Asymptotic analysis of periodically-perforated nonlinear media

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1 Introduction

A well-known result on the asymptotic behaviour of Dirichlet problems in perforated domains shows the appearance of a 'strange' extra term as the period of the perforation tends to 0. In a paper by Cioranescu and Murat [10] (see also e.g. earlier work by Marchenko Khrushlov [17]) the following result (among others) is proved. Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 3$ and for all $\delta > 0$ let Ω_δ be the *periodically perforated domain*

$$\Omega_\delta = \Omega \setminus \bigcup_{i \in \mathbb{Z}^n} \overline{B_i^\delta},$$

where B_i^δ denotes the open ball of centre $x_i^\delta = i\delta$ and radius $\delta^{n/(n-2)}$. Let $\phi \in H^{-1}(\Omega)$ be fixed, and let $u_\delta \in H_0^1(\Omega)$ be the solution of the problem

$$\begin{cases} -\Delta u_\delta = \phi \\ u \in H_0^1(\Omega_\delta), \end{cases}$$

extended to 0 outside Ω_δ. Then, as $\delta \to 0$, the sequence u_δ converges weakly in $H_0^1(\Omega)$ to the function u which solves the problem

$$\begin{cases} -\Delta u_\delta + Cu = \phi \\ u \in H_0^1(\Omega), \end{cases}$$

where C denotes the *capacity of the unit ball* in $\mathbb{R}^n$:

$$C = \inf\Big\{ \int_{\mathbb{R}^n} |D\zeta|^2 dx : \zeta \in H^1(\mathbb{R}^n),\ \zeta = 1 \text{ on } B_1(0) \Big\}.$$

This result can be easily translated in a equivalent variational form and set in the framework of Γ-convergence, since u_δ is the solution of the minimum problem

$$\min\Big\{ \int_\Omega |Dv|^2\, dx - 2\langle \phi, v\rangle : v \in H_0^1(\Omega),\ v = 0 \text{ on } \Omega \setminus \Omega_\delta \Big\},$$

and the limit function u solves

$$\min\Big\{\int_\Omega (|Dv|^2 + C|v|^2)\,dx - 2\langle\phi, v\rangle : v \in H^1_0(\Omega)\Big\}.$$

In this paper we give a direct proof of the non-linear vector-valued version of this variational problem under minimal assumptions. More precisely, let Ω be a bounded open set in $\mathbb{R}^n$ and let $m \geq 1$. Let $1 < p < n$ and for all $\delta > 0$ let Ω_δ be the *periodically perforated domain* defined as above, where now B_i^δ denotes the open ball of centre $x_i^\delta = i\delta$ and radius $\delta^{n/(n-p)}$ (for notational simplicity we do not treat the case $n = p$, which can be dealt with similarly; for the necessary changes in the statements see [10]). Note that this is the only meaningful scaling for the radii of the perforation, since other choices give trivial convergence results. Let $f : \mathbb{M}^{m\times n} \to [0, +\infty)$ be a Borel function satisfying a growth condition of order p, and let (δ_j) be a sequence of strictly positive numbers converging to 0 such that there exists the limit

$$g(A) = \lim_j \delta_j^{\frac{np}{n-p}} Qf\Big(\delta_j^{-\frac{n}{n-p}} A\Big)$$

for all $A \in \mathbb{M}^{m\times n}$, where Qf denotes the *quasiconvexification of* f. Note that this condition is not restrictive upon passing to a subsequence and is trivially satisfied if f is positively homogeneous of degree p. Then, if $\phi \in W^{-1,p'}(\Omega; \mathbb{R}^m)$ is fixed, the minimum values

$$m_j = \inf\Big\{\int_{\Omega_{\delta_j}} f(Du)\,dx + \langle\phi, u\rangle :\ u \in W_0^{1,p}(\Omega_{\delta_j}; \mathbb{R}^m)\Big\}$$

converge to the minimum value

$$m = \min\Big\{\int_\Omega \Big(Qf(Du) + \varphi(u)\Big)\,dx + \langle\phi, u\rangle :\ u \in W_0^{1,p}(\Omega; \mathbb{R}^m)\big\},$$

where φ is given by the *nonlinear capacitary formula*

$$\varphi(z) = \inf\Big\{\int_{\mathbb{R}^n} g(D\zeta)dx : \zeta - z \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\},$$

which agrees with those obtained in convex cases (see e.g. [2], [12], [19], [8]). Moreover, if $u_j \in W_0^{1,p}(\Omega_{\delta_j}; \mathbb{R}^m)$ is such that $\int_{\Omega_{\delta_j}} f(Du_j)\,dx + \langle\phi, u_j\rangle = m_j + o(1)$ as $j \to +\infty$, then, upon extending u_j to 0 outside Ω_{δ_j}, (u_j) admits a subsequence weakly converging in $W_0^{1,p}(\Omega; \mathbb{R}^m)$ to a solution of the problem defining m.

Note that we do not assume any structure or regularity condition on f. In the case of convex and differentiable f we may recover the corresponding result for systems contained in the paper by Casado Diaz and Garroni [8], where more arbitrary geometries are also considered. Note moreover that φ may depend on the subsequence (δ_j), and as a consequence the values m_j may not converge.

Furthermore, the function φ may not be positively homogeneous of degree p, as already observed by Casado Diaz and Garroni [9].

The proof of the result is based only on a direct Γ-convergence approach. The fundamental tool is a 'joining lemma for perforated domains' (Lemma 3.1), which, loosely speaking, allows us to restrict our attention to families of functions (u_δ), converging to a function u, which equal the constant $u(x_i^\delta)$ on suitable annuli surrounding B_i^δ. The contribution of these functions on such annuli easily leads to the formula defining φ. This method seems of interest also since it can be easily applied to sequences of integral functionals by considering minimum problems m_j where we replace $f(Du)$ by $f_j(x, Du)$, in the spirit of a recent result by Dal Maso and Murat [13]. In a parallel work [1], for example, we examine the case $f_j(x, z) = f(x/\varepsilon_j, z)$.

2 Statement of the main result

In all that follows $p > 1$, $m \geq 1$, $n > p$ are fixed ($m, n \in \mathbb{N}$), and Ω is a bounded open subset of $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$ is a Lebesgue-measurable set then $|E|$ is its Lebesgue measure. $B_\rho(x)$ is the open ball of centre x and radius ρ. We use standard notation for Lebesgue and Sobolev spaces. The letter c denotes a generic strictly positive constant.

With $\mathbb{M}^{m\times n}$ we denote the space of $m \times n$ matrices with real entries. If $h : \mathbb{M}^{m\times n} \to [0, +\infty)$ is a Borel function, the ($W^{1,p}$-)*quasiconvexification of* h is given by the formula

$$Qh(A) = \inf\Big\{ \int_{(0,1)^n} h(A + Du)\, dx : u \in W_0^{1,p}((0,1)^n; \mathbb{R}^m) \Big\} \tag{2.1}$$

for $A \in \mathbb{M}^{m\times n}$. We say that h is ($W^{1,p}$-)*quasiconvex* if $Qh = h$ (see [18], [3], [5]). We recall the following result.

Remark 2.1 If h is a Borel function as above, and there exist constants $c_1, c_2 > 0$ such that $c_1(|A|^p - 1) \leq h(A) \leq c_2(|A|^p + 1)$, then the function Qh is quasiconvex (see [5] Proposition 6.7) and the functional

$$\mathcal{H}(u) = \int_\Omega Qh(Du)\, dx$$

is the lower-semicontinuous envelope of the functional

$$H(u) = \int_\Omega h(Du)\, dx$$

on $W^{1,p}(\Omega; \mathbb{R}^m)$ with respect to the $L^p(\Omega; \mathbb{R}^m)$ convergence. In fact, e.g. by [5] Theorem 12.5, the lower-semicontinuous envelope $\overline{H}$ of H can be written in an integral form $\overline{H}(u) = \int_\Omega \psi(Du)\, dx$, with ψ quasiconvex. Since $\psi \leq h$ then $\psi =$

$Q\psi \leq Qh$ and $\overline{H} \leq \mathcal{H}$. On the other hand Qh is quasiconvex; hence, $\mathcal{H}$ is lower semicontinuous with respect to the $L^p(\Omega;\mathbb{R}^m)$ convergence (see e.g. [5] Theorem 5.16), so that $\mathcal{H} \leq \overline{H}$.

2.1 Γ-convergence

We recall the definition of Γ-convergence of a sequence (Φ_j) of functionals defined on $W^{1,p}(\Omega;\mathbb{R}^m)$ (with respect to the $L^p(\Omega;\mathbb{R}^m)$-convergence). We say that (Φ_j) Γ-converges to Φ_0 on $W^{1,p}(\Omega;\mathbb{R}^m)$ if for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we have:

(i) (*liminf inequality*) for all (u_j) sequences of functions in $W^{1,p}(\Omega;\mathbb{R}^m)$ converging to $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ in $L^p(\Omega;\mathbb{R}^m)$ we have

$$\Phi_0(u) \leq \liminf_j \Phi_j(u_j);$$

(ii) (*limsup inequality*) for all $\eta > 0$ there exists a sequence (u_j) of functions in $W^{1,p}(\Omega;\mathbb{R}^m)$ converging to $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ in $L^p(\Omega;\mathbb{R}^m)$ such that

$$\Phi_0(u) \geq \limsup_j \Phi_j(u_j) - \eta.$$

If (i) and (ii) hold we write $\Phi_0(u) = \Gamma\text{-}\lim_j \Phi_j(u)$

We also introduce the notation

$$\Gamma\text{-}\liminf_j \Phi_j(u) = \inf\Big\{\liminf_j \Phi_j(u_j) : u_j \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\Big\},$$

$$\Gamma\text{-}\limsup_j \Phi_j(u) = \inf\Big\{\limsup_j \Phi_j(u_j) : u_j \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\Big\},$$

so that the equality $\Gamma\text{-}\liminf_j \Phi_j(u) = \Gamma\text{-}\limsup_j \Phi_j(u)$ is equivalent to the existence of the $\Gamma\text{-}\lim_j \Phi_j(u)$.

We will say that a family (Φ_δ) Γ-converges to Φ_0 if for all sequences (δ_j) of positive numbers converging to 0 (i) and (ii) above are satisfied with Φ_{δ_j} in place of Φ_j.

We recall the following fundamental theorem (see e.g. [5] Theorem 7.2).

Theorem 2.2 *Let U be an open subset of $\mathbb{R}^n$ and let Φ_j Γ-converge to Φ_0 on $W^{1,p}(U;\mathbb{R}^m)$. Let there exist a compact set $K \subset W^{1,p}(U;\mathbb{R}^m)$, with respect to the $L^p(U;\mathbb{R}^m)$ convergence, such that* $\inf \Phi_j = \inf_K \Phi_j$ *for all* $j \in \mathbb{N}$*. Then there exists* $\min \Phi_0 = \lim_j \inf \Phi_j$*. Moreover, if (j_k) is an increasing sequence of integers and (u_k) is a converging sequence such that* $\lim_k \Phi_{j_k}(u_k) = \lim_j \inf \Phi_j$ *then its limit is a minimum point for Φ_0.*

For an introduction to Γ-convergence we refer to [11], [4] and Part II of [5].

2.2 Periodically perforated domains

For all $\delta > 0$ we consider the lattice $\delta\mathbb{Z}^n$ whose points will be denoted by $x_i^\delta = \delta i$ ($i \in \mathbb{Z}^n$). Moreover, for all $i \in \mathbb{Z}^n$

$$B_i^\delta = B_{\delta^{n/(n-p)}}(x_i^\delta)$$

denotes the ball of center x_i^δ and radius $\delta^{n/(n-p)}$. The main result of the paper is the following.

Theorem 2.3 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. Let $f : \mathbb{M}^{m\times n} \to [0,+\infty)$ be a Borel function such that $f(0) = 0$ and satisfying a* growth condition of order p: *there exist two constants $c_1, c_2 > 0$ such that*

$$c_1(|A|^p - 1) \le f(A) \le c_2(|A|^p + 1) \quad \text{for all } A \in \mathbb{M}^{m\times n}. \tag{2.2}$$

Let (δ_j) be a sequence of strictly positive numbers converging to 0. Then, upon possibly extracting a subsequence, for all $A \in \mathbb{M}^{m\times n}$ there exist the limit

$$g(A) = \lim_j \delta_j^{\frac{np}{n-p}} Qf\Big(\delta_j^{-\frac{n}{n-p}} A\Big), \tag{2.3}$$

where Qf denotes the quasiconvexification of f, so that the value

$$\varphi(z) = \inf\Big\{\int_{\mathbb{R}^n} g(D\zeta)dx : \zeta - z \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\} \tag{2.4}$$

is well defined for all $z \in \mathbb{R}^m$. Moreover, the functionals $F_j : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ defined by

$$F_j(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \text{if } u = 0 \text{ a.e. on } \bigcup_{i\in\mathbb{Z}^n} B_i^{\delta_j} \cap \Omega \\ +\infty & \text{otherwise} \end{cases} \tag{2.5}$$

Γ-converge to the functional $F : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$ defined by

$$F(u) = \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx. \tag{2.6}$$

Corollary 2.4 *If f is positively homogeneous of degree p then the limit is independent of the subsequence and*

$$\varphi(z) = \inf\Big\{\int_{\mathbb{R}^n} f(D\zeta)\,dx : \zeta - z \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\} \tag{2.7}$$

for all $z \in \mathbb{R}^m$.

Proof. It suffices to remark that in this case formula (2.3) gives $g = Qf$ and that we may replace Qf by f in (2.4) by using Remark 2.1. □

Corollary 2.5 (Convergence of minimum problems) *Let* (δ_j) *satisfy the thesis of Theorem* 2.3. *Then for all* $\phi \in W^{-1,p'}(\Omega;\mathbb{R}^m)$ *the minimum values*

$$m_j = \inf\Big\{F_j(u) + \langle\phi, u\rangle : \; u \in W_0^{1,p}(\Omega;\mathbb{R}^m)\Big\}$$

converge to

$$m = \min\Big\{F(u) + \langle\phi, u\rangle : \; u \in W_0^{1,p}(\Omega;\mathbb{R}^m)\Big\}.$$

Moreover, if u_j *is such that* $F_j(u_j) + \langle\phi, u_j\rangle = m_j + o(1)$ *as* $j \to +\infty$, *then it admits a subsequence weakly converging in* $W_0^{1,p}(\Omega;\mathbb{R}^m)$ *to a solution of the problem defining* m.

Proof. By a cut-off argument near $\partial\Omega$ (see [5] Section 11.3) if $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ then the sequences in (ii) of the definition of Γ-convergence can be taken in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ as well, while by the growth condition (2.2) we have $u_j \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. This fact, together with the continuity of $G(u) = \langle\phi, u\rangle$ with respect to the weak convergence in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, implies that the functionals

$$\Phi_j(u) = \begin{cases} F_j(u) + G(u) & \text{if } u \in W_0^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty \end{cases}$$

Γ-converge to

$$\Phi_0(u) = \begin{cases} F(u) + G(u) & \text{if } u \in W_0^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty \end{cases}$$

on $W^{1,p}(\Omega;\mathbb{R}^m)$. We can then apply Theorem 2.2 with $K = \{u \in W_0^{1,p}(\Omega;\mathbb{R}^m) : \|Du\|_{L^p(\Omega;\mathbb{R}^m)} \le c\}$ for a suitable $c > 0$. □

Remark 2.6 (*Non-spherical holes*) The results are easily extended to non-spherical geometries, by fixing any bounded set $E \subset \mathbb{R}^n$ and considering $x_i^\delta + \delta^{n/(n-p)}E$ in place of B_i^δ. The same conclusion follows, upon replacing $B_1(0)$ by E in the definition of φ.

Remark 2.7 In general, the function g depends on the subsequence (δ_j), and so does φ. In this case, the Γ-limit as $\delta \to 0$ of the functionals

$$F_\delta(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \text{if } u = 0 \text{ a.e. on } \bigcup_{i\in\mathbb{Z}^n} B_i^\delta \cap \Omega \\ +\infty & \text{otherwise} \end{cases} \tag{2.8}$$

does not exist.

The proof of Theorem 2.3 will be obtained in the next sections.

3 A joining lemma on varying domains

In this section we prove a technical result which allows to modify sequences of functions near the sets B_i^δ. Its proof is close in spirit to the method introduced by De Giorgi to match boundary conditions for minimizing sequences (see [14]). For future reference we state this lemma in a general form.

Let (δ_j) be a sequence of positive numbers converging to 0, and let $f_j : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to [0,+\infty)$ be Borel functions satisfying the growth conditions (2.2) uniformly in j. In the following sections we will simply take $f_j(x,z) = f(z)$.

Note that in this section and the following ones sometimes we simply write δ in place of δ_j not to overburden notation.

Lemma 3.1 *Let (u_j) converge weakly to u in $W^{1,p}(\Omega;\mathbb{R}^m)$, and let*

$$Z_j = \{i \in \mathbb{Z}^n : \ \mathrm{dist}\,(x_i^\delta, \mathbb{R}^n \setminus \Omega) > \delta_j\}. \tag{3.1}$$

Let $k \in \mathbb{N}$ be fixed. Let (ρ_j) be a sequence of positive numbers with $\rho_j < \delta_j/2$. For all $i \in Z_j$ there exists $k_i \in \{0,\dots,k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \Omega : \ 2^{-k_i-1}\rho_j < |x - x_i^\delta| < 2^{-k_i}\rho_j\Big\}, \tag{3.2}$$

$$u_j^i = |C_i^j|^{-1} \int_{C_i^j} u_j\,dx \quad (\textit{the } \text{mean value } \textit{of } u_j \textit{ on } C_i^j), \tag{3.3}$$

and

$$\rho_j^i = \frac{3}{4} 2^{-k_i}\rho_j \quad (\textit{the } \text{middle radius } \textit{of } C_i^j), \tag{3.4}$$

there exists a sequence (w_j), with $w_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$w_j = u_j \ \textit{on } \Omega \setminus \bigcup_{i\in Z_j} C_i^j \tag{3.5}$$

$$w_j(x) = u_j^i \ \textit{if } |x - x_i^\delta| = \rho_j^i \tag{3.6}$$

and

$$\Big|\int_\Omega (f_j(x,Dw_j) - f_j(x,Du_j))\,dx\Big| \le c\frac{1}{k}. \tag{3.7}$$

Moreover, if $\rho_j = o(\delta_j)$ and the sequence $(|Du_j|^p)$ is equi-integrable, then we can choose $k_i = 0$ for all $i \in Z_j$.

Proof. For all $j \in \mathbb{N}$, $i \in Z_j$ and $h \in \{0,...,k-1\}$ let

$$C_{i,h}^j = \Big\{x \in \Omega : \ 2^{-h-1}\rho_j < |x - x_i^\delta| < 2^{-h}\rho_j\Big\},$$

and let

$$u_j^{i,h} = |C_{i,h}^j|^{-1} \int_{C_{i,h}^j} u_j\,dx,$$

and

$$\rho_j^{i,h} = \frac{3}{4}2^{-h}\rho_j.$$

Consider a function $\phi = \phi_{i,h}^j \in C_0^\infty(C_{i,h}^j)$ such that $\phi = 1$ on $\partial B_{\rho_j^{i,h}}(x_i^\delta)$ and $|D\phi| \le c/2^{-h}\rho_j = c/\rho_j^{i,h}$. Let $w_j^{i,h}$ be defined on $C_{i,h}^j$ by

$$w_j^{i,h} = u_j^{i,h}\phi + (1-\phi)u_j \text{ on } C_{i,h}^j,$$

with $\phi = \phi_{i,h}^j$ as above. We then have, by the growth conditions on f_j,

$$\begin{aligned}\int_{C_{i,h}^j} f_j(x, Dw_j^{i,h})\,dx &= \int_{C_{i,h}^j} f_j(x, D\phi(u_j^{i,h} - u_j) + (1-\phi)Du_j))\,dx \\ &\le c\int_{C_{i,h}^j}(1 + |D\phi|^p|u_j - u_j^{i,h}|^p + |Du_j|^p)\,dx.\end{aligned}$$

By the Poincaré inequality and its scaling properties we have

$$\int_{C_{i,h}^j} |u_j - u_j^{i,h}|^p\,dx \le c(\rho_j^{i,h})^p\int_{C_{i,h}^j}|Du_j|^p\,dx, \tag{3.8}$$

so that, recalling that $|D\phi| \le c/\rho_j^{i,h}$,

$$\int_{C_{i,h}^j} f_j(x, Dw_j^{i,h})\,dx \le c\int_{C_{i,h}^j}(1 + |Du_j|^p)\,dx.$$

Since by summing up in h we trivially have

$$\sum_{h=0}^{k-1}\int_{C_{i,h}^j}(1 + |Du_j|^p)\,dx \le |B_{\rho_j}(x_i^\delta)| + \int_{B_{\rho_j}(x_i^\delta)}|Du_j|^p\,dx,$$

there exists $k_i \in \{0, \ldots, k-1\}$ such that

$$\int_{C_{i,k_i}^j}(1 + |Du_j|^p)\,dx \le \frac{1}{k}\Big(|B_{\rho_j}(x_i^\delta)| + \int_{B_{\rho_j}(x_i^\delta)}|Du_j|^p\,dx\Big), \tag{3.9}$$

There follows that

$$\int_{C_{i,k_i}^j} f_j(x, Dw_j^{i,k_i})\,dx \le \frac{c}{k}\Big(|B_{\rho_j}(x_i^\delta)| + \int_{B_{\rho_j}(x_i^\delta)}|Du_j|^p\,dx\Big). \tag{3.10}$$

By (3.9) and (3.10) we get

$$\begin{aligned}\int_{C_{i,k_i}^j}|f_j(x, Du_j) - f_j(x, Dw_j)|\,dx &\le \int_{C_{i,k_i}^j}(f_j(x, Du_j) + f_j(x, Dw_j))\,dx \\ &\le \frac{c}{k}\Big(|B_{\rho_j}(x_i^\delta)| + \int_{B_{\rho_j}(x_i^\delta)}|Du_j|^p\,dx\Big).\end{aligned}$$

Note that if $(|Du_j|^p)$ is equi-integrable and $\rho_j = o(\delta_j)$ then we do not need to use this argument, and may simply choose $k_i = 0$ for all $i \in Z_j$.

With this choice of k_i for all $i \in Z_j$, conditions (3.5)–(3.7) are satisfied by choosing $h = k_i$ in the definitions above, i.e. with $C_i^j = C_{i,k_i}^j$, $u_j^i = u_j^{i,k_i}$ $\rho_j^i = \rho_j^{i,k_i}$ and w_j defined by (3.5) and

$$w_j = u_j^i \phi + (1-\phi) u_j \text{ on } C_i^j,$$

with $\phi = \phi_{i,k_i}^j$.

Finally we prove the convergence of w_j to u in $L^p(\Omega;\mathbb{R}^m)$. By (3.8)

$$\begin{aligned}
\int_\Omega |w_j - u|^p\,dx &= \int_{\Omega\setminus\bigcup_{i\in Z_j} C_i^j} |u_j - u|^p\,dx \\
&\quad + \int_{\bigcup_{i\in Z_j} C_i^j} |u_j^i \phi_{i,k_i}^j + (1-\phi_{i,k_i}^j)u_j - u|^p\,dx \\
&\le \int_{\Omega\setminus\bigcup_{i\in Z_j} C_i^j} |u_j - u|^p\,dx \\
&\quad + c\sum_{i\in Z_j} \int_{C_i^j} |u_j - u_j^i|^p\,dx + c\int_{\bigcup_{i\in Z_j} C_i^j} |u_j - u|^p\,dx \\
&\le c\int_\Omega |u_j - u|^p\,dx + c\rho_j^p \sum_{i\in Z_j} \int_{C_i^j} |Du_j|^p\,dx \\
&\le c\int_\Omega |u_j - u|^p\,dx + c\rho_j^p \sup_j \int_\Omega |Du_j|^p\,dx.
\end{aligned}$$

Hence passing to the limit as j tends to $+\infty$ we get the desired convergence. In particular, since (w_j) is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$, we get that (w_j) weakly converges to u in $W^{1,p}(\Omega;\mathbb{R}^m)$. □

4 Some auxiliary energy densities

It will be convenient to approximate the function φ defined in (2.4) by suitable energy densities defined by minimum problems on bounded sets so as to use the properties of convergence of minima by Γ-convergence (Theorem 2.2). In this section we define such energies and list some of their properties.

We begin by proving in the following remark the existence of g in (2.3).

Remark 4.1 We can consider the functions $g_j : \mathbb{M}^{m\times n} \to [0,+\infty)$ defined by

$$g_j(A) = \delta_j^{\frac{np}{n-p}} Qf\Big(\delta_j^{-\frac{n}{n-p}} A\Big). \tag{4.1}$$

Since g_j are quasiconvex and satisfy uniformly a growth condition of order p they are equi-locally Lipschitz continuous on $\mathbb{M}^{m\times n}$: there exists C depending only on c_1, c_2, p such that

$$|g_j(A) - g_j(B)| \le C(1 + |A|^{p-1} + |B|^{p-1})|A - B| \tag{4.2}$$

for all $A, B \in \mathbb{M}^{m\times n}$ (see [5] Remark 4.13). Hence, there exists a subsequence (not relabeled) converging pointwise to some limit function g. We may therefore assume that (2.3) holds. Note that this convergence implies that for all subsets U of $\mathbb{R}^n$ the functionals $G_j(\cdot, U)$ defined on $W^{1,p}(U;\mathbb{R}^m)$ by

$$G_j(u, U) = \int_U g_j(Du)\,dx \tag{4.3}$$

Γ-converge to the functional $G(\cdot, U)$ defined on $W^{1,p}(U;\mathbb{R}^m)$ by

$$G(u, U) = \int_U g(Du)\,dx \tag{4.4}$$

(see [5] Proposition 12.8).

Using the notation of the remark above, we set

$$\varphi_{N,j}(z) = \inf\Big\{\int_{B_N(0)} g_j(D\zeta)\,dy : \ \zeta - z \in W_0^{1,p}(B_N(0);\mathbb{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\}. \tag{4.5}$$

Note that by the Γ-convergence in Remark 4.1 and Theorem 2.2, arguing as in the proof of Corollary 2.5, we easily deduce that $\varphi_{N,j}$ converge pointwise as $j \to +\infty$ to the function φ_N, defined by

$$\varphi_N(z) = \inf\Big\{\int_{B_N(0)} g(D\zeta)\,dy : \ \zeta - z \in W_0^{1,p}(B_N(0);\mathbb{R}^m),\ \zeta = 0 \text{ on } B_1(0)\Big\}. \tag{4.6}$$

We briefly examine some properties of the functions $\varphi_{N,j}$ and φ_N which are easily deduced from the growth conditions satisfied by g_j and g.

Remark 4.2 (i) For all $N \in \mathbb{N}$ and $\eta > 0$ there exists $c_{N,\eta}$ such that

$$\begin{aligned}|\varphi_{N,j}(z) - \varphi_{N,j}(w)| \ \le\ & c_{N,\eta}\,\delta_j^{n(p-1)/(n-p)}\,|z-w|(1 + |w|^{p-1} + |z|^{p-1}) \\ & + c|z-w|(|w|^{p-1} + |z|^{p-1})\end{aligned} \tag{4.7}$$

for all $|z|, |w| > \eta$ and j. This can be easily checked if we consider a linear similitude ϕ such that $\phi(z) = w$ and $\zeta \in z + W_0^{1,p}(B_N(0);\mathbb{R}^m)$ such that $\zeta = 0$ on $B_1(0)$ and

$$\varphi_{N,j}(z) = \int_{B_N(0)} g_j(D\zeta)\,dy\,.$$

The existence of ζ follows from the quasiconvexity of g_j. If we define $\tilde\zeta = \phi(\zeta)$ then $\tilde\zeta \in w + W^{1,p}_0(B_N(0);\mathbb{R}^m)$ and $\tilde\zeta = 0$ on $B_1(0)$. By using $\tilde\zeta$ as a test function we can estimate $\varphi_{N,j}(w)$ taking into account the following inequality

$$|g_j(A) - g_j(B)| \le C(\delta_j^{n(p-1)/(n-p)} + |A|^{p-1} + |B|^{p-1})|A - B|\,,$$

which refines (4.2). By a symmetric argument we deduce the estimate on $|\varphi_{N,j}(z) - \varphi_{N,j}(w)|$.

(ii) From (i) we deduce that $\varphi_{N,j} \to \varphi_N$ uniformly on compact sets of $\mathbb{R}^m\setminus\{0\}$ by Ascoli Arzela's Theorem.

(iii) By comparison with the well-known case $g_j(A) = |A|^p$, in which case we have $\varphi_{N,j}(z) = c|z|^p$, we deduce that

$$\varphi_{N,j}(z) \le c_N \delta_j^{np/(n-p)} + c|z|^p. \tag{4.8}$$

(iv) Note that $c_1|A|^p \le g(A) \le c_2|A|^p$, so that, again by comparison with the case $g(A) = |A|^p$, we have $c_1 c|z|^p \le \varphi_N(z) \le c_2 c|z|^p$. Taking this into account and arguing as in (i) for fixed $\eta > 0$ we also have

$$|\varphi_N(z) - \varphi_N(w)| \le c\,(\eta^p + |z - w|(|w|^{p-1} + |z|^{p-1})) \tag{4.9}$$

for all $w, z \in \mathbb{R}^m$.

(v) Arguing as in (ii) and taking (iv) into account, we deduce that $\varphi_N \to \varphi$ uniformly on compact sets of $\mathbb{R}^m$.

Proposition 4.3 *Let (u_j) be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$ converging to u weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, let (C^j_i) $(i \in Z_j)$ be a collection of annuli of the form* (3.2) *for an arbitrary choice of k_i, let u^i_j be defined by* (3.3), *and let ψ_j be defined by*

$$Q^\delta_i = x^\delta_i + \Big(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\Big)^n, \qquad \psi_j = \sum_{i\in Z_j} \varphi_{N,j}(u^i_j)\chi_{Q^\delta_i}. \tag{4.10}$$

Then we have

$$\lim_j \int_\Omega |\psi_j - \varphi_N(u)|\,dx = 0. \tag{4.11}$$

Proof. Let $\eta > 0$ be fixed. If $\eta < |z| \le \sup_j \|u_j\|_\infty$ then we have, by Remark 4.2(ii),

$$|\varphi_{N,j}(z) - \varphi_N(z)| \le o(1)$$

as $j \to +\infty$, uniformly in z, while, if $|z| < \eta$ then, by Remark 4.2(iii),

$$|\varphi_{N,j}(z) - \varphi_N(z)| \le c_N \delta_j^{np/(n-p)} + 2c\eta^p.$$

Set

$$\hat\psi_j = \sum_{i\in Z_j} \varphi_N(u^i_j)\chi_{Q^\delta_i}. \tag{4.12}$$

By the arbitrariness of η and the convergence of $\varphi_N(u_j)$ to $\varphi_N(u)$ in $L^1(\Omega)$, we deduce that the limit in (4.11) equals the limits

$$\begin{aligned}\lim_j \int_\Omega |\hat\psi_j - \varphi_N(u)|\,dx &= \lim_j \int_\Omega |\hat\psi_j - \varphi_N(u_j)|\,dx \\ &= \lim_j \sum_{i\in Z_j} \int_{Q_i^\delta} |\varphi_N(u_j^i) - \varphi_N(u_j)|\,dx \qquad (4.13)\\ &\le c\Big(\eta^p + \lim_j \Big(\sup_j \|u_j\|^p_{L^\infty(\Omega;\mathbb{R}^m)}\Big) \sum_{i\in Z_j}\int_{Q_i^\delta} |u_j^i - u_j| dx\Big)\end{aligned}$$

by (4.9). By Hölder's and Poincaré's inequalities, we have

$$\begin{aligned}\int_{Q_i^\delta} |u_j^i - u_j|\,dx &\le \delta_j^{n(p-1)/p}\Big(\int_{Q_i^\delta} |u_j^i - u_j|^p\,dx\Big)^{1/p}\\ &\le \delta_j^{n(p-1)/p} c\delta_j \Big(\int_{Q_i^\delta} |Du_j|^p\,dx\Big)^{1/p},\end{aligned}$$

so that

$$\sum_{i\in Z_j}\int_{Q_i^\delta} |u_j^i - u_j|\,dx \le c\delta_j \Big(\int_\Omega |Du_j|^p\,dx\Big)^{1/p},$$

which proves the convergence to 0 of the limits in (4.13) by the arbitrariness of η. □

5 Proof of the liminf inequality

Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and let $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ be such that $\sup_j F_j(u_j) < +\infty$. Note that by (2.2) $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$.

We can use a sequence (w_j) constructed as in Lemma 3.1 to estimate the liminf inequality for (F_j). We fix $k, N \in \mathbb{N}$ with $N > 2^k$, and define w_j as in Lemma 3.1 with

$$\rho_j = N\delta_j^{n/(n-p)}. \qquad (5.1)$$

Note that with this choice of ρ_j we always have $w_j = u_j = 0$ on B_i^δ. Let $E_j = E_j^{k,N}$ be given by

$$E_j = \bigcup_{i\in Z_j} B_i^j, \qquad \text{where} \qquad B_i^j = B_{\rho_j^i}(x_i^\delta)$$

for all $i \in Z_j$ (Z_j given by (3.1) and ρ_j^i by (3.4)). We first deal with the contribution of the part of Du_j outside the set E_j.

Proposition 5.1 *We have*

$$\liminf_j \int_{\Omega\setminus E_j} f(Du_j)\,dx \ge \int_\Omega Qf(Du)\,dx - \frac{c}{k} \qquad (5.2)$$

Proof. Let
$$v_j(x) = \begin{cases} u_j^i & \text{if } x \in B_i^j \\ w_j(x) & \text{if } x \in \Omega \setminus E_j. \end{cases}$$

Note that by Lemma 3.1 (v_j) is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ and that $\lim_j |\{x \in \Omega : u_j(x) \neq v_j(x)\}| = 0$. We deduce that $v_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ so that

$$\begin{aligned} \liminf_j \int_{\Omega\setminus E_j} f(Du_j)\,dx + \frac{c}{k} &\geq \liminf_j \int_{\Omega\setminus E_j} f(Dw_j)\,dx \\ &= \liminf_j \int_{\Omega} f(Dv_j)\,dx \geq \int_\Omega Qf(Du)\,dx, \end{aligned}$$

the last inequality following from Remark 2.1. □

We now turn to the estimate of the contribution on E_j. With fixed $j \in \mathbb{N}$ and $i \in Z_j$, let
$$\zeta(y) = w_j\Big(x_i^\delta + \delta_j^{n/(n-p)} y\Big)$$
be defined on $B_{\frac{3}{4}2^{-k_i}N}(0)$, and extended to u_j^i outside this ball. Note that

$$\zeta - u_j^i \in W_0^{1,p}(B_N(0);\mathbb{R}^m) \qquad \text{and} \qquad \zeta = 0 \text{ on } B_1(0). \tag{5.3}$$

By a change of variables we obtain

$$\int_{B_i^j} f(Dw_j)\,dx = \delta_j^n \int_{B_N(0)} \delta_j^{np/(n-p)} f(\delta_j^{-n/(n-p)} D\zeta)\,dx \geq \delta_j^n \varphi_{N,j}(u_j^i) \tag{5.4}$$

by (4.5); hence, to give the estimate on E_j we have to compute the limit

$$\liminf_j \sum_{i\in Z_j} \delta_j^n \varphi_{N,j}(u_j^i) = \liminf_j \int_\Omega \psi_j\,dx, \tag{5.5}$$

where ψ_j is defined as in (4.10).

Proposition 5.2 *We have*

$$\Gamma\text{-}\liminf_j F_j(u) \geq \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$.

Proof. Let $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$. We can assume, upon possibly passing to a subsequence, that there exists the limit

$$\lim_j F_j(u_j) < +\infty,$$

so that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. By [6] Lemma 3.5, upon passing to a further subsequence, for all $M \in \mathbb{N}$ and $\eta > 0$ there exists $R_M > M$ and a Lipschitz function Φ_M of Lipschitz constant 1 such that $\Phi_M(z) = z$ if $|z| < R_M$ and $\Phi_M(z) = 0$ if $|z| > 2R_M$, and

$$\lim_j F_j(u_j) \geq \liminf_j F_j(\Phi_M(u_j)) - \eta. \tag{5.6}$$

From Lemma 3.1, (5.5), and Proposition 4.3, applied to $(\Phi_M(u_j))$ in place of (u_j), we get that

$$\begin{aligned} \liminf_j \int_{E_j} f(D\Phi_M(u_j))\,dx + \frac{c}{k} &\geq \liminf_j \sum_{i\in Z_j} \delta_j^n \varphi_{N,j}((\Phi_M(u))_j^i) \\ &= \int_\Omega \varphi_N(\Phi_M(u))\,dx \\ &\geq \int_\Omega \varphi(\Phi_M(u))\,dx. \end{aligned} \tag{5.7}$$

Summing up (5.7) and (5.2) and by the arbitrariness of k, we then obtain

$$\liminf_j F_j(\Phi_M(u_j)) \geq \int_\Omega Qf(D\Phi_M(u))\,dx + \int_\Omega \varphi(\Phi_M(u))\,dx. \tag{5.8}$$

By (5.6) we then have

$$\lim_j F_j(u_j) + \eta \geq \int_\Omega Qf(D\Phi_M(u))\,dx + \int_\Omega \varphi(\Phi_M(u))\,dx.$$

We can let $M \to +\infty$ and note that $\Phi_M(u) \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ to get

$$\lim_j F_j(u_j) + \eta \geq \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx.$$

The thesis is obtained by letting $\eta \to 0$. □

6 Proof of the limsup inequality

The limsup inequality is obtained by suitably modifying a recovery sequence for the lower semicontinuous envelope of $\int_\Omega f(Du)\,dx$.

Proposition 6.1 *If $|\partial\Omega| = 0$ then we have*

$$\Gamma\text{-}\limsup_j F_j(u) \leq \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$.

Proof. Let $u \in W^{1,p}(\Omega)$ and let (v_j) be a sequence converging to u weakly in $W^{1,p}(\Omega)$ such that

$$\lim_j \int_\Omega f(Dv_j)\,dx = \int_\Omega Qf(Du)\,dx \tag{6.1}$$

We preliminarily note that we may assume that $(|Dv_j|^p)$ is equi-integrable on Ω (see e.g. [16], [5] Appendix C). With fixed $N \in \mathbb{N}$, by Lemma 3.1 applied with $u_j = v_j$,

$$\rho_j = \frac{4}{3} N \delta_j^{n/(n-p)},$$

and taking the equi-integrability of $|Dv_j|^p$ into account we may also suppose that v_j equals a constant v_j^i on $\partial B_{\rho'_j}(x_i^\delta)$ for all $i \in Z_j$, where

$$\rho'_j = N \delta_j^{n/(n-p)}.$$

Step 1. We first assume that in addition (v_j) is a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$.

Let $\eta > 0$ be fixed. We now modify the sequence (v_j) to obtain functions $u_j \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$u_j = v_j \text{ on } \Omega \setminus \bigcup_{i \in \mathbb{Z}^n} B_{\rho'_j}(x_i^\delta), \qquad u_j = 0 \text{ on } \Omega \cap \bigcup_{i \in \mathbb{Z}^n} B_i^\delta$$

and

$$\limsup_j \int_{\Omega \cap \bigcup_{i\in\mathbb{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx \le \int_\Omega \varphi(u)\,dx + \eta|\Omega|. \tag{6.2}$$

The sequence (u_j) will then be a recovery sequence for the limsup inequality. In fact, clearly $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ since $\lim_j |\{u_j \ne v_j\}| = 0$ and (u_j) is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$, and

$$\begin{aligned}
\limsup_j \int_\Omega f(Du_j)\,dx &\le \limsup_j \int_{\Omega \setminus \bigcup_{i\in\mathbb{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Dv_j)\,dx \\
&\quad + \limsup_j \int_{\Omega \cap \bigcup_{i\in\mathbb{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx \\
&\le \lim_j \int_\Omega f(Dv_j)\,dx + \int_\Omega \varphi(u)\,dx + \eta|\Omega| \\
&= \int_\Omega Qf(Du)\,dx + \int_\Omega \varphi(u)\,dx + \eta|\Omega|.
\end{aligned} \tag{6.3}$$

We now define u_j on each $B_{\rho'_j}(x_i^\delta) \cap \Omega$. We treat separately the cases $i \in Z_j$ and $i \in \mathbb{Z}^n \setminus Z_j$. We first treat the case $i \in Z_j$. Let

$$M = \sup_j \|v_j\|_{L^\infty(\Omega;\mathbb{R}^m)}.$$

By Remark 4.2(v) we can choose N such that

$$\varphi(z) \geq \varphi_N(z) - \frac{\eta}{3} \tag{6.4}$$

for all $|z| \leq M$. Recall moreover that $\varphi_{N,j}$ converges uniformly on compact sets of $\mathbb{R}^m$ to φ_N as $j \to +\infty$; we may therefore assume that

$$|\varphi_{N,j}(z) - \varphi_N(z)| \leq \frac{\eta}{3} \tag{6.5}$$

for all $|z| \leq M$ and $j \in \mathbb{N}$.

Let $\zeta_j^i \in v_j^i + W_0^{1,p}(B_N(0);\mathbb{R}^m)$ be such that $\zeta_j^i = 0$ on $B_1(0)$ and

$$\int_{B_N(0)} \delta_j^{np/(n-p)} f(\delta_j^{-n/(n-p)} D\zeta_j^i)\,dx \leq \varphi_{N,j}(v_j^i) + \frac{\eta}{3} \leq \varphi(v_j^i) + \eta, \tag{6.6}$$

the last inequality being a consequence of (6.4) and (6.5), taking into account that $|v_j^i| \leq M$.

We define u_j on $B_{\rho'_j}(x_i^\delta)$ by

$$u_j(x) = \zeta_j^i\Big((x - x_i^\delta)\delta_j^{-n/(n-p)}\Big).$$

By a change of variables we then have

$$\int_{B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx = \delta_j^n \int_{B_N(0)} \delta_j^{np/(n-p)} f(\delta_j^{-n/(n-p)} D\zeta_j^i)\,dx \leq \delta_j^n \varphi(v_j^i) + \delta_j^n \eta. \tag{6.7}$$

If $i \notin Z_j$ it is not possible to use the construction above since $B_{\rho'_j}(x_i^\delta)$ might intersect $\partial\Omega$. We then consider a scalar $\zeta \in W^{1,p}(B_N(0))$ such that $\zeta - 1 \in W_0^{1,p}(B_N(0))$, $0 \leq \zeta \leq 1$ and $\zeta = 0$ on $B_1(0)$, and simply define

$$u_j(x) = v_j(x)\,\zeta\Big((x - x_i^\delta)\delta_j^{-n/(n-p)}\Big)$$

on $B_{\rho'_j}(x_i^\delta) \cap \Omega$. We then have

$$\begin{aligned}
&\int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} f(Du_j)\,dx \\
\leq\ & c_2 \int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} (1 + |Du_j|^p)\,dx \\
\leq\ & c \int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} \Big(1 + |Dv_j|^p + \delta_j^{-np/(n-p)} \Big|D\zeta\Big((x - x_i^\delta)\delta_j^{-n/(n-p)}\Big)\Big|^p |v_j|^p\Big)\,dx \\
\leq\ & c\delta_j^n\Big(1 + M\int_{B_N(0)} |D\zeta|^p\,dx\Big) + c\int_{B_{\rho'_j}(x_i^\delta)\cap\Omega} |Dv_j^p|\,dx.
\end{aligned} \tag{6.8}$$

Let

$$E'_j = \bigcup_{i\in\mathbb{Z}^n\setminus Z_j} B_{\rho'_j}(x_i^\delta)\cap\Omega \qquad \text{and} \qquad \Omega'_j = \bigcup_{i\in\mathbb{Z}^n\setminus Z_j} Q_i^\delta .$$

Then (6.8) above implies that

$$\int_{E'_j} f(Du_j)\,dx \le c|\Omega'_j| + c\int_{E'_j} |Dv_j^p|\,dx = o(1), \tag{6.9}$$

by the equi-integrability of $(|Dv_j|^p)$ and the fact that $\lim_j |\Omega'_j| = |\partial\Omega| = 0$.

Taking (6.7) and (6.9) into account, we have

$$\limsup_j \int_{\Omega\cap\bigcup_{i\in\mathbb{Z}^n} B_{\rho'_j}(x_i^\delta)} f(Du_j)\,dx \le \limsup_j \sum_{i\in Z_j} \delta_j^n \varphi(v_j^i)\,dx + \eta|\Omega|,$$

so that (6.2) is proved by Proposition 4.3.

Step 2. We now remove the boundedness assumption. First assume that $u\in L^\infty(\Omega;\mathbb{R}^m)$. Then let $M = 4\|u\|_{L^\infty(\Omega;\mathbb{R}^m)}$ and let $\Phi:\mathbb{R}^m\to\mathbb{R}^m$ be a Lipschitz function of Lipschitz constant 1 such that $\Phi(z)=z$ if $|z|\le M/2$ and $\Phi(z)=0$ if $|z|\ge M$. Let (v_j) be a sequence converging to u weakly in $W^{1,p}(\Omega)$ such that (6.1) holds and $(|Dv_j|^p)$ is equi-integrable on Ω, and define $v_j^M = \Phi(v_j)$. We have $v_j^M \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $\lim_j |\{v_j \ne v_j^M\}| = 0$. Hence, by the equi-integrability of $(|Dv_j|^p)$, we obtain that

$$\lim_j \int_\Omega f(Dv_j^M)\,dx = \lim_j \int_\Omega f(Dv_j)\,dx = \int_\Omega Qf(Du)\,dx.$$

We can then repeat all the reasonings above with (v_j^M) in the place of (v_j).

Finally, for arbitrary $u\in W^{1,p}(\Omega;\mathbb{R}^m)$, simply note that it can be approximated by a sequence of functions $u_k\in W^{1,p}(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ with respect to the strong convergence of $W^{1,p}(\Omega;\mathbb{R}^m)$. By the lower semicontinuity of $F''(u) = \Gamma$-$\limsup_j F_j(u)$ with respect to the $L^p(\Omega;\mathbb{R}^m)$ convergence (see e.g. [5] Remark 7.8) we then have $F''(u) \le \liminf_k F''(u_k) = \lim_k F(u_k) = F(u)$ as desired. □

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