ON THE ISOPERIMETRIC DEFICIT IN THE GAUSS SPACE

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Abstract. We find a sharp quantitative estimate for the isoperimetric inequality in $\mathbb{R}^n$ with the Gaussian measure.

Contents

1. Introduction 1
2. Background and notation 4
3. The one-dimensional case 6
4. A geometric proof of the Gaussian isoperimetric theorem 10
5. Reduction to $(n-1)$-symmetric sets 18
6. The two-dimensional case 24
7. The n-dimensional case 37
References 41

1. Introduction

The Gauss space is the space $\mathbb{R}^n$, $n \geq 1$, endowed with the measure γ_n given by

$$\gamma_n(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}} dx\,, \tag{1.1}$$

for any (Lebesgue) measurable set $E \subseteq \mathbb{R}^n$. Notice that $(\mathbb{R}^n, \gamma_n)$ is, in fact, a probability space, since $\gamma_n(\mathbb{R}^n) = 1$. The Gauss space, and variants of it, plays a central role in various branches of Probability Theory. A result of geometric flavor is the isoperimetric inequality, which relates the measure $\gamma_n(E)$ of subsets E of $\mathbb{R}^n$ with their Gaussian perimeter, defined as

$$P_\gamma(E) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial^M E} e^{-\frac{|x|^2}{2}} d\mathscr{H}^{n-1}(x)\,. \tag{1.2}$$

Here, $\mathscr{H}^k$ denotes for $k \in [0, n]$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$, and $\partial^M E$ is the essential boundary of E (see Section 2), which agrees with the topological boundary ∂E whenever E is sufficiently regular.

The isoperimetric inequality in the Gauss space asserts that among all subsets of $\mathbb{R}^n$ with prescribed Gaussian measure, half-spaces have the least Gaussian perimeter. An analytic form of this statement reads as follows. Define the function $\Phi : \mathbb{R} \to (0, 1)$ as

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-\frac{t^2}{2}} dt\,, \qquad\qquad s \in \mathbb{R}\,, \tag{1.3}$$

and observe that Φ strictly decreases from 1 to 0. Hence, we will often consider the inverse function $\Phi^{-1} : [0,1] \to [-\infty, +\infty]$, where we set for simplicity $\Phi^{-1}(0) = +\infty$ and $\Phi^{-1}(1) = -\infty$. Denote by $H_{\nu,s}$ the half-space having the form

$$H_{\nu,s} = \{x \in \mathbb{R}^n : x \cdot \nu > s\}\,, \tag{1.4}$$

for some $\nu \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$, where $\cdot$ stands for the Euclidean scalar product on $\mathbb{R}^n$. Then, given any $s \in \mathbb{R}$, one has

$$\gamma_n(H_{\nu,s}) = \Phi(s)\,, \qquad\qquad P_\gamma(H_{\nu,s}) = e^{-\frac{s^2}{2}}\,,$$

for every $\nu \in \mathbb{S}^{n-1}$, owing to the invariance of γ_n and P_γ under rotations about the origin. The Gaussian perimeter of any half-space of Gaussian volume $r \in [0,1]$ is thus equal to $I(r)$, where we define $I : [0,1] \to [0,\infty)$ as

$$I(r) = e^{-\frac{\Phi^{-1}(r)^2}{2}}\,, \qquad\qquad r \in (0,1)\,, \tag{1.5}$$

and $I(0) = I(1) = 0$. Correspondingly, the Gaussian isoperimetric inequality tells us that

$$P_\gamma(E) \geq I(\gamma_n(E))\,, \tag{1.6}$$

for every measurable subset E of $\mathbb{R}^n$. In view of its role in (1.6), the function I is called the *isoperimetric function* in the Gauss space.

Independently established in [5] and [16], the isoperimetric inequality (1.6) has subsequently been recovered via different proofs, of probabilistic [2], [4], [14], [3] or geometric [10] nature. All these approaches imply (1.6) via limit arguments, which prevent from the discussion of equality cases. In fact, the characterization of half-spaces as the unique minimizers in the isoperimetric Gaussian problem is a relatively recent result of [6], relying on the theory of rearrangements as well as on probabilistic techniques involving the Mehler (Ornstein-Uhlenbeck) semigroup.

It is the aim of the present paper to investigate on the stability of the isoperimetric properties of half-spaces in (1.6). It is not difficult to show that, loosely speaking, if $P_\gamma(E)$ does not differ too much from the perimeter of an half-space having measure $\gamma_n(E)$, then E is nearly an half-space. More precisely, define the Gaussian isoperimetric deficit of a set E as

$$\delta(E) = P_\gamma(E) - I(\gamma_n(E))\,, \tag{1.7}$$

and the Gaussian asymmetry of E as

$$\lambda(E) = \min_{\nu \in \mathbb{S}^{n-1}} \gamma_n(E \Delta H_{\nu,\Phi^{-1}(\gamma_n(E))})\,, \tag{1.8}$$

where Δ denotes the symmetric difference between sets. As shown in the Continuity Lemma 4.7, Section 4, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\delta(E) \leq \delta$, then $\lambda(E) \leq \varepsilon$. A quantitative form of this property is considerably less straightforward, and is the content of our main result.

Theorem 1.1. *Let $n \geq 2$. For any $0 < r < 1$ there exists a positive constant $C(n,r)$ depending only on n and r such that*

$$P_\gamma(E) \geq I(r) + \frac{\lambda(E)^2}{C(n,r)} \tag{1.9}$$

for every measurable set $E \subseteq \mathbb{R}^n$ *such that* $\gamma_n(E) = r$*. Moreover, the power* 2 *of* $\lambda(E)$ *is sharp in* (1.9)*.*

Note that (1.9) is equivalent to the inequality

$$\lambda(E) \leq C(n,r)\sqrt{\delta(E)}\,. \tag{1.10}$$

When $n = 1$, the quantitative version of the isoperimetric inequality (1.6) takes a somewhat different form.

Theorem 1.2. *Let* $n = 1$*. Then for any* $0 < r < 1$ *there exists a positive constant* $C(r)$ *depending only on* r *such that*

$$P_\gamma(E) \geq I(r) + \frac{\lambda(E)}{C(r)}\sqrt{\log\left(\frac{1}{\lambda(E)}\right)}\,, \tag{1.11}$$

for every measurable set $E \subseteq \mathbb{R}$ *such that* $\gamma_1(E) = r$*. Moreover, the result is sharp, in the sense that the right hand side of* (1.11) *cannot be replaced by any function of* $\lambda(E)$ *decaying faster to* $I(r)$ *as* $\lambda(E) \to 0$*.*

Our approach to Theorem 1.1 (and of Theorem 1.2) is purely geometric, and avoids probabilistic considerations. In fact, part of this paper –Section 4– is devoted to an alternative proof, of possible independent interest, of the plain Gaussian isoperimetric inequality (1.6), leading at once to the characterization of the equality cases. Such a proof rests upon a careful study of the *Ehrhard symmetrization* [10], a Gaussian variant of the classical Steiner symmetrization, which, in turn, was already exploited by De Giorgi in his proof of the Euclidean isoperimetric theorem for sets of finite perimeter. Since non necessarily smooth sets are taken into account from the beginning, we have to resort to tools from Geometric Measure Theory. Nevertheless, we believe that the proof of (1.6) that will be presented involves quite transparent geometric ideas, and, on avoiding approximation arguments, has the advantage of directly characterizing half-spaces as the only extremals in (1.6). Furthermore, and more importantly in view of our purposes, it is flexible enough to serve as a starting point to derive the quantitative version contained in Theorem 1.1.

Indeed, the proof of Theorem 1.1 will be accomplished in several steps. We begin by considering the one-dimensional case (Section 3), which is substantially simpler, although not entirely trivial, and is one of the ingredients in the analysis of the higher-dimensional cases. The general strategy in this case is to reduce the proof of (1.9), via subsequent simplifications, to classes of sets E enjoying special additional geometric properties. A first key reduction consists in showing that it suffices to prove (1.9) for $(n-1)$-symmetric sets E, namely sets which are symmetric about $(n-1)$ mutually orthogonal hyperplanes containing the origin (Theorem 5.1, Section 5). Keeping in mind the equivalence between (1.9) and (1.10), this step is performed by showing that to any set of finite Gaussian perimeter E we can associate a $(n-1)$-symmetric set E', with the same Gaussian measure, and such that

$$\lambda(E) \leq C(n)\lambda(E')\,, \qquad\qquad \delta(E') \leq C(n)\delta(E)\,. \tag{1.12}$$

This is a delicate operation, since, in general, a symmetrization process is expected to decrease not only $\delta(E)$, but also $\lambda(E)$. The relevant operation is related to an argument from [13], where the counterpart of (1.1) for the Euclidean isoperimetric inequality was established (see also [15], and [11] for the anisotropic case).

In Section 6, as a first application of the results in Section 5, we prove Theorem 1.1 in dimension $n = 2$. The argument also involves ad hoc geometric constructions, together with a careful study of a particular class of almost optimal sets.

In Section 7, on combining Ehrhard symmetrization with the reduction result to $(n-1)$-symmetric sets, we show how to pass from a generic set E to a Cartesian product of $\mathbb{R}^{n-2}$ times an epigraph in $\mathbb{R}^2$. Thanks to the natural tensorial properties of the Gaussian density, this class of sets can be handled by the two-dimensional result.

2. Background and notation

In this section we collect a few definitions, notation and results from the theory of sets of finite perimeter that will be used throughout the paper.

Let U be an open subset of $\mathbb{R}^n$. A function $f \in L^1_{\rm loc}(U)$ is said *of locally bounded variation in U* if its distributional gradient $Df = (D_1 f, ..., D_n f)$ is an $\mathbb{R}^n$-valued Radon measure in U, which total variation we denote by $|Df|$. The space of functions of locally bounded variation in U is denoted by $\mathrm{BV}_{\rm loc}(U)$. For every $f \in \mathrm{BV}_{\rm loc}(U)$, we let ∇f stand for the density of Df with respect to the Lebesgue measure, so that the Sobolev space $\mathrm{W}^{1,1}_{\rm loc}(U)$ consists of those $f \in \mathrm{BV}_{\rm loc}(U)$ such that $Df = \nabla f dx$. The spaces $\mathrm{BV}(U)$ and $\mathrm{W}^{1,1}(U)$ are defined accordingly.

Let E be a measurable subset of $\mathbb{R}^n$ and let $\chi_{_E}$ denote its characteristic function. Then E is said to be *of locally finite perimeter* if $D\chi_{_E} \in \mathrm{BV}_{\rm loc}(\mathbb{R}^n)$. In this case we denote by ν^E the derivative of the measure $D\chi_{_E}$ with respect to $|D\chi_{_E}|$. Hence

$$\nu^E(x) = (\nu^E_1(x), ..., \nu^E_n(x)) = \lim_{r\to 0} \frac{D\chi_{_E}(B_r(x))}{|D\chi_{_E}|(B_r(x))}\,,$$

at every $x \in \mathbb{R}^n$ such that the limit exists. The *reduced boundary* $\partial^* E$ of E is the set of all points $x \in \mathbb{R}^n$ such that the vector $\nu^E(x)$, called the *generalized inner normal* to E at x, exists and satisfies $|\nu^E(x)| = 1$. One has

$$|D\chi_{_E}| = \mathscr{H}^{n-1} \lfloor \partial^* E\,. \tag{2.1}$$

Let E be any measurable subset of $\mathbb{R}^n$ and let $0 \le d \le 1$. The set E^d of all points where E has density d is defined as

$$E^d = \left\{ x \in \mathbb{R}^n : \lim_{\rho\to 0} \frac{\mathscr{L}^n(E \cap Q_\rho(x))}{\mathscr{L}^n(Q_\rho(x))} = d \right\} \tag{2.2}$$

where $\mathscr{L}^n$ stands for the Lebesgue measure in $\mathbb{R}^n$ and $Q_\rho(x)$ denotes the cube centered at x, whose sides have length 2ρ and are parallel to the coordinate axes. The *essential boundary* $\partial^M E$ is defined by

$$\partial^M E = \mathbb{R}^n \setminus (E^0 \cup E^1)\,,$$

and, for any set of locally finite perimeter E in $\mathbb{R}^n$, one has

$$\partial^* E \subseteq \partial^M E\,, \qquad\qquad \mathscr{H}^{n-1}(\partial^M E \setminus \partial^* E) = 0\,. \tag{2.3}$$

Given a set of locally finite perimeter E and a Borel subset A of $\mathbb{R}^n$, the *relative perimeter* of E in A is defined as

$$P(E;A) = |D\chi_E|(A)\,,$$

and we write for simplicity $P(E) = P(E;\mathbb{R}^n)$; if $P(E) < \infty$, then E is called *of finite perimeter*. Owing to the characterization (2.1) and to (2.3), one has

$$P(E;A) = \mathscr{H}^{n-1}(\partial^* E \cap A) = \mathscr{H}^{n-1}(\partial^M E \cap A)\,, \tag{2.4}$$

for any Borel set A of $\mathbb{R}^n$. Conversely, if E is a measurable subset of $\mathbb{R}^n$ such that $\mathscr{H}^{n-1}(\partial^M E \cap U) < \infty$ for any open bounded subset U of $\mathbb{R}^n$, then E is of locally finite perimeter in $\mathbb{R}^n$, and (2.4) holds for every Borel set $A \subseteq \mathbb{R}^n$.

The Gaussian perimeter can be defined in analogy with (2.4). Given any $k \in \mathbb{N}$, $0 \le k \le n$, we denote by $\mathscr{H}^k_\gamma$ the *k-dimensional Gaussian Hausdorff measure*, defined for any Borel set $A \subseteq \mathbb{R}^n$ as

$$\mathscr{H}^k_\gamma(A) = \frac{1}{(2\pi)^{k/2}} \int_A e^{-\frac{|x|^2}{2}} d\mathscr{H}^k(x)\,.$$

Then the *Gaussian perimeter* of a measurable set E of $\mathbb{R}^n$ is defined as

$$P_\gamma(E) = \mathscr{H}^{n-1}_\gamma(\partial^M E)\,,$$

and we say E is a set of finite Gaussian perimeter if $P_\gamma(E) < \infty$. Accordingly, the relative Gaussian perimeter of E with respect to a Borel set $A \subseteq \mathbb{R}^n$ is given by

$$P_\gamma(E;A) = \mathscr{H}^{n-1}_\gamma(\partial^M E \cap A)\,.$$

Notice that, since E agrees with E^1 up to a set of Lebesgue measure zero, then $\partial^M E = \partial^M(E^1)$, and hence

$$P_\gamma(E;A) = P_\gamma(E^1;A)\,,$$

for any Borel set A of $\mathbb{R}^n$. In the special case when A is an open set $U \subseteq \mathbb{R}^n$, one has

$$P_\gamma(E;U) = \sup\left\{ \int_E \mathrm{div}\varphi(x) - x\cdot\varphi(x)\, d\gamma_n(x) : \varphi \in C^1_0(U;\mathbb{R}^n)\,, \sup_U |\varphi| \le 1 \right\}. \tag{2.5}$$

Note the analogy of (2.5) with the parallel formula for the relative Euclidean perimeter, in which case the term $x\cdot\varphi(x)$ is missing. Equation (2.5) entails that the Gaussian perimeter is lower semicontinuous with respect to L^1_{loc} convergence of sets, namely

$$P_\gamma(E;U) \le \liminf_{h\to\infty} P_\gamma(E_h;U)\,.$$

whenever the sequence of measurable sets E_h is such that $\chi_{E_h} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

3. The one-dimensional case

We are concerned here with the proof of Theorem 1.2, dealing with the quantitative isoperimetric inequality in the one-dimensional Gauss space. We begin with a proof of the isoperimetric inequality (1.6) in the case $n = 1$.

Proposition 3.1. *Let E be any measurable subset of $\mathbb{R}$. Then*

$$P_\gamma(E) \geq I(\gamma_1(E))\,. \tag{3.1}$$

Moreover, equality holds in (3.1) *if and only if either $\gamma_1(E) = 0$ or $\gamma_1(E) = 1$ or, up to a set of measure zero and for some $\sigma \in \mathbb{R}$, $E = (-\infty, -\sigma)$ or $E = (\sigma, \infty)$.*

Our proof of Proposition 3.1 makes use of a quite intuitive argument based on the following definition and lemma:

Definition 3.2. *Let (a, b) be any interval of $\mathbb{R}$, with $-\infty < a < b < \infty$. We call a* right-shifted *of (a, b) any interval (a', b') such that $a < a' < b' \leq \infty$ and $\gamma_1((a, b)) = \gamma_1((a', b'))$.* Left-shifted *intervals are defined analogously.*

The convenience of these notions is immediately clarified by the following lemma, which follows by a trivial computation.

Lemma 3.3. *If $a + b \geq 0$ (resp., $a + b \leq 0$), then $P_\gamma((a, b)) > P_\gamma((a', b'))$ for any right-shifted (resp., left-shifted) of (a, b).*

Proof of Propositon 3.1. It suffices to show that, given any set E with $0 < \gamma_1(E) < 1$, one has $P_\gamma(E) > P_\gamma((\sigma, \infty))$ unless $E = (\sigma, \infty)$ or $E = (-\infty, -\sigma)$, where clearly $\sigma = \Phi^{-1}(\gamma_1(E))$.

We begin with the following remark. Let $F = (a, b) \cup (c, d)$ with $a < b < c < d$ and $a + b \geq 0$, and let $\widetilde{F} = (a', d)$, where a' is determined by the right-shifted (a', b') of (a, b) such that $b' = c$. By definition we have

$$\gamma_1\big(\widetilde{F}\big) = \gamma_1\big((a', b')\big) + \gamma_1\big((c, d)\big) = \gamma_1\big((a, b)\big) + \gamma_1\big((c, d)\big) = \gamma_1(F)\,,$$

while as an immediate consequence of Lemma 3.3 it is also

$$P_\gamma\big(\widetilde{F}\big) < P_\gamma\big((a', b')\big) + P_\gamma\big((c, d)\big) < P_\gamma\big((a, b)\big) + P_\gamma\big((c, d)\big) = P_\gamma(F)\,.$$

Of course, an analogous argument may be performed if $c + d \leq 0$, up to replace the right-shifting of (a, b) with the suitable left-shifting of (c, d).

Now, let E be any set of finite Gaussian perimeter with $0 < \gamma_1(E) < 1$. As $\mathscr{H}^0_\gamma(\partial^M E) = P_\gamma(E) < \infty$, there is a countable set H such that, up to a set of a measure zero, $E = \bigcup_{h \in H}(a_h, b_h)$, where $-\infty \leq a_h < b_h \leq \infty$ and each interval (a_h, b_h) is a strictly positive distance apart from $E \setminus (a_h, b_h)$. On iterating the argument applied to the set F above, namely by right-shifting all the intervals such that $a_h + b_h \geq 0$ and left-shifting the remaining ones, we easily obtain a limit set $E_1 = (-\infty, a) \cup (b, +\infty)$, with $-\infty \leq a < b \leq \infty$, such that

$$\gamma_1\big(E_1\big) = \gamma_1(E)\,, \qquad\qquad P_\gamma\big(E_1\big) \leq P_\gamma(E)\,,$$

where the inequality is strict unless (up to sets of measure zero) $E_1 = E$. Define now the set $E_2 = \mathbb{R} \setminus E_1 = [a, b]$, so that $\gamma_1(E_2) = 1 - \gamma_1\big(E_1\big)$ and $P_\gamma(E_2) = P_\gamma\big(E_1\big)$. By applying Lemma 3.3 to E_2 we find a half-line E_3 such that

$$\gamma_1(E_3) = \gamma_1(E_2) = 1 - \gamma_1\big(E_1\big)\,, \qquad\qquad P_\gamma(E_3) \le P_\gamma(E_2)\,,$$

where the second inequality is strict unless $E_3 = E_2$. Eventually, we consider the half-line $E_4 = \mathbb{R} \setminus E_3$, that satisfies

$$\gamma_1\big(E_4\big) = \gamma_1\big(E_1\big) = \gamma_1(E)\,, \qquad\qquad P_\gamma\big(E_4\big) = P_\gamma(E_3) \le P_\gamma(E)\,,$$

with strict inequality unless $E = E_4$, and the conclusion follows. □

The proof of Theorem 1.2 will require the following lemmas. We start with

Lemma 3.4. *We have*

$$\int_s^{+\infty} e^{-\frac{t^2}{2}}\,dt < \frac{e^{-\frac{s^2}{2}}}{s}\,, \qquad\qquad s > 0\,, \tag{3.2}$$

and

$$\frac{e^{-\frac{s^2}{2}}}{4s} < \int_s^{+\infty} e^{-\frac{t^2}{2}}\,dt\,, \qquad\qquad s \ge 1\,. \tag{3.3}$$

Proof. One inequality is given by

$$\int_s^{+\infty} e^{-\frac{t^2}{2}}\,dt < \int_s^{+\infty} \frac{t}{s}\,e^{-\frac{t^2}{2}}\,dt = \left[-\frac{e^{-\frac{t^2}{2}}}{s}\right]_s^{+\infty} = \frac{e^{-\frac{s^2}{2}}}{s}\,.$$

Concerning the other one, if $s \ge 1$ then we can evaluate

$$\begin{aligned}\int_s^{+\infty} e^{-\frac{t^2}{2}}\,dt &> \int_s^{\sqrt{s^2+2\ln 2}} e^{-\frac{t^2}{2}}\,dt > e^{-\frac{s^2+2\ln 2}{2}}\Big(\sqrt{s^2+2\ln 2} - s\Big)\\ &= \frac{e^{-\frac{s^2}{2}}}{2}\,\frac{2\ln 2}{\sqrt{s^2+2\ln 2}+s} \ge \frac{e^{-\frac{s^2}{2}}}{2s}\,\frac{2\ln 2}{\sqrt{1+2\ln 2}+1} > \frac{e^{-\frac{s^2}{2}}}{4s}\,.\end{aligned}$$

□

Now we prove Theorem 1.2 in the case of the simplest possible perturbation of an optimal set.

Lemma 3.5. *For $r \in (0,1)$ and $\varepsilon > 0$, let $\sigma = \Phi^{-1}(r)$ and define $K_\varepsilon \in \mathbb{R}$ so that*

$$E_\varepsilon = (-\infty, -\sigma - \varepsilon) \cup (K_\varepsilon, +\infty) \tag{3.4}$$

satisfies $\gamma_1(E_\varepsilon) = \gamma_1((-\infty, -\sigma)) = r$. There exists $\varepsilon(r) > 0$ such that

$$\delta(E_\varepsilon) \ge \lambda(E_\varepsilon)\sqrt{\ln\left(\frac{1}{\lambda(E_\varepsilon)}\right)}\,,$$

for every $\varepsilon \in (0, \varepsilon(r))$.

Proof. By construction, $K_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$. Therefore we may assume that

$$K_\varepsilon \geq \sigma + \varepsilon\,, \qquad K_\varepsilon > -\sigma\,, \qquad K_\varepsilon \geq 1\,, \qquad \sqrt{\frac{\pi}{2}}\Big(K_\varepsilon - \sigma - \varepsilon\Big) \geq K_\varepsilon\,, \tag{3.5}$$

provided $\varepsilon \in (0, \varepsilon(r))$ and $\varepsilon(r)$ is chosen suitably small. The first assumption in (3.5) ensures that $\gamma_1\big(E_\varepsilon \cap \{t \leq 0\}\big) \geq \gamma_1\big(E_\varepsilon \cap \{t \geq 0\}\big)$, which in turns gives that $\lambda(E_\varepsilon) = \gamma_1\big(E_\varepsilon \Delta (-\infty, -\sigma)\big)$: this is clear once one keeps in mind that the only optimal sets of one-dimensional Gaussian volume equal to r are $(-\infty, -\sigma)$ and $(\sigma, +\infty)$. The second assumption in (3.5) tells us that $(-\infty, -\sigma) \cap (K_\varepsilon, +\infty) = \emptyset$, hence

$$\begin{aligned} \frac{\lambda(E_\varepsilon)}{2} &= \gamma_1\big((-\infty, -\sigma) \setminus E_\varepsilon\big) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma-\varepsilon}^{-\sigma} e^{-\frac{t^2}{2}}\, dt\,, \\ \frac{\lambda(E_\varepsilon)}{2} &= \gamma_1\big(E_\varepsilon \setminus (-\infty, -\sigma)\big) = \frac{1}{\sqrt{2\pi}} \int_{K_\varepsilon}^{+\infty} e^{-\frac{t^2}{2}}\, dt\,. \end{aligned} \tag{3.6}$$

By the second characterization of (3.6), by (3.2) and (3.3), and thanks to the third assumption in (3.5), we deduce that

$$\frac{1}{4\sqrt{2\pi}}\, \frac{e^{-\frac{K_\varepsilon^2}{2}}}{K_\varepsilon} < \frac{\lambda(E_\varepsilon)}{2} < \frac{1}{\sqrt{2\pi}}\, \frac{e^{-\frac{K_\varepsilon^2}{2}}}{K_\varepsilon}\,. \tag{3.7}$$

Concerning the isoperimetric deficit, we immediately compute

$$\delta(E_\varepsilon) = e^{-\frac{(-\sigma-\varepsilon)^2}{2}} + e^{-\frac{K_\varepsilon^2}{2}} - e^{-\frac{(-\sigma)^2}{2}}\,; \tag{3.8}$$

however, by the first characterization of (3.6),

$$e^{-\frac{(-\sigma-\varepsilon)^2}{2}} - e^{-\frac{(-\sigma)^2}{2}} = \int_{-\sigma-\varepsilon}^{-\sigma} t e^{-\frac{t^2}{2}}\, dt \geq (-\sigma - \varepsilon) \int_{-\sigma-\varepsilon}^{-\sigma} e^{-\frac{t^2}{2}}\, dt = (-\sigma - \varepsilon)\sqrt{\frac{\pi}{2}}\, \lambda(E_\varepsilon)\,; \tag{3.9}$$

On combining (3.9), (3.7) and (3.8) together, we find

$$\delta(E_\varepsilon) \geq \sqrt{\frac{\pi}{2}}\, \lambda(E_\varepsilon)\Big(K_\varepsilon - \sigma - \varepsilon\Big) \geq \lambda(E_\varepsilon) K_\varepsilon\,, \tag{3.10}$$

where the last inequality follows from the last assumption in (3.5). From the first inequality in (3.7) we deduce that, whenever $\varepsilon \in (0, \varepsilon(r))$ and up to possibly consider a smaller value for $\varepsilon(r)$, it is $\lambda(E_\varepsilon) \geq e^{-K_\varepsilon^2}$, which can be also written as $K_\varepsilon \geq \sqrt{\log\big(\lambda_\varepsilon^{-1}\big)}$. This remark, combined with (3.10), proves the lemma. □

We can now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. The sharpness of (1.11) is easily deduced by looking at the sets E_ε defined in (3.4) and studied in Lemma 3.5. We are thus left to show its validity. To this end, let E be such that $\delta(E) < \infty$ and $\gamma_1(E) = r \in (0, 1)$. It suffices to prove that, for a suitable positive constant $\delta(r) > 0$, we have

$$\delta(E) \geq \lambda(E) \sqrt{\log\left(\frac{1}{\lambda(E)}\right)}\,, \tag{3.11}$$

whenever $\delta(E) \leq \delta(r)$ (the complementary case $\delta(E) > \delta(r)$ is then easily solved by taking into account that we always have $\lambda(E) \leq 1$). By the Continuity Lemma 4.7, that we will show in Section 4 in the general $n-$dimensional case, but which is very easy in the one-dimensional case,

we can also assume $\lambda(E) \le \varepsilon(r)$ for a given $\varepsilon(r) > 0$, provided we choose a suitably small value of $\delta(r)$.

Possibly replacing E with $\mathbb{R} \setminus E$, we may assume that $r \in (0, 1/2]$. Moreover, up to a reflection with respect to the origin, we may consider the case that

$$\lambda(E) = \gamma_1\big((-\infty, \sigma)\Delta E\big)\,.$$

where $\sigma = -\Phi^{-1}(r) \le 0$. As observed in the proof of Proposition 3.1, since $P_\gamma(E) < \infty$ then E is equivalent, up to a set of measure zero, to a countable and locally finite union of disjoint intervals (a_h, b_h). Similarly to the construction of Proposition 3.1, we right-shift the intervals (a_h, b_h) with $a_h + b_h \ge 0$ and $a_h \ge \sigma$, we left-shift those ones with $a_h + b_h < 0$ and $b_h \le \sigma$, and claim that the resulting set E_1 satisfies

$$\lambda(E_1) = \lambda(E)\,, \qquad\qquad \delta(E_1) \le \delta(E)\,.$$

The second property is an immediate consequence of Lemma 3.3. The first property is clearly valid if the set $(-\infty, \sigma)$ realizes the minimum in the definition of $\lambda(E_1)$ too: in turn, this happens if $\lambda(E) \le r$, as we can assume up to take $\varepsilon(r) \le r$. As shown in Figure 1, E_1 has necessarily the form

$$E_1 = I_1 \cup J \cup \left(\bigcup\nolimits_{h=1}^{N} Y_h\right) \cup I_2\,,$$

where I_1 is a left unbounded interval, and where I_2, J and $\{Y_h = (a_h, b_h)\}_{h=1}^N$, $N \in \mathbb{N}$, are (possibly empty) intervals, with I_2 unbounded from the right, with $\sigma \in J$, and with $\sigma < a_h$, $a_h + b_h \le 0$ for every $h = 1, ..., N$. Now, for any interval $Y_h = (a_h, b_h)$, we consider its

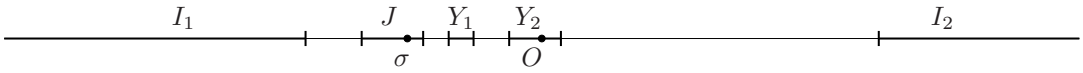

FIGURE 1. The set E_1

reflection $\widetilde{Y}_h = (-b_h, -a_h)$ with respect to the origin. The interval $\widetilde{Y}_h$ cannot intersect J, because otherwise Y_h would intersect J, which is false by construction; a fortiori, $\widetilde{Y}_h$ cannot intersect I_1; finally, $\widetilde{Y}_h$ cannot intersect I_2, as $I_2 \subseteq (-\sigma, +\infty)$ while $\widetilde{Y}_h \subseteq (-\infty, -\sigma)$. Therefore, the set $E_2 = I_1 \cup J \cup \big(\cup_{h=1}^N \widetilde{Y}_N\big) \cup I_2$ is such that

$$\lambda(E_2) = \lambda(E_1) = \lambda(E)\,, \qquad\qquad \delta(E_2) = \delta(E_1) \le \delta(E)\,.$$

The intervals $\widetilde{Y}_h = (\tilde{a}_h, \tilde{b}_h)$ satisfy $\tilde{a}_h + \tilde{b}_h \ge 0$ and $\tilde{a}_h \ge \sigma$. By iteratively right-shifting each $\widetilde{Y}_h$ towards I_2, we construct a (possibly empty) right unbounded interval I_3 (with $I_2 \subseteq I_3$), such that the set $E_3 = I_1 \cup J \cup I_3$ satisfies

$$\lambda(E_3) = \lambda(E_2) = \lambda(E)\,, \qquad\qquad \delta(E_3) \le \delta(E_2) \le \delta(E)\,.$$

The final step consists in showing that we can reduce from E_3 to one of the sets E_ε defined in Lemma 3.5 above. To this purpose, let us observe that

$$\frac{\lambda(E)}{2} = \frac{\lambda(E_3)}{2} = \gamma_1\big(E_3 \setminus (-\infty, \sigma)\big) = \gamma_1\big(J \setminus (-\infty, \sigma)\big) + \gamma_1(I_3)\,,$$

and consider the following cases separately (let us set $J = (a, b)$).

Case I. *We have* $\gamma_1\big(J \setminus (-\infty, \sigma)\big) \le \lambda(E)/4$.

If $a+b \le 0$, then we directly left-shift J up to connect it to I_1; if else $a+b > 0$, we first consider the reflection $\widetilde{J} = (-b, -a)$ of J with respect to the origin, and then left-shift $\widetilde{J}$ up to connect it to I_1. In both cases, the resulting set has the form E_ε given in (3.4) for some ε which is as small as we wish provided $\varepsilon(r) \ll 1$, and satisfies

$$\gamma_1(E) = \gamma_1(E_3) = \gamma_1(E_\varepsilon)\,, \qquad\qquad \delta(E_\varepsilon) \le \delta(E_3) \le \delta(E)\,, \tag{3.12}$$

thanks to Lemma 3.3. Moreover, by our assumption,

$$\lambda(E) = \lambda(E_3) \ge \lambda(E_\varepsilon) \ge \frac{\lambda(E_3)}{2} = \frac{\lambda(E)}{2}\,. \tag{3.13}$$

Provided $\varepsilon(r)$ is small enough, (3.11) holds for E_ε by Lemma 3.5, and then follows for E by (3.12) and (3.13).

Case II. *We have* $\gamma_1(I_3) \le \lambda(E)/4$.

In this case, we first right-shift $I_1 = (-\infty, b_1) \subseteq E_3$ up to reach J, thus obtaining the interval $\widetilde{I}_1 = (\tilde{a}_1, \tilde{b}_1]$ with $\tilde{b}_1 = a$: of course, by Lemma 3.3, we know that

$$P_\gamma\big(\widetilde{I}_1\big) \ge P_\gamma(I_1)\,;$$

but for the same reason we have also that

$$P_\gamma\big(\widetilde{I}_1 \cup J\big) \le P_\gamma\big(I_1 \cup J\big)\,, \tag{3.14}$$

since this is equivalent to say that the interval (b_1, a) lowers its perimeter when is left-shifted up to become $(-\infty, \tilde{a}_1)$. Let $\widetilde{I}_3$ be the reflection of I_3 through the origin. The volume constraint ensures that $\widetilde{I}_3$ does not intersect $\widetilde{I}_1 \cup J$, thus $E_4 = \widetilde{I}_3 \cup \widetilde{I}_1 \cup J$ satisfies

$$\lambda(E) = \lambda(E_3) \ge \lambda(E_4) \ge \frac{\lambda(E_3)}{2} = \frac{\lambda(E)}{2}\,, \qquad\qquad \delta(E_4) \le \delta(E_3) \le \delta(E)\,.$$

If we eventually right-shift $\widetilde{I}_3$ up to reach $\widetilde{I}_1$, then we lower the perimeter for the same reason as in (3.14) and conserve the asymmetry, thus reducing to a set of the form $\mathbb{R} \setminus E_\varepsilon$ again for some ε which can be assumed arbitrarily small if $\varepsilon(r)$ is small enough. We deduce (3.11) for E by Lemma 3.5, and then the thesis is obtained. □

4. A geometric proof of the Gaussian isoperimetric theorem

The present section is devoted to the proof of the isoperimetric inequality (1.6) in the case $n \ge 2$.

Theorem 4.1 (Isoperimetric Theorem)**.** *Let $n \ge 2$. Then $P_\gamma(E) \ge I(\gamma_n(E))$ for every measurable subset E of $\mathbb{R}^n$. Moreover, equality holds if and only if E is equivalent to an half-space.*

In Section 4.1 we study the Ehrhard symmetrization on sets of finite perimeter, in Section 4.2 we prove Theorem 4.1, and in Section 4.3 we derive by a compactness argument a mild stability property of the isoperimetric inequality (1.6).

4.1. **Ehrhard symmetrization on sets of finite perimeter.** As anticipated in the introduction, the proof that will be offered rests upon a symmetrization argument.

We start with some results about orthogonal sections of sets of finite perimeter. We use the notation $x = (z, y)$ for $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. If $f \in L^1_{\rm loc}(\mathbb{R}^n)$ we denote $D_n f$ by $D_y f$ and, similarly, if ν is a vector, we denote by ν_y the n-th component of ν. Let E be a measurable set in $\mathbb{R}^n$, $n \geq 2$. The section $E_z \subseteq \mathbb{R}$ of E is the set

$$E_z = \{y \in \mathbb{R} : (z, y) \in E\}\,, \qquad\qquad z \in \mathbb{R}^{n-1}\,,$$

while the essential projection of E is defined as

$$\pi_+(E) = \left\{z \in \mathbb{R}^{n-1} : \gamma_1(E_z) > 0\right\}\,.$$

We recall the following theorem by Vol'pert (see [17], [1, Theorem 3.108]).

Theorem 4.2 (Vol'pert)**.** *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter, $n \geq 2$. Then there exists a Borel set $B_E \subseteq \pi_+(E)$ with $\mathscr{L}^{n-1}(\pi_+(E) \setminus B_E) = 0$, and such that, for every $z \in B_E$,*

(i) *E_z is a set of locally finite perimeter in $\mathbb{R}$;*
(ii) *$(\partial^M E)_z = \partial^M(E_z) = \partial^*(E_z) = (\partial^* E)_z$;*
(iii) *$\nu_y^E(z, y) \neq 0$ for every y such that $(z, y) \in \partial^* E_z$.*

We are also going to use the following particular instance of the Coarea Formula for sets of finite perimeter: for every Borel function $g : \mathbb{R}^n \to [0, \infty]$, we have

$$\int_{\partial^* E} g(x)|\nu_y^E(x)|d\mathscr{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} dz \int_{(\partial^* E)_z} g(z, y)d\mathscr{H}^0(y)\,. \tag{4.1}$$

Let us now introduce the notion of *Ehrhard symmetrization*. Let $v_E : \mathbb{R}^{n-1} \to [0, 1]$ be defined as

$$v_E(z) = \gamma_1(E_z)\,.$$

By Fubini Theorem, $v_E \in L^1_{\rm loc}(\mathbb{R}^{n-1})$. The *Ehrhard symmetral* E^s of E with respect to the y-direction is defined as

$$E^s = \{(z, y) \in \mathbb{R}^n : y > y(z)\}\,, \qquad\qquad y(z) = \Phi^{-1}(v_E(z))\,.$$

Recall that $\Phi^{-1}(0) = +\infty$ and $\Phi^{-1}(1) = -\infty$, and of course the inequality $y > -\infty$ (resp. $y > +\infty$) is true (resp. false) for any $y \in \mathbb{R}$. By construction, E^s is the $(n-1)$-dimensional epigraph of a possibly infinity valued function, see Figure 2, and $v_E \equiv v_{E^s}$. Therefore Fubini Theorem implies that $\gamma_n(E) = \gamma_n(E^s)$. Moreover, the Gaussian perimeter of the one-dimensional sections is decreased under Ehrhard symmetrization: indeed, if we let $p_E : \mathbb{R}^{n-1} \to [0, \infty]$ be defined as

$$p_E(z) = \mathscr{H}^0_\gamma(\partial^M E_z)\,, \qquad\qquad z \in \mathbb{R}^{n-1}\,,$$

then we have, thanks to Proposition 3.1,

$$p_E(z) \geq p_{E^s}(z)\,, \qquad\qquad \forall z \in \mathbb{R}^{n-1}\,. \tag{4.2}$$

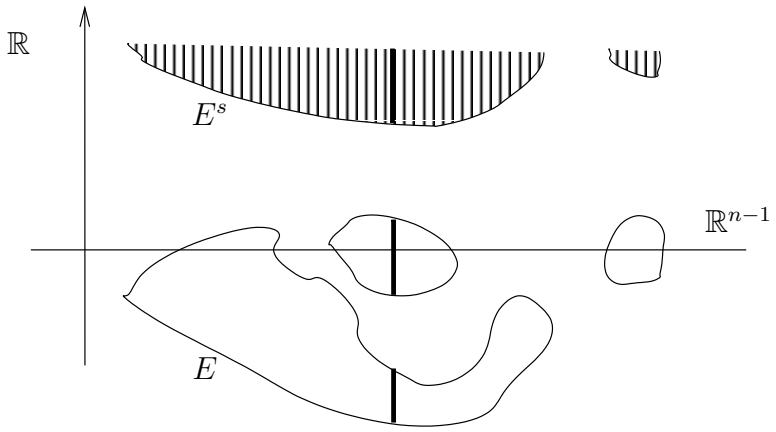

FIGURE 2. A set E and its Ehrhard symmetral E^s.

In fact Ehrhard symmetrization decreases the Gaussian perimeter of E itself, as explained in the next theorem.

Theorem 4.3. *Let $n \geq 2$ and let E be a set of finite Gaussian perimeter in $\mathbb{R}^n$. Then, for every Borel set $B \subseteq \mathbb{R}^{n-1}$ we have*

$$P_\gamma\big(E^s; B \times \mathbb{R}\big) \leq P_\gamma\big(E; B \times \mathbb{R}\big)\,, \tag{4.3}$$

and in particular $P_\gamma(E^s) \leq P_\gamma(E)$. Assume in addition that

$$\mathscr{H}^{n-1}_\gamma\Big(\big\{x \in \partial^* E : \nu^E_y(x) = 0\big\}\Big) = 0\,. \tag{4.4}$$

Then the same holds for E^s, i.e.

$$\mathscr{H}^{n-1}_\gamma\Big(\big\{x \in \partial^* E^s : \nu^{E^s}_y(x) = 0\big\}\Big) = 0\,, \tag{4.5}$$

and, moreover, $v_E \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ with

$$P_\gamma(E^s) = \int_{\mathbb{R}^{n-1}} \sqrt{p^2_{E^s}(z) + |\nabla v_E(z)|^2}\, d\gamma_{n-1}(z)\,. \tag{4.6}$$

The proof of Theorem 4.3 is split in several lemmas. In the first lemma we collect some properties of v_E and its distributional derivative, that can be easily inferred on arguing along the lines of [7, Lemmas 3.1 and 3.2].

Lemma 4.4. *Let $n \geq 2$ and let E be any set of finite Gaussian perimeter in $\mathbb{R}^n$. Then $v_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{n-1})$, and*

$$\frac{1}{(2\pi)^{(n-1)/2}} \int_B e^{-\frac{|x|^2}{2}} d|Dv_E|(x) \leq P_\gamma(E; B \times \mathbb{R})\,, \tag{4.7}$$

for every Borel set $B \subseteq \mathbb{R}^{n-1}$. Moreover,

$$D_i v_E(z) = \int_{(\partial^* E)_z} \frac{\nu^E_i(z,y)}{|\nu^E_y(z,y)|}\, d\mathscr{H}^0_\gamma(y)\,, \qquad\qquad i = 1, ..., n-1\,, \tag{4.8}$$

for $\mathscr{L}^{n-1}$-a.e. $z \in B_E$, where B_E is the set appearing in Vol'pert Theorem 4.2.

Since $v_E = v_{E^s}$, Lemma 4.4 in the case $E = E^s$ implies that

$$D_i v_{E^s}(z) = e^{-\frac{y(z)^2}{2}} \frac{\nu_i^{E^s}(z, y(z))}{|\nu_y^{E^s}(z, y(z))|}\,, \qquad\qquad i = 1, ..., n-1\,, \tag{4.9}$$

for $\mathscr{L}^{n-1}$-a.e. $z \in B_{E^s}$. The following lemma presents a further property of E^s.

Lemma 4.5. *Let $n \geq 2$ and let E be a set of finite Gaussian perimeter in $\mathbb{R}^n$. Then, for every Borel set $B \subseteq \mathbb{R}^{n-1}$,*

$$P_\gamma(E^s; B \times \mathbb{R}) \leq \frac{1}{(2\pi)^{(n-1)/2}} \int_B e^{-\frac{|z|^2}{2}} d|Dv_E|(z) + \int_B p_{E^s}(z) d\gamma_{n-1}(z)\,, \tag{4.10}$$

In particular (4.3) *holds true whenever $\mathscr{L}^{n-1}(B \cap B_E) = 0$.*

Proof. By a standard approximation argument, in the proof of (4.10) we may replace B by a bounded smooth open set $\Omega \subseteq \mathbb{R}^{n-1}$. Consider any function $v \in \mathrm{C}^1(\Omega)$ satisfying $0 < v(z) < 1$ for all $z \in \Omega$, and let $\varphi \in \mathrm{C}^1_0(\Omega \times \mathbb{R})$ such that $|\varphi| \leq 1$. If we define

$$F := \big\{(z, y) \in \Omega \times \mathbb{R} :\, y > y(z)\big\}\,, \qquad\qquad y(z) = \Phi^{-1}(v(z))\,, \tag{4.11}$$

then (here $x = (z, y)$)

$$\int_F \operatorname{div} \varphi - \varphi \cdot x \, d\gamma_n(x) = \sum_{i=1}^{n-1} \int_F \frac{\partial \varphi_i}{\partial z_i} - z_i \varphi_i \, d\gamma_n(x) + \int_F \frac{\partial \varphi_n}{\partial y} - y \varphi_n \, d\gamma_n(x)\,. \tag{4.12}$$

Let us now estimate the first term in the right hand side of (4.12). As $y(z) = \Phi^{-1}(v(z))$ we have

$$\nabla v(z) = -\frac{1}{\sqrt{2\pi}}\, e^{-\frac{y(z)^2}{2}} \nabla y(z)\,,$$

and therefore

$$\begin{aligned}
\sum_{i=1}^{n-1} \int_F \frac{\partial \varphi_i}{\partial z_i} - z_i \varphi_i \, d\gamma_n(x) &= \int_\Omega d\gamma_{n-1}(z) \int_{y(z)}^\infty \sum_{i=1}^{n-1} \frac{\partial \varphi_i}{\partial z_i} - z_i \varphi_i \, d\gamma_1(y) \\
&= \frac{1}{(2\pi)^{n/2}} \int_\Omega e^{-\frac{|z|^2}{2}}\, dz \int_{y(z)}^\infty e^{-\frac{y^2}{2}} \Big(\sum_{i=1}^{n-1} \frac{\partial \varphi_i}{\partial z_i} - z_i \varphi_i \Big)\, dy \\
&= \frac{1}{(2\pi)^{n/2}} \int_\Omega \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \Big(e^{-\frac{|z|^2}{2}} \int_{y(z)}^\infty e^{-\frac{y^2}{2}} \varphi_i \, dy \Big)\, dz + \\
&\qquad + \frac{1}{(2\pi)^{n/2}} \int_\Omega \sum_{i=1}^{n-1} e^{-\frac{|z|^2}{2}} e^{-\frac{y(z)^2}{2}} \varphi_i(z, y(z)) \frac{\partial y}{\partial z_i}(z)\, dz \\
&= \frac{1}{(2\pi)^{n/2}} \int_\Omega \sum_{i=1}^{n-1} e^{-\frac{|z|^2}{2}} e^{-\frac{y(z)^2}{2}} \varphi_i(z, y(z)) \frac{\partial y}{\partial z_i}(z)\, dz \\
&= -\frac{1}{(2\pi)^{(n-1)/2}} \int_\Omega \sum_{i=1}^{n-1} e^{-\frac{|z|^2}{2}} \varphi_i(z, y(z)) \frac{\partial v}{\partial z_i}(z)\, dz \\
&\leq \int_\Omega \big|\nabla v(z)\big|\, d\gamma_{n-1}(z)\,.
\end{aligned} \tag{4.13}$$

Let us now consider a set E as in the statement. Since $v_E \in \mathrm{BV}(\Omega)$ and $0 \le v_E \le 1$, we may find [1, Theorem 3.9] a sequence of functions $v_h \in \mathrm{C}^1(\Omega)$, $0 < v_h < 1$, such that $v_h \to v_E$ in $L^1(\Omega)$ and a.e. in Ω and

$$\lim_{h\to\infty} \int_\Omega |\nabla v_h(z)|\, dz = \big|Dv_E\big|(\Omega)\,.$$

On denoting by F_h the sets defined as in (4.11) with v_h replacing v, we have $\chi_{F_h} \to \chi_{E^s}$ a.e. in Ω. Then we consider (4.12) with E^s in place of F, and find that

$$\begin{aligned}
\int_{E^s} \operatorname{div}\varphi - \varphi\cdot x\, d\gamma_n(x) &= \sum_{i=1}^{n-1} \int_{E^s} \frac{\partial\varphi_i}{\partial z_i} - z_i\varphi_i\, d\gamma_n(x) + \int_{E^s} \frac{\partial}{\partial y}\left(\varphi_n e^{-\frac{|x|^2}{2}}\right) dx \\
&= \lim_{h\to\infty} \sum_{i=1}^{n-1} \int_{F_h} \frac{\partial\varphi_i}{\partial z_i} - z_i\varphi_i\, d\gamma_n(x) - \int_{\partial^* E^s} \varphi_n \nu_y^E\, d\mathscr{H}^{n-1}_\gamma(x) \\
&\le \lim_{h\to\infty} \int_\Omega |\nabla v_h(z)|\, d\gamma_{n-1}(z) + \int_{\partial^* E^s\cap(\Omega\times\mathbb{R})} |\nu_y^E|\, d\mathscr{H}^{n-1}_\gamma(x) \\
&= \frac{1}{(2\pi)^{(n-1)/2}} \int_\Omega e^{-\frac{|z|^2}{2}}\, d|Dv_E|(z) + \int_\Omega \mathscr{H}^0_\gamma(\partial^*(E^s)_z)\, d\gamma_{n-1}(z)\,,
\end{aligned}$$

where we have applied the Divergence Theorem on E^s, (4.13), the Coarea Formula (4.1) and Reshetnjak Continuity Theorem (see for instance [1, Theorem 2.39]). Since $\mathscr{H}^0_\gamma(\partial^*(E^s)_z) = p_{E^s}(z)$ for $z \in \Omega$, taking the supremum among functions φ yields (4.10). □

We now prove a lower bound on $P_\gamma(E)$ in terms of p_E and ∇v_E.

Lemma 4.6. *Let E be a set of finite Gaussian perimeter in $\mathbb{R}^n$, $n \ge 2$. Then*

$$P_\gamma(E; B\times\mathbb{R}) \ge \int_B \sqrt{p_E(z)^2 + |\nabla v_E(z)|^2}\, d\gamma_{n-1}(z)\,, \tag{4.14}$$

for every Borel set $B \subseteq B_E$. Equality holds in (4.14) *if $E \cap (B\times\mathbb{R}) = E^s \cap (B\times\mathbb{R})$.*

Proof. By the Coarea Formula (4.1) and Vol'pert Theorem

$$P_\gamma\big(E; B\times\mathbb{R}\big) = \int_{\partial^* E\cap(B\times\mathbb{R})} \frac{|\nu_y^E|}{|\nu_y^E|}\, d\mathscr{H}^{n-1}_\gamma = \int_B d\gamma_{n-1}(z) \int_{\partial^* E_z} \frac{1}{|\nu_y^E|}\, d\mathscr{H}^0_\gamma\,.$$

From this equality, Jensen's inequality and the formula (4.8) for ∇v_E we get

$$\begin{aligned}
P_\gamma(E; B\times\mathbb{R}) &= \int_B p_E(z) d\gamma_{n-1}(z) \fint_{\partial^* E_z} \sqrt{1 + \frac{\sum_{i=1}^{n-1} |\nu_{z_i}^E|^2}{|\nu_y^E|^2}}\, d\mathscr{H}^0_\gamma \\
&\ge \int_B p_E(z) \sqrt{1 + \sum_{i=1}^{n-1} \left(\fint_{\partial^* E_z} \frac{\nu_{z_i}^E}{|\nu_y^E|}\, d\mathscr{H}^0_\gamma\right)^2}\, d\gamma_{n-1}(z) \\
&= \int_B \sqrt{p_E(z)^2 + |\nabla v_E(z)|^2}\, d\gamma_{n-1}(z)\,,
\end{aligned}$$

and inequality (4.14) follows. In particular, if $E^s \cap (B\times\mathbb{R}) = E \cap (B\times\mathbb{R})$, then for $\mathscr{L}^{n-1}$-a.e. $z \in B$ the set $\partial^*(E_z)$ is either empty or consists of a single point. Thus equality holds in Jensen's inequality and also in (4.14). □

Lemmas 4.4, 4.5 and 4.6 provide us with the key technical material for the proof of Theorem 4.3.

Proof of Theorem 4.3. We start proving (4.3). Thanks to Lemma 4.5, we can directly consider the case of a Borel set $B \subseteq B_E \cap B_{E^s}$. By (4.14) and thanks to the one-dimensional Gaussian isoperimetric inequality,

$$\begin{aligned} P_\gamma(E; B \times \mathbb{R}) &\geq \int_B \sqrt{p_E(z)^2 + |\nabla v_E(z)|^2}\, d\gamma_{n-1}(z) \\ &\geq \int_B \sqrt{p_{E^s}(z)^2 + |\nabla v_{E^s}(z)|^2}\, d\gamma_{n-1}(z) = P_\gamma(E^s; B \times \mathbb{R})\,, \end{aligned} \tag{4.15}$$

and (4.3) follows. Let us now show that if (4.4) is in force, then (4.5) holds true. To this end, let us define for brevity

$$C_E = \left\{x \in \partial^* E : \nu_y^E(x) = 0\right\}, \qquad C_{E^s} = \left\{x \in \partial^* E^s : \nu_y^{E^s}(x) = 0\right\}.$$

By (iii) in Vol'pert Theorem 4.2 and by Lemma 4.5, we have

$$\begin{aligned} \mathscr{H}^{n-1}_\gamma\big(C_{E^s}\big) &\leq P_\gamma\Big(E^s; \big(\mathbb{R}^{n-1} \setminus B_{E^s}\big) \times \mathbb{R}\Big) \leq P_\gamma\Big(E; \big(\mathbb{R}^{n-1} \setminus B_{E^s}\big) \times \mathbb{R}\Big) \\ &\leq P_\gamma\big(E; C_E\big) + P_\gamma\Big(E; \big((\mathbb{R}^{n-1} \setminus B_{E^s}) \times \mathbb{R}\big) \setminus C_E\Big)\,. \end{aligned}$$

By (4.4), $P_\gamma(E; C_E) = 0$. As for the second term, by definition of C_E and by the Coarea formula, it equals

$$\int_{\mathbb{R}^{n-1} \setminus B_{E^s}} d\mathscr{H}^{n-1}_\gamma(z) \int_{(\partial^* E \setminus C_E)_z} \frac{1}{|\nu_y^E(z,y)|}\, d\mathscr{H}^0_\gamma(y)\,,$$

that in turn is zero by Vol'pert Theorem 4.2. Thus $\mathscr{H}^{n-1}_\gamma(C_{E^s}) = 0$, i.e. (4.5) holds true, as desired. Moreover, the above considerations show that, under the assumption (4.4), $P_\gamma(E^s) = P_\gamma(E^s; B_{E^s} \times \mathbb{R})$, thus by Lemma 4.6 and by Vol'pert Theorem, (4.6) holds.

We are left to show that $v_E \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$. Thanks to (4.7), it suffices to prove that, if $\mathscr{H}^{n-1}(B) = 0$, then $P_\gamma(E; B \times \mathbb{R}) = 0$. This last assertion is again a consequence of the Coarea Formula, since

$$P_\gamma(E; B \times \mathbb{R}) = P_\gamma(E; (B \times \mathbb{R}) \setminus C_E) = \int_B d\mathscr{H}^{n-1}_\gamma(z) \int_{(\partial^* E \setminus C_E)_z} \frac{1}{|\nu_y^E(z,y)|}\, d\mathscr{H}^0_\gamma(y) = 0\,.$$

□

4.2. **Proof of Theorem 4.1.** We have to show that, fixed any $r \in (0,1)$, half-spaces having Gaussian measure r are the only minimizers of perimeter among all sets of Gaussian measure r. We split the proof into four steps.

Step I. Existence of an optimal set H.

Fix any $r \in (0,1)$ and consider any sequence E_h of measurable sets of $\mathbb{R}^n$ with $\gamma_n(E_h) = r$ such that

$$\lim_{h\to\infty} P_\gamma(E_h) = \inf\left\{P_\gamma(E) : \gamma_n(E) = r\right\}.$$

Since for any bounded open set U in $\mathbb{R}^n$ the total variations $|D\chi_{E_h}|(U)$ are uniformly bounded in h, the compactness Theorem for BV functions (see for instance [1, Theorem 3.23]) implies the

existence of a Borel set $H \subseteq \mathbb{R}^n$ such that $\gamma_n(H) = r$ and, up to a subsequence, $\chi_{E_h} \to \chi_H$ in $L^1_{\rm loc}(\mathbb{R}^n)$. By the lower semicontinuity of the Gaussian perimeter, $P_\gamma(H) \le \liminf_{h\to\infty} P_\gamma(E_h)$, and therefore $P_\gamma(H) \le P_\gamma(E)$ for every E such that $\gamma_n(E) = r$. In the rest of the proof, we will then denote by H an optimal set in the Gaussian isoperimetric problem.

Step II. Almost every one-dimensional section of H is a half-line.

Theorem 4.3 and the minimality of H ensure that $P_\gamma(H) = P_\gamma(H^s)$. Hence, by (4.3),

$$P_\gamma(H; B \times \mathbb{R}) = P_\gamma(H^s; B \times \mathbb{R})$$

for every Borel set $B \subseteq \mathbb{R}^{n-1}$; in particular,

$$P_\gamma(H; (B_H \cap B_{H^s}) \times \mathbb{R}) = P_\gamma(H^s; (B_H \cap B_{H^s}) \times \mathbb{R})\,.$$

Hence, equality holds in (4.15) with $B = B_H \cap B_{H^s}$. By Vol'pert Theorem, this implies that $p_H(x) = p_{H^s}(z)$ for $\mathscr{L}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$ and clearly, the same argument also applies to the Ehrhard symmetral of H about any other hyperplane. Thus, by Proposition 3.1, we deduce that $\mathscr{H}^{n-1}$-a.e. one-dimensional section of H orthogonal to any hyperplane in $\mathbb{R}^n$ is equivalent to an half-line.

Step III. H is open and convex.

Consider the set H^1 of point of density 1 of H, and recall that $\gamma_n(H \Delta H^1) = 0$, $P_\gamma(H) = P_\gamma(H^1)$. We claim that H^1 is convex. Indeed, take any $x_1, x_2 \in H^1$. Up to a rotation, we may assume that $x_1 = (\bar z, y_1)$ and $x_2 = (\bar z, y_2)$ for some $\bar z \in \mathbb{R}^{n-1}$ and $y_1 < y_2$; then, we fix a generic point $\bar x = (\bar z, \bar y)$ with $y_1 < \bar y < y_2$ and we aim to show that $\bar x \in H^1$. As each x_i has density 1 with respect to H, and hence also with respect to H^1, for every $\varepsilon > 0$ there exists ρ_ε such that for any $0 < \rho < \rho_\varepsilon$ one has

$$\frac{\mathscr{L}^n(H^1 \cap Q_\rho(x_i))}{\mathscr{L}^n(Q_\rho(x_i))} > 1 - \varepsilon\,, \qquad\qquad i = 1, 2\,.$$

Here, as in (2.2), $Q_\rho(x)$ denotes the cube centered in x, with half-side equal to ρ and whose sides are parallel to the coordinate axes. Therefore, by Fubini Theorem we have

$$\begin{aligned}(2\rho)^n(1-\varepsilon) < \mathscr{L}^n(H^1 \cap Q_\rho(x_i)) &= \int_{\mathbb{R}^{n-1}} \mathscr{L}^1\Big(\big(H^1 \cap Q_\rho(x_i)\big)_z\Big)\, d\mathscr{L}^{n-1}(z)\\ &\le 2\rho \mathscr{L}^{n-1}\Big(\pi_+\big(H^1 \cap Q_\rho(x_i)\big)\Big)\,.\end{aligned}$$

Thus, for every $\rho < \rho_\varepsilon$ and $i = 1, 2$,

$$\mathscr{L}^{n-1}\Big(\pi_+(H^1 \cap Q_\rho(x_i))\Big) > (2\rho)^{n-1}(1-\varepsilon)\,.$$

As the two essential projections $\pi_+(H^1 \cap Q_\rho(x_i))$ are both contained in the same $(n-1)-$dimensional cube of side 2ρ centered in $\bar z$, we have

$$\mathscr{L}^{n-1}\Big(\pi_+(H^1 \cap Q_\rho(x_1)) \cap \pi_+(H^1 \cap Q_\rho(x_2))\Big) > (2\rho)^{n-1}(1-2\varepsilon)\,. \tag{4.16}$$

By Step II, for $\mathscr{L}^{n-1}$-a.e. $z \in \pi_+(H^1 \cap Q_\rho(x_1)) \cap \pi_+(H^1 \cap Q_\rho(x_2))$ the set $(H^1)_z$ is $\mathscr{L}^1-$equivalent to an half-line. Therefore,

$$\mathscr{L}^1\Big(\big(H^1 \cap Q_\rho(x_i)\big)_z\Big) > 0\,, \qquad\qquad i = 1,\, 2\,.$$

Notice that the half-line $(H^1)_z$ contains both points with height in $(y_1-\rho, y_1+\rho)$ and points with height in $(y_2-\rho, y_2+\rho)$. Therefore, by connectedness, it contains the whole segment $(y_1+\rho, y_2-\rho) \supseteq (\bar y-\rho, \bar y+\rho)$, provided ρ is small enough (depending on $\bar y$). This gives

$$\mathscr{L}^1\Big(\big(H^1 \cap Q_\rho(\bar x)\big)_z\Big) = 2\rho\,.$$

Thanks to (4.16), we may conclude that

$$\mathscr{L}^n\Big(H^1 \cap Q_\rho(\bar x)\Big) \geq (2\rho)^n(1-2\varepsilon)\,,$$

and thus, that $\bar x \in H^1$. Summarizing, we have proved that H^1 is convex, and since any convex set having all the points of density 1 is necessarily open, this step is concluded.

Step IV. The set H is equivalent to an half-space.

Since H is a minimizer of $\{P_\gamma(E) : \gamma_n(E) = r\}$, then $\mathbb{R}^n \setminus H$ is a minimizer of $\{P_\gamma(E) : \gamma_n(E) = 1-r\}$. By Step III, both H^1 and $(\mathbb{R}^n \setminus H)^1$ are open and convex. Since they are disjoint and their union has full measure in the Gauss space, they are both half-spaces.

4.3. **A mild stability property of half-spaces.** In Theorem 4.1 and Proposition 3.1 we have proved the Gaussian isoperimetric inequality in every dimension $n \geq 1$, also characterizing the equality cases. By a compactness argument we next deduce a mild stability property of the Gaussian isoperimetric inequality, that we have already used in the proof of Theorem 1.2.

Lemma 4.7 (Continuity Lemma)**.** *Let $n \geq 1$. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if E is any measurable set fulfilling $\delta(E) \leq \delta(\varepsilon)$, then $\lambda(E) \leq \varepsilon$.*

Proof. Let us assume, by contradiction, that the statement is false for some $\varepsilon > 0$. Then there exists a sequence of measurable sets E_h in $\mathbb{R}^n$, and a sequence of half-spaces $H_h = \{(z,y) \in \mathbb{R}^n : y > y_h\}$ with $-\infty \leq y_h \leq +\infty$, such that $\gamma_n(E_h) = \gamma_n(H_h)$ for all $h \in \mathbb{N}$, $P_\gamma(E_h) - P_\gamma(H_h) \to 0$ as $h \to \infty$, and $\lambda(E_h) > \varepsilon$. By the same compactness argument used the proof of the Isoperimetric Theorem 4.1, there exist a measurable set E and $\bar y \in [-\infty, +\infty]$ such that, up to a subsequence, $\chi_{E_h} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$, $y_h \to \bar y$ and thus $\chi_{H_h} \to \chi_H$ in $L^1_{\rm loc}(\mathbb{R}^n)$, where $H = \{(z,y) \in \mathbb{R}^n : y > \bar y\}$. We have that $\gamma_n(E) = \gamma_n(H)$ and, by the lower semicontinuity of the Gaussian perimeter, that

$$P_\gamma(E) \leq \liminf_{h\to\infty} P_\gamma(E_h) = \lim_{h\to\infty} P_\gamma(H_h) = P_\gamma(H)\,.$$

Thus E is optimal in the isoperimetric inequality (1.6), and therefore, by the Isoperimetric Theorem 4.1, $\lambda(E) = 0$. Let us now use the following inequality, which is immediately seen to hold for any two sets F_1 and F_2, of possibly different volumes:

$$|\lambda(F_1) - \lambda(F_2)| \leq 2\gamma_n(F_1 \Delta F_2)\,. \tag{4.17}$$

By this estimate we get the contradiction

$$\varepsilon \leq \liminf_{h\to\infty} \lambda(E_h) \leq 2 \lim_{h\to\infty} \gamma_n(E_h \Delta E) = 0\,.$$

□

5. Reduction to $(n-1)$-symmetric sets

Our task in this section is to show that, in order to prove Theorem 1.1, we can reduce to consider sets E from the special class of $(n-1)$-*symmetric sets*, namely those sets which are symmetric by reflection with respect to $(n-1)$-mutually orthogonal hyperplanes passing through the origin. This is the content of the following theorem.

Theorem 5.1. *Let $0 < r < 1$, and let E be a set with finite Gaussian perimeter in $\mathbb{R}^n$ and with Gaussian volume $\gamma_n(E) = r$. Then there exists an $(n-1)$-symmetric set E' such that $\gamma_n(E) = \gamma_n(E')$ and*

$$\lambda(E) \le C(n,r)\lambda(E')\,, \qquad\qquad \delta(E') \le 2^{n-1}\delta(E)\,, \tag{5.1}$$

being $C(n,r)$ a geometric constant depending only on n and r.

An analogous "symmetrization theorem" was one of the key ingredients in the proof of the quantitative Euclidean isoperimetric inequality given in [13]. In that case we were looking for n-symmetric sets, instead of $(n-1)$-symmetric sets, due to the different group of symmetries acting on isoperimetric sets. The next result ensures that if a set E is symmetric about some linear subspace S of $\mathbb{R}^n$, then we can essentially restrict the class of admissible hyperplanes in the definition of $\lambda(E)$ to those which are symmetric about S.

Lemma 5.2. *Let S be a linear subspace of $\mathbb{R}^n$ of dimension $1 \le \dim(S) \le n-1$, and denote by $\pi : \mathbb{R}^n \to \mathbb{R}^n$ the reflection with respect to S. Let E be any measurable subset of $\mathbb{R}^n$, with $\pi(E) = E$. Then there exists an half-space H such that $\pi(H) = H$, $\gamma_n(H) = \gamma_n(E)$ and*

$$\lambda(E) \le \gamma_n(E\Delta H) \le 3\lambda(E)\,.$$

Proof. Without loss of generality we may assume that $\gamma_n(E) \le 1/2$. Let H_1 be an half-space with $\gamma_n(H_1) = \gamma_n(E)$ and $\lambda(E) = \gamma_n(E\Delta H_1)$, and set $H_2 = \pi(H_1)$. Since $\pi(E) = E$, it is clearly $\lambda(E) = \gamma_n(E\Delta H_2)$. Notice that, for some $\nu \in \mathbb{S}^{n-1}$, we have

$$H_1 = \big\{x \in \mathbb{R}^n :\ x\cdot\nu > s\big\}\,, \qquad\qquad H_2 = \big\{x \in \mathbb{R}^n :\ x\cdot\pi(\nu) > s\big\}\,,$$

where $s = \Phi^{-1}(\gamma_n(E))$. Let us set $\nu_0 = (\nu+\pi(\nu))/|\nu+\pi(\nu)|$ if $\nu + \pi(\nu) \neq 0$, or let ν_0 be any

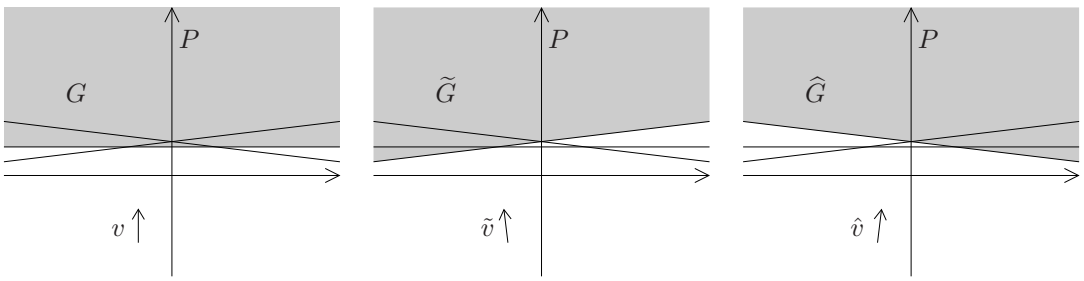

FIGURE 3. The sets H, H_1 and H_2.

vector in $S \cap \mathbb{S}^{n-1}$ if $\nu + \pi(\nu) = 0$. We define $H = \big\{x \in \mathbb{R}^n :\ x\cdot\nu_0 > s\big\}$, see Figure 3. By

construction, $\gamma_n(H) = \gamma_n(E)$ and $\pi(H) = H$. Moreover, since $\gamma_n(E) \leq 1/2$ and thus $s \geq 0$, $H_1 \setminus H \subseteq H_1 \setminus H_2$. By the triangular inequality,

$$\begin{aligned}\gamma_n(E\Delta H) &\leq \gamma_n(E\Delta H_1) + \gamma_n(H_1\Delta H) \leq \gamma_n(E\Delta H_1) + \gamma_n(H_1\Delta H_2)\\ &\leq \gamma_n(E\Delta H_1) + \gamma_n(H_1\Delta E) + \gamma_n(E\Delta H_2) = 3\lambda(E)\,,\end{aligned}$$

from which the thesis follows. □

We come now to prove Theorem 5.1. It is useful to isolate the following inequalities: if H_1 and H_2 are half-spaces orthogonal to the unit vectors ν_1 and ν_2 respectively, and such that $\gamma_n(H_1) = \gamma_n(H_2) = r$, then

$$c_1(r)|\nu_1 - \nu_2| \leq \gamma_n(H_1\Delta H_2) \leq c_2(r)|\nu_1 - \nu_2|\,, \tag{5.2}$$

where c_1 and c_2 are positive constants. Moreover, we recall (4.17), i.e.

$$|\lambda(F_1) - \lambda(F_2)| \leq 2\gamma_n(F_1\Delta F_2)\,.$$

which holds for every pair of sets F_1 and F_2 in $\mathbb{R}^n$. According with the notaion set in (1.4), through the proof we denote by $H_{\nu,s}$ the half-space

$$H_{\nu,s} = \{x : x\cdot\nu > s\}\,, \qquad\qquad s = \Phi^{-1}(r)\,,$$

so that $\gamma_n(E) = r = \gamma_n(H_{\nu,s})$.

Proof of Theorem 5.1. We may assume that $\delta(E) \leq \bar\delta$, for a suitable constant $\bar\delta$ to be determined later. Indeed, fixed any E', symmetric with respect to $(n-1)$-orthogonal hyperplanes but different from an half-space, and such that $\delta(E') \leq 2^{n-1}\bar\delta$. Then (5.1) holds with $C = 1/\lambda(E')$ on every E such that $\delta(E) \geq \bar\delta$.

We will perform an inductive construction in $(n-1)$-steps, finding a set with a new hyperplane of symmetry at each step. We start setting $E_0 = E$. Given $j \in \mathbb{N}$ such that $0 \leq j \leq n-2$, assume that we have already exhibited a set E_j fulfilling $\gamma_n(E_j) = r$ and

$$E_j \text{ is symmetric with respect to } j\text{-orthogonal hyperplanes } S_1,\, S_2,\, \dots\,,\, S_j \tag{5.3a}$$

$$\lambda(E) \leq M^j\lambda(E_j)\,, \qquad\qquad \delta(E_j) \leq 2^j\delta(E)\,, \tag{5.3b}$$

for a suitable constant $M = M(n,r)$ to be chosen later. Then, our step consists in building an hyperplane S_{j+1} and a set E_{j+1} such that (5.3a) and (5.3b) hold true with $j+1$ replacing j. Clearly this will prove the thesis with the constant $C(n,r) = M^{n-1}$.

In general, the hyperplane $S = S(\nu) = \{x : x\cdot\nu = 0\}$, $\nu \in \mathbb{S}^{n-1}$, is orthogonal to all the S_i if and only if $\nu \in X$, where

$$X = \mathbb{S}^{n-1} \cap \bigcap\nolimits_{i=1}^{j} S_i\,.$$

The set X is connected, since it is homeomorphic to $\mathbb{S}^{n-j-1}$ and $j \leq n-2$. At the same time, the function $\psi : X \to \mathbb{R}$ defined as $\psi(\nu) = \gamma_n(E_j \cap H_{\nu,0}) - \gamma_n(E_j \setminus H_{\nu,0})$, being $H_{\nu,0} = \{x : x\cdot\nu > 0\}$ as in (1.4), is continuous and odd. Therefore there exists $\nu \in X$ for which $\psi(\nu) = 0$, that is,

$$\gamma_n(E_j \cap H_{\nu,0}) = \gamma_n(E_j \setminus H_{\nu,0}) = \frac{\gamma_n(E_j)}{2}\ :$$

to fix the ideas, we consider $\nu = e_1$; moreover, for simplicity we write $H = H_{\nu,0} = \{x : x \cdot e_1 > 0\}$ (the right half-space in Figure 4). Hence, the hyperplane $S = \{x : x \cdot e_1 = 0\}$ is orthogonal to each S_i, $1 \le i \le j$, and it splits E_j in two subsets of equal Gaussian measure.

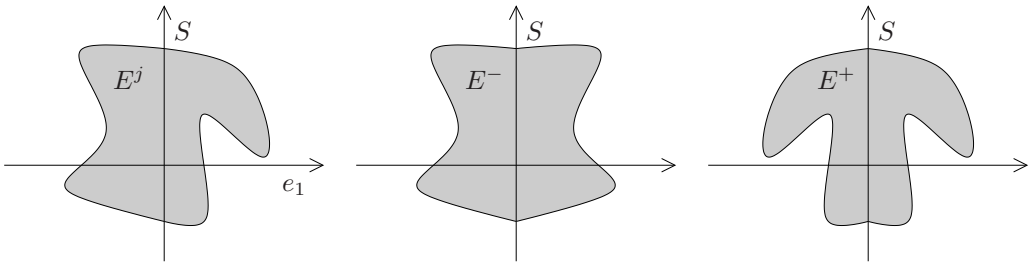

FIGURE 4. The sets E_j, E^- and E^+

Let now $\pi : \mathbb{R}^n \to \mathbb{R}^n$ denote the reflection with respect to S, and define two sets E^- and E^+ as

$$E^- = (E^j \setminus H) \cup \pi(E^j \setminus H)\,, \qquad\qquad E^+ = (E^j \cap H) \cup \pi(E^j \cap H)\,.$$

The situation is as shown in Figure 4. The sets E^+ and E^- are both symmetric with respect to each S_i, $1 \le i \le j$, and with respect to S. By construction one has $\gamma_n(E^+) = \gamma_n(E^-) = \gamma_n(E)$. Moreover, since $P_\gamma(E_j) \ge \big(P_\gamma(E^+) + P_\gamma(E^-)\big)/2$, with strict inequality if $\mathscr{H}^{n-1}_\gamma(\partial^* E_j \cap S) > 0$, we deduce that

$$\delta(E^+) \le 2\delta(E_j)\,, \qquad\qquad \delta(E^-) \le 2\delta(E_j)\,.$$

Then, if either

$$\lambda(E^+) \ge \frac{1}{M}\,\lambda(E_j) \qquad\qquad \text{or} \qquad\qquad \lambda(E^-) \ge \frac{1}{M}\,\lambda(E_j)\,,$$

we can choose $S_{j+1} = S$ and $E_{j+1} \in \{E^+, E^-\}$, fulfilling (5.3a) and (5.3b) and thus concluding the proof. We are therefore left to consider the case

$$\lambda(E^+) < \frac{1}{M}\,\lambda(E_j) \qquad\qquad \text{and} \qquad\qquad \lambda(E^-) < \frac{1}{M}\,\lambda(E_j)\,. \tag{5.4}$$

Roughly speaking, in this case the sets E^+ and E^- are much closer to be half-spaces than E is. In order to conclude the proof we will construct another pair of sets.

To begin, we apply Lemma 5.2 and find two half-spaces $H_{\nu_1,s}$ and $H_{\nu_2,s}$, symmetric with respect to the hyperplanes S_i, $1 \le i \le j$, as well as to S, such that

$$\lambda(E^-) \le \gamma_n(E^- \Delta H_{\nu_1,s}) \le 3\lambda(E^-)\,, \qquad \lambda(E^+) \le \gamma_n(E^+ \Delta H_{\nu_2,s}) \le 3\lambda(E^+)\,. \tag{5.5}$$

In view of the property (5.4), by (5.5) we can conclude that E_j is close to the special set K defined as

$$K = (H_{\nu_1,s} \setminus H) \cup (H_{\nu_2,s} \cap H)\,.$$

Indeed, one has

$$\gamma_n\big(E_j \Delta K\big) = \frac{\gamma_n(E^- \Delta H_{\nu_1,s}) + \gamma_n(E^+ \Delta H_{\nu_2,s})}{2} \le \frac{3}{2}\,\Big(\lambda(E^-) + \lambda(E^+)\Big) < \frac{3}{M}\,\lambda(E_j)\,. \tag{5.6}$$

There are three possibilities that, in principle, have to be considered:

$$\nu_1 = \nu_2\,, \qquad \nu_1 = -\nu_2\,, \qquad \nu_1 \text{ and } \nu_2 \text{ are linearly independent}\,. \tag{5.7}$$

The first two possibilities can be excluded a priori, provided $M > 3$ and $\bar\delta$ is assumed sufficiently small with respect to n and $\gamma_n(E)$. Indeed, if $\nu_1 = \nu_2$ then $K = H_{\nu_1,s} = H_{\nu_2,s}$ is an half-space. Thus, by (5.6),

$$\lambda(E_j) \le \gamma_n(E_j \Delta K) < \frac{3}{M}\,\lambda(E_j)\,,$$

against the assumption $M > 3$. In the case $\nu_1 = -\nu_2$ the set K is uniquely determined by r. In particular, its asymmetry $\lambda(K)$ is positive and depends only on r. By (4.17) and (5.6), as $M > 3$, we find

$$\lambda(E_j) \ge \lambda(K) - 2\gamma_n\big(E_j\Delta K\big) \ge \lambda(K) - \frac{6}{M}\,\lambda(E_j) \Longrightarrow \lambda(E_j) \ge \frac{\lambda(K)}{3}\,.$$

Thanks to the Continuity Lemma 4.7, this easily leads to a contradiction if we consider $\bar\delta$ small enough with respect to $\lambda(K)$. Thus we assume that ν_1 and ν_2 are linearly independent. Note that this forces $n \ge 3$. By a continuity argument analogous to the one we have used to prove the existence of ν, we can show the existence of a vector $\nu_0 \in \mathrm{Span}\{\nu_1, \nu_2\}$ such that

$$\gamma_n(E_j \cap H_0) = \gamma_n(E_j \setminus H_0) = \frac{\gamma_n(E_j)}{2}\,, \qquad H_0 = H_{\nu_0,0} = \{x : x \cdot \nu_0 > 0\}\,.$$

Again to fix the ideas, we assume that $\nu_0 = e_2$ and that $\mathrm{Span}\{\nu_1, \nu_2\} = \mathrm{Span}\{e_2, e_3\}$. Note that, provided $\bar\delta$ is small, the vectors ν_1 and ν_2 are close to each other and they are both close to be orthogonal to e_2; we assume then, again without loss of generality, also that ν_1 and ν_2 are close to e_3 and that

$$\big|\nu_2 - e_3\big| \le \big|\nu_1 - e_3\big|\,. \tag{5.8}$$

On denoting by $\pi_0 : \mathbb{R}^n \to \mathbb{R}^n$ the reflection with respect to $\{x : x \cdot e_2 = 0\}$, we define

$$F = (E_j \cap H_0) \cup \pi_0(E_j \cap H_0)\,.$$

We claim that the choice $S_{j+1} = \{x : x_2 = 0\} = \partial H_0$, $E_{j+1} = F$ satisfies the required properties. Indeed F is symmetric with respect to each S_i, $1 \le i \le j$, and with respect to ∂H_0, therefore (5.3a) holds true. And concerning (5.3b), we clearly have $\delta(F) \le 2\delta(E^j)$ as before while, as we are now going to show, provided we choose suitable values of $\bar\delta$ and M,

$$\lambda(F) \ge \frac{1}{M}\,\lambda(E_j)\,. \tag{5.9}$$

We are going to compare F with the set

$$K' = (K \cap H_0) \cup \pi_0(K \cap H_0)\,.$$

To this end we first derive some preliminary estimates on K and K'.

Step I. Estimate for K.

We prove that

$$\frac{c_1(r)}{6}|\,\nu_1 - \nu_2| \le \lambda(K) \le c_2(r)|\nu_1 - \nu_2|\,, \tag{5.10}$$

where c_1, c_2 were defined in (5.2). To this end, let us set $\nu_3 = (\nu_1+\nu_2)/|\nu_1+\nu_2|$, a vector which is close to e_3 if $\bar\delta$ is small. It is easily checked that K is invariant by reflection with respect to the line $\{t\nu_3 : t \in \mathbb{R}\}$. Therefore, we claim that

$$\lambda(K) \le \gamma_n(K\Delta H_{\nu_3,s}) \le 3\lambda(K)\,. \tag{5.11}$$

Indeed, first recall that by construction $\gamma_n(K) = r$, so that $\gamma_n(K) = \gamma_n(H_{\nu_3,s})$. Moreover, the only half-spaces invariant by reflection with respect to the line $\{t\nu_3,\, t \in \mathbb{R}\}$ and with Gaussian volume $r = \gamma_n(K)$ are $H_{\nu_3,s}$ and $H_{-\nu_3,s}$; finally, provided that $\bar\delta$ is small, the set K is very close to E_j which is very close to $H_{\nu_3,s}$ and very far from $H_{-\nu_3,s}$, so Lemma 5.2 gives (5.11). By definition of K we have that

$$\gamma_n(K\Delta H_{\nu_3,s}) = \frac{\gamma_n(H_{\nu_1,s}\Delta H_{\nu_3,s}) + \gamma_n(H_{\nu_2,s}\Delta H_{\nu_3,s})}{2}\,.$$

On combining this identity with (5.2) and (5.11) and taking into account that $|\nu_1-\nu_2| \le |\nu_1-\nu_3| + |\nu_2-\nu_3| \le 2|\nu_1-\nu_2|$, we prove (5.10).

Step II. Estimate for K'.

We prove that, for a positive constant $c_3 = c_3(r)$, we have

$$\lambda(K') \ge c_3(r)|\nu_1-\nu_2|\,. \tag{5.12}$$

By construction, K' is symmetric with respect to every $\{x : x_j = 0\}$ for $j = 2$ and for each $j > 3$. Therefore, by Lemma 5.2, we can find an half-space H' symmetric with respect to the same hyperplanes and such that

$$\lambda(K') \ge \frac{\gamma_n(K'\Delta H')}{3}\,.$$

We continue with the rough estimate

$$\lambda(K') \ge \frac{\gamma_n\big((K'\Delta H')\cap\{x : x_1 < 0\}\big)}{3} = \frac{1}{3}\int_{-\infty}^0 \mathscr{H}^{n-1}_\gamma\big(K'_t\Delta H'_t\big)d\gamma_1(t)\,, \tag{5.13}$$

denoting by K'_t and H'_t the intersections of K' and H' with $\{x : x_1 = t\}$. We note that $K'\setminus H$ is obtained by reflecting the set $H_{\nu_1,s}$ in the quadrant $\{x : x_1 < 0,\, x_2 > 0\}$ with respect to $S_{j+1} = \{x : x_2 = 0\}$. As a consequence, keeping in mind that inside the half-space $\mathbb{R}^n\setminus H$ the set $K = H_{\nu_1,s}$ is an half-space orthogonal to ν_1, and that ν_1 is orthogonal to e_1, we derive that the section K'_t is independent from $t < 0$, and in fact it only depends on $|\nu_1 - e_3|$ and $s = \Phi^{-1}(r)$, as shown in Figure 5. On the other hand, thanks to the symmetries of H', we have that $H'_t = \{x : x_1 = t,\, x_3 \ge \alpha(t)\}$ for some affine function $\alpha : \mathbb{R}\to\mathbb{R}$. In particular, as it appears evident from Figure 6, there is a constant $c_4 = c_4(r)$ such that

$$\mathscr{H}^{n-1}_\gamma(K'_t\Delta H'_t) \ge c_4(r)|\nu_1 - e_3|\,,$$

for every $t < 0$. Indeed, $K'_t\Delta H'_t$ is the dark region of the figure and, depending on the value of $\alpha(t)$, either the triangle or the two unbounded regions have Gaussian volume bigger than $|\nu_1-e_3|$ times a constant depending only on r. On combining this estimate with (5.13) we find that

$$\lambda(K') \ge \frac{\sqrt{2\pi}}{6}\,c_4(r)|\nu_1-e_3|\,,$$

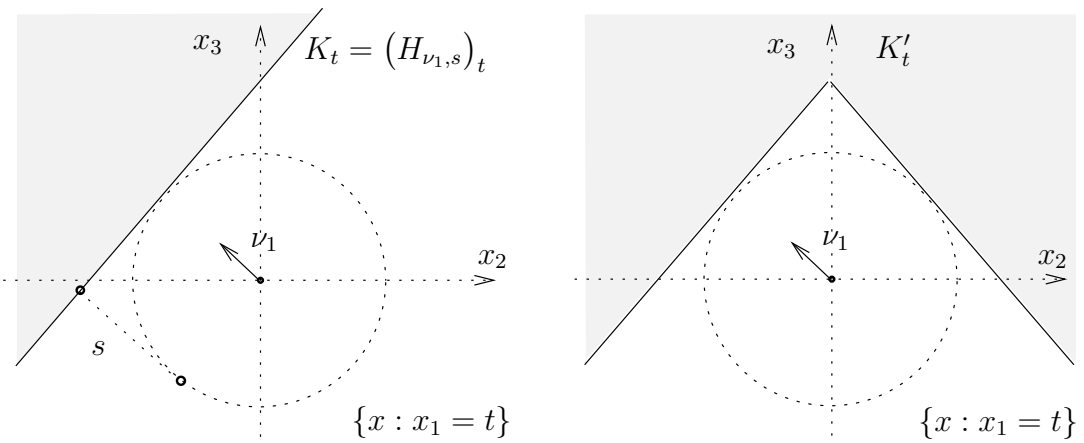

FIGURE 5. The section K'_t only depends on $|\nu_1 - e_3|$, and in particular it is independent from $t < 0$.

and therefore, as $|\nu_1 - \nu_2| \le |\nu_1 - e_3| + |\nu_2 - e_3| \le 2|\nu_1 - e_3|$, we come to (5.12).

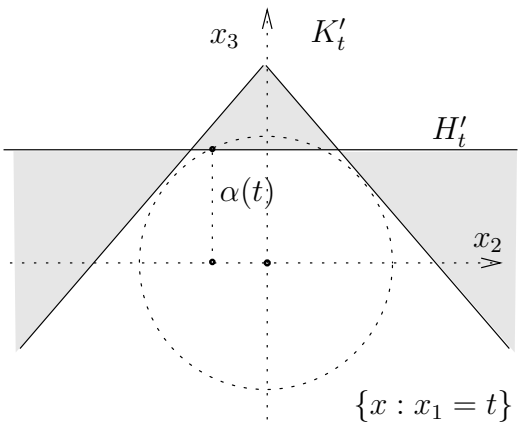

FIGURE 6. The measure of $\mathscr{H}^{n-1}_\gamma(K'_t \Delta H'_t)$ is at least $c_4(r)|\nu_1 - e_3|$, whatever t, therefore $\alpha(t)$, is.

With (5.10) and (5.12) in force, we immediately conclude the proof of (5.9). Indeed, by (4.17), (5.12), (5.10) and (5.6) we have

$$\begin{aligned}
\lambda(F) &\ge \lambda(K') - 2\gamma_n(K'\Delta F) \ge c_3(r)|\nu_1 - \nu_2| - 3\gamma_n((K\Delta E_j) \cap H_0) \\
&\ge \frac{c_3(r)}{c_2(r)}\,\lambda(K) - 3\gamma_n(K\Delta E_j) \ge \frac{c_3(r)}{c_2(r)}\Big(\lambda(E_j) - 2\gamma_n(K\Delta E_j)\Big) - 3\gamma_n(K\Delta E_j) \\
&\ge \frac{c_3(r)}{c_2(r)}\,\lambda(E_j) - \Big(\frac{2c_3(r)}{c_2(r)} + 3\Big)\gamma_n(K\Delta E_j) \ge \left(\frac{c_3(r)}{c_2(r)} - \frac{6c_3(r)}{c_2(r)M} - \frac{12}{M}\right)\lambda(E_j)\,.
\end{aligned}$$

Therefore (5.9) follows on choosing a suitably large value of M, and we achieve the proof of Theorem 5.1. □

6. The two-dimensional case

In this section we prove Theorem 1.1 in the two dimensional case $n = 2$. Through all the section, we will denote by $x = (z, y)$ the generic point in $\mathbb{R}^2$. In Section 6.1 we prepare the proof of the theorem in the case of the Ehrhard symmetral E^s of E, while in Section 6.2 we finalize the argument for E^s (see Step II) and explain how to deduce the theorem for E starting from this case.

6.1. **A particular class of sets.** Before coming to the development of the strategy sketched above, we consider a special class of sets for the isoperimetric problem in the Gauss plane, and evaluate their deficit and asymmetry. We are concerned with the class $\mathcal{E}$ of those smooth open sets $E \subseteq \mathbb{R}^2$ such that

$$E_0 = \{x \in \mathbb{R}^2 : z = 0\} = (\hat{y}, \infty)\,, \qquad \gamma_1((\partial E)_0) = 0\,,$$

for some $\hat{y} \in \mathbb{R}$. Notice that by the smoothness assumption, one has $\partial E = \partial^* E$. We will collect a series of lemmas on sets E from the class $\mathcal{E}$, that we are going to use as building steps in the proof of Theorem 1.1 in the Gauss plane. Before starting, we need to focus our attention on the following particular class of sets: given $\hat{z} \in [0, \infty)$ and $\hat{y} \in \mathbb{R}$, for every $\varepsilon \in \mathbb{R}$ we define a set E_ε as

$$E_\varepsilon = E_\varepsilon(\hat{z}, \hat{y}) = \big\{x \in \mathbb{R}^2 : |z| \geq \hat{z},\, y > \hat{y} - \varepsilon(|z| - \hat{z})\big\} \cup \big\{x \in \mathbb{R}^2 : |z| \leq \hat{z},\, y > \hat{y}\big\}\,, \tag{6.1}$$

see Figure 7. It is useful to compare E_ε with the half-plane $H = H(\hat{y}) = \{x \in \mathbb{R}^2 : y \geq \hat{y}\}$. To this end, we introduce the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(\varepsilon) = \gamma_2(E_\varepsilon \setminus H) - \gamma_2\big(H \setminus E_\varepsilon\big)\,. \tag{6.2}$$

This is a strictly increasing function on $\mathbb{R}$, whose image is a bounded interval, and in particular

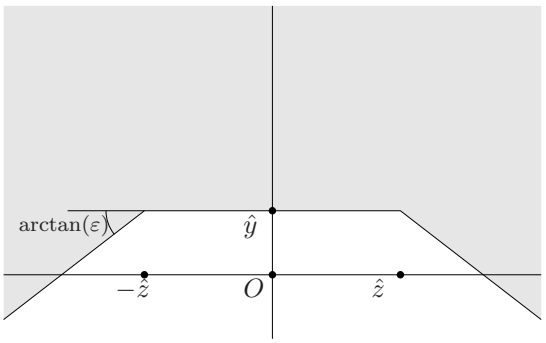

Figure 7. The set $E_\varepsilon(\hat{z}, \hat{y})$.

we have that $f(\varepsilon) = \gamma_2(E_\varepsilon \setminus H) > 0$ if $\varepsilon > 0$, while $f(\varepsilon) = -\gamma_2(H \setminus E_\varepsilon) < 0$ if $\varepsilon < 0$. In any case $|f(\varepsilon)| = \gamma_2(E_\varepsilon \Delta H)$. In the next lemma we study $P_\gamma(E_\varepsilon)$ in the regime $\varepsilon \to 0$.

Lemma 6.1. *Let $\hat{z} \geq 0$, $\hat{y} \in \mathbb{R}$ and let $\beta \in f(\mathbb{R})$, being f as in (6.2). Then there exists a smooth function $c = c(z)$ with $c(0) = 0$, $c(z) > 0$ for $z > 0$, such that*

$$P_\gamma(E_{f^{-1}(\beta)}) = P_\gamma(H) + \sqrt{2\pi}\,\hat{y}\,\beta + e^{\frac{\hat{y}^2}{2}}\,c(\hat{z})\beta^2 + o(\beta^2)\,, \tag{6.3}$$

as $\beta \to 0$. Here $o(\beta^2)$ may depend on $\hat{z}$ and $\hat{y}$.

Proof. By definition of E_ε, H, γ_2 and P_γ we have that

$$f(\varepsilon) = \sqrt{\frac{2}{\pi}} \int_{\hat{z}}^{\infty} \left(\Phi(\hat{y} - \varepsilon(z - \hat{z})) - \Phi(\hat{y})\right) e^{-\frac{z^2}{2}} dz$$

and that

$$P_\gamma(E_\varepsilon) = P_\gamma(H) - 2e^{-\frac{\hat{y}^2}{2}}\,\Phi(\hat{z}) + 2\sqrt{1+\varepsilon^2}\,g(\varepsilon)\,, \tag{6.4}$$

where we have set

$$g(\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{\hat{z}}^{\infty} e^{-\frac{(\hat{y} - \varepsilon(z-\hat{z}))^2}{2}}\, e^{-\frac{z^2}{2}} dz\,.$$

Moreover, we have

$$f'(\varepsilon) = \sqrt{\frac{2}{\pi}} \int_{\hat{z}}^{\infty} \frac{e^{-\frac{(\hat{y} - \varepsilon(z-\hat{z}))^2}{2}}}{\sqrt{2\pi}}\,(z - \hat{z})\, e^{-\frac{z^2}{2}} dz\,,$$
$$f''(\varepsilon) = \sqrt{\frac{2}{\pi}} \int_{\hat{z}}^{\infty} \frac{e^{-\frac{(\hat{y} - \varepsilon(z-\hat{z}))^2}{2}}}{\sqrt{2\pi}}\,(\hat{y} - \varepsilon(z - \hat{z}))\,(z - \hat{z})^2\, e^{-\frac{z^2}{2}} dz\,.$$

We can easily express $f'(0)$ and $f''(0)$ thanks to the following definitions.

$$\begin{aligned} a(\hat{z}) :=& \frac{1}{\sqrt{2\pi}} \int_{\hat{z}}^{\infty} (z - \hat{z}) e^{-\frac{z^2}{2}}\, dz\,, \\ b(\hat{z}) :=& \frac{1}{\sqrt{2\pi}} \int_{\hat{z}}^{\infty} (z - \hat{z})^2 e^{-\frac{z^2}{2}}\, dz\,. \end{aligned} \tag{6.5}$$

This notation is particularly useful when writing the following Taylor expansion for f,

$$f(\varepsilon) = \sqrt{\frac{2}{\pi}}\, e^{-\frac{\hat{y}^2}{2}}\, a(\hat{z})\,\varepsilon + \sqrt{\frac{2}{\pi}}\,\hat{y}\, e^{-\frac{\hat{y}^2}{2}}\, b(\hat{z})\,\frac{\varepsilon^2}{2} + o(\varepsilon^2)\,. \tag{6.6}$$

as $\varepsilon \to 0$. It is to be noticed that both a and b are strictly decreasing functions from $[0, +\infty)$ to $\mathbb{R}$ which tend to 0 as $\hat{z} \to +\infty$, and that $a(0) = 1/\sqrt{2\pi}$ and $b(0) = 1/2$. In particular we can underline, for future reference, the estimate

$$f(\varepsilon) \leq \frac{1}{\pi}\, e^{-\frac{\hat{y}^2}{2}}\,\varepsilon + o(\varepsilon)\,. \tag{6.7}$$

Concerning g, we have that

$$g'(\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{\hat{z}}^{\infty} e^{-\frac{(\hat{y} - \varepsilon(z-\hat{z}))^2}{2}}\,(\hat{y} - \varepsilon(z - \hat{z}))\,(z - \hat{z})\, e^{-\frac{z^2}{2}} dz\,,$$
$$g''(\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{\hat{z}}^{\infty} e^{-\frac{(\hat{y} - \varepsilon(z-\hat{z}))^2}{2}} \left(\left(\hat{y} - \varepsilon(z - \hat{z})\right)^2 - 1\right)(z - \hat{z})^2\, e^{-\frac{z^2}{2}} dz\,.$$

On taking (6.6) into account,

$$\begin{aligned} g(\varepsilon) &= e^{-\frac{\hat y^2}{2}}\,\Phi(\hat z) + e^{-\frac{\hat y^2}{2}}\,\hat y\,a(\hat z)\,\varepsilon + e^{-\frac{\hat y^2}{2}}\,(\hat y^2-1)\,b(\hat z)\,\frac{\varepsilon^2}{2} + o(\varepsilon^2) \\ &= e^{-\frac{\hat y^2}{2}}\,\Phi(\hat z) + \sqrt{\frac{\pi}{2}}\,\hat y\,f(\varepsilon) - e^{-\frac{\hat y^2}{2}}\,b(\hat z)\,\frac{\varepsilon^2}{2} + o(\varepsilon^2)\,. \end{aligned}$$

Eventually, as $2\sqrt{1+\varepsilon^2}g(\varepsilon) = (2+\varepsilon^2)g(\varepsilon) + o(\varepsilon^2)$, an integration by parts starting from (6.5) immediately yields

$$\begin{aligned} 2\sqrt{1+\varepsilon^2}\,g(\varepsilon) &= 2e^{-\frac{\hat y^2}{2}}\,\Phi(\hat z) + \sqrt{2\pi}\,\hat y\,f(\varepsilon) + e^{-\frac{\hat y^2}{2}}\varepsilon^2\Big(-b(\hat z)+\Phi(\hat z)\Big) + o(\varepsilon^2) \\ &= 2e^{-\frac{\hat y^2}{2}}\,\Phi(\hat z) + \sqrt{2\pi}\,\hat y\,f(\varepsilon) + e^{-\frac{\hat y^2}{2}}\,\hat z\,a(\hat z)\,\varepsilon^2 + o(\varepsilon^2)\,. \end{aligned} \tag{6.8}$$

Plugging (6.8) into (6.4), we find that

$$P_\gamma(E_\varepsilon) = P_\gamma(H) + \sqrt{2\pi}\,\hat y\,f(\varepsilon) + e^{-\frac{\hat y^2}{2}}\,\hat z\,a(\hat z)\,\varepsilon^2 + o(\varepsilon^2)\,. \tag{6.9}$$

Recall now that $\beta = f(\varepsilon)$, so that by construction $\varepsilon \to 0$ if and only if $\beta \to 0$. By (6.6),

$$\varepsilon^2 = \frac{\pi}{2}\frac{e^{\hat y^2}}{a(\hat z)^2}\,\beta^2 + o(\beta^2)\,.$$

Thus, we deduce from (6.9) that

$$P_\gamma(E_{f^{-1}(\beta)}) = P_\gamma(H) + \sqrt{2\pi}\,\hat y\,\beta + e^{\frac{\hat y^2}{2}}\,c(\hat z)\,\beta^2 + o(\beta^2)\,,$$

where we have set

$$c(\hat z) = \frac{\pi}{2}\frac{\hat z}{a(\hat z)}\,.$$

This remark concludes the proof of the lemma. □

In the next lemma we compare a generic set $E \in \mathcal{E}$ with a suitable E_ε.

Lemma 6.2. *Let E be a smooth open set in $\mathbb{R}^2$ such that $E_{\hat z} = (\hat y, \infty)$ and $\gamma_1\big((\partial E)_{\hat z}\big) = 0$ for some $(\hat z, \hat y) \in \mathbb{R}^2$, and let $\varepsilon \in \mathbb{R}$ be such that the set $E^+ \subseteq \mathbb{R}^2$ defined by*

$$E^+ = \{x \in E : z \le \hat z\} \cup \{x \in \mathbb{R}^2 : z > \hat z\,,\, y > \hat y - \varepsilon(z-\hat z)\}\,.$$

satisfies $\gamma_2(E^+) = \gamma_2(E)$. Then $P_\gamma(E^+) \le P_\gamma(E)$. In particular, if $E \in \mathcal{E}$, E is symmetric with respect to the y-axis and $\hat z = 0$, then $P_\gamma(E_\varepsilon(0,\hat y)) \le P_\gamma(E)$.

Proof. Let us define a set $E^- \subseteq \mathbb{R}^2$ and an half-plane H as

$$E^- = \{x \in E : z \ge \hat z\} \cup \{x \in \mathbb{R}^2 : z < \hat z\,,\, y > \hat y - \varepsilon(z-\hat z)\}\,,\quad H = \{x \in \mathbb{R}^2 : y > \hat y - \varepsilon(z-\hat z)\}\,.$$

We have $\gamma_2(H) = \gamma_2(E^-)$ (while, in general, $\gamma_2(H) \neq \gamma_2(E)$) therefore, by the Isoperimetric Theorem 4.1, $P_\gamma(E^-) \ge P_\gamma(H)$. Notice now that, since E is a smooth open set, then $(\partial^* E^+)_{\hat z} \subseteq (\partial E)_{\hat z}$, so in particular $\gamma_1\big((\partial^* E^+)_{\hat z}\big) = 0$. Thus we have that

$$\begin{aligned} P_\gamma(E) - P_\gamma(E^+) &= P_\gamma(E; z < \hat z) + P_\gamma(E; z > \hat z) - P_\gamma(E^+; z < \hat z) - P_\gamma(E^+; z > \hat z) \\ &= P_\gamma(E; z > \hat z) - P_\gamma(E^+; z > \hat z) = P_\gamma(E^-) - P_\gamma(H) \ge 0\,, \end{aligned}$$

as desired. The second part of the lemma follows just applying twice this argument. □

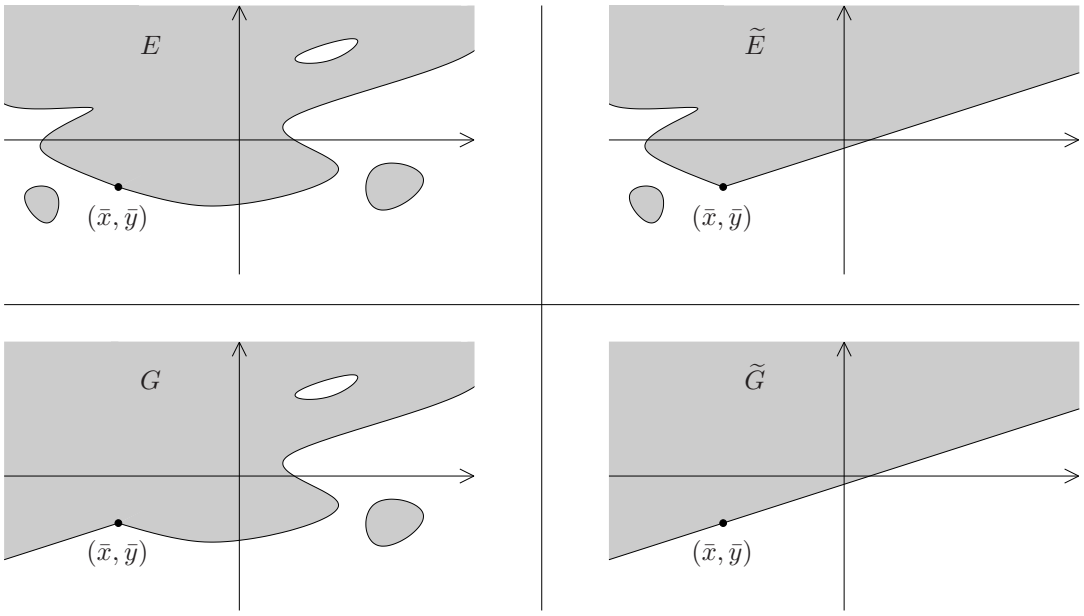

FIGURE 8. The sets E, E^+, E^- and H in Lemma 6.2

We investigate further on sets $E \in \mathcal{E}$ that are symmetric with respect to the y-axis.

Lemma 6.3. *Let $E \in \mathcal{E}$ be symmetric with respect to the y-axis, with $E_0 = (\hat{y}, \infty)$. Setting $\bar{y} = \Phi^{-1}\big(\gamma_2(E)\big)$, one has*

$$\delta(E) \geq \frac{e^{-\frac{\bar{y}^2}{2}}}{2}\big(\hat{y}-\bar{y}\big)^2 + o\big((\hat{y}-\bar{y})^2\big)$$

as $|\hat{y}-\bar{y}| \to 0$, where $o\big((\hat{y}-\bar{y})^2\big)$ depends only on $\gamma_2(E)$.

Proof. Let $\varepsilon \in \mathbb{R}$ be such that $E_\varepsilon = E_\varepsilon(0,\hat{y})$ satisfies $\gamma_2(E) = \gamma_2(E_\varepsilon)$, and define for brevity

$$\widehat{H} = \{x \in \mathbb{R}^2 : y > \hat{y}\}, \qquad \overline{H} = \{x \in \mathbb{R}^2 : y > \bar{y}\}.$$

First of all we recall that, by Lemma 6.2,

$$P_\gamma(E_\varepsilon) \leq P_\gamma(E). \tag{6.10}$$

Let now $\tau = \hat{y} - \bar{y}$ and notice that $\varepsilon \to 0$ if and only if $\tau \to 0$. Therefore, as $\gamma_2(E) = \gamma_2(E_\varepsilon) = \gamma_2(\overline{H})$ and E_ε either contains or is contained in $\widehat{H}$, we have that

$$\begin{aligned} f(\varepsilon) &= \gamma_2(E_\varepsilon \setminus \widehat{H}) - \gamma_2(\widehat{H} \setminus E_\varepsilon) = \gamma_2(\overline{H} \setminus \widehat{H}) - \gamma_2(\widehat{H} \setminus \overline{H}) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\bar{y}}^{\hat{y}} e^{-\frac{y^2}{2}}\,dy = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{\bar{y}^2}{2}}\int_0^\tau e^{-t\bar{y}-\frac{t^2}{2}}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\,e^{-\frac{\bar{y}^2}{2}}\int_0^\tau 1 - t\bar{y} + o(\tau)\,dt = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{\bar{y}^2}{2}}\Big(\tau - \frac{\bar{y}\,\tau^2}{2} + o\big(\tau^2\big)\Big), \end{aligned}$$

Similarly, we have that

$$P_\gamma(\widehat{H}) = e^{-\frac{\hat{y}^2}{2}} = e^{-\frac{\bar{y}^2}{2}} - \int_{\bar{y}}^{\hat{y}} e^{-\frac{y^2}{2}}\,y\,dy = P_\gamma(\overline{H}) + e^{-\frac{\bar{y}^2}{2}}\Big(-\bar{y}\tau - \big(1-\bar{y}^2\big)\frac{\tau^2}{2}\Big) + o\big(\tau^2\big).$$

On combining these two Taylor expansions together with (6.10) and (6.3) (note that $\hat{z} = 0$, therefore $c(\hat{z}) = 0$ in that formula), we find

$$\delta(E) = P_\gamma(E) - P_\gamma(\overline{H}) \geq P_\gamma(E_\varepsilon) - P_\gamma(\overline{H}) = P_\gamma(\widehat{H}) - P_\gamma(\overline{H}) + \sqrt{2\pi}\,\hat{y}\,f(\varepsilon) + o(f(\varepsilon)^2)$$

$$= e^{-\frac{\bar{y}^2}{2}}\left(\Big(-\bar{y}\,\tau - \big(1-\bar{y}^2\big)\frac{\tau^2}{2}\Big) + \Big(\hat{y}\,\tau - \hat{y}\bar{y}\frac{\tau^2}{2}\Big)\right) + o\big(\tau^2\big) = \frac{e^{-\frac{\bar{y}^2}{2}}}{2}\,\tau^2 + o\big(\tau^2\big)\,,$$

and the thesis follows. □

Eventually, we consider a set $E \in \mathcal{E}$ as in the previous lemma, with the further property of being equal to its Ehrhard symmetral E^s, and prove Theorem 1.1 for such a set under a suitable geometric assumption.

Lemma 6.4. *Let $E \in \mathcal{E}$ be symmetric with respect to the y-axis, with $E = E^s$ and $E_0 = (\hat{y}, \infty)$. Moreover, define $\bar{y} = \Phi^{-1}(\gamma_2(E))$. If $|\hat{y} - \bar{y}| \leq \lambda(E)/4$, then*

$$\lambda(E) \leq C(\gamma_2(E))\sqrt{\delta(E)}\,,$$

for a suitable constant $C = C(\gamma_2(E))$ depending only on $\gamma_2(E)$.

Proof. Let $\overline{H}$ and $\widehat{H}$ be defined as in the proof of Lemma 6.3, and consider the two sets

$$E_1 := E \cup \widehat{H}, \qquad\qquad E_2 := E \cap \widehat{H}\,.$$

We need to evaluate $P_\gamma(E_1)$ and $P_\gamma(E_2)$, as we have that

$$P_\gamma(E) = P_\gamma(E_1) + P_\gamma(E_2) - P_\gamma(\widehat{H})\,. \tag{6.11}$$

Up to change E with $\mathbb{R}^2 \setminus E$, and up to reflect the resulting set with respect to the z-axis (this

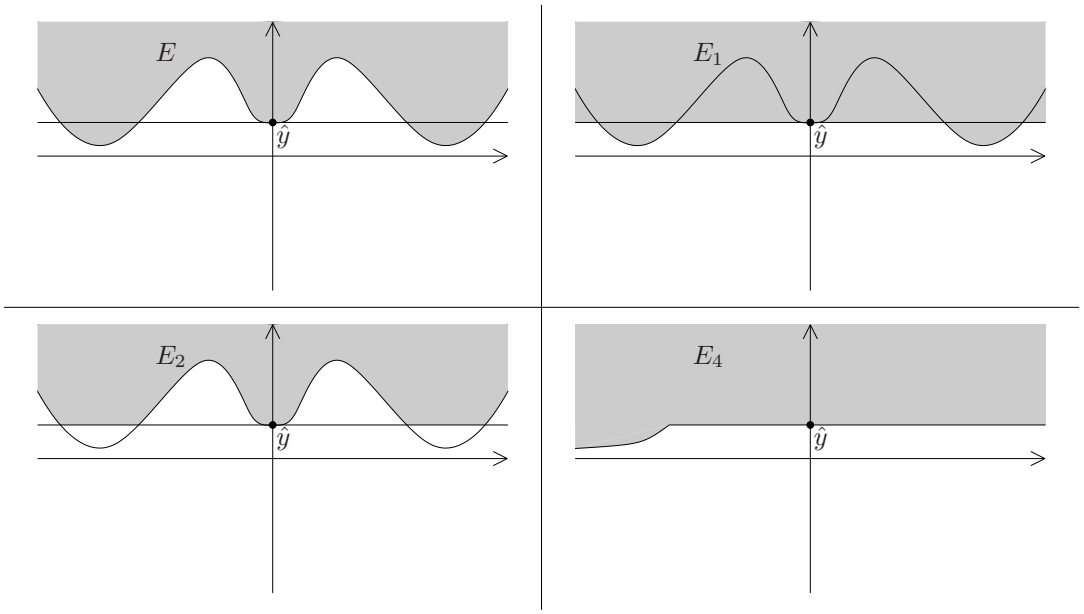

FIGURE 9. The sets E, E_1, E_2 and E_4

procedure leaves invariant the deficit, the asymmetry and our set of assumptions on E), we may assume that

$$\mathscr{H}^1_\gamma(\partial^* E_1 \cap \{x : y = \hat{y}\}) \geq \mathscr{H}^1_\gamma(\partial^* E_2 \cap \{x : y = \hat{y}\})\,. \tag{6.12}$$

On applying Lemma 6.2 to E_2 we find that $P_\gamma(E_2) \geq P_\gamma(E_{\varepsilon_2}(0,\hat{y}))$, where $\varepsilon_2 \in \mathbb{R}$ is such that $f(\varepsilon_2) = -\gamma_2(\widehat{H} \setminus E_2)$. By (6.3), we deduce that

$$P_\gamma(E_2) \geq P_\gamma(\widehat{H}) + \sqrt{2\pi}\,\hat{y}\,f(\varepsilon_2) + o(f(\varepsilon_2)^2)\,, \tag{6.13}$$

as $\varepsilon_2 \to 0$. In order to estimate $P_\gamma(E_1)$, we introduce two comparison sets: the first one, E_3, is defined as

$$E_3 := \Big(E_1 \cap \{x : z \leq 0\}\Big) \cup \Big(\widehat{H} \cap \{x : z \geq 0\}\Big)\Big)\,.$$

The second one, E_4, is obtained by an Ehrhard symmetrization of E_3 in the z-direction, that is,

$$E_4 := \{x \in \mathbb{R}^2 : z < -\Phi^{-1}(\gamma_1((E_3)^y))\}\,,$$

where $(E_3)^y = \{z \in \mathbb{R} : (z,y) \in E_3\}$. Then we have

$$P_\gamma(E_1) = 2P_\gamma(E_3) - P_\gamma(\widehat{H})\,. \tag{6.14}$$

By the assumption (6.12), and since $E = E^s$, we have that

$$E_4 \cap \{x : z \geq -\hat{z}\} = \widehat{H} \cap \{x : z \geq -\hat{z}\}$$

where $\hat{z}$ is the positive real number such that

$$\int_{-\infty}^{-\hat{z}} e^{-\frac{t^2}{2}}\,dt = \int_{-\hat{z}}^{0} e^{-\frac{t^2}{2}}\,dt\,.$$

Thanks to Lemma 6.2 we have that

$$P_\gamma(E_4) \geq P_\gamma\bigg(\Big(E_{\varepsilon_1}(\hat{z},\hat{y}) \cap \{x : z \leq 0\}\Big) \cup \Big(\widehat{H} \cap \{x : z \geq 0\}\Big)\bigg)\,,$$

where $\varepsilon_1 \in \mathbb{R}$ is such that

$$f(\varepsilon_1) = 2\gamma_2(E_4 \setminus \widehat{H}) = 2\gamma_2(E_3 \setminus \widehat{H}) = \gamma_2(E_1 \setminus \widehat{H})\,.$$

Therefore, by (6.14),

$$\begin{aligned} P_\gamma(E_1) &= 2P_\gamma(E_3) - P_\gamma(\widehat{H}) \geq 2P_\gamma(E_4) - P_\gamma(\widehat{H}) \\ &\geq 2P_\gamma\bigg(\Big(E_{\varepsilon_1}(\hat{z},\hat{y}) \cap \{x : z \leq 0\}\Big) \cup \Big(\widehat{H} \cap \{x : z \geq 0\}\Big)\bigg) - P_\gamma(\widehat{H}) \\ &= P_\gamma\big(E_{\varepsilon_1}(\hat{z},\hat{y})\big)\,. \end{aligned}$$

From (6.3) we obtain

$$P_\gamma(E_1) \geq P_\gamma(\widehat{H}) + \sqrt{2\pi}\,\hat{y}\,f(\varepsilon_1) + e^{\frac{\hat{y}^2}{2}}\,c(\hat{z})\,f(\varepsilon_1)^2 + o(f(\varepsilon_1)^2)\,, \tag{6.15}$$

as $\varepsilon_1 \to 0$.

Before using the estimates (6.13) and (6.15) together with (6.11), we need to evaluate $f(\varepsilon_1)$ and $f(\varepsilon_2)$. To this aim, let us define

$$m_1 := \gamma_2\Big(\{x : \bar{y} \leq y \leq \hat{y}\} \cap E\Big) - \gamma_2\Big(\{x : \hat{y} \leq y \leq \bar{y}\} \cap E\Big)$$

$$m_2 := \gamma_2\Big(\{x : \bar y \le y \le \hat y\} \setminus E\Big) - \gamma_2\Big(\{x : \hat y \le y \le \bar y\} \setminus E\Big),$$
$$\rho := \frac{\gamma_2(E\Delta\overline{H})}{2} = \gamma_2(E \setminus \overline{H}) = \gamma_2(\overline{H} \setminus E)\,.$$

By construction $f(\varepsilon_1) = \rho + m_1$, while $f(\varepsilon_2) = -\rho + m_2$. On defining $\tau = |\hat y - \bar y| \le \lambda(E)/4$, by Lemma 5.2 and since $E = E^s$ it is

$$f(\varepsilon_1) = \gamma_2\big(E \setminus \widehat{H}\big) \ge \gamma_2\big(E \setminus \overline{H}\big) - \gamma_2\big(\widehat{H} \setminus \overline{H}\big) \ge \frac{\lambda(E)}{2} - \tau \ge \frac{\lambda(E)}{4}\,,$$
$$f(\varepsilon_1) = \gamma_2\big(E \setminus \widehat{H}\big) \le \gamma_2\big(E \setminus \overline{H}\big) + \gamma_2\big(\overline{H} \setminus \widehat{H}\big) \le \frac{3}{2}\,\lambda(E) + \tau \le 2\lambda(E)\,,$$
$$f(\varepsilon_2) = \gamma_2\big(\widehat{H} \setminus E\big) \le \gamma_2\big(\widehat{H} \setminus \overline{H}\big) + \gamma_2\big(\overline{H} \setminus E\big) \le \tau + \frac{3}{2}\,\lambda(E) \le 2\lambda(E)\,.$$

The second and third estimates ensure that $o(f(\varepsilon_i)^2) = o(\lambda(E)^2)$ for $i = 1, 2$. Thus from the first estimate, using (6.11), (6.15) and (6.13), we get

$$\begin{aligned} P_\gamma(E) &= P_\gamma(E_1) + P_\gamma(E_2) - P_\gamma(\widehat{H}) \\ &\ge P_\gamma(\widehat{H}) + \sqrt{2\pi}\hat y\, f(\varepsilon_1) + e^{\frac{\hat y^2}{2}} c(\hat z) f(\varepsilon_1)^2 + \sqrt{2\pi}\,\hat y\, f(\varepsilon_2) + o(\lambda(E)^2) \\ &\ge e^{-\frac{\hat y^2}{2}} + \sqrt{2\pi}\,\hat y\,(m_1 + m_2) + \frac{e^{-\frac{\hat y^2}{2}} c(\hat z)}{16}\,\lambda(E)^2 + o(\lambda(E)^2)\,. \end{aligned}$$

Eventually, since we have

$$e^{-\frac{\hat y^2}{2}} + \sqrt{2\pi}\,\hat y\,(m_1 + m_2) = e^{-\frac{\bar y^2}{2}} - \int_{\bar y}^{\hat y} t\, e^{-\frac{t^2}{2}}\, dt + \hat y \int_{\bar y}^{\hat y} e^{-\frac{t^2}{2}}\, dt \ge e^{-\frac{\bar y^2}{2}} = P_\gamma(\overline{H})\,,$$

we conclude that

$$P_\gamma(E) \ge P_\gamma(\overline{H}) + \frac{e^{\frac{\hat y^2}{2}}\, c(\hat z)}{16}\,\lambda(E)^2 + o(\lambda(E)^2)\,.$$

Since $\delta(E) = P_\gamma(E) - P_\gamma(\overline{H})$, the thesis follows from the above inequality if $\lambda(E) \le \bar\lambda(r)$ is smaller than a suitable constant $\bar\lambda(r)$ depending only on $r = \gamma_2(E)$. On the other hand, by the Continuity Lemma 4.7, if $\lambda(E) \ge \bar\lambda(r)$ then $\delta(E) \ge \bar\delta(r)$ for some positive constant $\bar\delta(r)$. And in this case the thesis is trivially true since one has always $\lambda(E) \le 1$. □

6.2. **The proof of Theorem 1.1 in dimension $n = 2$.** We are finally in the position to give the proof of Theorem 1.1 in the two-dimensional case.

Proof of Theorem 1.1, $n = 2$. We divide the proof in several steps.

Step I. Preparatory remarks.

By a standard approximation argument, E can be assumed to be a smooth open set such that (4.4) is in force, which in the two-dimensional case reads as

$$\mathscr{H}^1_\gamma(\{x \in \partial E : \nu_y^E = 0\}) = 0\,. \tag{6.16}$$

Without loss of generality, we assume $r = \gamma_2(E) \in (0, 1/2)$. Let $\varepsilon(r)$ be a positive constant, to be chosen later depending only on r. By the Continuity Lemma 4.7, there exists a positive

constant $\delta(r) \leq 1$ such that, if $\delta(E) \leq \delta(r)$, then $\lambda(E) \leq \varepsilon(r)$. We can directly reduce ourselves to consider the case $\delta(E) \leq \delta(r)$. Indeed, if $\delta(E) > \delta(r)$, then

$$\lambda(E) \leq 1 \leq \frac{1}{\sqrt{\delta(r)}} \sqrt{\delta(E)}\,,$$

and so it suffices to take $C(r) \geq 1/\sqrt{\delta(r)}$ to achieve the thesis.

By Theorem 5.1, we can consider the case when E is symmetric with respect to the y-axis. We also let $H = \{x \in \mathbb{R}^2 : y \geq \bar{y}\}$ for $\bar{y} = \Phi^{-1}(r)$ so that, by Lemma 5.2 and up to a reflection with respect to the z-axis, we can assume that

$$\lambda(E) \leq \gamma_2(E \Delta H) \leq 3\lambda(E \Delta H)\,. \tag{6.17}$$

Eventually, let us consider the Ehrhard symmetral E^s of E

$$E^s = \{x \in \mathbb{R}^2 :\ y > \Phi^{-1}(v_E(z))\}\,, \qquad\qquad v_E(z) = \gamma_1(E_z)\,,$$

and let $\hat{y} \in \mathbb{R}$ be such that $(E^s)_0 = \{x \in E^s : z = 0\} = (\hat{y}, \infty)$.

Step II. The estimate for the Ehrhard symmetral.

In this step we aim to prove that, if $E = E^s$,

$$\lambda(E) \leq C(r) \sqrt{\delta(E)} \tag{6.18}$$

for a suitable positive constant $C(r)$.

If $|\bar{y} - \hat{y}| \leq \lambda(E)/4$, then (6.18) follows from Lemma 6.4. If $|\bar{y} - \hat{y}| \geq \lambda(E)/4$ and $|\bar{y} - \hat{y}| \leq \eta(r)$ for a sufficiently small constant $\eta(r)$, then (6.18) follows by Lemma 6.3. Therefore we are left to show that, provided $\delta(r)$ is small enough, then $|\bar{y} - \hat{y}| \leq \eta(r)$ holds true. If this were not the case, we could find a sequence of sets $\{F_h\}_{h \in \mathbb{N}}$ with $F_h = (F_h)^s$, $\gamma_2(F_h) = r$, and such that each F_h is symmetric with respect to the y-axis, (6.16) holds true, $\lim_{h \to \infty} \delta(F_h) = 0$, and finally calling y_h the number so that $(F_h)_0 = (y_h, +\infty)$ it is

$$|y_h - \bar{y}| \geq \eta(r)\ \forall h\,. \tag{6.19}$$

Let $v_h(z) = \gamma_1((F_h)_z)$, $z \in \mathbb{R}$, and note that $v_h \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R})$ as (6.16) is in force. By the symmetry of F_h and Lemma 5.2, we have $3\lambda(F_h) \geq \gamma_2(F_h \Delta H)$. Moreover we have that $\lambda(F_h) \to 0$ by the Continuity Lemma 4.7, therefore $\gamma_2(F_h \Delta H) \to 0$ as $h \to \infty$. By Fubini's Theorem, possibly passing to a subsequence, this implies that

$$\lim_{h \to \infty} v_h(z) = \Phi^{-1}(\bar{y})\,, \qquad\qquad \text{for } \gamma_1\text{-a.e. } z \in \mathbb{R}. \tag{6.20}$$

For all such z, one has clearly

$$\liminf_{h \to \infty} p_h(z) \geq \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\bar{y}^2}{2}}$$

where we have set $p_h(z) = \mathscr{H}^0_\gamma(\partial^*(F_h)_z)$. As a consequence, also recalling Theorem 4.3 and Fatou's Lemma we can evaluate

$$\begin{aligned} P_\gamma(H) &= \lim_{h \to \infty} P_\gamma(F_h) = \lim_{h \to \infty} \int_{\mathbb{R}} \sqrt{|v_h'(z)|^2 + p_h(z)^2}\, d\gamma_1(z) \\ &= \lim_{h \to \infty} \int_{\mathbb{R}} \sqrt{|v_h'(z)|^2 + p_h(z)^2} - p_h(z)\, d\gamma_1(z) + \int_{\mathbb{R}} p_h(z)\, d\gamma_1(z) \end{aligned}$$

$$\geq \liminf_{h\to\infty} \int_{\mathbb{R}} \frac{|v_h'(z)|^2}{\sqrt{|v_h'(z)|^2+p_h(z)^2}+p_h(z)}\, d\gamma_1(z) + \liminf_{h\to\infty} \int_{\mathbb{R}} p_h(z)\, d\gamma_1(z)$$
$$\geq \liminf_{h\to\infty} \frac{1}{2} \int_{\mathbb{R}} \frac{|v_h'(z)|^2}{\sqrt{|v_h'(z)|^2+p_h(z)^2}}\, d\gamma_1(z) + P_\gamma(H)\,.$$

on the other hand, again by (4.6) and by Hölder inequality,

$$\begin{aligned}\left(\int_{\mathbb{R}} |v_h'(z)|\, d\gamma_1(x)\right)^2 &\leq \int_{\mathbb{R}} \sqrt{|v_h'(z)|^2+p_h(z)^2}\, d\gamma_1(z) \int_{\mathbb{R}} \frac{|v_h'(z)|^2}{\sqrt{|v_h'(z)|^2+p_h(z)^2}}\, d\gamma_1(z)\\ &= P_\gamma(F_h) \int_{\mathbb{R}} \frac{|v_h'(z)|^2}{\sqrt{|v_h'(z)|^2+p_h(z)^2}}\, d\gamma_1(z)\,.\end{aligned} \tag{6.21}$$

Therefore, since $P_\gamma(F_h) \to P_\gamma(H) > 0$, once we show that

$$\liminf_{h\to\infty} \int_{\mathbb{R}} |v_h'(z)| d\gamma_1(z) \geq \eta'(r) \tag{6.22}$$

for a suitably small $\eta'(r) > 0$, we will obtain a contradiction and then get (6.18). To this aim notice that, by (6.19), we have

$$|v_h(0) - \Phi(\bar y)| = |\Phi(y_h) - \Phi(\bar y)| \geq \eta''(r) \tag{6.23}$$

for a positive constant $\eta''(r)$ depending only on r (recall that $\bar y = \Phi^{-1}(r)$ depends only on r). Fix now $h \in \mathbb{N}$ and $t > 0$: one can evaluate

$$\begin{aligned}\sqrt{2\pi} \int_{\mathbb{R}} |v_h'(z)|\, d\gamma_1(z) &= \int_{\mathbb{R}} |v_h'(z)| e^{-\frac{z^2}{2}}\, dz \geq \int_0^t |v_h'(z)| e^{-\frac{z^2}{2}} dz \geq \left| \int_0^t v_h'(z) e^{-\frac{z^2}{2}}\, dz \right|\\ &= \left| \int_0^t (v_h(z) e^{-\frac{z^2}{2}})' + z v_h(z) e^{-\frac{z^2}{2}} dz \right|\\ &\geq |e^{-\frac{t^2}{2}} v_h(t) - v_h(0)| - \sqrt{2\pi} \int_0^t z v_h(z) d\gamma_1(z)\\ &\geq |e^{-\frac{t^2}{2}} v_h(t) - v_h(0)| - \sqrt{2\pi}\, t\, \gamma_2(F_h) = |e^{-\frac{t^2}{2}} v_h(t) - v_h(0)| - \sqrt{2\pi}\, t r\,.\end{aligned}$$

Finally, recall that for a.e. t one has $v_h(t) \to \Phi(\bar y)$ as noticed in (6.20). Hence, using (6.23), it suffices to take a very small such t and a very large h to get (6.22) with a positive $\eta'(r)$. As we remarked, this implies (6.18) and this step is concluded.

Step III. A comparison between E and E^s.

Let us consider the mutually disjoint subsets of $\mathbb{R}$ given by

$$A = \left\{ z \in \mathbb{R} : |v_E(z) - r| \geq \frac{r}{2} \right\},$$
$$B = \left\{ z \in \mathbb{R} \setminus A : \gamma_1(E_z \cap (-\infty, 0)) \geq \frac{r}{4} \right\},$$
$$C = \mathbb{R} \setminus (A \cup B)\,.$$

As E is close to $H = \{x \in \mathbb{R}^2 : y \geq \bar y\}$ and $0 < r \leq 1/2$, the sets A and B should have small Gaussian measure, i.e. we expect C to exhaust most of $\mathbb{R}$. We first prove that

$$\gamma_2\big((E\Delta H) \cap (A \times \mathbb{R})\big) \leq C(r) \sqrt{\delta(E)}\,, \tag{6.24}$$

$$\gamma_2\big((E\Delta H) \cap (C \times \mathbb{R})\big) \leq C(r) \sqrt{\delta(E)}\,. \tag{6.25}$$

Indeed, if $z \in A$, then $\gamma_1(E^s_z \Delta(\bar{y}, \infty)) \geq r/2$. By Step II,

$$\begin{aligned} C(r)\sqrt{\delta(E)} \geq C(r)\sqrt{\delta(E^s)} \geq \lambda(E^s) \geq \frac{\gamma_2(E^s \Delta H)}{3} \geq \frac{1}{3}\int_A \gamma_1(E^s_z \Delta(\bar{y}, \infty))\, d\gamma_1(z) \\ \geq \frac{r}{6}\,\gamma_1(A)\,, \end{aligned} \tag{6.26}$$

and (6.24) is immediately deduced since

$$\gamma_2\big((E\Delta H) \cap (A \times \mathbb{R})\big) \leq \gamma_2(A \times \mathbb{R}) = \gamma_1(A)\,.$$

Let us pass now to the set C. Concerning the one-dimensional asymmetry of a section E_z for $z \in C$, we recall that either

$$\lambda_1(E_z) = \gamma_1\big(E_z \Delta E^s_z\big) = \gamma_1\Big(E_z \Delta\big(\Phi^{-1}(\gamma_1(E_z)), +\infty\big)\Big)\,, \tag{6.27a}$$

or

$$\lambda_1(E_z) = \gamma_1\Big(E_z \Delta\big(-\infty, -\Phi^{-1}(\gamma_1(E_z))\big)\Big)\,, \tag{6.27b}$$

and we claim that this easily implies

$$\lambda_1(E_z) \geq \frac{1}{4}\,\gamma_1(E_z \Delta E^s_z)\,, \qquad\qquad \forall z \in C\,. \tag{6.28}$$

Indeed, (6.28) is trivially true if (6.27a) holds. And on the other hand, if (6.27b) holds, recalling that $r \leq 1/2$ one gets immediately

$$\lambda_1(E_z) \geq \gamma_1\big(E_z \cap (0, +\infty)\big) \geq \frac{1}{2}\,\gamma_1(E_z) \geq \frac{1}{4}\,\gamma_1(E_z \Delta E^s_z)\,.$$

We go further noticing that by Theorem 4.3 one has

$$\begin{aligned} \delta(E) &\geq P_\gamma(E) - P_\gamma(E^s) \geq P_\gamma(E; C \times \mathbb{R}) - P_\gamma(E^s; C \times \mathbb{R}) \\ &\geq \int_C \sqrt{p_E(z)^2 + v'_E(z)^2} - \sqrt{p_{E^s}(z)^2 + v'_E(z)^2}\, d\gamma_1(z) \\ &= \int_C \frac{p_E(z) - p_{E^s}(z)}{\sqrt{p_E(z)^2 + v'_E(z)^2} + \sqrt{p_{E^s}(z)^2 + v'_E(z)^2}}\,\big(p_E(z) + p_{E^s}(z)\big)\, d\gamma_1(z) \\ &\geq \int_C \frac{p_E(z) - p_{E^s}(z)}{\sqrt{p_E(z)^2 + v'_E(z)^2}}\,\big(p_{E^s}(z)\big)\, d\gamma_1(z) \geq \eta(r)\int_C \frac{p_E(z) - p_{E^s}(z)}{\sqrt{p_E(z)^2 + v'_E(z)^2}}\, d\gamma_1(z)\,, \end{aligned} \tag{6.29}$$

where in the last inequality we have used that for all $z \in C$ the volume of E^s_z is at least $r/2$ and then the perimeter $p_{E^s}(z)$ is greater than a positive constant $\eta(r)$ depending only on r. On the other hand, since the one-dimensional Theorem 1.2 ensures that the one-dimensional deficit of E_z verifies

$$\delta_1(E_z) = p_E(z) - p_{E^s}(z) \geq \frac{1}{C(r)}\,\lambda_1(E_z)\,,$$

again Hölder Theorem allows to derive from (6.29) that

$$\begin{aligned} \Big(\int_C \lambda_1(E_z)\, d\gamma_1(z)\Big)^2 \leq \Big(\int_C \sqrt{\lambda_1(E_z)}\, d\gamma_1(z)\Big)^2 &\leq C(r)\Big(\int_C \sqrt{p_E(z) - p_{E^s}(z)}\, d\gamma_1(z)\Big)^2 \\ &\leq C(r)\int_C \frac{p_E(z) - p_{E^s}(z)}{\sqrt{p_E(z)^2 + v'_E(z)^2}}\, d\gamma_1(z)\ \int_C \sqrt{p_E(z)^2 + v'_E(z)^2}\, d\gamma_1(z) \\ &\leq C(r)\,\delta(E) P_\gamma(E) \leq C(r)\,\delta(E)\,, \end{aligned}$$

using that $\delta(E)$ is bounded from above thus the same is true for $P_\gamma(E)$ (note that the constant $C(r)$ may change from line to line, still remaining a constant depending only on r). To conclude, we combine this last estimate with (6.28), getting

$$\begin{aligned} C(r)\sqrt{\delta(E)} &\geq \int_C \lambda_1(E_z)\, d\gamma_1(z) \geq \frac{1}{4}\int_C \gamma_1(E_z \Delta E^s_z)\, d\gamma_1(z) = \frac{1}{4}\,\gamma_2\Big(\big(E\Delta E^s\big)\cap\big(C\times\mathbb{R}\big)\Big) \\ &\geq \frac{1}{4}\left(\gamma_2\Big(\big(E\Delta H\big)\cap\big(C\times\mathbb{R}\big)\Big) - \gamma_2\big(E^s\Delta H\big)\right). \end{aligned}$$

Since by Step II

$$\gamma_2(E^s\Delta H) = \lambda(E^s) \leq C(r)\sqrt{\delta(E^s)} \leq C(r)\sqrt{\delta(E)}\,,$$

we finally deduce (6.25).

Step IV. A lower bound for the perimeter of E over B.

If $\gamma_1(B) = 0$ we have achieved the proof of the theorem. In the complementary case, let us consider

$$z_1 = \inf\{z > 0 : z \in B\} \geq 0\,.$$

As (6.16) is in force, $w(z) = \gamma_1(E_z \cap (-\infty, 0))$ is continuous and $w(z_1) = r/4$. If $\varepsilon(r)$ is small enough then

$$\left\{z < z_1 : w(z) < \frac{r}{8}\right\}$$

is non empty, for otherwise we would obtain a contradiction starting from

$$3\lambda(E) \geq \gamma_2(E\Delta H) \geq \gamma_2(E\cap[(-\infty, z_1)\times(-\infty,0)]) \geq \frac{r}{16}\,.$$

Correspondingly, let us consider

$$z_0 = \sup\left\{z < z_1 : w(z) < \frac{r}{8}\right\},$$

so that again by continuity it is $w(z_0) = r/8$, and set $I = (z_0, z_1)$. Let now $M(r)$ be a fixed large constant, to be determined later: the aim of this step is to show that, provided $\varepsilon(r)$ is suitably small,

$$P_\gamma(E; I\times\mathbb{R}) \geq M(r)\gamma_1(B)\,. \tag{6.30}$$

To this end, we first prove that

$$P_\gamma(E; I\times\mathbb{R}) \geq c(r)\min\left\{e^{-\frac{z_0^2}{2}}, e^{-\frac{z_1^2}{2}}\right\}, \tag{6.31}$$

for a positive constant $c(r)$. Indeed, let us consider the projection $q : \mathbb{R}^2 \to \mathbb{R}$ defined by $q((z,y)) = y$. If it happens that

$$q\left(\partial E \cap \left(I\times\left(-\frac{r}{16}, 0\right)\right)\right) = \left(-\frac{r}{16}, 0\right), \tag{6.32}$$

then we have

$$P_\gamma(E; I\times\mathbb{R}) \geq P_\gamma\Big(E; I\times\Big(-\frac{r}{16}, 0\Big)\Big) \geq \gamma_1\left(\left(-\frac{r}{16}, 0\right)\right)\frac{1}{\sqrt{2\pi}}\,\min\left\{e^{-\frac{z_0^2}{2}}, e^{-\frac{z_1^2}{2}}\right\},$$

and (6.31) follows. If (6.32) fails to be true, then there exists $y_0 \in (-(r/16), 0)$ such that $\partial E \cap (I \times \{y_0\}) = \emptyset$. In this case we set

$$\widetilde{E} = \begin{cases} E \cup [I \times (y_0, \infty)], & \text{if } I \times \{y_0\} \subseteq E, \\ E \setminus [I \times (y_0, \infty)], & \text{if } I \times \{y_0\} \subseteq \mathbb{R}^2 \setminus E. \end{cases}$$

In both cases $\partial^* \widetilde{E} \cap (I \times \mathbb{R}) \subseteq \partial E \cap (I \times \mathbb{R})$ and therefore, also thanks to Theorem 4.3,

$$P_\gamma(E; I \times \mathbb{R}) \geq P_\gamma(\widetilde{E}; I \times \mathbb{R}) \geq \int_{z_0}^{z_1} |v'_{\widetilde{E}}(z)| d\gamma_1(z) \geq \frac{r}{16} \frac{1}{\sqrt{2\pi}} \min\left\{ e^{-\frac{z_0^2}{2}}, e^{-\frac{z_1^2}{2}} \right\}$$

where we have used the fact that $|v_{\widetilde{E}}(z_1) - v_{\widetilde{E}}(z_0)| \geq r/16$. This establishes (6.31). Let us now remark that

$$3\lambda(E) \geq \gamma_2(E \Delta H) \geq \gamma_2\Big(E \cap \Big(I \times (-\infty, 0)\Big)\Big) \geq \frac{r}{8} \gamma_1(I), \tag{6.33}$$

as $w(z) \geq r/8$ for every $z \in I$. An identical argument shows that $3\lambda(E) \geq (r/4)\gamma_1(B)$, therefore we have

$$\max\{\gamma_1(I), \gamma_1(B)\} \leq \frac{24}{r} \varepsilon(r). \tag{6.34}$$

We are now in the position to derive (6.30) from (6.31). If

$$z_1 > \frac{M(r)}{c(r)} \sqrt{\frac{2}{\pi}}$$

then, up to take $\varepsilon(r)$ small enough, thanks to (6.34) we can assume that the minimum in (6.31) is achieved by $e^{-\frac{z_1^2}{2}}$. Correspondingly, by the definition of z_1 and by (3.2),

$$\gamma_1(B) \leq 2\gamma_1\big([z_1, +\infty)\big) = \sqrt{\frac{2}{\pi}} \int_{z_1}^{\infty} e^{-\frac{z^2}{2}} dz \leq \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{z_1^2}{2}}}{z_1} \leq \frac{P_\gamma(E; I \times \mathbb{R})}{M(r)},$$

as desired. If else

$$z_1 \leq \frac{M(r)}{c(r)} \sqrt{\frac{2}{\pi}},$$

then by (6.34) we know that, if $\varepsilon(r)$ is small enough, then both z_0 and z_1 are bounded from above by a constant depending only on r. Hence, also $\min\left\{ e^{-\frac{z_0^2}{2}}, e^{-\frac{z_1^2}{2}} \right\}$ is bounded from above by a strictly positive constant and finally by (6.31) $P_\gamma(E; I \times \mathbb{R}) \geq c'(r)$ for some positive $c'(r)$. Thanks to (6.34) again, by taking

$$\varepsilon(r) \leq \frac{r c'(r)}{24 M(r)}$$

we immediately deduce (6.30).

Step V. Conclusion.

We can now get to conclude the proof of the Theorem. First, recall that for any $z \in C$ one has $0 < r/2 \leq v_E(z) \leq 3r/2 < 1$. Hence,

$$\left| e^{-\frac{\Phi^{-1}(v_E(z))^2}{2}} - e^{-\frac{\Phi^{-1}(r)^2}{2}} \right| \leq C(r) |v_E(z) - r|.$$

Therefore,

$$\begin{aligned} P_\gamma\big(E;(C\setminus I)\times\mathbb{R}\big)-P_\gamma\big(H;(C\setminus I)\times\mathbb{R}\big) &\geq \int_{C\setminus I} p_E(z)-p_H(z)\,d\gamma_1(z)\\ &\geq \int_{C\setminus I} p_{E^s}(z)-p_H(z)\,d\gamma_1(z)=\frac{1}{\sqrt{2\pi}}\int_{C\setminus I} e^{-\frac{\Phi^{-1}(v_E(z))^2}{2}}-e^{-\frac{\Phi^{-1}(r)^2}{2}}\,d\gamma_1(z)\\ &\geq -C(r)\int_{C\setminus I}|v_E(z)-r|\,d\gamma_1(z)\geq -C(r)\gamma_2\big((E\Delta H)\cap[(C\setminus I)\times\mathbb{R}]\big)\\ &\geq -C(r)\sqrt{\delta(E)}\,, \end{aligned} \tag{6.35}$$

where in the last inequality we have made use of (6.25).

On the other hand we have also that

$$\begin{aligned} P_\gamma(H) &\leq \gamma_1(A)+\gamma_1(B)+\gamma_1(I)+P_\gamma(H;(C\setminus I)\times\mathbb{R})\\ &\leq C(r)\sqrt{\delta(E)}+\gamma_1(B)+\frac{24}{r}\lambda(E)+P_\gamma(H;(C\setminus I)\times\mathbb{R})\,, \end{aligned} \tag{6.36}$$

thanks to (6.26) and (6.33). Putting together (6.35) and (6.36) we obtain now

$$\begin{aligned} P_\gamma(H) &\leq C(r)\sqrt{\delta(E)}+\gamma_1(B)+\frac{24}{r}\lambda(E)+P_\gamma(E;(C\setminus I)\times\mathbb{R})\\ &\leq C(r)\sqrt{\delta(E)}+\gamma_1(B)+\frac{24}{r}\lambda(E)+P_\gamma(E)-P_\gamma(E;I\times\mathbb{R})\,, \end{aligned}$$

and since $P_\gamma(E)-P_\gamma(H)=\delta(E)\leq\sqrt{\delta(E)}$, being $\delta(E)\leq\delta(r)\leq 1$, we deduce

$$C(r)\sqrt{\delta(E)}\geq P_\gamma(E;I\times\mathbb{R})-\gamma_1(B)-\frac{24}{r}\lambda(E)\geq\big(M(r)-1\big)\gamma_1(B)-\frac{24}{r}\lambda(E)\,, \tag{6.37}$$

using also (6.30).

We are now close to the conclusion of the proof, as

$$\lambda(E)\leq\gamma_2(E\Delta H)=\gamma_2\big((E\Delta H)\cap(B\times\mathbb{R})\big)+\gamma_2\big((E\Delta H)\cap(A\cup C\times\mathbb{R})\big)\,.$$

Indeed, if $\gamma_2((E\Delta H)\cap(B\times\mathbb{R}))\leq\gamma_2((E\Delta H)\cap(A\cup C\times\mathbb{R}))$, then the thesis follows by (6.24) and (6.25). In the complementary case

$$\frac{\lambda(E)}{2}\leq\gamma_2\big((E\Delta H)\cap(B\times\mathbb{R})\big)\leq\gamma_1(B)\,,$$

therefore we deduce from (6.37) that

$$C(r)\sqrt{\delta(E)}\geq\left(\frac{M(r)-1}{2}-\frac{24}{r}\right)\lambda(E)\,,$$

and conclude the proof by choosing a suitably large value of $M(r)$ so that the term in the parentheses is positive. □

7. The n-dimensional case

In this section we show how the proof of Theorem 1.1 can be reduced to the two-dimensional case, that was discussed and solved in the previous section.

Our approach starts again with an Ehrhard symmetrization, different from the one considered before since we will now use $(n-1)$-dimensional instead of 1-dimensional sections. In dimension $n \geq 3$ we factorize $\mathbb{R}^n$ as $\mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$, correspondingly writing $x = (z, w, y)$, with $z \in \mathbb{R}^{n-2}$ and w, $y \in \mathbb{R}$. Given a measurable set $E \subseteq \mathbb{R}^n$ we consider its $(n-1)$-dimensional sections $E^y = \{(z, w) \in \mathbb{R}^{n-1} : (z, w, y) \in E\}$, $y \in \mathbb{R}$, and define a function $v_E : \mathbb{R} \to [0, 1]$ by

$$v_E(y) = \gamma_{n-1}(E^y)\,, \qquad\qquad y \in \mathbb{R}\,.$$

Thanks to Fubini Theorem, $v_E \in L^1_{\rm loc}(\mathbb{R})$. We denote by $\widehat{E}$ the Ehrhard symmetral of E defined as

$$\widehat{E} := \left\{x \in \mathbb{R}^n :\, w > \Phi^{-1}(v_E(y))\right\}, \tag{7.1}$$

see Figure 10. Note that $\widehat{E}$ is essentially a two-dimensional set, as $\widehat{E} = \mathbb{R}^{n-2} \times F$, where F is

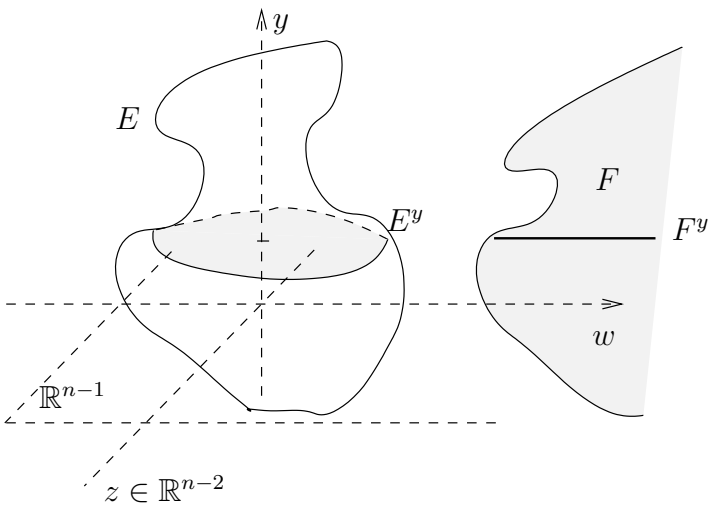

FIGURE 10. The set $\widehat{E} = \mathbb{R}^{n-2} \times F$. The one-dimensional section $F^y = \{w \in \mathbb{R} : (w, y) \in F\}$ satisfies $\gamma_1(F^y) = \gamma_{n-1}((\widehat{E})^y) = \gamma_{n-1}(E^y)$, and $\gamma_2(F) = \gamma_n(\widehat{E}) = \gamma_n(E)$.

defined as

$$F = \{(w, y) \in \mathbb{R}^2 : w > \Phi^{-1}(v_E(y))\}\,.$$

By Fubini Theorem, as $v_E = v_{\widehat{E}}$, we have $\gamma_n(E) = \gamma_n(\widehat{E}) = \gamma_2(F)$. Moreover, the Gaussian isoperimetric inequality in $\mathbb{R}^{n-1}$ ensures that, if we let

$$p_E(y) = \mathscr{H}^{n-2}_\gamma(\partial^* E^y)\,, \qquad\qquad y \in \mathbb{R}\,,$$

then $p_E(y) \geq p_{\widehat{E}}(y)$ for $\mathscr{L}^1$-a.e. $y \in \mathbb{R}$. The theory set forth in Section 4 for the Ehrhard symmetral E^s holds as well for $\widehat{E}$, and in particular the following Theorem holds true (cf. with Theorem 4.3 and Lemma 4.6).

Theorem 7.1. *Let E be a set of finite Gaussian perimeter in $\mathbb{R}^n$, $n \geq 3$. Then $v_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ and for any Borel set $B \subseteq \mathbb{R}$ one has*

$$P_\gamma\big(E;\mathbb{R}^{n-1}\times B\big) \geq P_\gamma\big(\widehat{E};\mathbb{R}^{n-1}\times B\big)\,;$$

in particular, $P_\gamma(E) \geq P_\gamma(\widehat{E})$. Moreover, if

$$\mathscr{H}^{n-1}_\gamma\Big(\big\{x\in\partial^* E:\,\nu^E_y=\pm 1\big\}\Big)=0\,, \tag{7.2}$$

then $v_E \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and

$$\begin{aligned} P_\gamma\big(E;\mathbb{R}^{n-1}\times B\big) &\geq \int_B \sqrt{p_E(y)^2+v_E'(y)^2}\,d\gamma_1(y)\\ &\geq \int_B \sqrt{p_{\widehat{E}}(y)^2+v_E'(y)^2}\,d\gamma_1(y) = P_\gamma\big(\widehat{E};\mathbb{R}^{n-1}\times B\big)\,. \end{aligned}$$

The proof of this result can be obtained on arguing exactly as in Section 4, with the obvious modifications due to the fact that we now work with $(n-1)$-dimensional sections in place of one-dimensional sections. In particular, one needs the Gaussian isoperimetric inequality in dimension $n-1 \geq 2$ (proved in Theorem 4.1), as well as the analogous version of Vol'pert Theorem 4.2 in codimension one (a proof of which can be found, for example, in [12]). In fact, the same strategy could be adopted to prove the fully analogous results about the behavior of the Gaussian perimeter under the Ehrhard symmetrizations of any dimension, making use of the Vol'pert Theorem of the corresponding codimension (again proved in [12]). We omit the details.

Before coming to the proof of Theorem 1.1 we need a last lemma, in which it is proved that n-symmetric sets are necessarily far from being optimal in the Gaussian isoperimetric inequality.

Lemma 7.2. *Let E be a set of finite Gaussian perimeter in $\mathbb{R}^n$, symmetric with respect to the coordinate hyperplanes, and such that $1/3 \leq \gamma_n(E) \leq 2/3$. Then one has $\delta(E) \geq c(n)$ for a suitable positive constant $c = c(n)$.*

Proof. Up to replace E with $\mathbb{R}^n \setminus E$ we may assume that $\gamma_n(E) \leq 1/2$. Let H be an half-space such that $\lambda(E) = \gamma_n(E\Delta G)$. By the symmetries of E, $x \in E$ if and only if $-x \in E$. Moreover, as $\gamma_n(H) = \gamma_n(E) \leq 1/2$, we also have that if $x \in H$ then $-x \notin H$. Thus

$$\lambda(E) = \gamma_n(E\Delta G) \geq \gamma_n(E) \geq \frac{1}{3}\,.$$

The thesis is then inferred by means of the Continuity Lemma 4.7. □

We can now conclude with the proof Theorem 1.1 in the $n-$dimensional case.

Proof of Theorem 1.1. In the case $n = 2$ the theorem has already been proved in Section 6. Therefore we assume that $n \geq 3$, and factorize $\mathbb{R}^n$ as $\mathbb{R}^{n-2} \times \mathbb{R}^2$ with $x = (z, w, y)$, $z \in \mathbb{R}^{n-2}$, $(w, y) \in \mathbb{R}^2$. By Theorem 5.1 we may assume without loss of generality that E is symmetric with respect to the first $(n-1)$-coordinates, namely $z_1, \ldots, z_{n-2}, w$. By a density argument, we can also assume that (7.2) holds for E in our coordinate system. By Lemma 5.2, if we set

$$H = \{x \in \mathbb{R}^n : y > \bar{y}\}\,, \qquad\qquad \bar{y} = \Phi^{-1}(r)\,,$$

then, up to reflect E with respect to the y-direction, we have

$$\lambda(E) \geq \frac{\gamma_n(E\Delta H)}{3}\,.$$

Eventually, we can directly consider as usual the case that $\delta(E) \leq \delta(r)$ for a suitably small constant $\delta(r) > 0$. Let us consider the Ehrhard symmetral of E, that is the set $\widehat{E} = \mathbb{R}^{n-2} \times F$ defined in (7.1). It is immediately seen that

$$\begin{aligned}\delta(E) &= P_\gamma(E) - I(\gamma_n(E)) \geq P_\gamma(\widehat{E}) - I(\gamma_n(\widehat{E})) = \mathscr{H}^{n-1}_\gamma(\partial^*(\mathbb{R}^{n-2}\times F)) - I(\gamma_n(\widehat{E}))\\ &= \mathscr{H}^1_\gamma(\partial^* F) - I(\gamma_2(F)) = \delta_2(F)\,,\end{aligned} \tag{7.3}$$

where, to avoid confusion, we have denoted by δ_2 the isoperimetric deficit in $\mathbb{R}^2$. Therefore, in order to conclude the proof of the theorem, it suffices to show that

$$\lambda(E) \leq C(n,r)\Big(\lambda_2(F) + \sqrt{\delta(E)}\Big)\,, \tag{7.4}$$

where λ_2 denotes the asymmetry in $\mathbb{R}^2$. Indeed, the thesis then easily follows on combining (7.3), (7.4) and the two-dimensional case of the theorem proved in Section 6.

In order to prove (7.4), we consider an half-plane H_F in $\mathbb{R}^2$ such that $\lambda_2(F) = \gamma_2(F\Delta H_F)$. Then we can write

$$H_F = \{(w,y) \in \mathbb{R}^2 : y > \bar{y} + \varepsilon(w - \widehat{w})\}\,, \tag{7.5}$$

for some $\varepsilon \in \mathbb{R}$, $\widehat{w} \in \mathbb{R}$. Note that, by the Continuity Lemma 4.7, up to choose $\delta(r)$ sufficiently small we may assume $|\varepsilon| \leq \varepsilon(r)$, where $\varepsilon(r)$ is a small constant to be specified later. The half-space $\widehat{H} = \mathbb{R}^{n-2} \times H_F$ is such that $\gamma_2(F\Delta H_F) = \gamma_n(\widehat{E}\Delta\widehat{H})$. Then (7.4) follows at once if

$$|\gamma_n(E \setminus H) - \gamma_n(\widehat{H} \setminus H)| \geq \frac{\lambda(E)}{4}\,,$$

for, in this case, thanks to the fact that $\gamma_n(E \setminus H) = \gamma_n(\widehat{E} \setminus H)$, we have

$$\frac{\lambda(E)}{4} \leq |\gamma_n(\widehat{E} \setminus H) - \gamma_n(\widehat{H} \setminus H)| \leq \gamma_n((\widehat{E}\Delta\widehat{H}) \setminus H) \leq \gamma_2(F\Delta H_F) = \lambda_2(F)\,.$$

Therefore we can assume from now on that

$$|\gamma_n(E \setminus H) - \gamma_n(\widehat{H} \setminus H)| < \frac{\lambda(E)}{4}\,. \tag{7.6}$$

In this case, let us consider $F^y = \{w \in \mathbb{R} : (w,y) \in F\}$ and $(H_F)^y = \{w \in \mathbb{R} : (w,y) \in H_F\}$, and define

$$\begin{aligned}A &:= \Big\{y \in \mathbb{R} :\ \frac{1}{3} \leq v_E(y) \leq \frac{2}{3}\Big\}\,,\\ B &= \Big\{y \in \mathbb{R} \setminus A : \frac{4}{9} \leq \gamma_1((H_F)^y) \leq \frac{5}{9}\Big\}\,.\end{aligned}$$

If $y \in B$, then $\gamma_1(F^y) = \gamma_{n-1}((\widehat{E})^y) = v_E(y) \in [0,1/3) \cup (2/3,1]$. Therefore

$$\lambda_2(F) = \gamma_2(F\Delta H_F) \geq \int_B \gamma_1(F^y\Delta(H_F)^y)d\gamma_1(y) \geq \frac{\gamma_1(B)}{9}\,. \tag{7.7}$$

From (7.5), up to take $\varepsilon(r)$ small enough, we have

$$\gamma_1(B) \geq \gamma_1\left(\left\{y \in \mathbb{R} : \frac{4}{9} \leq \gamma_1((H_F)^y) \leq \frac{5}{9}\right\}\right) - \gamma_1(A) \geq c(r)|\varepsilon| - \gamma_1(A)\,,$$

for a positive constant $c = c(r)$. Therefore, by (7.7),

$$C(r)\big(\gamma_1(A) + \lambda_2(F)\big) \geq |\varepsilon|\,. \tag{7.8}$$

We pass now to give an upper bound for $\gamma_1(A)$ in terms of $\sqrt{\delta(E)}$ and a lower bound for $|\varepsilon|$ in terms of $\lambda(E)$. Concerning $\gamma_1(A)$, let us notice that, for every $y \in A$, the set E^y is a $(n-1)$-symmetric subset of $\mathbb{R}^{n-1}$, with $1/3 \leq \gamma_{n-1}(E^y) \leq 2/3$: hence, by Lemma 7.2, the Gaussian isoperimetric deficit (relatively to $\mathbb{R}^{n-1}$) of each E^y is bounded from below by a positive constant $c(n)$, i.e.

$$p_E(y) - p_{\widehat{E}}(y) = \mathscr{H}^{n-2}_\gamma(\partial^* E^y) - \mathscr{H}^{n-2}_\gamma(\partial(\widehat{E}^y)) \geq c(n)$$

for any $y \in A$. As a consequence, using Theorem 7.1, we get

$$\begin{aligned}
\delta(E) &\geq P_\gamma(E) - P_\gamma(\widehat{E}) \geq P_\gamma(E;\mathbb{R}^{n-1}\times A) - P_\gamma(\widehat{E};\mathbb{R}^{n-1}\times A)\\
&\geq \int_A \sqrt{p_E(y)^2 + v'_E(y)^2} - \sqrt{p_{\widehat{E}}(y)^2 + v'_E(y)^2}\, d\gamma_1(y)\\
&= \int_A \frac{p_E(y)^2 - p_{\widehat{E}}(y)^2}{\sqrt{p_E(y)^2 + v'_E(y)^2} + \sqrt{p_{\widehat{E}}(y)^2 + v'_E(y)^2}}\, d\gamma_1(y)\\
&\geq c(n)\int_A \frac{p_{\widehat{E}}(y)}{\sqrt{p_E(y)^2 + v'_E(y)^2}}\, d\gamma_1(y) \geq \frac{1}{C(n)}\int_A \frac{1}{\sqrt{p_E(y)^2 + v'_E(y)^2}}\, d\gamma_1(y)\,,
\end{aligned} \tag{7.9}$$

where the last inequality follows by the lower bound on $p_{\widehat{E}}(y)$ which holds since $1/3 \leq v_E(y) \leq 2/3$ for all $y \in A$. Applying now Hölder inequality, Theorem 7.1, and the fact that $P_\gamma(E) = P_\gamma(H) + \delta(E) \leq C(r)$, we obtain

$$\begin{aligned}
\gamma_1(A) = \int_A 1\, d\gamma_1(y) &\leq \bigg(\int_A \sqrt{p_E(y)^2 + v'_E(y)^2}\, d\gamma_1(y)\bigg)^{1/2}\bigg(\int_A \frac{1}{\sqrt{p_E(y)^2 + v'_E(y)^2}}\, d\gamma_1(y)\bigg)^{1/2}\\
&\leq \sqrt{P_\gamma(E)}\bigg(\int_A \frac{1}{\sqrt{p_E(y)^2 + v'_E(y)^2}}\, d\gamma_1(y)\bigg)^{1/2}\\
&\leq C(r)\bigg(\int_A \frac{1}{\sqrt{p_E(y)^2 + v'_E(y)^2}}\, d\gamma_1(y)\bigg)^{1/2},
\end{aligned}$$

so (7.9) gives

$$\gamma_1(A) \leq C(n,r)\sqrt{\delta(E)}\,. \tag{7.10}$$

Concerning the lower bound on $|\varepsilon|$, let us notice that, thanks to (7.6),

$$\gamma_n(\widehat{H}\setminus H) \geq \gamma_n(E\setminus H) - \frac{\lambda(E)}{4} = \frac{\gamma_n(E\Delta H)}{2} - \frac{\lambda(E)}{4} \geq \frac{\lambda(E)}{4}\,.$$

Thus, calling $H_E = \{(w,y) : y \geq \bar{y}\}$,

$$\lambda(E) \leq 4\,\gamma_n(\widehat{H}\setminus H) = 4\,\gamma_2(H_F \setminus H_E) = 2\gamma_2(E_{|\varepsilon|}(|\widehat{w}|,\bar{y})\setminus H_E) = 2f(|\varepsilon|)\,,$$

where the set $E_{|\varepsilon|}(|\widehat{w}|,\bar{y})$ and the function $f(|\varepsilon|)$ are defined as in (6.1) and in (6.2). By (6.7), provided $\varepsilon(r)$ is small enough, we conclude that

$$\lambda(E) \leq 2f(|\varepsilon|) \leq C(r)|\varepsilon|\,. \tag{7.11}$$

On putting together (7.11), (7.8) and (7.10), we obtain (7.4) and so we conclude the proof of the theorem. □

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