C'' is obtained translating C first upward and then to the left.

lies at positive distance $\delta_0 > 0$ from the boundary of F. Observe also that there exist $s \in \mathbb{R}$ and $0 < \delta_1 < \min\{\beta - \widetilde{x}, \widetilde{x} - \overline{x}\}$ such that if $0 < \delta < \delta_1$, then

$$\frac{f\left(\overline{x}+\delta\right)-f\left(\overline{x}\right)}{\delta} < s < \min\left\{\frac{f\left(x+\delta\right)-f\left(x\right)}{\delta} : x \in [\widetilde{x}-\delta, \beta-\delta]\right\} . \tag{6.16}$$

Therefore, if we choose

$$0 < \delta < \min\left\{\frac{\delta_0}{\sqrt{1+s^2}}, \delta_1\right\} ,$$

we may first translate C in the vertical direction by the vector $(0, -s\delta)$ to obtain C', and then translate C' in the horizontal direction by $(-\delta, 0)$ thus obtaining a new set $C'' \subset \mathbb{R}^2 \setminus F$ containing a. Indeed, after these translations the points of γ' have been moved to a distance equal to $\delta\sqrt{1+s^2} < \delta_0$, hence in their final position they are still away from ∂F. Note also that after these translations the graph of f has been moved to the graph of the function $\overline{f} : [\alpha-\delta, \beta-\delta] \to \mathbb{R}$ defined by $\overline{f}\,(x) := f\,(x+\delta) - s\delta$. By (6.16), it follows that $\overline{f}\,(\overline{x}) < f\,(\overline{x})$, hence $a \in C''$, provided that δ is so small that $f(\overline{x}) < g(\overline{x}+\delta) - s\delta$. Indeed, if $f(\overline{x}) < g(\overline{x})$, this choice of δ is obviously possible, otherwise, if $\overline{x} = \alpha$ and $g(\alpha) = f(\alpha)$, this choice of δ is possible if one chooses s satisfying $f'_+(\alpha) < s < g'_+(\alpha)$ in addition to (6.16). Finally, we have that for every $x \in [\widetilde{x}-\delta, \beta-\delta]$, $\overline{f}\,(x) > f\,(x)$. Therefore we may conclude that $C'' \subset \mathbb{R}^2 \setminus F$ and this is again a contradiction.

Case 4: Assume now that f is affine in some maximal interval $(\overline{x}, x')$ where $x_0 \le x' \le \beta$, and let L be the line containing the graph of f above $(\overline{x}, x')$. In this case we can slide C in the left direction along L in such a way that the point $(x', f\,(x'))$ has been moved to the point z_0. Note that this is possible because, while sliding C, the set $\partial C \setminus \gamma$ cannot touch the boundary of F otherwise, by Proposition 6.4, there would be an arc in ∂C contained ∂F and containing a. Let C' be the resulting set. Note that now $\partial C' \cap \partial F = \{z_0\}$, $a \in \partial C'$, and $C' \subset \mathbb{R}^2 \setminus F$. Therefore, with the same argument as before, we may slide also C' slightly to the left along L, thus getting a new set $C'' \subset \mathbb{R}^2 \setminus F$ such that $\partial C'' \cap \partial F = \emptyset$ and $a \in \partial C''$. Finally, by translating C'' downward we obtain some set $C''' \subset \mathbb{R}^2 \setminus F$ containing a. This contradiction concludes the proof. □

7. Regularity in the polygonal case

Throughout this section we will assume that W is a polygon with internal angles greater than $\pi/2$, and we are going to prove that if (F, u) is a minimizer of the constrained problem (3.10), then the boundary of F is the union of finitely many Lipschitz graphs. In particular, this will imply that the number of cut segments is at most finite. The essential tool used to prove this regularity result is the uniform exterior Wulff condition established in the previous section. As a first step, we show that

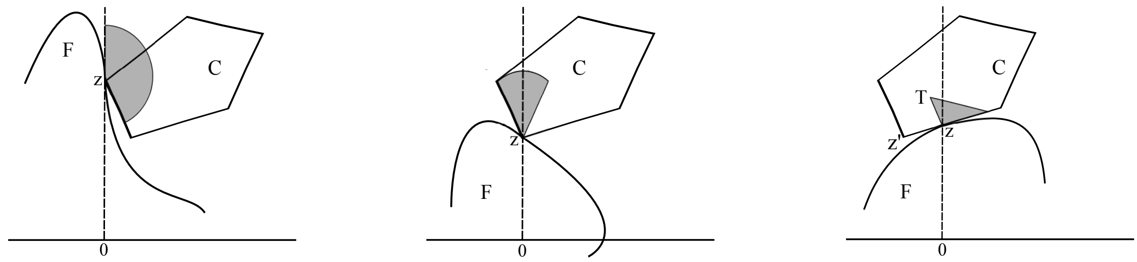

Fig. 5. At every point $z \in \partial F$ there exists an exterior sector with a radius and an angle uniformly bounded from below.

this condition implies the existence of uniform exterior sectors at every point of ∂F, where the three exterior sectors at a point $z_0 = r_0\sigma(\theta_0)$ in $\mathbb{R}^2 \setminus \{0\}$ determined by $h > 0$ and $\alpha \in (0, \pi]$ are defined by

$$\begin{aligned}
\mathcal{S}^{+}_{\alpha,h}(z_0) &:= z_0 + \left\{ r\sigma(\theta) \in \mathbb{R}^2 : \theta_0 < \theta < \theta_0 + \alpha \,,\, 0 < r < h \right\} , \\
\mathcal{S}^{-}_{\alpha,h}(z_0) &:= z_0 + \left\{ r\sigma(\theta) \in \mathbb{R}^2 : \theta_0 - \alpha < \theta < \theta_0 \,,\, 0 < r < h \right\} , \\
\mathcal{S}_{\alpha,h}(z_0) &:= z_0 + \left\{ r\sigma(\theta) \in \mathbb{R}^2 : \theta_0 - \alpha < \theta < \theta_0 + \alpha \,,\, 0 < r < h \right\} .
\end{aligned}$$

Proposition 7.1. *Assume that the Wulff set W (see (6.3)) is a polygon with internal angles greater than $\frac{\pi}{2}$. Let $(F, u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$ (see Proposition 6.1). Then there exist $\alpha > \frac{\pi}{2}$, $\beta > 0$, and $h > 0$ such that for all $z \in \partial F \setminus \{0\}$ at least one of the three exterior sectors $\mathcal{S}^{+}_{\alpha,h}(z)$, $\mathcal{S}^{-}_{\alpha,h}(z)$, $\mathcal{S}_{\beta,h}(z)$ is contained in $\mathbb{R}^2 \setminus F$.*

Proof. Let $\alpha_0 > \frac{\pi}{2}$ be the minimum of the internal angles of W and $\frac{\pi}{2} < \alpha_1 < \alpha_0$. Let $z \in \partial F \setminus \{0\}$, and let $C := w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ be such that $z \in \partial C$, where ϱ_0 is as in Theorem 6.5. Without loss of generality we may assume that z lies on the positive y-axis so that $z = (0, r)$ with $r > 0$ (see Figure 5).

Consider first the case in which z is a vertex of C. If the y-axis lies to the left of C, then there exists an angle $\alpha \geq \alpha_0$ greater than or equal to the internal angle of C at z such that $\mathcal{S}^{-}_{\alpha,h_0}(z) \subset \mathbb{R}^2 \setminus F$ for $h_0 := \varrho_0 s_W$, where s_W denotes the length of the shortest side of W. Similarly, $\mathcal{S}^{+}_{\alpha,h_0}(z) \subset \mathbb{R}^2 \setminus F$ if the y-axis lies to the right of C. It remains to consider the case in which the y-axis crosses the interior of C. In this case, either $\mathcal{S}_{\alpha_0-\alpha_1,h_0}(z) \subset \mathbb{R}^2 \setminus F$ or at least one of the two sectors $\mathcal{S}^{+}_{\alpha_1,h_0}(z)$ and $\mathcal{S}^{-}_{\alpha_1,h_0}(z)$ is contained in $\mathbb{R}^2 \setminus F$.

Next suppose that z belongs to one of the sides of C, which we denote by S. Let z' be the vertex on S closest to z (if z is the middle point of S then take any of the two). Then a triangle T with one vertex in z and two sides of length $h_0/2$ departing from z and parallel to the two sides of C that intersect at z' is contained in C. Note that the angle of T at z is the same angle of C at z', and so it is greater than or equal to α_0. Since the y-axis crosses the interior of T, we may argue as before to conclude that either $\mathcal{S}_{\alpha_0-\alpha_1,\frac{h_0}{2}}(z_0) \subset \mathbb{R}^2 \setminus F$ or at least one of the two sectors $\mathcal{S}^{+}_{\alpha_1,\frac{h_0}{2}}(z)$ and $\mathcal{S}^{-}_{\alpha_1,\frac{h_0}{2}}(z)$ is contained in $\mathbb{R}^2 \setminus F$.

Hence, the proposition holds with $\alpha := \alpha_1$, $\beta := \alpha_0 - \alpha_1$, and $h := h_0/2$. $\square$

Remark 7.2. In view of the uniformity of the size of the sectors, we can extend Proposition 7.1 to the case $z = 0$ as follows. If 0 belongs to ∂F, then there exists θ_0 such that one of the three sectors $\mathcal{S}^{+}_{\alpha,h,\theta_0}(0)$, $\mathcal{S}^{-}_{\alpha,h,\theta_0}(0)$, $\mathcal{S}_{\beta,h,\theta_0}(0)$ is contained in $\mathbb{R}^2 \setminus F$, where

$$\mathcal{S}^{+}_{\alpha,h,\theta_0}(0) := \left\{ r\sigma(\theta) \in \mathbb{R}^2 : \theta_0 < \theta < \theta_0 + \alpha \,,\, 0 < r < h \right\}$$

and the two other sectors are defined similarly. Indeed, consider a sequence $\{z_n\} \subset \partial F \setminus \{0\}$ converging to 0. Applying Proposition 7.1 to each z_n, we find that for every n at least one of the three exterior sectors $\mathcal{S}^+_{\alpha,h}(z_n)$, $\mathcal{S}^-_{\alpha,h}(z_n)$, $\mathcal{S}_{\beta,h}(z_n)$ is contained in $\mathbb{R}^2 \setminus F$. Therefore, there exists a subsequence (not relabeled) such that, say, $\mathcal{S}^+_{\alpha,h}(z_n)$ is contained in $\mathbb{R}^2 \setminus F$ for every n. Moreover, we can assume that $z_n/|z_n| \to \sigma(\theta_0)$ for some $\theta_0 \in [0, 2\pi)$. We claim that $\mathcal{S}^+_{\alpha,h,\theta_0}(0)$ is contained in $\mathbb{R}^2 \setminus F$. If not, then there would exist $w \in \mathcal{S}^+_{\alpha,h,\theta_0}(0) \cap F$. Since $\mathcal{S}^+_{\alpha,h,\theta_0}(0)$ is open, then for n large enough $w \in \mathcal{S}^+_{\alpha,h}(z_n)$, which is a contradiction.

Remark 7.3. If W is a polygon with internal angles greater than or equal to $\frac{\pi}{2}$, denote by $(L_1, L_1'),\ldots, (L_k, L_k')$ the pairs of adjacent sides of W forming an internal angle of $\frac{\pi}{2}$ and denote by $(\sigma_1, \sigma_1'),\ldots, (\sigma_k, \sigma_k')$ their corresponding directions (observe that, by the convexity of W, k can be at most 4). If $[\sigma', \sigma'']$ does not contain any of the directions σ_i, σ_i', $i = 1, \ldots, k$, then the conclusion of the previous proposition holds for all $z \in \partial F \cap A(\sigma', \sigma'')$ (with parameters α, β and h depending on σ' and σ''). It also holds for $z = 0$, provided that there exists a sequence $z_n \in \partial F \cap A(\sigma', \sigma'')$ converging to 0.

The following lemma will also be used in the next section.

Lemma 7.4. *Let $F \in \mathcal{A}$ and let $z \in \partial F^+ \setminus \{0\}$ (see (2.8)). Assume that there exist $\delta > 0$, $\nu \in \mathbb{S}^1$, and $\eta > 0$ such that for every $z' \in \partial F \cap B_\delta(z)$ and for every $\nu' \in \mathbb{S}^1$ satisfying $\nu \cdot \nu' \geq \eta$, the segment $\{z' + t\nu' : 0 < t < 2\delta\}$ is contained in $\mathbb{R}^2 \setminus F$. Then there exist $\overline{\delta}_1, \overline{\delta}_2 \in (0, \delta)$ such that*

$$\partial F \cap \left\{ z + t_1 \nu^\perp + t_2 \nu : -\overline{\delta}_i < t_i < \overline{\delta}_i \right\}$$

is the graph of a Lipschitz function.

Proof. *Step 1.* Let L_1 be the line through z orthogonal to ν oriented in the direction $-\nu^\perp$, and let L_2 be the line through z oriented in the direction ν. We claim that the set $\partial F \cap B_\delta(z)$ is contained in the graph of a Lipschitz function defined on L_1 in an open neighborhood of z. Let Π and $\Pi^\perp$ be the projection of $\mathbb{R}^2$ onto L_1 and L_2, respectively.

Let z_1, $z_2 \in \partial F \cap B_\delta(z)$ and, without loss of generality, assume that $\Pi^\perp(z_2) \geq \Pi^\perp(z_1)$. Let $S := z_1 + \{r\nu : r \geq 0\}$, and consider the two half-lines S_1 and S_2 with endpoint z_1 and forming on both sides of S an angle of $\arccos \eta$. By assumption, the open sector of radius 2δ with center at z_1, bounded by the half-lines S_1 and S_2, and intersecting S, is contained in $\mathbb{R}^2 \setminus F$. Hence, since $z_2 \in \partial F$, we have that z_2 does not belong to this sector, and so

$$\left|\Pi^\perp(z_2) - \Pi^\perp(z_1)\right| \leq m \left|\Pi(z_2) - \Pi(z_1)\right| ,$$

where $m := \tan\left(\frac{\pi}{2} - \arccos \eta\right)$. Note that this inequality implies that if z_1, $z_2 \in \partial F \cap B_\delta(z)$ and $\Pi(z_1) = \Pi(z_2)$, then $z_1 = z_2$. Therefore, setting $P := \Pi(\partial F \cap B_\delta(z))$, it follows that $\Pi_{|\partial F \cap B_\delta(z)}$ is one-to-one, and the function $f : P \to L_2$, defined by $f(w) := \Pi^\perp\left(\left(\Pi_{|\partial F \cap B_\delta(z)}\right)^{-1}(w)\right)$, is Lipschitz continuous with Lipschitz constant less than or equal to m.

Step 2. To complete the proof it suffices to show that P contains an open neighborhood of z in L_1. Write $z = r\sigma_0$ with $r > 0$, and assume without loss of generality that $\rho_F^+(\sigma_0) = \rho_F(\sigma_0-)$. Take $\delta > 0$ so small that $0 < \delta < r$, and in such a way that if $A(\sigma_1, \sigma_2)$ is the smallest sector containing $B_\delta(z)$, then $\rho_F^-(\sigma) > r/2$ for all $\sigma \in [\sigma_1, \sigma_2]$. As in the proof of Lemma 2.2, we have that $\partial F \cap A[\sigma_1, \sigma_0]$ is a compact connected set. Consequently (see Theorem 4.46 in [24]), there exists a curve γ_1 contained in $\partial F \cap A[\sigma_1, \sigma_0]$ connecting $\rho_F(\sigma_1)\sigma_1$ to z. Similarly, there exists a curve γ_2 contained in $\partial F \cap A[\sigma_0, \sigma_2]$ connecting $\rho_F(\sigma_2)\sigma_2$ to z. Observe that the two curves γ_1 and γ_2 intersect only at the point z. By Step 1, we deduce that $\Pi(\gamma_1 \cap B_\delta(z))$ contains a left or right open neighborhood N_1 of z in L_1, while $\Pi(\gamma_2 \cap B_\delta(z))$ contains an opposite side open neighborhood N_2. We conclude that $N_1 \cup N_2$ is a neighborhood of z in L_1. □

Remark 7.5. Arguing as in the previous proof, one can also show a one sided version of the lemma. More precisely, let $z = r\sigma_0$, for some $\sigma_0 \in \mathbb{S}^1$, $r \geq 0$. Assume that there exist $\delta > 0$, $\nu \in \mathbb{S}^1$, and $\eta > 0$

such that for every $z' \in \overline{\partial F \cap A(\sigma_0, \sigma_0 + \delta)}$ and for every $\nu' \in \mathbb{S}^1$ satisfying $\nu \cdot \nu' \geq \eta$, the segment

$$\{z' + t\nu' : 0 < t < \delta\}$$

is contained in $\mathbb{R}^2 \setminus F$. Then there exists $0 < \overline{\delta} < \delta$ such that $\overline{\partial F \cap A(\sigma_0, \sigma_0 + \overline{\delta})}$ is the graph of a Lipschitz function.

We are now in position to prove the regularity of ∂F.

Theorem 7.6. *Assume that the Wulff set W is a polygon with internal angles greater than $\frac{\pi}{2}$. Let $(F, u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$. Then ∂F is the union of finitely many Lipschitz graphs. Precisely, ∂F contains finitely many cut segments, i.e., S_F is finite, and there exists a finite set $\Gamma_{\text{sing}} \subset \partial F^+$ such that:*

(i) if $z \in \partial F^+ \setminus \Gamma_{\text{sing}}$, then there exists a neighborhood $\mathcal{N}(z)$ of z such that $\partial F \cap \mathcal{N}(z)$ is the graph of a Lipschitz function;

(ii) if $z = r_0\sigma_0 \in \Gamma_{\text{sing}} \setminus \{0\}$, then there exists a neighborhood $\mathcal{N}(z)$ of z such that $\big(\partial F \cap \mathcal{N}(z)\big) \setminus \Gamma_{\text{cut}}$ is the union of two graphs of Lipschitz functions intersecting only at z;

(iii) if $0 \in \Gamma_{\text{sing}}$, then there exists a neighborhood $\mathcal{N}_0$ of 0 such that $\partial F \cap \mathcal{N}_0$ is the union of at most six graphs of Lipschitz functions intersecting only at 0.

Proof. Let $\alpha > \frac{\pi}{2}$, $\beta > 0$, and $h > 0$ be as in Proposition 7.1. Then we can write

$$\partial F \setminus \{0\} = \Gamma^+ \cup \Gamma^- \cup \Gamma^0 \,,$$

where

$$\begin{aligned} \Gamma^+ &:= \Big\{ z \in \partial F \setminus \{0\} : \mathcal{S}^+_{\alpha,h}(z) \subset \mathbb{R}^2 \setminus F \Big\} \,, \\ \Gamma^- &:= \Big\{ z \in \partial F \setminus \{0\} : \mathcal{S}^-_{\alpha,h}(z) \subset \mathbb{R}^2 \setminus F \Big\} \,, \\ \Gamma^0 &:= \big\{ z \in \partial F \setminus \{0\} : \mathcal{S}_{\beta,h}(z) \subset \mathbb{R}^2 \setminus F \big\} \,. \end{aligned}$$

Step 1. Let $z \in \partial F^+$ and assume that $z \neq 0$. We now consider all possible cases.

Case 1: Either $z \in \Gamma^+ \setminus \big(\Gamma^- \cup \Gamma^0\big)$ or $z \in \Gamma^- \setminus \big(\Gamma^+ \cup \Gamma^0\big)$. We only consider the first case, since the other one is analogous. We claim that there exists $\delta > 0$ such that $\partial F \cap B_\delta(z) \subset \Gamma^+$. Indeed, if this were not true, then there would exist a sequence $\{z_n\} \subset \Gamma^- \cup \Gamma^0$ converging to z, i.e., for infinitely many n's either $\mathcal{S}^-_{\alpha,h}(z_n) \subset \mathbb{R}^2 \setminus F$ or $\mathcal{S}_{\beta,h}(z_n) \subset \mathbb{R}^2 \setminus F$. Passing to the limit, either $\mathcal{S}^-_{\alpha,h}(z) \subset \mathbb{R}^2 \setminus F$ or $\mathcal{S}_{\beta,h}(z) \subset \mathbb{R}^2 \setminus F$, which contradicts the fact that $z \in \Gamma^+ \setminus \big(\Gamma^- \cup \Gamma^0\big)$. Let $\nu \in \mathbb{S}^1$ be the unit vector parallel to the vector that bisects the sector $S^+_{\alpha,h}(z)$ and points towards $\mathbb{R}^2 \setminus F$. By taking δ smaller if necessary, the assumptions of Lemma 7.4 are satisfied in $\partial F \cap B_\delta(z)$ for some $\eta > 0$. Hence $\partial F \cap \mathcal{N}(z)$ is the graph of a Lipschitz function for some open neighborhood $\mathcal{N}(z)$ of z.

Case 2: Either $z \in \big(\Gamma^+ \cap \Gamma^0\big) \setminus \Gamma^-$ or $z \in \big(\Gamma^- \cap \Gamma^0\big) \setminus \Gamma^+$. Again, we only consider the first case, since the other is analogous. The same continuity argument as before shows that there exists $\delta > 0$ such that $\partial F \cap B_\delta(z) \subset \Gamma^+ \cup \Gamma^0$. Therefore at each point $z' \in \partial F \cap B_\delta(z)$ we have $\mathcal{S}^+_{\beta',h}(z') \subset \mathbb{R}^2 \setminus F$ with $\beta' := \min\{\beta, \alpha\}$. Then we can argue as in the previous case to conclude that $\partial F \cap \mathcal{N}(z)$ is the graph of a Lipschitz function for some open neighborhood $\mathcal{N}(z)$ of z.

Case 3: $z \in \Gamma^0 \setminus (\Gamma^+ \cup \Gamma^-)$. Still by a continuity argument there exists $\delta > 0$ such that $\partial F \cap B_\delta(z) \subset \Gamma^0 \setminus (\Gamma^+ \cup \Gamma^-)$. The conclusion then follows as in Case 1.

Case 4: Assume that $z = r\sigma(\theta) \in \Gamma^+ \cap \Gamma^-$. Since $z \in \partial F^+$ we have $r = (\rho^*_F)^+(\theta)$. We shall prove that there exists a neighborhood $\mathcal{N}(z)$ of z such $(\partial F \cap \mathcal{N}(z)) \setminus \{r'\sigma(\theta) : r < r' \leq \rho^*_F(\theta)\}$ is the union of two Lipschitz graphs intersecting only at z.

First we show that $\Gamma^+ \cap \Gamma^- \cap \partial F^+$ contains at most finitely many points. Indeed, assume that $z_0 = r_0\sigma(\theta_0)$ and $z_1 = r_1\sigma(\theta_1)$ are two distinct points in $\Gamma^+ \cap \Gamma^- \cap \partial F^+$. We claim that $|z_0 - z_1| \geq h$

or $|\theta_0 - \theta_1| \geq \min\{2\alpha - \pi, \frac{\pi}{4}\}$ from which the conclusion follows. To prove the claim, assume that $|z_0 - z_1| < h$ and $|\theta_0 - \theta_1| < \min\{2\alpha - \pi, \frac{\pi}{4}\}$. Observe that z_i does not belong to $\mathcal{S}^+_{\alpha,h}(z_j) \cup \mathcal{S}^-_{\alpha,h}(z_j)$ if $i \neq j$, and $\sigma(\theta_0) \neq \sigma(\theta_1)$ since $z_0, z_1 \notin \Gamma_{\rm cut}$. Consider the triangle of vertices 0, z_1 and z_0. Setting κ to be the interior angle of this triangle at z_1, we have that $\kappa > \pi - \alpha$. Consequently, $z_0 \in \mathcal{S}^+_{\alpha,h}(z_1) \cup \mathcal{S}^-_{\alpha,h}(z_1) \subset \mathbb{R}^2 \setminus F$, which is impossible.

Assume first that $(\rho^*_F)^-(\theta) = (\rho^*_F)^+(\theta)$. If there were a sequence $z_n = r_n\sigma(\theta_n) \in \Gamma^+$ converging to z counterclockwise, we would have for n sufficiently large $|\theta_n - \theta| < \min\{2\alpha - \pi, \frac{\pi}{4}\}$ and thus we would conclude, arguing as in the proof of the previous claim, that $z \in \mathcal{S}^+_{\alpha,h}(z_n)$, which is impossible. Therefore, there exists $\varepsilon > 0$ such that $\partial F \cap A\,[\sigma(\theta - \varepsilon), \sigma(\theta)] \subset \Gamma^- \cup \Gamma^0$. Arguing as in the previous cases and using Remark 7.5, we conclude that $\overline{\partial F \cap A\,(\sigma(\theta - \varepsilon), \sigma(\theta))}$ is the graph of a Lipschitz function for ε sufficiently small. A similar argument shows that $\overline{\partial F \cap A\,(\sigma(\theta), \sigma(\theta + \varepsilon'))}$ is the graph of a Lipschitz function for a suitable small $\varepsilon' > 0$. In conclusion,

$$(\partial F \cap A\,[\sigma(\theta - \varepsilon), \sigma(\theta + \varepsilon')]) \setminus \{r'\sigma(\theta) : r < r' \leq \rho^*_F(\theta)\}$$

is the union of two Lipschitz graphs intersecting only at z.

Finally, if $(\rho^*_F)^-(\theta) < (\rho^*_F)^+(\theta)$, we assume without loss of generality that $(\rho^*_F)^+(\theta) = \rho_F(\sigma(\theta)-)$. Reasoning as in the case $(\rho^*_F)^-(\theta) = (\rho^*_F)^+(\theta)$, we deduce that $\overline{\partial F \cap A\,(\sigma(\theta - \varepsilon), \sigma(\theta))}$ is the graph of a Lipschitz function, while the jump segment gives the second graph.

Step 2. Assume that $0 \in \partial F$, i.e., $\rho^-_F(\sigma) = 0$ for some $\sigma \in \mathbb{S}^1$. We claim that the open set $\{\sigma \in \mathbb{S}^1 : \rho^-_F(\sigma) > 0\}$ has at most three connected components. Indeed, let (σ_0, σ_1) be a connected component. Then $\rho^-_F(\sigma_1) = 0$. Let $\{r_n\}$ be a sequence of positive numbers converging to 0. Then the points $z_n = r_n\sigma_1$ are all contained in $\mathbb{R}^2 \setminus \mathring{F}$ and, by (6.13) and (6.14), there exist $C_n = w_n + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ such that $z_n \in \overline{C}_n$. Arguing as in the proof of Theorem 6.5, letting $n \to \infty$ we conclude that there exists $C = w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ such that $0 \in \partial C$ and the half line $\{r\sigma_1 : r \geq 0\}$ crosses $\overline{C}$. Since $C_n \subset \mathbb{R}^2 \setminus F$ for every n, we also derive that there exists $\sigma_2 > \sigma_1$ such that $A(\sigma_1, \sigma_2) \subset \mathbb{R}^2 \setminus F$ and the angle between σ_1 and σ_2 is greater than $\pi/2$. In particular, $\rho^-_F(\sigma) = 0$ for every $\sigma \in [\sigma_1, \sigma_2]$. Therefore the distance on $\mathbb{S}^1$ between two connected component of $\{\sigma \in S^1 : \rho^-_F(\sigma) > 0\}$ is greater than $\pi/2$, which proves the claim.

Now let $(\sigma(\theta_0), \sigma(\theta_1))$ be one of the connected components of $\{\rho^-_F(\sigma) > 0\}$. Note that $\rho^+_F(\sigma(\theta_0)) = \rho_F(\sigma(\theta_0)+)$ and $\rho^+_F(\sigma(\theta_1)) = \rho_F(\sigma(\theta_1)-)$ since $\rho_F(\sigma(\theta_0)-) = \rho_F(\sigma(\theta_1)+) = 0$. Arguing as in Remark 7.2 we have that at least one of the three sectors $\mathcal{S}^+_{\alpha,h,\theta_0}(0)$, $\mathcal{S}_{\beta,h,\theta_0}(0)$, $\mathcal{S}^-_{\alpha,h,\theta_0}(0)$ is contained in $\mathbb{R}^2 \setminus F$. But since the first two intersect F, we conclude that $\mathcal{S}^-_{\alpha,h,\theta_0}(0) \subset \mathbb{R}^2 \setminus F$. If $\rho^+_F(\sigma(\theta_0)) = 0$, arguing as in the proof of Case 1 in the previous step, we get that there exists $\varepsilon > 0$ such that $\mathcal{S}^-_{\alpha,h}(z) \subset \mathbb{R}^2 \setminus F$ for all $z \in \partial F \cap A[\sigma(\theta_0), \sigma(\theta_0 + \varepsilon)]$. Therefore, by Remark 7.5, we conclude that $\overline{\partial F \cap A\,(\sigma(\theta_0), \sigma(\theta_0 + \varepsilon))}$ is the graph of a Lipschitz function, for ε sufficiently small. On the other hand, from the exterior Wulff condition and the fact that the interior angles of W are greater than $\frac{\pi}{2}$, we have that $\sigma(\theta_0) \notin S_F$, and thus $\partial F \cap A\,[\sigma(\theta_0), \sigma(\theta_0 + \varepsilon)]$ is the graph of a Lipschitz function. If $\rho^+_F(\sigma(\theta_0)) > 0$, then the segment from 0 to $\rho^+_F(\sigma(\theta_0))\sigma(\theta_0)$ provides the desired graph. A similar argument applies at the angle θ_1, thus providing another Lipschitz graph intersecting the previous one only at 0.

Step 3. It remains to prove that the set S_F is finite. Let $\sigma \in S_F$ and assume that $\rho^+_F(\sigma) > 0$. Since $\sigma \in S_F$, ∂F does not coincide with the graph of a Lipschitz function in any neighborhood of $\rho^+_F(\sigma)\sigma$. In view of Step 1, we then have $\rho^+_F(\sigma)\sigma \in \Gamma^+ \cap \Gamma^- \cap \partial F^+$, and thus $\{\sigma \in S_F : \rho^+_F(\sigma) > 0\}$ is finite thanks to Case 4 of Step 1.

Next by Step 2 we have that the interior of $\{\sigma \in \mathbb{S}^1 : \rho^+_F(\sigma) = 0\})$ is the union of at most finitely many open arcs. Consider one such open arc (σ_0, σ_1), and observe that σ_0 and σ_1 do not belong to S_F again by Step 2. Then assume that there exist $\sigma_2, \sigma_3 \in S_F \cap (\sigma_0, \sigma_1)$. Arguing as in Step 2, we derive that the angle between the σ_i's, $i = 0, 1, 2, 3$, are larger than $\pi/2$. Consequently, the set (σ_0, σ_1) contains at most two elements in S_F, and the proof is complete. □

Remark 7.7. From the proof of the previous theorem it is clear that Γ_{sing} is precisely given by the finite set $\Gamma^{+} \cap \Gamma^{-} \cap \partial F^{+}$ to which one has to add the origin if more than one Lipschitz graph departs from there.

Remark 7.8. If W is a polygon with internal angles greater than or equal to $\frac{\pi}{2}$, the conclusions of the previous theorem hold for $\partial F \cap A(\sigma', \sigma'')$ whenever $[\sigma', \sigma'']$ does not contain any of the angles σ_i, σ_i', $i = 1, \ldots, k$ considered in Remark 7.3.

8. Regularity in the strictly convex case

Throughout this section we assume that φ satisfies (H2) and that

(H3)' the sublevel set $\{z \in \mathbb{R}^2 : \varphi(z) \leq 1\}$ is strictly convex.

A condition under which (H3)' holds is the following: there exist $\varepsilon > 0$ and a nonnegative positively 1-homogenous convex function ψ such that

$$\varphi(a) = \varepsilon |a| + \psi(a) \tag{8.1}$$

for all $a \in \mathbb{R}^2$. A function φ satisfying (8.1) is said to be *elliptic*. We refer to [26] and [34] for a detailed analysis of this class of surface energies and their relevance in the physical literature.

We emphasize that, under assumptions (H2) and (H3)', the function φ is convex (see Proposition 8.1 below), and thus the results of Section 6 do hold. We shall prove that if $(F, u) \in X$ is a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$, then, apart from a finite singular set, $\partial F \setminus \{0\}$ is a C^1-manifold. Moreover, the singular set may possibly contain the origin, from which at most two Lipschitz branches of ∂F may depart.

We begin with some auxiliary results. The next one will be proved in the appendix.

Proposition 8.1. *Assume that φ satisfies $(H2)$ and $(H3)'$. Then φ is convex and there exists a modulus of continuity $\omega : [0, 2] \to [0, \infty)$ such that*

$$\varphi(a) + \varphi(b) \geq \varphi(a + b) + \min\{|a|, |b|\}\, \omega\left(1 - \frac{a}{|a|} \cdot \frac{b}{|b|}\right)$$

for all a, $b \in \mathbb{R}^2 \setminus \{0\}$.

Proposition 8.2. *Assume that φ satisfies $(H2)$ and $(H3)'$. Then W is a C^1 open set.*

Proof. By Theorem 3.7.3 in [38] (see also Proposition 3.3(2) in [26]), W has a unique tangent line at any point of its boundary. Fix $z \in \partial W$. Then, in a neighborhood of z, the boundary of W is a graph over the tangent line at z of a convex function f that is differentiable at every point. By well-known properties of convex functions, it follows that f is actually of class C^1. □

Lemma 8.3. *Let φ satisfy (H2) and (H3)'. For every $0 < \varepsilon < 1$ there exists $\delta_0 > 0$ such that for every $z \in \partial W$ and $\nu \in \mathbb{S}^1$ satisfying $\nu \cdot \nu_W(z) > \varepsilon$, the point $z - \delta\nu$ belongs to W for all $0 < \delta \leq \delta_0$.*

Proof. Since ∂W is a compact set, it is enough to show that for every $z_0 \in \partial W$ and $0 < \varepsilon < 1$ there exist a neighborhood of z_0 and $\delta = \delta(\varepsilon, z_0) > 0$ such that the statement holds in this neighborhood. Up to a translation and a rotation, we may assume that $z_0 = 0$ and that there exist a neighborhood U of the origin and a nonnegative convex function $f \in C^1([-a, a])$ for some $a > 0$ such that $f(0) = f'(0) = 0$ and

$$\begin{aligned} \partial W \cap U &= \{(x, f(x)) : x \in (-a, a)\}, \\ W \cap U &\supset \{(x, y) : x \in (-a, a),\ f(x) < y < \eta\} \end{aligned} \tag{8.2}$$

for some $\eta > 0$.

Let $0 < \delta < \min(\frac{a}{2}, \frac{\eta}{2})$ be such that if $|x| < 2\delta$, then $|f'(x)| < \frac{\varepsilon}{2}$ and $|f(x)| < \frac{\eta}{2}$. Fix $x_0 \in (-\delta, \delta)$ and $\nu \in \mathbb{S}^1$ satisfying $\nu \cdot \nu_W(x_0, f(x_0)) > \varepsilon$, or, equivalently,

$$\nu_1 f'(x_0) - \nu_2 > \varepsilon \sqrt{1 + |f'(x_0)|^2}\,,$$

where $\nu = (\nu_1, \nu_2)$. Then for all $x \in (-2\delta, 2\delta)$ we have

$$\nu_1 f'(x) - \nu_2 = \nu_1 f'(x_0) - \nu_2 + \nu_1 (f'(x) - f'(x_0)) > \varepsilon \sqrt{1 + |f'(x_0)|^2} - \varepsilon > 0\,. \tag{8.3}$$

We claim that

$$f(x_0 - \delta\nu_1) < f(x_0) - \delta\nu_2 < \eta\,,$$

The second inequality is satisfied by the choice of δ. To prove the first inequality, we use (8.3) and the convexity of f, thus getting

$$f(x_0) - f(x_0 - \delta\nu_1) \geq f'(x_0 - \delta\nu_1)\,\delta\nu_1 > \delta\nu_2\,.$$

Thus the claim holds, and so by (8.2) we have $(x_0, f(x_0)) - \delta\nu \in W$. This concludes the proof. $\square$

In the next proposition we study cuts segments. The additional hypothesis $(H3)'$ will allow us to obtain a result stronger than the one obtained in Proposition 7.1 for the polygonal case.

Proposition 8.4. *Let φ satisfy (H2) and (H3)'. Let $(F, u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$ and let $\sigma \in S_F$ and $z = r\sigma \in \partial F$ be such that $\rho_F^+(\sigma) \leq r \leq \rho_F(\sigma)$. Then there exist $C = w + \varrho_0 W$, $C' = w' + \varrho_0 W$, where ϱ_0 is given in Theorem 6.5, such that C, $C' \subset \mathbb{R}^2 \setminus F$, $z \in \partial C \cap \partial C'$, $\nu_C(z) \cdot \sigma = 0$, $\nu_{C'}(z) \cdot \sigma = 0$ and $\nu_{C'}(z) = -\nu_C(z)$.*

Proof. Without loss of generality, we may assume that $\rho_F^+(\sigma) < r < \rho_F(\sigma)$. The cases $r \in \{\rho_F^+(\sigma), \rho_F(\sigma)\}$ follow by a continuity argument.

Let $\{\sigma_n\}$ be a sequence converging to σ, with $\sigma_n < \sigma$, so that for n large $z_n = r\sigma_n \notin F$. Arguing as in the first part of the proof of Theorem 6.5, there exist $C_n = w_n + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ such that $z_n \in C_n$ and C_n converges in the Hausdorff metric to some $C = w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ with $z \in \partial C$. Since C is of class C^1, we have that $\nu_C(z) \cdot \sigma = 0$. We claim that $\nu_C(z) = \sigma^\perp$. Indeed, if $\nu_C(z) = -\sigma^\perp$, then for $t > 0$ sufficiently small, the point $w_* := z + t\sigma^\perp$ belongs to C and, writing $w_* = r_*\sigma_*$, we may assume that $\rho_F^+(\sigma) < r_* < \rho_F(\sigma)$. Note that $\sigma_* > \sigma$. By Hausdorff convergence, $w \in C_n$ for all n sufficiently large, and since C_n is convex, the segment S_n of endpoints w_* and z_n is contained in C_n. Using the facts that $\sigma_n < \sigma < \sigma_*$ and that $\rho_F^+(\sigma) < r$, $r_* < \rho_F(\sigma)$, it follows that S_n intersects the segment $\{r'\sigma : \rho_F^+(\sigma) < r' < \rho_F(\sigma)\} \subset F$. This contradicts the fact that C_n is contained in $\mathbb{R}^2 \setminus F$ and proves the claim.

In a similar way, considering $\{\sigma_n\}$ converging to σ, with $\sigma_n > \sigma$, we prove that there exists $C' = w' + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ such that $z \in \partial C'$ and $\nu_{C'}(z) = -\sigma^\perp$. $\square$

Definition 8.5 (Cusp points). Given $(F, u) \in X$ and $\sigma \in \mathbb{S}^1$, a point $z = r\sigma \in \partial F^+$ is called a *cusp point* if there exist $C = w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$, $C' = w' + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ such that $z \in \partial C \cap \partial C'$ and $\nu_C(z) \cdot \sigma = \nu_{C'}(z) \cdot \sigma = 0$ and $\nu_C(z) = -\nu_C'(z)$, where ϱ_0 is as in Theorem 6.5. The set of cusp points in ∂F^+ is denoted by Γ_{cusp}.

Remark 8.6. In view of Proposition 8.4, if $\sigma \in S_F$ then $\rho_F^+(\sigma)\sigma \in \Gamma_{\text{cusp}}$. Moreover, we note that the origin cannot be a cusp point. Indeed, if 0 were a cusp point, since the sets C and C' given in Definition 8.5 are C^1 and F is starshaped, it would follow that F lies in the line through 0 in the direction σ. This would contradict the fact that $|F| > 0$. In particular, by Proposition 8.4 the origin cannot be the endpoint of a cut segment, i.e., if $\rho_F^+(\sigma) = 0$ then $\rho_F(\sigma) = 0$.

Next we show that at every point of ∂F^+ there exist left and right (classical) tangent vectors according to the counterclockwise orientation, and that the number of cusp points is finite.

Proposition 8.7. *Let φ satisfy (H2) and (H3)$'$. Let $(F, u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$, and let z be a point on ∂F^+.*

(i) If $z = r\sigma \in \Gamma_{\rm cusp}$, then ∂F^+ has a left tangent at z equal to σ, and a right tangent equal to $-\sigma$.

(ii) If $z \in \partial F^+ \setminus (\Gamma_{\rm cusp} \cup \Gamma_{\rm jump})$ and $z \neq 0$, then ∂F has a left and right tangent at z, while if $z = 0$ then there exist at most two tangents forming an angle of at least π.

Moreover, ∂F contains only finitely many cut segments and finitely many cusp points, i.e., the sets S_F and $\Gamma_{\rm cusp}$ are finite.

Proof. *(i).* Let $z \in \partial F^+$ be a cusp point and let C and C' be given as in Definition 8.5. By Remark 8.6, $z \neq 0$. Thus, up to a rotation, we may assume that $z = (0, y)$ with $y > 0$ and that $\nu_C(z) = (-1, 0)$. Without loss of generality, we may also assume that

$$\lim_{\sigma' \to \sigma^-} \rho_F(\sigma') = \rho_F^+(\sigma) \,. \tag{8.4}$$

Take a sequence $\{z_n\} \subset \partial F$ converging to z from the left (i.e., counterclockwise). Hence, if $z_n = (x_n, y_n)$, then $x_n > 0$ and $y_n < y$. Since C is C^1, the segment joining z_n and z intersects ∂C at some point $w_n = (x'_n, y'_n)$, with $0 < x'_n < x_n$ and $y_n < y'_n < y$. Then

$$\frac{z - z_n}{|z - z_n|} = \frac{z - w_n}{|z - w_n|} \to (0, 1) = \sigma \,.$$

Thus, ∂F^+ has the left tangent σ at z. If $\rho_F^-(\sigma) = \rho_F^+(\sigma)$, a similar argument shows that the right tangent at z is $-\sigma$. If instead $\rho_F^-(\sigma) < \rho_F^+(\sigma)$, then σ is a jump direction and the right tangent is again $-\sigma$.

(ii). Assume first that $z \neq 0$ and, without loss of generality, that $z = \rho_F^+(\sigma)\,\sigma$, $\sigma = (0, 1)$ and that (8.4) holds. We argue by contradiction and assume that ∂F does not admit a right tangent at z. Then there exist $0 < \alpha < \beta < \pi$ such that, denoting by M and L the two half-lines

$$M := z + \{r'\sigma(\pi/2 - \alpha) : r' \geq 0\} \,, \qquad L := z + \{r'\sigma(\pi/2 - \beta) : r' \geq 0\} \,,$$

there exist two sequences $\{z'_n\}$, $\{z_n\} \subset \partial F$ converging to z such that

$$\frac{z'_n - z}{|z'_n - z|} \to \sigma(\pi/2 - \alpha) \,, \quad \frac{z_n - z}{|z_n - z|} \to \sigma(\pi/2 - \beta) \,.$$

By replacing α and β with $0 < \alpha < \alpha' < \beta' < \beta < \pi$, if necessary, and using the fact that ∂F is pathwise connected (see Lemma 2.2), without loss of generality, we may assume that $z'_n \in \partial F \cap M$ and $z_n \in \partial F \cap L$, so that $\frac{z'_n - z}{|z'_n - z|} = \sigma(\pi/2 - \alpha)$ and $\frac{z_n - z}{|z_n - z|} = \sigma(\pi/2 - \beta)$. Denote by $\tau_L := \sigma(\pi/2 - \beta)$ the tangential direction of L. We claim that there exists $C := w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ such that $z \in \partial C \cap \partial F$ and $\nu_C(z) = -\tau_L^\perp$.

To prove the claim we argue as follows. For every n, let $C_n := w_n + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ be such that $z_n \in \partial C_n$. Up to a subsequence, $\{C_n\}$ converges in the Hausdorff metric to some $C = w + \varrho_0 W$ such that $z \in \partial C$. Fix $\varepsilon \in (0, 1)$. If $\nu_{C_n}(z_n) \cdot \tau_L > \varepsilon$, then by Lemma 8.3, $z = z_n - |z - z_n| \tau_L \in C_n$ whenever $|z_n - z| < \delta$, which is impossible. If $\nu_{C_n}(z_n) \cdot \tau_L < -\varepsilon$, then by Lemma 8.3, $z_m = z_n + |z_m - z_n| \tau_L \in C_n$ whenever $|z - z_n| < |z - z_m| < \delta$, which is again impossible. Therefore, $|\nu_{C_n}(z_n) \cdot \tau_L| \leq \varepsilon$ for all n large enough. Since W is C^1, we have $\nu_{C_n}(z_n) \to \nu_C(z)$ as $n \to \infty$, and consequently $\nu_C(z) \cdot \tau_L = 0$ by the arbitrariness of ε. On the other hand, since $0 < \beta < \pi$, Lemma 8.3 and the starshapedness of F with respect to 0 imply that $\nu_C(z) = -\tau_L^\perp$.

From this last equality, since $0 < \alpha < \beta < \pi$, setting $\tau_M := \sigma(\pi/2 - \alpha)$, we have that $\nu_C(z) \cdot \tau_M < -\varepsilon$ for some $\varepsilon > 0$. Therefore, by Lemma 8.3, $z'_n = z + |z'_n - z| \tau_M \in C$ whenever $|z - z'_n| < \delta$ which is impossible. This shows that α must coincide with β, and so there exists a unique tangent line to the left of z.

To prove the existence of a unique tangent line to the right of z, as before there are two possible cases. If $\rho_F^-(\sigma) = \rho_F^+(\sigma)$, we can repeat the argument just used above. If $\rho_F^-(\sigma) < \rho_F^+(\sigma)$, then the existence of a unique tangent line at z from the right is trivial since in a small right neighborhood of z, ∂F is a segment contained in the segment $[\rho_F^-(\sigma)\,\sigma, z]$.

If $z=0$, i.e., $\rho_F^-(\sigma)=0$ for some $\sigma\in\mathbb{S}^1$, we argue as in the Step 2 of the proof of Theorem 7.6 to prove that the open set $\{\sigma\in\mathbb{S}^1:\rho_F^-(\sigma)>0\}$ has exactly one connected component. Indeed, setting (σ_0,σ_1) to be such a connected component, there exists $C=w+\varrho_0 W\subset\mathbb{R}^2\setminus F$ such that $0\in\partial C$ and C belongs to the right of the direction σ_1. As consequence, there exists $\sigma_2>\sigma_1$ such that $A(\sigma_1,\sigma_2)\subset\mathbb{R}^2\setminus F$, and since C is of class C^1, the angle between σ_1 and σ_2 is greater than or equal to π. Therefore the distance on $\mathbb{S}^1$ between two connected components of $\{\sigma\in\mathbb{S}^1:\rho_F^-(\sigma)>0\}$ is greater than or equal to π and the conclusion follows, i.e., $\{\sigma\in\mathbb{S}^1:\rho_F^-(\sigma)>0\}=(\sigma_0,\sigma_1)$. Then, the two vectors σ_0 and $-\sigma_1$ are the two required tangents.

To prove the last part of the statement, we argue again by contradiction and we assume first that there exist infinitely many cusps. Let $z_n=r_n\sigma_n\in\Gamma_{\mathrm{cusp}}$ converging to some point $z\in\partial F^+$, $\sigma_n\to\sigma$ with, say, $\sigma_n<\sigma$, and let $\{C_n\}$, $\{C_n'\}\subset\mathbb{R}^2\setminus F$ be translated sequences of $\varrho_0 W$ such that $z_n\in\partial C_n\cap\partial C_n'$, $\nu_{C_n}(z_n)=-\sigma_n^\perp$ and $\nu_{C_n'}(z_n)=\sigma_n^\perp$. Passing to the limit, we conclude that there exist C, $C'\subset\mathbb{R}^2\setminus F$, translations of $\varrho_0 W$, such that $z\in\partial C\cap\partial C'$, $\nu_C(z)=-\sigma^\perp$ and $\nu_{C'}(z)=\sigma^\perp$, i.e., $z\in\Gamma_{\mathrm{cusp}}$. In particular $z\neq 0$ by Remark 8.6 so that $z=|z|\sigma$. The same argument used in part *(i)* shows that $\frac{z-z_n}{|z-z_n|}\to\sigma$. On the other hand, $\{r\sigma:r\geq 0\}\cap C_n\neq\emptyset$ whenever n is large enough, and consequently $|z|\leq\inf\{r:r\sigma\in C_n\}$. Then arguing as in part *(i)*, we deduce that $\frac{z-z_n}{|z-z_n|}\to-\sigma$ which is a contradiction.

Finally, by Remark 8.6, for any $\sigma\in S_F$ we have $\rho_F^+(\sigma)\sigma\in\Gamma_{\mathrm{cusp}}$, and thus S_F is finite, i.e., ∂F contains finitely many cut segments. □

We now state the main regularity result for ∂F.

Theorem 8.8. *Let φ satisfy (H2) and (H3)$'$. Let $(F,u)\in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$ and $z_0\in\partial F^+$.*

(i) If $z_0\notin\Gamma_{\mathrm{cusp}}$ and $z_0\neq 0$, there exists a neighborhood $\mathcal{N}(z_0)$ of z_0 such that $\partial F\cap\mathcal{N}(z_0)$ coincides with the graph of a Lipschitz function.

(ii) If $0\in\partial F$, there exists a neighborhood $\mathcal{N}$ of 0 such that $\partial F\cap\mathcal{N}$ is the union of at most two graphs of Lipschitz functions intersecting only at 0.

(iii) If $z_0=r_0\sigma_0\in\Gamma_{\mathrm{cusp}}$, there exist $\delta>0$ and two Lipschitz functions $h,g:(r_0-\delta,r_0]\to\mathbb{R}$ satisfying $g\leq 0\leq h$, $h(r_0)=g(r_0)=0$, $h(r)>g(r)$ for $r\in(r_0-\delta,r_0)$ and $h_-'(r_0)=g_-'(r_0)=0$, and such that

$$\left\{r\sigma_0+g(r)\sigma_0^\perp:r\in(r_0-\delta,r_0]\right\}\cup\left\{r\sigma_0+h(r)\sigma_0^\perp:r\in(r_0-\delta,r_0]\right\}$$

coincides with $\partial F\setminus\Gamma_{\mathrm{cut}}$ in an open neighborhood of z_0.

Proof. *(i)*. Given $z=r\sigma\in\partial F$, $r>0$, we observe that the set

$$N(z):=\left\{\nu_C(z):C=w+\varrho_0 W\subset\mathbb{R}^2\setminus F\,,\,z\in\partial C\right\}$$

is closed in $\mathbb{S}^1$. Note also that if $\nu\in N(z)$, then $\nu\cdot\sigma\leq 0$. Indeed, if $\nu\cdot\sigma>0$, then let $C=w+\varrho_0 W\subset\mathbb{R}^2\setminus F$ be such that $z\in\partial C$ and $\nu_C(z)=\nu$. By Lemma 8.3 we obtain that for some small $\delta>0$ the point $z-\delta\sigma$ lies inside C, which is impossible.

Fix $z_0=r_0\sigma_0\in\partial F^+\setminus\Gamma_{\mathrm{cusp}}$ with $z_0\neq 0$, and let $\nu_-(z_0)$and $\nu_+(z_0)$ denote the smallest and largest element in $N(z_0)$, respectively. Note that since z_0 is not a cusp point, the distance in $\mathbb{S}^1$ between $\nu_-(z_0)$ and $\nu_+(z_0)$ is strictly smaller than π and that $N(z_0)$ is contained in the smallest arc in $\mathbb{S}^1$ with endpoints $\nu_-(z_0)$ and $\nu_+(z_0)$.

Let $I=(\nu_1,\nu_2)$ be an open arc in $\mathbb{S}^1$ containing $\nu_-(z_0)$ and $\nu_+(z_0)$ with $\mathcal{H}^1(I)<\pi$. We observe that there exists $\delta>0$ such that if $|z-z_0|<\delta$, $z\in\partial F$, then for all $\nu\in N(z)$ we have that $\nu\in I$.

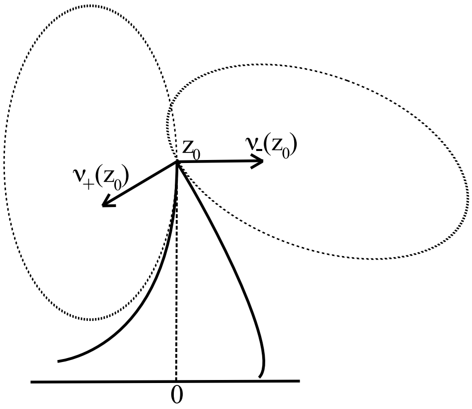

Fig. 6. ν_- and ν_+ are the endpoints of $N(z_0)$.

Indeed, if not then there would exist $\{z_n\} \subset \partial F$ converging to z_0 and $\nu_n \in N(z_n) \setminus I$. But then, up to a sequence, $\{\nu_n\}$ would converge to some $\nu \in N(z_0) \setminus I$, which is impossible.

Let $\overline{\nu}$ be the midpoint of I (see Figure 6). Then the angle $\alpha := \widehat{\nu_1 \overline{\nu}} = \widehat{\overline{\nu} \nu_2}$ is strictly smaller than $\frac{\pi}{2}$. Set $\eta := 1 - \frac{1}{8}\cos^2\alpha \in (0,1)$. We claim that there exists $0 < \overline{\delta} < \delta$ such that if $\nu \in \mathbb{S}^1$ satisfies $\nu \cdot \overline{\nu} \geq \eta$, then $\{z - t\nu : 0 < t < \overline{\delta}\} \subset \mathbb{R}^2 \setminus F$ for all $z \in \partial F \cap B_{\overline{\delta}}(z_0)$. Note that if the claim holds, then by Lemma 7.4 (applied to $-\overline{\nu}$ and $-\nu$ in place of ν and ν' in the lemma) we conclude that $\partial F \cap \mathcal{N}(z_0)$ is the graph of a Lipschitz function for some neighborhood $\mathcal{N}(z_0)$ of z_0. The claim follows from Lemma 8.3, provided we show that for any such ν and z we have $\nu \cdot \nu_C(z) > \frac{1}{2}\cos\alpha$. To see this, note that $\nu_C(z) \in I$, since $N(z) \subset I$, and so

$$\nu \cdot \nu_C(z) = \overline{\nu} \cdot \nu_C(z) + (\nu - \overline{\nu}) \cdot \nu_C(z) > \cos\alpha - |\nu - \overline{\nu}| \, .$$

In turn, $|\nu - \overline{\nu}|^2 = 2\,(1 - \nu \cdot \overline{\nu}) \leq 2\,(1 - \eta) = \frac{1}{4}\cos^2\alpha$. Therefore, $\nu \cdot \nu_C(z) > \frac{1}{2}\cos\alpha$.

(ii). Assume that $0 \in \partial F$. Then from the proof of Proposition 8.7 we know that the set $\{\rho_F^-(\sigma) > 0\}$ has just one connected component $(\sigma(\theta'), \sigma(\theta''))$, with $0 < \theta'' - \theta' \leq \pi$. If $(\rho_F^*)^+(\theta') = 0$ set

$$N_{\theta'}(0) := \left\{ \nu = \lim_{n\to\infty} \nu_n : \nu_n \in N(z_n) \, , \, z_n = r_n\sigma(\theta_n) \in \partial F \setminus \{0\} \, , \, r_n \to 0^+ \, , \, \theta_n \to \theta' \right\} \, .$$

Arguing as in the proof of *(i)* (use $N_{\theta'}(0)$ in place of $N(z_0)$ and apply Remark 7.5 instead of Lemma 7.4), we conclude that there exists $\varepsilon > 0$ such that $\overline{\partial F \cap A(\sigma(\theta'), \sigma(\theta' + \varepsilon))}$ is the graph of a Lipschitz function. The case $(\rho_F^*)^+(\theta') > 0$ is trivial.

A similar argument shows the existence of another Lipschitz graph departing from 0 and contained in some sector $A[\sigma(\theta'' - \varepsilon), \sigma(\theta'')]$. Then the conclusion follows from Remark 8.6 which excludes the possibility of cut segments starting from the origin.

(iii). Assume first that $\rho_F^-(\sigma_0) = \rho_F^+(\sigma_0)$. Observe that, since z_0 a cusp point, $\nu_-(z_0) = -\sigma_0^\perp$ and $\nu_+(z_0) = \sigma_0^\perp$ form an angle equal to π and thus we cannot argue as before. Fix an open arc I in $\mathbb{S}^1$ containing $\nu_-(z_0)$ with $\mathcal{H}^1(I) < \pi$, and note that there exists a right neighborhood of z_0 (according to the counterclockwise orientation) such that for all $z \in \partial F$ in this neighborhood and for all $\nu \in N(z)$, we have $\nu \in I$. Indeed, from Proposition 8.7 it follows that if $\{z_n\} \subset \partial F$ converges to z_0 from the right and $\nu_n \in N(z_n)$, then $\nu_n \to \nu_-(z_0)$. The same argument used in part *(i)* (with the one-sided version of Lemma 7.4 given in Remark 7.5) shows that there exists $\sigma_1 > \sigma_0$ such that $\overline{\partial F \cap A(\sigma_0, \sigma_1)}$ coincides in a neighborhood of z_0 with the graph of a Lipschitz function h defined in $\{r\sigma_0 : r \in [r_0 - \delta, r_0]\}$. Similarly, there exists $\sigma_2 < \sigma_0$ such that $\overline{\partial F \cap A(\sigma_2, \sigma_0)}$ coincides in a neighborhood of z_0 with the graph of a Lipschitz function g. The fact that $h'_-(r_0) = g'_-(r_0)$ is again an immediate consequence of Proposition 8.7.

Finally if $\rho_F^-(\sigma_0) < \rho_F^+(\sigma_0)$ the proof is even simpler since one of two Lipschitz graphs now coincides with the jump segment with endpoints $\rho_F^-(\sigma_0)\sigma_0$ and $\rho_F^+(\sigma_0)\sigma_0$. $\square$

In the remainder of this paper we assume that $\mathcal{W}$ is the bulk energy density of a linearly isotropic material, i.e.,

$$\mathcal{W}(\mathbf{E}) = \frac{1}{2}\lambda\left[\operatorname{tr}(\mathbf{E})\right]^2 + \mu \operatorname{tr}\left(\mathbf{E}^2\right) ,$$

where λ and μ are the (constant) Lamé moduli with

$$\mu > 0 , \quad \mu + \lambda > 0 .$$

The proof of following theorem is similar to the one of Theorem 3.12 in [15] and thus we omit it. Note that Step 5 in that theorem is not needed in our case.

Theorem 8.9 (Blow-Up). *Let φ satisfy (H2) and (H3)$'$. Let $(F,u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$. Assume that $z_0 \in \partial F \cap B_0 \setminus (\Gamma_{\mathrm{cut}} \cup \Gamma_{\mathrm{cusp}})$. Then there exist a constant $c > 0$, a radius r_0 and an exponent $\frac{1}{2} < \alpha < 1$ such that*

$$\int_{B_r(z_0)\setminus F} |\nabla u|^2 \, dz \leq c r^{2\alpha} \tag{8.5}$$

for all $0 < r < r_0$.

From Theorem 8.9 we now obtain an improved regularity of ∂F near its regular points.

Theorem 8.10. *Let φ satisfy (H2) and (H3)$'$. Let $(F,u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$. Assume that $z_0 \in \partial F \cap B_0 \setminus (\Gamma_{\mathrm{cut}} \cup \Gamma_{\mathrm{cusp}})$ and $z_0 \neq 0$. Then ∂F coincides in a neighborhood of z_0 with the graph of a function of class C^1.*

Proof. By Theorem 8.8 there exists an open neighborhood $\mathcal{N}$ of z_0 such that $\partial F \cap \mathcal{N}$ is the graph of a Lipschitz function with Lipschitz constant L. Fix $r_1 > 0$ such that $\overline{B}_{r_1}(z_0) \subset B_0 \cap \mathcal{N}$. By a standard extension argument we may extend u in $B_{r_1}(z_0)$ to a function $\widetilde{u}$ in such a way that for all $0 < r < r_1$,

$$\int_{B_r(z_0)} |\nabla \widetilde{u}|^2 \, dz \leq c(L) \int_{B_r(z_0)\setminus F} |\nabla u|^2 \, dz , \tag{8.6}$$

where the constant $c(L)$ is independent of r and only depends on L. We also recall that by Proposition 8.7, ∂F admits a left and a right tangent vector at z_0, respectively τ_l and τ_r. To fix the ideas, we assume without loss of generality that $z_0 = (x_0, 0)$ with $x_0 > 0$. Then $\tau_l = \sigma(\theta_l)$ and $\tau_r = \sigma(\theta_r)$ for some $\theta_l, \theta_r \in [0, \pi]$. From the exterior Wulff condition and the fact that W is C^1, we infer that $\theta_l \leq \theta_r$. Moreover, since ∂F is a Lipschitz graph in a neighborhood of z_0, we have $\theta_r - \theta_l < \pi$. Now we assume by contradiction that the two tangents are distinct, i.e., $\theta := \theta_r - \theta_l > 0$ (see Figure 7).

For $r > 0$ (sufficiently small) $\partial F \cap \partial B_r(z_0)$ contains exactly two points, say, z_r', z_r''. Let γ_r', γ_r'' be the open curves on $\partial F \cap B_r(z_0)$ with endpoints z_r' and z_0, and z_r'' and z_0, respectively. Denote by S_r the open segment (z_r', z_r''), and α_r the angle $\widehat{z_r' z_0 z_r''}$, which is converging to $\pi - \theta$ as $r \to 0^+$. Define a competing set $\widetilde{F}$ such that

$$\partial\widetilde{F} \setminus B_r(z_0) = \partial F \setminus (\gamma_r' \cup \gamma_r'') , \quad \partial\widetilde{F} \cap B_r(z_0) = S_r . \tag{8.7}$$

Note that the assumption $z_0 \neq 0$ is necessary for $\widetilde{F}$ to be starshaped with respect to the origin. One may easily check that $\widetilde{F} \in \mathcal{A}$. Let ν_r denote the normal to S_r interior to the triangle of vertices z_r', z_0 and z_r'', and let ν_r', respectively ν_r'', be the normal to the segment $[z_r', z_0]$, respectively $[z_r'', z_0]$, pointing

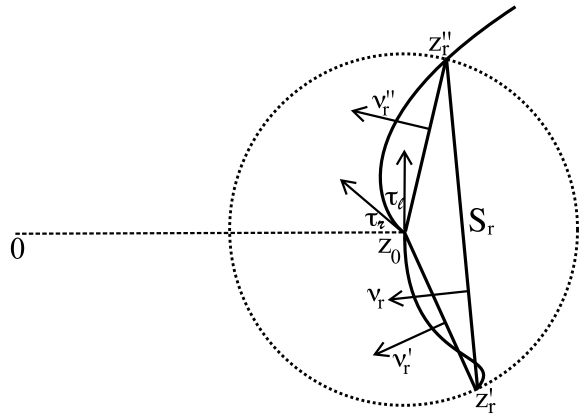

Fig. 7. The construction of $\widetilde{F}$ in the proof of Theorem 8.10.

toward the exterior of the same triangle. We observe that $|z_r' - z_0|\nu_r' + |z_r'' - z_0|\nu_r'' = |z_r' - z_r''|\nu_r$. Then, using Lemma 6.2, Proposition 8.1, Theorem 8.9, and (8.6), we have

$$
\begin{aligned}
\mathcal{F}_{\ell_0}(F,u) - \mathcal{F}_{\ell_0}(\widetilde{F},\widetilde{u}) &\geq \int_{\gamma_r' \cup \gamma_r''} \varphi\left(\nu_F^i\right)\, d\mathcal{H}^1 - |z_r' - z_r''|\,\varphi(\nu_r) - \ell_0\big|\widetilde{F}\Delta F\big| - c\int_{B_r(z_0)} |\nabla\widetilde{u}|^2\, dz \\
&\geq |z_r' - z_0|\,\varphi(\nu_r') + |z_r'' - z_0|\,\varphi(\nu_r'') - |z_r' - z_r''|\,\varphi(\nu_r) \\
&\qquad\qquad - \pi\ell_0 r^2 - c\int_{B_r(z_0)\setminus F} |\nabla u|^2\, dz \\
&\geq r\omega\,(1 - \nu_r'\cdot\nu_r'') - \pi\ell_0 r^2 - c r^{2\alpha}
\end{aligned}
$$

for a constant $c > 0$ independent of r. Since $\alpha > \frac{1}{2}$ and $\nu_r'\cdot\nu_r'' \to \cos\theta < 1$ as $r \to 0^+$, for r sufficiently small we have $\mathcal{F}_{\ell_0}(F,u) - \mathcal{F}_{\ell_0}(\widetilde{F},\widetilde{u}) > 0$, which is a contradiction to the minimality of (F,u). This contradiction proves the existence of a unique tangent line.

Since ∂W is C^1, using the exterior Wulff condition we infer that there exists a unique $C = w + \varrho_0 W \subset \mathbb{R}^2 \setminus F$ such that $z_0 \in \partial C$ and $\nu_C(z_0) = -\nu_F(z_0)$. Then the continuity of $\nu_F(z_0)$ in a neighborhood of z_0 easily follows. □

We now show that if φ is elliptic, then the regularity of ∂F can be further improved.

Theorem 8.11. *Assume that φ satisfies* (8.1). *Let $(F,u) \in X$ be a minimizer for the penalized functional $\mathcal{F}_{\ell_0}$. Assume that $z_0 \in \partial F \cap B_0 \setminus (\Gamma_{\text{cut}} \cup \Gamma_{\text{cusp}})$ and $z_0 \neq 0$. Then ∂F coincides in a neighborhood of z_0 with the graph of a function of class $C^{1,\alpha}$ for every $0 < \alpha < \frac{1}{2}$.*

Proof. By Theorem 8.10 we have that ∂F coincides in a neighborhood of z_0 with the graph of a C^1 function. Hence (8.5) holds in a stronger form, see Theorem 3.16 in [15], namely for every $\beta \in \left(\frac{1}{2}, 1\right)$ there exist a neighborhood $U \subset\subset B_0 \setminus \{0\}$ of z_0, and two constants $c_0 > 0$, $r_0 > 0$ such that for every $z \in \partial F \cap U$ and for every $0 < r < r_0$,

$$\int_{B_r(z)\setminus F} |\nabla u|^2\, dw \leq c_0 r^{2\beta}\,. \tag{8.8}$$

Since $\partial F \cap U$ is a graph of a C^1–function, we can find $0 < r_0' < r_0$ and extend as in the proof of Theorem 8.10 the function u to a function $\widetilde{u}$ defined in an open neighborhood $U' \subset\subset U$ of z_0 in such a way that for all $z \in \partial F \cap U'$ and all $0 < r < r_0'$,

$$\int_{B_r(z)} |\nabla\widetilde{u}|^2\, dw \leq c(L)\int_{B_r(z)\setminus F} |\nabla u|^2\, dw \tag{8.9}$$

for some constants $c(L) > 0$ independent of z and r. Moreover, by taking r_0' smaller if necessary, we may assume that ∂F crosses $\partial B(z,r)$ at exactly two points for all $z \in \partial F \cap U'$ and all $0 < r < r_0'$.

Fix $z \in \partial F \cap U'$, and for every $0 < r < r_0'$ let z_r', z_r'' be the two points in $\partial F \cap \partial B(z,r)$. Then let γ_r', γ_r'' be the two arcs of endpoints z_r' and z, and z_r'' and z respectively, such that $\gamma_r' \cup \gamma_r'' = \partial F \cap B(z,r)$. Define $\widetilde{F}$ as in (8.7) and let ν_r', ν_r'' be the normals to the segments (z_r', z) and (z_r'', z) respectively, pointing toward the exterior of the triangle of vertices z_r', z_0 and z_r''. Then, using Lemma 6.2, Proposition 8.1, (8.8) and (8.9), we estimate

$$\begin{aligned} 0 \geq \mathcal{F}_{\ell_0}(F,u) - \mathcal{F}_{\ell_0}(\widetilde{F},\widetilde{u}) \geq & \int_{\gamma_r' \cup \gamma_r''} \varphi\left(\nu_F^i\right) d\mathcal{H}^1 - |z_r' - z|\,\varphi(\nu_r') - |z_r'' - z|\,\varphi(\nu_r'') \\ & - \ell_0 \big|\widetilde{F} \Delta F\big| - c\int_{B_r(z)} |\nabla \widetilde{u}|^2\, dw \\ \geq & \,\varepsilon \mathcal{H}^1(\gamma_r' \cup \gamma_r'') - 2\varepsilon r + \int_{\gamma_r' \cup \gamma_r''} \psi\left(\nu_F^i\right) d\mathcal{H}^1 - |z_r' - z|\,\psi(\nu_r') \\ & - |z_r'' - z|\,\psi(\nu_r'') - \pi \ell_0 r^2 - c\int_{B_r(z)\setminus F} |\nabla u|^2\, dw \\ \geq & \,\varepsilon \mathcal{H}^1(\gamma_r' \cup \gamma_r'') - 2\varepsilon r - \pi\ell_0 r^2 - c r^{2\beta} \end{aligned}$$

for a constant $c > 0$ independent of z and r. Hence $\mathcal{H}^1(\partial F \cap B_r(z)) - 2r \leq C r^{2\beta}$ for r sufficiently small uniformly in $z \in U'$. By Proposition 6.4 in [5] and the proof of Theorem 6.1 in [5], this fact implies that $\partial F \cap U'$ is of class $C^{1,\alpha}$ with $\alpha = \beta - \frac{1}{2}$. □

9. Appendix

Let us consider

$$f(\theta,p,q) := \Phi(\sigma(\theta),p,q)\,,$$

where Φ is the function defined in (3.4), and note that the biconjugate $f^{**}(\theta,p,\cdot)$ of $f(\theta,p,\cdot)$ coincides with

$$f^{**}(\theta,p,q) = \overline{\Phi}(\sigma(\theta),p,q)\,, \tag{9.1}$$

where $\overline{\Phi}$ is given by (3.5).

Observe that if $F \in \mathcal{A}_{\mathrm{Lip}}$, then

$$\int_{\partial F} \varphi(\nu_F^i)\, d\mathcal{H}^1 = \int_0^{2\pi} f(\theta, \rho_F(\theta), \rho_F'(\theta))\, d\theta\,,$$

where we have used Lemma 2.5, the area formula, and the 1-homogeneity of φ.

Lemma 9.1. *Let φ satisfy (H2). Then for all (θ_0,p_0) and for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|f^{**}(\theta,p,q) - f^{**}(\theta_0,p_0,q)| \leq \varepsilon(1 + |q|)$$

for all (θ,p) with $|\theta - \theta_0| < \delta$ and $|p - p_0| < \delta$ and all $q \in \mathbb{R}$.

Proof. Since $f^{**}(\theta,p,\cdot)$ coincides with the convex envelope of $f(\theta,p,\cdot)$, we have

$$f^{**}(\theta,p,q) = \inf\left\{\lambda f(\theta,p,q_1) + (1-\lambda) f(\theta,p,q_2) : \lambda \in [0,1]\,,\, q_1\,,\, q_2 \in \mathbb{R}\,,\, \lambda q_1 + (1-\lambda) q_2 = q\right\}.$$

Fix (θ_0,p_0) and $\varepsilon > 0$, and let $q \in \mathbb{R}$. Find $\lambda \in [0,1]$, q_1, $q_2 \in \mathbb{R}$ such that $\lambda q_1 + (1-\lambda) q_2 = q$ and

$$f^{**}(\theta_0,p_0,q) \geq \lambda f(\theta_0,p_0,q_1) + (1-\lambda) f(\theta_0,p_0,q_2) - \frac{\varepsilon}{2}\,. \tag{9.2}$$

By (H2) we have

$$m\left(\lambda|q_1| + (1-\lambda)|q_2|\right) - \frac{\varepsilon}{2} \leq f^{**}(\theta_0,p_0,q) \leq f(\theta_0,p_0,q) \leq M(|p_0| + |q|)\,. \tag{9.3}$$

From (9.2) and (9.3) we obtain, writing σ and σ_0 instead of $\sigma(\theta), \sigma(\theta_0)$, and setting $L = \mathrm{Lip}\ \varphi$,

$$\begin{aligned} f^{**}(\theta,p,q) - f^{**}(\theta_0,p_0,q) &\le \lambda f(\theta,p,q_1) + (1-\lambda) f(\theta,p,q_2) - \lambda f(\theta_0,p_0,q_1) \\ &\qquad - (1-\lambda) f(\theta_0,p_0,q_2) + \frac{\varepsilon}{2} \\ &\le L\left(|p\sigma - p_0\sigma_0| + (\lambda|q_1| + (1-\lambda)|q_2|)\left|\sigma^\perp - {\sigma_0}^\perp\right|\right) + \frac{\varepsilon}{2} \\ &\le L\left(|p\sigma - p_0\sigma_0| + \frac{M}{m}(|p_0| + |q|)\left|\sigma^\perp - {\sigma_0}^\perp\right| + \frac{\varepsilon}{2m}\right) + \frac{\varepsilon}{2}\,. \end{aligned}$$

Then the result follows by taking δ sufficiently small and by interchanging the roles of (θ,p,q) and (θ_0,p_0,q). □

Theorem 9.2. *Let φ be a Lipschitz continuous function satisfying $(H2)$. Then for every nonnegative 2π-periodic Lipschitz function ρ,*

$$\int_0^{2\pi} f^{**}(\theta,\rho(\theta),\rho'(\theta))\,d\theta = \inf\left\{\liminf_{n\to\infty}\int_0^{2\pi} f(\theta,\rho_n(\theta),\rho_n'(\theta))\,d\theta :\ \rho_n \in W^{1,\infty}(\mathbb{R}),\right.$$
$$\left.\rho_n \ge 0\,,\ \rho_n \text{ is } 2\pi\text{-periodic, } \rho_n \overset{*}{\rightharpoonup} \rho \text{ in } W^{1,\infty}(\mathbb{R})\right\}. \quad (9.4)$$

Proof. Note that since $\rho \ge 0$ and $f(\theta,0,0) = 0$, by a truncation argument, the infimum on the right hand side of (9.4) coincides with the one obtained by removing the constraint $\rho_n \ge 0$. Thus, the representation (9.4) follows directly from Theorem 3.8 in [25]. □

We conclude with the proof of Proposition 8.1.

Proof of Proposition 8.1. Since the set $K := \{\varphi \le 1\}$ is strictly convex, and φ is positively 1-homogeneous, for any $a, b \in \mathbb{R}^2 \setminus \{0\}$ with $a \ne b$, the point

$$\frac{a+b}{\varphi(a)+\varphi(b)} = \frac{\varphi(a)}{\varphi(a)+\varphi(b)}\frac{a}{\varphi(a)} + \frac{\varphi(b)}{\varphi(a)+\varphi(b)}\frac{b}{\varphi(b)}$$

belongs to the interior of K unless $b = ta$ for some $t > 0$. Hence, still by homogeneity,

$$\varphi(a+b) < \varphi(a) + \varphi(b) \quad (9.5)$$

for $a, b \in \mathbb{R}^2$ unless $a = 0$ or $b = ta$ for some $t \ge 0$. In this later case, the inequality above is an equality, and the convexity of φ follows.

Since φ satisfies (9.5), a compactness argument shows that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $a,\ b \in \mathbb{R}^2$ are such that $\varphi(a) = \varphi(b) = 1$ and $\varphi(a-b) \ge \varepsilon$, then

$$\varphi\left(\frac{a+b}{2}\right) < 1 - \delta\,.$$

Hence there exists a modulus of continuity $\omega_1 : [0,2] \to [0,\infty)$ such that if $\varphi(a) = \varphi(b) = 1$ then

$$\varphi(a) + \varphi(b) \ge \varphi(a+b) + \omega_1(\varphi(a-b))\,.$$

Fix $\lambda \in \left[0, \frac{1}{2}\right]$. The previous inequality and the convexity of φ yield

$$\begin{aligned} \varphi(\lambda a + (1-\lambda) b) = \varphi\left((1-2\lambda) b + 2\lambda\frac{a+b}{2}\right) &\le (1-2\lambda)\varphi(b) + \lambda\varphi(a+b) \\ &\le (1-2\lambda)\varphi(b) + \lambda(\varphi(a)+\varphi(b)) - \lambda\omega_1(\varphi(a-b)) = 1 - \lambda\omega_1(\varphi(a-b))\,. \end{aligned}$$

Similarly, for $\lambda \in \left[\frac{1}{2}, 1\right]$ we get

$$\varphi(\lambda a + (1-\lambda) b) \le 1 - (1-\lambda)\omega_1(\varphi(a-b))\,.$$

From the last two inequalities we infer that if $\varphi(a) = \varphi(b) = 1$ and $\lambda \in [0,1]$, then

$$\lambda\varphi(a) + (1-\lambda)\varphi(b) = 1 \geq \varphi(\lambda a + (1-\lambda)b) + \min\{\lambda, 1-\lambda\}\omega_1(\varphi(a-b)) .$$

Now let a, $b \in \mathbb{R}^2 \setminus \{0\}$. By the previous inequality and (H2), setting $\lambda = \frac{\varphi(a)}{\varphi(a)+\varphi(b)}$ we derive that

$$\begin{aligned}\varphi(a) + \varphi(b) &= \big(\varphi(a)+\varphi(b)\big)\left(\lambda\varphi\left(\frac{a}{\varphi(a)}\right) + (1-\lambda)\varphi\left(\frac{b}{\varphi(b)}\right)\right) \\ &\geq \big(\varphi(a)+\varphi(b)\big)\left(\varphi\left(\frac{a+b}{\varphi(a)+\varphi(b)}\right) + \min\{\lambda,1-\lambda\}\omega_1\left(\varphi\left(\frac{a}{\varphi(a)} - \frac{b}{\varphi(b)}\right)\right)\right) \\ &= \varphi(a+b) + \min\{\varphi(a), \varphi(b)\}\omega_1\left(\varphi\left(\frac{a}{\varphi(a)} - \frac{b}{\varphi(b)}\right)\right) \\ &\geq \varphi(a+b) + m\min\{|a|, |b|\}\omega_1\left(m\left|\frac{a}{\varphi(a)} - \frac{b}{\varphi(b)}\right|\right) .\end{aligned}$$

To conclude it remains to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\left|\frac{a}{\varphi(a)} - \frac{b}{\varphi(b)}\right| \geq \varepsilon$, then $1 - \frac{a}{|a|}\cdot\frac{b}{|b|} \geq \delta$. We argue by contrdiction assuming that this is not true. Then there exist $\varepsilon > 0$ and a_n, $b_n \in \mathbb{R}^2 \setminus \{0\}$ such that $\left|\frac{a_n}{\varphi(a_n)} - \frac{b_n}{\varphi(b_n)}\right| \geq \varepsilon$, $\frac{a_n}{|a_n|}\cdot\frac{b_n}{|b_n|} \to 1$, $\frac{a_n}{\varphi(a_n)} \to a$, $\frac{b_n}{\varphi(b_n)} \to b$. Hence, $\varphi(a) = \varphi(b) = 1$, $|a-b| \geq \varepsilon$, and $\frac{a_n}{|a_n|}\cdot\frac{b_n}{|b_n|} \to \frac{a}{|a|}\cdot\frac{b}{|b|} = 1$. The last condition implies that $a = tb$ for some $t > 0$. But since $\varphi(a) = \varphi(b) = 1$, we have that $t = 1$, which contradicts the fact $|a-b| \geq \varepsilon$. □

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