Eulerian calculus for the displacement convexity in the Wasserstein distance

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Abstract

In this paper we give a new proof of the (strong) displacement convexity of a class of integral functionals defined on a compact Riemannian manifold satisfying a lower Ricci curvature bound. Our approach does not rely on existence and regularity results for optimal transport maps on Riemannian manifolds, but it is based on the Eulerian point of view recently introduced by OTTO-WESTDICKENBERG in [19] and on the metric characterization of the gradient flows generated by the functionals in the Wasserstein space.

Keywords: Gradient flows, displacement convexity, heat and porous medium equation, nonlinear diffusion, optimal transport, Kantorovich-Rubinstein-Wasserstein distance, Riemannian manifolds with a lower Ricci curvature bound.

1 Introduction

In this paper we give a new proof, based on a gradient flow approach and on the Eulerian point of view introduced by [19], of the so called "displacement convexity" for integral functionals as

$$\mathscr{E}(\mu) := \int_{\mathbb{M}} e(\rho)\,\mathrm{dV} + e'(\infty)\,\mu^\perp(\mathbb{M}), \quad \rho = \frac{\mathrm{d}\mu}{\mathrm{dV}}, \tag{1.1}$$

where μ is a Borel probability measure on a compact, connected Riemannian manifold without boundary $(\mathbb{M}, \mathsf{g})$, V is the volume measure on $\mathbb{M}$ induced by the metric tensor g, $\mu^\perp$ is the singular part of μ with respect to V, $e : [0, +\infty) \to \mathbb{R}$ is a smooth convex function satisfying the so called McCann conditions (see (1.7) below), and $e'(\infty) = \lim_{r\to+\infty} \frac{e(r)}{r}$. When e has a superlinear growth, $e'(\infty) = +\infty$ so that μ should be absolutely continuous with respect to V when $\mathscr{E}(\mu)$ is finite.

Displacement convexity for integral functionals. The notion of *displacement convexity* has been introduced by MCCANN [15] to study the behavior of integral functionals like (1.1) along optimal transportation paths, i.e. geodesics in the space of Borel probability measures $\mathscr{P}(\mathbb{M})$ endowed with the L^2-Kantorovich-Rubinstein-Wasserstein distance.

Recall that (the square of) this distance can be defined by the following optimal transport problem

$$\begin{aligned} W_2^2(\mu^0, \mu^1) := \min\Big\{ \int_{\mathbb{M}\times\mathbb{M}} \mathsf{d}^2(x, y)\,\mathrm{d}\boldsymbol{\sigma}(x, y) : \boldsymbol{\sigma} \in \mathscr{P}(\mathbb{M}\times\mathbb{M}),\\ \boldsymbol{\sigma}(\mathbb{M}\times B) = \mu^0(B),\ \boldsymbol{\sigma}(B\times\mathbb{M}) = \mu^1(B) \quad \forall B \text{ Borel set in } \mathbb{M}\Big\}, \end{aligned} \tag{1.2}$$

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for the cost function induced by the Riemannian distance d on the manifold $\mathbb{M}$. We keep the usual notation to denote by $\mathscr{P}_2(\mathbb{M})$ the metric space $(\mathscr{P}(\mathbb{M}), W_2)$, that is called Wasserstein space; being $\mathbb{M}$ compact, W_2 induces the topology of the weak convergence of probability measures (i.e., the weak* topology associated to the duality of $\mathscr{P}(\mathbb{M})$ with $C^0(\mathbb{M})$).

As in any metric space, (minimal, constant speed) *geodesics* can be defined as curves $\mu : s \in [0,1] \mapsto \mu^s \in \mathscr{P}_2(\mathbb{M})$ between μ^0 and μ^1 satisfying

$$W_2(\mu^r, \mu^s) = |s-r|\, W_2(\mu^0, \mu^1) \quad \forall\ 0 \le r \le s \le 1. \tag{1.3}$$

A functional $\mathscr{E} : \mathscr{P}(\mathbb{M}) \to (-\infty, +\infty]$ is then *(strongly) displacement convex* (or, more generally, *displacement* λ*-convex* for some $\lambda \in \mathbb{R}$) if, *for all* Wasserstein geodesics $\{\mu^s\}_{0\le s\le 1} \subset \mathscr{P}_2(\mathbb{M})$, we have

$$\mathscr{E}(\mu^s) \le (1-s)\mathscr{E}(\mu^0) + s\mathscr{E}(\mu^1) - \frac{\lambda}{2} s(1-s) W_2^2(\mu^0, \mu^1), \quad \forall s \in [0,1]. \tag{1.4}$$

A weaker notion is also often considered: one can ask that there exists *at least one* geodesic connecting μ^0 to μ^1 along which (1.4) holds.

The term "displacement convexity" arises from the strictly related concept of "displacement interpolation" introduced by [15] in the Euclidean case $\mathbb{M} = \mathbb{R}^d$; in a general metric setting, property (1.4) is simply called, as in the Riemannian case, "$\lambda-$geodesic convexity" (or "geodesic convexity" if $\lambda = 0$).
It is possible to show [4] that the measures μ^s can also be defined through the formula

$$\mu^s(B) := \boldsymbol{\sigma}\big(\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : (1-s)x + sy \in B\}\big), \quad \text{where } \boldsymbol{\sigma} \text{ is a minimizer of (1.2).} \tag{1.5}$$

A similar construction can also be performed in a Riemannian manifold [14, 20, 13]: the segments $s \mapsto (1-s)x + sy$ should be substituted by a Borel map $\boldsymbol{\gamma} : \mathbb{M} \times \mathbb{M} \to C^0([0,1]; \mathbb{M})$ that at each couple $(x,y) \in \mathbb{M} \times \mathbb{M}$ associate a (minimal, constant speed) geodesic $s \mapsto \boldsymbol{\gamma}^s(x,y)$ in $\mathbb{M}$ connecting x to y. We have the representation formula

$$\mu^s(B) := \boldsymbol{\sigma}\big(\{(x,y) \in \mathbb{M} \times \mathbb{M} : \boldsymbol{\gamma}^s(x,y) \in B\}\big), \quad \text{where } \boldsymbol{\sigma} \text{ is a minimizer of (1.2).} \tag{1.6}$$

After the pioneering paper [15], the notion of displacement convexity for integral functionals found applications in many different fields, as Functional inequalities [18, 2, 9], generation, contraction, and asymptotic properties of diffusion equations and Gradient flows [17, 1, 19, 4, 8, 5], Riemannian Geometry and synthetic study of Metric-Measure spaces [20, 14].

In the context of Riemannian manifolds it turns out that displacement λ-convexity of certain classes of entropy functionals is equivalent to a lower bound for the Ricci curvature of the manifold. The connection between displacement convexity and Ricci curvature, introduced by [18], was then further deeply studied by [18, 9, 10, 20]; the equivalence has been proved by Sturm and Von Renesse in [23], who considered the case in which the domain of the functional consists only of measures that are absolutely continuous with respect to the volume measure, and then completed by Lott and Villani [14] (with the remarks made in [12], where convexity in the strong form has been proved), who extended the previous results to the functionals defined by (1.1) on all $\mathscr{P}(\mathbb{M})$. We refer to the forthcoming monograph [22] for further references, details, and discussions.

The strategy followed by the authors of [9] (and by all the following contributions) in order to characterize the displacement convexity of entropy functionals relies on a characterization of optimal transportation and Wasserstein geodesics [16] and on a careful study of the Jacobian properties of the exponential function which are crucial to estimate the integral functionals along this class of curves. The lack of regularity of Wasserstein geodesics and the lack of global smoothness of the squared distance function d^2 on the manifold $\mathbb{M}$ (due to the existence of the cut-locus) require a careful use of non-smooth analysis arguments and non trivial approximation processes to extend the results to geodesics between arbitrary measures (see [14, 12]).

The main result is the following

Theorem 1.1 (I) *If $e \in C^\infty(0,+\infty)$ satisfies the* McCann *conditions:*

$$U(\rho) := \rho e'(\rho) - \big(e(\rho) - e(0_+)\big) \geq 0, \qquad \rho U'(\rho) - \Big(1 - \frac{1}{n}\Big) U(\rho) \geq 0, \quad n := \dim(\mathbb{M}) > 1 \quad (1.7)$$

and $\mathbb{M}$ *has* nonnegative *Ricci curvature, then the functional* $\mathscr{E}$ *defined by* (1.1) *is* (strongly) displacement convex.

(II) *If $\mathscr{E}$ is the relative entropy functional, corresponding to $e(\rho) = \rho\log\rho$ (which satisfies* (1.7) *in any dimension) in* (1.1), *and there exists $\lambda \in \mathbb{R}$ such that*

$$\mathrm{Ric}_x(\xi,\xi) \geq \lambda\langle\xi,\xi\rangle_{\mathrm{g}_x} \quad \forall x \in \mathbb{M}, \quad \forall \xi \in T_x\mathbb{M}, \quad (1.8)$$

then the functional $\mathscr{E}$ defined by (1.1) *is* (strongly) displacement λ-convex.

Remark 1.2 Besides the logarithmic entropy corresponding to $e(\rho) = \rho\log\rho$ (and $U(\rho) = \rho$), typical examples of functionals that satisfy properties (1.7) are

$$e(\rho) = \tfrac{1}{m-1}\rho^m, \quad U(\rho) = \rho^m, \quad m \geq 1 - \tfrac{1}{n}. \quad (1.9)$$

We recall that assumptions (1.7) imply the convexity of the function $\rho \mapsto e(\rho)$ (since the dimension n is greater than 1, they are in fact more restrictive).

Aim of the paper: an Eulerian approach to displacement convexity. In this paper we present an alternative proof of Theorem 1.1, which does not rely on the existence and smoothness of optimal transport maps and geodesics for the Wasserstein distance.

Our strategy can be described in three steps:

1. Following the approach suggested by Otto-Westdickenberg in [19], we work in the subspace $\mathscr{P}_2^{ar}(\mathbb{M})$ of measures with smooth and positive densities and we use the "Riemannian" formula for the Wasserstein distance, originally introduced in the Euclidean framework by Benamou-Brenier [6]: if $\mu^i = \rho^i\,\mathrm{V} \in \mathscr{P}_2^{ar}(\mathbb{M})$, $i = 0, 1$, then [19, Prop. 4.3]

$$W_2^2(\mu^0,\mu^1) = \inf_{\mathscr{C}(\mu^0,\mu^1)} \Big\{ \int_0^1 \int_M |\nabla\phi^s|^2 \rho^s \,\mathrm{dV}\,\mathrm{d}s \Big\} \qquad \forall\, \mu^0,\mu^1 \in \mathscr{P}_2^{ar}(\mathbb{M}) \quad (1.10)$$

   where

$$\begin{aligned}\mathscr{C}(\mu^0,\mu^1) = \Big\{ (\rho,\phi) : \rho \in C^\infty([0,1]\times\mathbb{M};\mathbb{R}_+), \quad \phi \in C^\infty([0,1]\times\mathbb{M}) \\ \partial_s\rho^s + \nabla\cdot(\rho^s\nabla\phi^s) = 0 \text{ in } (0,1)\times\mathbb{M}, \quad \mu^i = \rho^i\,\mathrm{V} \Big\}.\end{aligned} \quad (1.11)$$

   Even though the Wasserstein space can't be endowed with a smooth Riemannian structure, (1.11) still shows a "Riemannian" characterization of the Wasserstein distance on $\mathscr{P}_2^{ar}(\mathbb{M})$.

2. The second important fact, originally showed by the so-called "Otto calculus" in [17], is that the nonlinear diffusion equation

$$\partial_t\rho_t - \Delta_{\mathrm{g}}U(\rho_t) = 0 \quad \text{in } [0,+\infty)\times\mathbb{M}, \qquad \rho_{|t=0} = \rho_0, \quad (1.12)$$

   where $U : \mathbb{R}^+ \to \mathbb{R}$ is the function defined in (1.7) and Δ_{g} is the Laplace-Beltrami operator on $\mathbb{M}$, is the gradient flow of the functional (1.1) in $\mathscr{P}_2(\mathbb{M})$. Indeed, (1.12) corresponds to the heat equation if U is the logarithmic entropy and to the porous medium equation if U is defined by (1.9).

   Starting directly from (1.10) and owing to the fact that the flow generated by (1.12) preserves smooth and positive densities, when $\mathrm{Ric}(\mathbb{M}) \geq 0$ we shall show that the measures $\mu_t = \rho_t\mathrm{V} \in \mathscr{P}_2^{ar}(\mathbb{M})$ associated to the solutions of (1.12) also solve the Evolution Variational Inequality (E.V.I.)

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} W_2^2(\nu,\mu_t) \leq \mathscr{E}(\nu) - \mathscr{E}(\mu_t) \quad \forall\, t \geq 0,\ \nu \in \mathscr{P}_2^{ar}(\mathbb{M}), \quad (1.13)$$

which has been introduced in [4] as a purely metric characterization of the gradient flows of geodesically convex functionals in metric spaces (and in particular in $\mathscr{P}_2(\mathbb{R}^d)$); here

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\zeta(t) = \limsup_{h\downarrow 0} \frac{\zeta(t+h)-\zeta(t)}{h} \tag{1.14}$$

for every real function $\zeta : [0,+\infty) \to \mathbb{R}$.

When $\mathrm{Ric}\,(\mathbb{M}) \geq \lambda$ (a shorthand for (1.8)), we also show that the solutions of the heat equation satisfy the modified inequality

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\nu,\mu_t) + \frac{\lambda}{2}W_2^2(\nu,\mu_t) \leq \mathscr{E}(\nu) - \mathscr{E}(\mu_t) \quad \forall\, t \geq 0,\, \nu \in \mathscr{P}_2^{ar}(\mathbb{M}), \tag{1.15}$$

where $\mathscr{E}$ is the relative entropy functional whose integrand function is $e(\rho) = \rho\log\rho$. Note that (1.15) reduces to (1.13) when $\lambda = 0$. In order to prove (1.13) and (1.15), we propose an "Eulerian" strategy which could be adapted to more general situations.

3. The third crucial fact is the following: whenever a functional $\mathscr{E}$ satisfies (1.13) (or, more generally, (1.15)) for a given semigroup $\mathsf{S}_t : \mu_0 = \rho_0\mathrm{V} \mapsto \mu_t = \rho_t\mathrm{V}$ in $\mathscr{P}_2^{ar}(\mathbb{M})$, $\mathscr{E}$ is displacement convex (resp. displacement λ-convex). Thus the question of the behavior of $\mathscr{E}$ along geodesics can be reduced to a differential estimate of $\mathscr{E}$ along the smooth and positive solutions of its gradient flow.

Plan of the paper. In Section 2 we present the main ideas of our approach in the simplified (finite-dimensional and smooth) setting of geodesically convex functions on Riemannian manifolds. We think that these ideas are sufficiently general to be useful in other circumstances, at least for distances which admits a Riemannian characterization as (1.10), see e.g. [11, 7]

After a brief review of the definition of (gradient) λ-flows in arbitrary metric spaces (basically following the ideas of [4]), we present in Section 3 our first result, showing that the existence of a flow satisfying the E.V.I. (1.15) (even on a dense subset of initial data, such as $\mathscr{P}_2^{ar}(\mathbb{M})$) entails the (strong) displacement λ-convexity of the functional $\mathscr{E}$.

Following the strategy explained in the second section, in the last two sections we prove the differential estimates showing that (1.12) satisfies (1.13) (in Section 4) or, in the case of the Heat equation, (1.15) (in Section 5).

2 Gradient flows and geodesic convexity in a smooth setting

Contraction semigroups and action integrals. In order to explain the main point of our strategy, let us first consider the simple setting of a smooth function $F : \mathrm{X} \to \mathbb{R}$ on a complete Riemannian manifold X with metric $\langle\cdot,\cdot\rangle_g$, (squared) norm $|\xi|_g^2 = \langle\xi,\xi\rangle_g$, and the endowed Riemannian distance

$$d^2(u,v) := \min\Big\{\int_0^1 |\dot\gamma^s|_g^2\,\mathrm{d}s, \quad \gamma : [0,1] \to \mathrm{X},\ \gamma^0 = v,\ \gamma^1 = u\Big\}. \tag{2.1}$$

In a smooth setting, the geodesic λ-convexity of F can be expressed through the differential condition

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}F(\gamma^s) \geq \lambda\,|\dot\gamma^s|_g^2 \tag{2.2}$$

along any geodesic curve γ minimizing (2.1). As we discussed in the introduction, the direct computation of (2.2) could be difficult in a non-smooth, infinite dimensional setting; it is therefore important to find equivalent conditions which avoid twofold differentiation along geodesics. One possibility, suggested in [19], is to find equivalent conditions to geodesic λ-convexity in terms of the gradient flow generated by F.

Let us recall that the gradient flow of F is a continuous semigroup of (time-dependent) maps $\mathsf{S}_t : \mathrm{X} \to \mathrm{X}$, $t \in [0, +\infty)$, which at every initial datum u associate the curve $u_t := \mathsf{S}_t(u)$ solution of the differential equation

$$\dot{u}_t = -\nabla F(u_t) \quad \forall\, t \geq 0, \quad u_0 = u. \tag{2.3}$$

It is well known that, when F is geodesically λ-convex, S_t is λ-contracting, i.e.

$$d^2(\mathsf{S}_t(u), \mathsf{S}_t(v)) \leq e^{-2\lambda t} d^2(u, v) \qquad \forall\, u, v \in \mathrm{X}. \tag{2.4}$$

By the semigroup property, (2.4) is also equivalent to the differential inequality (see (1.14))

$$\frac{\mathrm{d}^+}{\mathrm{d}t} d^2(\mathsf{S}_t(u), \mathsf{S}_t(v))\Big|_{t=0} \leq -2\lambda\, d^2(u, v) \quad \forall\, u, v \in \mathrm{X}. \tag{2.5}$$

[19] reverts this argument and observes that it could be easier to directly prove (2.5) by a differential estimate involving only the action of the semigroup along smooth curves; as a byproduct, one should obtain the convexity of F. To this aim, they consider a smooth curve γ^s, $s \in [0, 1]$, connecting v to u, and the action integral $\mathscr{A}_t$ associated to its smooth perturbation

$$\gamma_t^s := \mathsf{S}_t(\gamma^s), \quad A_t^s := \left|\partial_s \gamma_t^s\right|_g^2, \quad \mathscr{A}_t := \int_0^1 A_t^s \,\mathrm{d}s, \tag{2.6}$$

where $\partial_s \gamma$ denotes the tangent vector in $T_\gamma \mathrm{X}$ obtained by differentiating w.r.t. s. Since, by the very definition of d,

$$d^2(\mathsf{S}_t(v), \mathsf{S}_t(u)) \leq \mathscr{A}_t \tag{2.7}$$

and for every $\varepsilon > 0$ one can always find a curve γ^s so that $\mathscr{A}_0 \leq d^2(u, v) + \varepsilon$ (in a smooth setting one can take $\varepsilon = 0$), (2.5) surely holds if one can prove that

$$\frac{\mathrm{d}^+}{\mathrm{d}t} \mathscr{A}_t\Big|_{t=0} \leq -2\lambda\, \mathscr{A}_0, \quad \text{or its pointwise version} \quad \frac{\partial^+}{\partial t}\Big|_{t=0} A_t^s \leq -2\lambda\, A_0^s. \tag{2.8}$$

Having obtained the contraction property from (2.8), it still remains open how to deduce that F is geodesically convex. Notice that along an arbitrary curve η^s

$$\frac{\partial}{\partial s} F(\eta^s) = \langle \nabla F(\eta^s), \partial_s \eta^s \rangle_g = -\langle \partial_r \mathsf{S}_r(\eta^s)\big|_{r=0}, \partial_s \eta^s \rangle_g; \tag{2.9}$$

applied to $\eta^s := \gamma_t^s$, (2.9) and the semigroup property $\mathsf{S}_r(\gamma_t^s) = \gamma_{t+r}^s$ yield

$$\frac{\partial}{\partial s} F(\gamma_t^s) = -\langle \partial_t \gamma_t^s, \partial_s \gamma_t^s \rangle_g. \tag{2.10}$$

In a smooth setting we can assume that γ^s is a minimal geodesic; operating a further differentiation with respect to s, we obtain

$$\frac{\partial^2}{\partial s^2} F(\gamma^s) \overset{(2.9)}{=} -\frac{\partial}{\partial s} \langle \partial_t \gamma_t^s, \partial_s \gamma_t^s \rangle_g \Big|_{t=0} = -\langle D_{\partial_s} \partial_t \gamma_t^s, \partial_s \gamma_t^s \rangle_g - \langle \partial_t \gamma_t^s, D_{\partial_s} \partial_s \gamma_t^s \rangle_g \Big|_{t=0} \tag{2.11}$$

$$= -\langle D_{\partial_s} \partial_t \gamma_t^s, \partial_s \gamma_t^s \rangle_g \Big|_{t=0} = -\langle D_{\partial_t} \partial_s \gamma_t^s, \partial_s \gamma_t^s \rangle_g \Big|_{t=0} = -\frac{1}{2} \frac{\partial}{\partial t} \langle \partial_s \gamma_t^s, \partial_s \gamma_t^s \rangle_g \Big|_{t=0}$$

$$\overset{(2.6)}{=} -\frac{1}{2} \frac{\partial}{\partial t}\Big|_{t=0} A_t^s \overset{(2.8)}{\geq} \lambda \left|\partial_s \gamma^s\right|_g^2, \tag{2.12}$$

where we used the standard properties of the covariant differentiations $D_{\partial_s}, D_{\partial_t}$ and, in (2.11), the fact that at $t = 0$ $D_{\partial_s} \partial_s \gamma_t^s = 0$, being $\gamma_t^s = \gamma^s$ a geodesic.

A metric derivation of convexity. Even if the previous differential argument shows that (2.8) implies geodesic λ-convexity, it still requires nice smooth properties on geodesics and covariant differentiation, which could be hard to extend to a non smooth setting.

This is not at all surprising, since the contraction property (2.5) and its action-differential characterization (2.8) do not carry all the information linking the semigroup S to F: in order to conclude the argument in (2.11) we had therefore to insert the information coming from (2.9).

To overcome these difficulties, we shall deal with a more precise metric characterization of S than (2.4). As it has been proposed and studied in [4], gradient flows of geodesically λ-convex functionals in "almost" Euclidean settings should satisfy a purely metric formulation in terms of the Evolution Variational Inequality

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}d^2(\mathsf{S}_t(u),v)+\frac{\lambda}{2}d^2(\mathsf{S}_t(u),v)+F(\mathsf{S}_t(u))\le F(v),\quad \forall\, v\in \mathrm{X},\ t>0. \tag{2.13}$$

It can be proved (see [5]) that (2.13) characterizes S and implies the contractivity property (2.4).

As we discussed before, here we invert the usual procedure (starting from a convex functional, construct its gradient flow) and we suppose that there exists a smooth flow S_t satisfying (2.13). The following result, whose proof will be postponed (in a more general form) to Theorem 3.2 in the next Section, shows that F is geodesically λ-convex.

Theorem 2.1 *Suppose that there exists a continuous semigroup of maps* $\mathsf{S}_t\in C^0(\mathrm{X};\mathrm{X})$*,* $t\ge 0$*, satisfying* (2.13)*. Then for every (minimal, constant speed) geodesic* $\gamma:[0,1]\to\mathrm{X}$

$$F(\gamma^s)\le(1-s)F(\gamma^0)+sF(\gamma^1)-\frac{\lambda}{2}s(1-s)d^2(\gamma^0,\gamma^1),\quad \forall\, s\in[0,1] \tag{2.14}$$

i.e. F *is (strongly) geodesically* λ*-convex.*

E.V.I. through action-differential estimates. Thanks to Theorem 2.1, it is possible to prove the geodesic λ-convexity of F by exhibiting a flow S satisfying the E.V.I. (2.13). According to the general strategy suggested by [19], we want to reduce (2.13) to a suitable family of differential inequalities satisfied by the action A^s_t of (2.6).

The idea here is to consider a different family of perturbations of a given smooth curve $\gamma:[0,1]\to\mathrm{X}$, still induced by the semigroup S. In fact, differently from the contraction estimate (2.5) where we are flowing both the points u,v through S_t, in (2.13) we want to keep the point $v:=\gamma^0$ fixed and to vary only $u:=\gamma^1$. If γ^s is a smooth curve connecting them, it is then natural to consider the new families (see Figure 1)

$$\tilde\gamma^s_t:=\mathsf{S}_{st}(\gamma^s)=\gamma^s_{st},\quad \tilde F^s_t:=F(\tilde\gamma^s_t)\qquad s\in[0,1],\ t\ge 0. \tag{2.15}$$

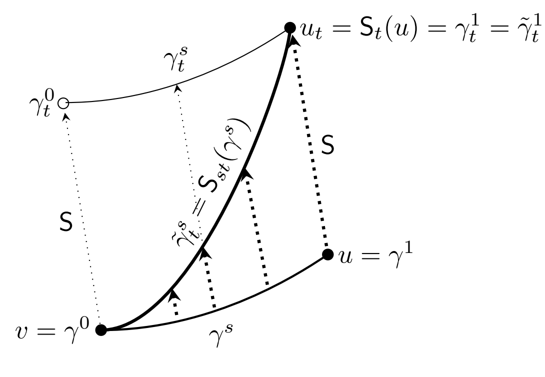

Figure 1: variation of the curve γ^s under the action of the semigroup S.

Notice that $\tilde\gamma^s_0 = \gamma^s$, $\tilde\gamma^0_t = \gamma^0 = v$, $\tilde\gamma^1_t = \mathsf{S}_t(\gamma^1) = \mathsf{S}_t(u)$. As before, we introduce the quantities

$$\tilde A^s_t := \big|\partial_s \tilde\gamma^s_t\big|^2_g, \quad \tilde{\mathscr{A}}_t := \int_0^1 \tilde A^s_t \,\mathrm{d}s. \tag{2.16}$$

Theorem 2.2 (A differential inequality linking action and flow) *Suppose that for every smooth curve $\gamma : [0,1] \to \mathrm{X}$ the quantities $\tilde A^s_t, \tilde F^s_t$ induced by the flow S through* (2.15),(2.16) *satisfy*

$$\frac{1}{2}\frac{\partial}{\partial t}\tilde A^s_t + \frac{\partial}{\partial s}\tilde F^s_t \le -\lambda\, s\, \tilde A^s_t, \quad \forall\, t \ge 0. \tag{2.17}$$

Then S satisfies (2.13), *it is the gradient flow of F, and F is geodesically λ-convex. Moreover, it is sufficient to check* (2.17) *at $t = 0$.*

Proof. Let us first observe that (2.17) yields, after an integration with respect to s in $[0,1]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{A}}_t + \tilde F^1_t - \tilde F^0_t \le -\lambda \int_0^1 s\tilde A^s_t \,\mathrm{d}s. \tag{2.18}$$

By the semigroup property, it is sufficient to prove (2.13) at $t = 0$. We choose a geodesic γ^s connecting v to u and we consider the curves given by (2.15). Since

$$d^2(v, \mathsf{S}_t(u)) \le \int_0^1 \tilde A^s_t \,\mathrm{d}s = \tilde{\mathscr{A}}_t, \quad d^2(v,u) = \int_0^1 \tilde A^s_0 \,\mathrm{d}s = \tilde{\mathscr{A}}_0, \quad \tilde F^1_t = F(\mathsf{S}_t(u)), \quad \tilde F^0_t = F(v), \tag{2.19}$$

by (2.18) at $t = 0$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} d^2(\mathsf{S}_t(u), v)\Big|_{t=0} + F(u) - F(v) \le -\lambda \int_0^1 s\, \tilde A^s_0 \,\mathrm{d}s = -\frac{\lambda}{2} d^2(u,v), \tag{2.20}$$

where in the last identity we used the fact that γ^s is a geodesic and therefore $\tilde A^s_0 = |\partial_s \gamma^s|^2_g$ is constant in $[0,1]$ and takes the value $d^2(\gamma^0,\gamma^1) = d^2(v,u)$.

Since $\tilde\gamma^s_{t_0+t} = \mathsf{S}_{st}\tilde\gamma^s_{t_0}$ by the semigroup property, if S satisfies (2.17) at the initial time $t = 0$ for an arbitrary smooth curve γ, then it also satisfies (2.17) for $t > 0$. □

Our last result provides a simple criterion to check (2.17):

Theorem 2.3 *Suppose that $\mathsf{S} : [0,+\infty) \times \mathrm{X} \to \mathrm{X}$ is the "differential" gradient flow of F satisfying* (2.9) *for any smooth curve γ^s, let $\gamma^s_t, \tilde\gamma^s_t, A^s_t, \tilde A^s_t, \tilde F^s_t$ be defined as in* (2.6), (2.15), *and* (2.16), *and let us set*

$$\tilde D^s_r := \frac{1}{2}\lim_{h\downarrow 0} h^{-1}\Big(\big|\partial_s \gamma^s_{sr+h}\big|^2_g - \big|\partial_s \gamma^s_{sr}\big|^2_g\Big). \tag{2.21}$$

Then

$$\frac{1}{2}\frac{\partial}{\partial t}\tilde A^s_t + \frac{\partial}{\partial s}\tilde F^s_t = s\tilde D^s_t. \tag{2.22}$$

Furthermore, if (2.8) *holds, then*

$$\tilde D^s_t \le -\lambda\, \tilde A^s_t \tag{2.23}$$

and (2.17) *holds, too, so that F is geodesically λ-convex, and S is also its "metric" gradient flow, characterized by the E.V.I.* (2.13).

Proof. Let us set

$$\tilde\gamma^s_{t,\tau} := \mathsf{S}_\tau \tilde\gamma^s_t = \gamma^s_{st+\tau}, \quad \tilde A^s_{t,\tau} := \big|\partial_s \tilde\gamma^s_{t,\tau}\big|^2_g, \tag{2.24}$$

so that

$$\tilde\gamma^s_{t+h} = \tilde\gamma^s_{t,sh}, \quad \partial_s \tilde\gamma^s_{t+h} = \partial_s \tilde\gamma^s_{t,\tau} + h\partial_\tau \tilde\gamma^s_{t,\tau}\Big|_{\tau=sh}, \quad \tilde D^s_t = \frac{1}{2}\frac{\partial}{\partial\tau}\tilde A^s_{t,\tau}\Big|_{\tau=0}. \tag{2.25}$$

Observe that the identity

$$|x+y|^2_g = 2\langle x+y, y\rangle_g + |x|^2_g - |y|^2_g, \qquad \forall\, x,y \in T_\gamma X \tag{2.26}$$

yields

$$
\begin{aligned}
\tilde{A}^s_{t+h} &= \big|\partial_s\tilde{\gamma}^s_{t+h}\big|^2_g \stackrel{(2.25)}{=} \big|\partial_s\tilde{\gamma}^s_{t,\tau} + h\partial_\tau\tilde{\gamma}^s_{t,\tau}\big|^2_g\Big|_{\tau=sh} \\
&\stackrel{(2.26)}{=} \Big[2h\langle\partial_s\tilde{\gamma}^s_{t,\tau} + h\partial_\tau\tilde{\gamma}^s_{t,\tau}, \partial_\tau\tilde{\gamma}^s_{t,\tau}\rangle + \big|\partial_s\tilde{\gamma}^s_{t,\tau}\big|^2_g - h^2\big|\partial_\tau\tilde{\gamma}^s_{t,\tau}\big|^2_g\Big]_{\tau=sh} \\
&= 2h\,\langle\partial_s\tilde{\gamma}^s_{t+h}, \partial_\theta\mathsf{S}_\theta(\tilde{\gamma}^s_{t+h}))\rangle\Big|_{\theta=0} + \tilde{A}^s_{t,sh} - o(h) \stackrel{(2.9)}{=} -2h\,\frac{\partial}{\partial s}F(\tilde{\gamma}^s_{t+h}) + \tilde{A}^s_{t,sh} - o(h).
\end{aligned}
$$

We thus get

$$
\frac{1}{2h}\big(\tilde{A}^s_{t+h} - \tilde{A}^s_t\big) + \frac{\partial}{\partial s}F(\tilde{\gamma}^s_{t+h}) = \frac{1}{2h}\big(\tilde{A}^s_{t,sh} - \tilde{A}^s_t\big) - o(1), \tag{2.27}
$$

so that, passing to the limit as $h \downarrow 0$ we get (2.22). □

Remark 2.4 Notice that the remainder term $o(1)$ in (2.27) is non-negative, so it can be simply neglected, if one is just interested to the inequality (2.17).

3 Gradient flows and geodesic convexity in a metric setting

In this section we will briefly recall some basic definitions and properties of gradient flows in a metric setting and we will prove Theorem 2.1 in a slightly more general framework.

Let (X, d) be a metric space (not necessarily complete) and let $F : \mathrm{X} \to (-\infty, +\infty]$ be a lower semicontinuous functional, whose proper domain $D(F) := \big\{w \in \mathrm{X} : F(w) < +\infty\big\}$ is dense in X (otherwise we can always restrict all the next statements to the closure of $D(F)$ in X). We also assume that F is bounded from below, i.e. $F_{\inf} := \inf_{u\in\mathrm{X}} F(u) > -\infty$.

A C^0-semigroup S in $C^0(\mathrm{X};\mathrm{X})$ is a family S_t, $t \ge 0$, of continuous maps in X such that

$$
\mathsf{S}_{t+h}(u) = \mathsf{S}_h\big(\mathsf{S}_t(u)\big), \quad \lim_{t\downarrow 0}\mathsf{S}_t(u) = \mathsf{S}_0(u) = u \quad \forall\, u \in \mathrm{X},\ t, h \ge 0. \tag{3.1}
$$

Given a real number $\lambda \in \mathbb{R}$, we say that S is the λ-(gradient) flow of F if it satisfies

$$
\mathsf{S}_t(\mathrm{X}) \subset D(F) \text{ for every } t > 0; \tag{3.2a}
$$

$$
\text{the map } t \mapsto F(\mathsf{S}_t(u)) \text{ is not increasing in } (0, +\infty); \tag{3.2b}
$$

$$
\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}d^2(\mathsf{S}_t(u), v) + \frac{\lambda}{2}d^2(\mathsf{S}_t(u), v) + F(\mathsf{S}_t(u)) \le F(v), \quad \forall\, u \in \mathrm{X},\, v \in D(F),\, t \ge 0. \tag{3.2c}
$$

Clearly, if S is a λ-flow for F, then it is also a λ'-flow for every $\lambda' \le \lambda$. The next proposition collects some useful properties of λ-flows.

Proposition 3.1 (Integral characterization of flows and contraction) *A C^0-semigroup S satisfies (3.2a, b, c) if and only if it satisfies the following integrated form*

$$
\frac{e^{\lambda(t_1-t_0)}}{2}d^2(\mathsf{S}_{t_1}(u), v) - \frac{1}{2}d^2(\mathsf{S}_{t_0}(u), v) \le \mathsf{E}_\lambda(t_1 - t_0)\Big(F(v) - F(\mathsf{S}_{t_1}(u))\Big) \quad \forall\, 0 \le t_0 < t_1, \tag{3.3}
$$

for every $u \in \mathrm{X},\ v \in D(F)$, where $\mathsf{E}_\lambda(t) := \int_0^t e^{\lambda r}\,\mathrm{d}r = \begin{cases} \frac{e^{\lambda t}-1}{\lambda} & \text{if } \lambda \ne 0, \\ t & \text{if } \lambda = 0. \end{cases}$

In particular S satisfies the uniform regularization bound

$$
F(\mathsf{S}_t(u)) \le F(v) + \frac{1}{2\,\mathsf{E}_\lambda(t)}d^2(u, v) \quad \forall\, u \in \mathrm{X},\ v \in D(F),\ t > 0, \tag{3.4}
$$

the uniform continuity estimate

$$
d^2(\mathsf{S}_{t_1}(u), \mathsf{S}_{t_0}(u)) \le 2\mathsf{E}_{-\lambda}(t_1 - t_0)\Big(F(\mathsf{S}_{t_0}u) - F_{\inf}\Big) \quad \forall\, u \in D(F),\ 0 \le t_0 \le t_1, \tag{3.5}
$$

and the λ-contraction property, i.e.

$$
d(\mathsf{S}_t(u), \mathsf{S}_t(v)) \le e^{-\lambda t}d(u, v) \quad \forall\, u, v \in \mathrm{X},\ t \ge 0. \tag{3.6}
$$

Proof. Clearly (3.3) yields (3.2a), being $D(F) \neq \emptyset$; (3.2b) and (3.5) follow by taking $v := \mathsf{S}_{t_0}(u)$ and (3.2c) can be proved by dividing both sides of (3.3) by $t_1 - t_0$ and passing to the limit as $t_1 \downarrow t_0$. In order to prove the converse implication, let us first observe that for a continuous real function $\zeta : [0, +\infty) \to \mathbb{R}$

$$\liminf_{h \downarrow 0} \frac{\zeta(t+h) - \zeta(t)}{h} \leq 0 \quad \forall t > 0 \qquad \Longrightarrow \qquad \zeta \text{ is not increasing.} \tag{3.7}$$

In fact, if $0 \leq t_0 < t_0 + \tau$ existed with $\delta := \tau^{-1}\big(\zeta(t_0 + \tau) - \zeta(t_0)\big) > 0$, then a minimum point $\bar t \in [t_0, t_0 + \tau)$ of $t \mapsto \zeta(t) - \zeta(t_0) - \delta(t - t_0)$ would satisfy

$$\liminf_{h \downarrow 0} \frac{\zeta(\bar t + h) - \zeta(\bar t)}{h} - \delta \geq 0, \quad \text{which contradicts (3.7).}$$

(3.3) then follows by (3.2c), after a multiplication by $e^{\lambda t}$ and choosing

$$\zeta(t) := \frac{e^{\lambda t}}{2} d^2(\mathsf{S}_t(u), v) + \int_{\bar t}^{t} e^{\lambda r}\big(F(\mathsf{S}_r(u)) - F(v)\big)\,\mathrm{d}r, \quad \bar t > 0,$$

and recalling the monotonicity property (3.2b). A similar argument shows that

$$\frac{1}{2} d^2(\mathsf{S}_{t_1}(u), v) - \frac{1}{2} d^2(\mathsf{S}_{t_0}(u), v) + \frac{\lambda}{2} \int_{t_0}^{t_1} d^2(\mathsf{S}_r(u), v)\,\mathrm{d}r \leq (t_1 - t_0)\Big(F(v) - F(\mathsf{S}_{t_1}(u))\Big), \tag{3.8}$$

for every $0 \leq t_0 < t_1$, $u \in \mathrm{X}$, and $v \in D(F)$. In order to prove the λ-contracting property, we apply (3.8) obtaining

$$\begin{aligned} &d^2(\mathsf{S}_h(u), \mathsf{S}_h(v)) - d^2(u, v) = d^2(\mathsf{S}_h(u), \mathsf{S}_h(v)) - d^2(\mathsf{S}_h(u), v) + d^2(\mathsf{S}_h(u), v) - d^2(u, v) \\ &\leq -\lambda \int_0^h \Big(d^2(\mathsf{S}_h(u), \mathsf{S}_r(v)) + d^2(\mathsf{S}_r(u), v)\Big)\,\mathrm{d}r + 2h\Big(F(v) - F(\mathsf{S}_h(v))\Big). \end{aligned}$$

We divide this inequality by h and we pass to the limit as $h \downarrow 0$; the continuity of S_t, the lower semicontinuity of F, and the semigroup property of S yield

$$\frac{\mathrm{d}^+}{\mathrm{d}t} d^2(\mathsf{S}_t(u), \mathsf{S}_t(v)) \leq -2\lambda\, d^2(u, v) \qquad \forall\, u, v \in \mathrm{X},\ t > 0, \tag{3.9}$$

which yields (3.6) thanks to (3.7). □

We can now prove the main result of this section: if a functional F admits a λ-flow, then F is geodesically λ-convex.

Theorem 3.2 (Geodesic convexity via E.V.I.) *Let us suppose that S is a λ-flow for the functional F, according to (3.2a,b,c), and let $\gamma : [0,1] \to \mathrm{X}$ be a Lipschitz curve satisfying*

$$d(\gamma^r, \gamma^s) \leq L\,|r - s|, \quad L^2 \leq d^2(\gamma^0, \gamma^1) + \varepsilon^2 \quad \forall\, r, s \in [0,1], \tag{3.10}$$

for some constant $\varepsilon \geq 0$. Then for every $t > 0$ and $s \in [0,1]$

$$F(\mathsf{S}_t(\gamma^s)) \leq (1-s)F(\gamma^0) + sF(\gamma^1) - \frac{\lambda}{2} s(1-s) d^2(\gamma^0, \gamma^1) + \frac{\varepsilon^2}{2\mathsf{E}_\lambda(t)} s(1-s). \tag{3.11}$$

In particular, when γ is a geodesic (i.e. γ satisfies (3.10) with $L = d(\gamma^0, \gamma^1)$, $\varepsilon = 0$), we have

$$F(\gamma^s) \leq (1-s)F(\gamma^0) + sF(\gamma^1) - \frac{\lambda}{2} s(1-s) d^2(\gamma^0, \gamma^1), \tag{3.12}$$

i.e. F is (strongly) geodesically λ-convex.

Proof. Let γ be satisfying (3.10) and let us set $\gamma_t^s := \mathsf{S}_t(\gamma^s)$. Choosing $t_0 = 0$, $t_1 = t$, $u := \gamma^s$, and taking a convex combination of (3.3) written for $v := \gamma^0$, and $v := \gamma^1$, we get

$$\frac{e^{\lambda t}}{2}\Big((1-s)\,d^2(\gamma_t^s,\gamma^0) + s\,d^2(\gamma_t^s,\gamma^1)\Big) - \frac{1}{2}\Big((1-s)\,d^2(\gamma^s,\gamma^0) + s\,d^2(\gamma^s,\gamma^1)\Big) \tag{3.13}$$

$$\leq \mathsf{E}_\lambda(t)\Big((1-s)F(\gamma^0) + sF(\gamma^1) - F(\gamma_t^s)\Big). \tag{3.14}$$

We now observe that the elementary inequality

$$(1-s)a^2 + sb^2 \geq s(1-s)(a+b)^2 \quad \forall\, a,b \in \mathbb{R}, \quad s \in [0,1], \tag{3.15}$$

and the triangular inequality yield

$$(1-s)d^2(\gamma_t^s,\gamma^0) + sd^2(\gamma_t^s,\gamma^1) \overset{(3.15)}{\geq} s(1-s)\Big(d(\gamma_t^s,\gamma^0) + d(\gamma_t^s,\gamma^1)\Big)^2 \geq s(1-s)d(\gamma^0,\gamma^1)^2. \tag{3.16}$$

On the other hand, (3.10) yields

$$(1-s)\,d^2(\gamma^s,\gamma^0) + s\,d^2(\gamma^s,\gamma^1) \leq L^2 s(1-s). \tag{3.17}$$

Inserting (3.17) and (3.16) in (3.14) we obtain

$$\frac{e^{\lambda t}-1}{2}s(1-s)d^2(\gamma^0,\gamma^1) - \frac{\varepsilon^2}{2}s(1-s) \leq \mathsf{E}_\lambda(t)\Big((1-s)F(\gamma^0) + sF(\gamma^1) - F(\gamma_t^s)\Big). \tag{3.18}$$

Dividing then both sides of (3.18) by $\mathsf{E}_\lambda(t)$ we get (3.11); when $\varepsilon = 0$ we can pass to the limit as $t \downarrow 0$ obtaining (3.12). □

We conclude this section by considering the case when the flow S is only defined on a *dense* subset X_0 of $D(F)$. In order to prove the geodesic convexity of F in X by Theorem 3.2 we first have to extend S to the whole space X. This can be achieved by a density argument, if X is complete and the lower semicontinuous functional F satisfies the following approximation property:

$$\forall\, u \in \mathrm{X} \quad \exists\, u_n \in \mathrm{X}_0\colon \qquad \lim_{n\to\infty} d(u_n,u) = 0, \quad \lim_{n\to\infty} F(u_n) = F(u). \tag{3.19}$$

We state the precise extension result in the next theorem.

Theorem 3.3 *Suppose that the functional F and the subset $\mathrm{X}_0 \subset D(F)$ satisfy (3.19) and let S be a λ-flow for F in X_0. If X is complete, S can be extended to a unique λ-flow $\bar{\mathsf{S}}$ in X and therefore F is (strongly) geodesically λ-convex in X.*

Proof. Given $u \in \mathrm{X}$ and a sequence $u_n \in \mathrm{X}_0$ as in (3.19), we can define

$$\bar{\mathsf{S}}_t(u) := \lim_{n\to\infty} \mathsf{S}_t(u_n) \quad \forall\, t > 0, \tag{3.20}$$

where it is clear that the limit in (3.20) exists (being X complete and S_t Lipschitz by (3.6)) and does not depend on the particular sequence u_n we used to approximate u. Moreover $\bar{\mathsf{S}}_t$ is a semigroup and satisfies the estimate (3.5) and the λ-contracting property (3.6); being $D(F)$ dense in X, it is not difficult to combine (3.5), (3.6) and (3.19) to show that $\lim_{t\downarrow 0} \mathsf{S}_t(u) = u$ for every $u \in \mathrm{X}$.

In order to prove that $\bar{\mathsf{S}}$ is still a λ-flow for F in X we have to check (3.3) in X: we fix $v \in D(F)$ and a sequence $v_n \in \mathrm{X}_0$ converging to v with $F(v_n) \to F(v)$ and we pass to the limit as $s \to \infty$ in the inequalities

$$\frac{e^{\lambda(t_1-t_0)}}{2}d^2(\mathsf{S}_{t_1}(u_n),v_n) - \frac{1}{2}d^2(\mathsf{S}_{t_0}(u_n),v_n) \leq \mathsf{E}_\lambda(t_1-t_0)\big(F(v_n) - F(\mathsf{S}_{t_1}(u_n)\big), \tag{3.21}$$

using the lower semicontinuity of F. □

4 Nonlinear diffusion equations as gradient flows of entropy functionals in $\mathscr{P}_2(\mathbb{M})$

We apply the strategy described in the Section 2 to prove the geodesic convexity of the integral functional (1.1) in the case of a Riemannian manifold of nonnegative Ricci curvature. We therefore exhibit a smooth flow (induced by the nonlinear diffusion equation (1.12) on the dense subset $\mathscr{P}_2^{ar}(\mathbb{M})$) which satisfies the Evolution Variational Inequality (1.13).

Before stating the main theorem of this section let us recall a fundamental result on this kind of evolution equations, that can be found in [21, 19]:

Theorem 4.1 (Classical solutions of nonlinear diffusion equations) *Let $e \in C^\infty(\mathbb{R}^+)$ and U be functions that satisfy the assumptions (1.7) of Theorem 1.1. For every $\rho_0 \in C^\infty(\mathbb{M})$ with $\rho_0 > 0$, there exists a unique smooth positive solution $\rho \in C^\infty([0,+\infty) \times \mathrm{X})$ to the Cauchy problem*

$$\partial_t \rho_t = \Delta_{\mathrm{g}} U(\rho_t), \qquad \rho_{|t=0} = \lim_{t \downarrow 0} \rho_t = \rho_0. \tag{4.1}$$

Moreover, given a one parameter family of positive initial data $s \mapsto \rho_0^s \in C^\infty([0,1] \times \mathbb{M})$, the corresponding solutions ρ_t^s of the equation (4.1) depend smoothly on s,t.

For every $\mu_0 = \rho_0 \mathrm{V} \in \mathscr{P}_2^{ar}(\mathbb{M})$ we denote by $\mathsf{S}_t(\mu_0) \in \mathscr{P}_2^{ar}(\mathbb{M})$ the measure $\mu_t = \rho_t \mathrm{V}$.

The main result that we show in this section is the following:

Theorem 4.2 *Let $e \in C^\infty(\mathbb{R}^+)$ and U be functions that satisfy the assumptions (1.7) of Theorem 1.1 and let us suppose that*

$$\mathrm{Ric}(x) \geq 0 \quad \forall\, x \in \mathbb{M}. \tag{4.2}$$

The semigroup S induced by (4.1) in $\mathscr{P}_2^{ar}(\mathbb{M})$ is a 0-flow in $\mathscr{P}_2^{ar}(\mathbb{M})$ for the functional

$$\mathscr{E}(\mu) = \int_{\mathbb{M}} e(\rho)\, \mathrm{dV}, \quad \forall\, \mu = \rho \mathrm{V} \in \mathscr{P}_2^{ar}(\mathbb{M}). \tag{4.3}$$

In particular, for every $\mu_0 = \rho_0 \mathrm{V}, \nu \in \mathscr{P}_2^{ar}(\mathbb{M})$, the measures $\mu_t = \mathsf{S}_t(\mu_0) = \rho_t \mathrm{V} \in \mathscr{P}_2^{ar}(\mathbb{M})$ solving (4.1) satisfy the E.V.I.

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} W_2^2(\nu, \mu_t) \leq \mathscr{E}(\nu) - \mathscr{E}(\mu_t) \quad \forall\, t \in [0,+\infty). \tag{4.4}$$

In order to prove Theorem 4.2, thanks to the "Riemannian-like" characterization of the Wasserstein distance provided by (1.10), we can follow the strategy presented in Section 2, in particular we want to prove the differential inequality of Theorem 2.2. Following OTTO's formalism, we collect in the next table the formal correspondences between the various objects:

X, Riemannian manifold, with distance d	$\mathscr{P}_2^{ar}(\mathbb{M})$ with distance W_2
a smooth curve γ^s in X	a smooth family $\mu^s = \rho^s \mathrm{V} \in \mathscr{P}_2^{ar}(\mathbb{M})$
the tangent vector $\partial_s \gamma^s$ in $T_{\gamma^s}\mathrm{X}$	the vector field $\nabla \phi^s$ where $-\nabla \cdot (\rho^s \nabla \phi^s) = \frac{\partial}{\partial s} \rho^s$
$\left\|\partial_s \gamma^s\right\|_g^2$	$\int_{\mathbb{M}} \left\|\nabla \phi^s(x)\right\|_{\mathrm{g}}^2 \rho^s(x)\, \mathrm{dV}(x)$
$\gamma_t^s := \mathsf{S}_t(\gamma^s)$, $\tilde\gamma_t^s := \gamma_{st}^s = \mathsf{S}_{st}(\gamma^s)$	$\mu_t^s = \rho_t^s \mathrm{V} := \mathsf{S}_t(\mu^s)$, $\tilde\mu_t^s = \tilde\rho_t^s \mathrm{V} := \mu_{st}^s = \mathsf{S}_{st}(\mu^s)$
$\tilde A_t^s = \left\|\partial_s \tilde\gamma_t^s\right\|_g^2$	$\int_{\mathbb{M}} \left\|\nabla \tilde\phi_t^s(x)\right\|_{\mathrm{g}}^2 \tilde\rho_t^s(x)\, \mathrm{dV}(x)$
$F(\gamma^s)$	$\mathscr{E}(\mu^s) = \int_{\mathbb{M}} e(\rho^s)\, \mathrm{dV}$
$\left(\partial_\theta \mathsf{S}_\theta \gamma^s\right)_{\|\theta=0} = -\nabla F(\gamma^s)$	$-\nabla U(\rho^s)/\rho^s = -\nabla e'(\rho^s)$.

The core of the proof of Theorem 4.2 lies in the following lemma:

Lemma 4.3 *Let* $\mu^s = \rho^s \mathrm{V}$, $s \in [0,1]$, *be a smooth family of measures in* $\mathscr{P}_2^{ar}(\mathbb{M})$ *and let* $\tilde\mu^s_t = \tilde\rho^s_t \mathrm{V} = \mathsf{S}_{st}(\mu^s)$ *be obtained by flowing* ρ^s *along the flow* (4.1), *i.e.* $\tilde\rho^s_t = \rho^s_{st}$ *where* ρ^s_t *satisfies*

$$\frac{\partial}{\partial t}\rho^s_t - \Delta_{\mathrm{g}}\, U(\rho^s_t) = 0 \ \ in\ \mathbb{M}, \quad \forall\, s\in[0,1],\ t>0; \qquad \rho^s_{t=0} = \rho^s. \tag{4.5}$$

Let $\tilde\phi^s_t \in C^\infty([0,1]\times[0,+\infty)\times\mathbb{M})$ *be the functions defined by the equation*

$$-\nabla\cdot(\tilde\rho^s_t \nabla\tilde\phi^s_t) = \partial_s \tilde\rho^s_t \quad\ in\ \mathbb{M}, \qquad \int_{\mathbb{M}} \tilde\phi^s_t(x)\,\mathrm{dV}(x) = 0 \quad \forall\, s\in[0,1],\ t\in[0,+\infty), \tag{4.6}$$

and let us set

$$\begin{aligned}\tilde A^s_t &:= \int_{\mathbb{M}} |\nabla\tilde\phi^s_t(x)|^2_{\mathrm{g}}\,\tilde\rho^s_t(x)\,\mathrm{dV}(x),\\ \tilde D^s_t &:= -\int_{\mathbb{M}}\Big[\Big(|\mathrm{Hess}\,\tilde\phi^s_t|^2_{\mathrm{g}} + \mathrm{Ric}\,(\nabla\tilde\phi^s_t,\nabla\tilde\phi^s_t)\Big)\,U\,(\tilde\rho^s_t) + (\Delta_{\mathrm{g}}\,\tilde\phi^s_t)^2\,\big(\tilde\rho^s_t U'(\tilde\rho^s_t) - U\,(\tilde\rho^s_t)\big)\Big]\mathrm{dV}.\end{aligned} \tag{4.7}$$

Then, we have the formula

$$\frac{\partial}{\partial t}\frac{1}{2}\tilde A^s_t + \frac{\partial}{\partial s}\mathscr{E}(\tilde\rho^s_t\mathrm{V}) = s\tilde D^s_t, \quad \forall\, t\in[0,+\infty),\ \forall\, s\in[0,1]. \tag{4.8}$$

In particular, if $\mathbb{M}$ *has nonnegative Ricci curvature, then* $\tilde D^s_t \le 0$ *and therefore*

$$\frac{\partial}{\partial t}\frac{1}{2}\tilde A^s_t + \frac{\partial}{\partial s}\mathscr{E}(\tilde\rho^s_t\mathrm{V}) \le 0. \tag{4.9}$$

Proof. Being $\tilde\rho^s_t := \rho^\sigma_\tau|_{\sigma=s,\tau=st}$ we get

$$\tfrac{\partial}{\partial s}\tilde\rho^s_t = \Big(\tfrac{\partial}{\partial\sigma}\rho^\sigma_\tau + t\tfrac{\partial}{\partial\tau}\rho^\sigma_\tau\Big)_{\sigma=s,\tau=st}, \qquad \tfrac{\partial}{\partial t}\tilde\rho^s_t = s\partial_\tau\rho^s_\tau|_{\tau=st} = s\Delta_{\mathrm{g}}\,U\,(\tilde\rho^s_t), \tag{4.10}$$

$$\tfrac{\partial^2}{\partial t\,\partial s}\tilde\rho^s_t \overset{(4.6)}{=} -\nabla\cdot(\tfrac{\partial}{\partial t}\tilde\rho^s_t\,\nabla\tilde\phi^s_t) - \nabla\cdot(\tilde\rho^s_t\,\tfrac{\partial}{\partial t}\nabla\tilde\phi^s_t), \tag{4.11}$$

$$\tfrac{\partial^2}{\partial s\,\partial t}\tilde\rho^s_t \overset{(4.10)}{=} s\Delta_{\mathrm{g}}\Big(U'(\tilde\rho^s_t)\,\tfrac{\partial}{\partial s}\tilde\rho^s_t\Big) + \Delta_{\mathrm{g}}\,U\,(\tilde\rho^s_t) \overset{(4.6)}{=} -s\Delta_{\mathrm{g}}\Big(U'(\tilde\rho^s_t)\,\nabla\cdot(\tilde\rho^s_t\nabla\tilde\phi^s_t)\Big) + \Delta_{\mathrm{g}}\,U\,(\tilde\rho^s_t). \tag{4.12}$$

Differentiation and integration by parts yield

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{M}}\frac{1}{2}|\nabla\tilde\phi^s_t|^2_{\mathrm{g}}\,\tilde\rho^s_t\,\mathrm{dV} &= \int_{\mathbb{M}}\langle\tfrac{\partial}{\partial t}\nabla\tilde\phi^s_t,\nabla\tilde\phi^s_t\rangle_{\mathrm{g}}\,\tilde\rho^s_t\,\mathrm{dV} + \tfrac12\int_{\mathbb{M}}|\nabla\tilde\phi^s_t|^2_{\mathrm{g}}\,\tfrac{\partial}{\partial t}\tilde\rho^s_t\,\mathrm{dV} =\\ &= -\int_{\mathbb{M}}\nabla\cdot(\tilde\rho^s_t\tfrac{\partial}{\partial t}\nabla\tilde\phi^s_t)\,\tilde\phi^s_t\,\mathrm{dV} \overset{(4.10)}{+} \frac12 s\int_{\mathbb{M}}\Delta_{\mathrm{g}}\,(|\nabla\tilde\phi^s_t|^2_{\mathrm{g}})\,U(\tilde\rho^s_t)\,\mathrm{dV} =\\ &\overset{(4.11)}{=} \int_{\mathbb{M}}\tfrac{\partial^2}{\partial t\partial s}\tilde\rho^s_t\,\tilde\phi^s_t\,\mathrm{dV} + \int_{\mathbb{M}}\Big(\nabla\cdot(\tfrac{\partial}{\partial t}\tilde\rho^s_t\nabla\tilde\phi^s_t)\Big)\,\tilde\phi^s_t\,\mathrm{dV} + \frac12 s\int_{\mathbb{M}}\Delta_{\mathrm{g}}\,(|\nabla\tilde\phi^s_t|^2_{\mathrm{g}})\,U(\tilde\rho^s_t)\,\mathrm{dV} =\\ &\overset{(4.12)}{=} \int_{\mathbb{M}}\Big(\Delta_{\mathrm{g}}\,U\,(\tilde\rho^s_t) - s\Delta_{\mathrm{g}}\Big(U'(\tilde\rho^s_t)\,\nabla\cdot(\tilde\rho^s_t\nabla\tilde\phi^s_t)\Big)\,\tilde\phi^s_t\,\mathrm{dV}\\ &\qquad - s\int_{\mathbb{M}}\Delta_{\mathrm{g}}\,U(\tilde\rho^s_t)\,|\nabla\tilde\phi^s_t|^2_{\mathrm{g}}\,\mathrm{dV} + \frac{s}{2}\int_{\mathbb{M}}\Delta_{\mathrm{g}}\Big(|\nabla\tilde\phi^s_t|^2_{\mathrm{g}}\Big)\,U(\tilde\rho^s_t)\,\mathrm{dV} =\\ &= \int_{\mathbb{M}}U(\tilde\rho^s_t)\,\Delta_{\mathrm{g}}\,\tilde\phi^s_t\,\mathrm{dV} - s\int_{\mathbb{M}}\Big(\big\langle\nabla U(\tilde\rho^s_t),\nabla\tilde\phi^s_t\big\rangle_{\mathrm{g}}\,\Delta_{\mathrm{g}}\,\tilde\phi^s_t + \tilde\rho^s_t\,U'(\tilde\rho^s_t)\big(\Delta_{\mathrm{g}}\,\tilde\phi^s_t\big)^2\Big)\,\mathrm{dV}\\ &\qquad - \frac{s}{2}\int_{\mathbb{M}}\Delta_{\mathrm{g}}\,\big(|\nabla\tilde\phi^s_t|^2_{\mathrm{g}}\big)\,U(\tilde\rho^s_t)\,\mathrm{dV}\end{aligned} \tag{4.13}$$

$$\begin{aligned}&= -\int_{\mathbb{M}}\big\langle\nabla U\,(\tilde\rho^s_t),\nabla\tilde\phi^s_t\big\rangle_{\mathrm{g}}\,\mathrm{dV} + s\int_{\mathbb{M}}\Big[-\frac12\Delta_{\mathrm{g}}\,(|\nabla\tilde\phi^s_t|^2_{\mathrm{g}}) + \langle\nabla\tilde\phi^s_t,\nabla\Delta_{\mathrm{g}}\,\tilde\phi^s_t\rangle_{\mathrm{g}}\Big]U\,(\tilde\rho^s_t)\,\mathrm{dV}+\\ &\qquad + s\int_{\mathbb{M}}\big(\Delta_{\mathrm{g}}\,\tilde\phi^s_t\big)^2\Big(U\,(\tilde\rho^s_t) - \tilde\rho^s_t U'(\tilde\rho^s_t)\Big)\,\mathrm{dV}\end{aligned} \tag{4.14}$$

Applying Bochner formula:

$$\langle\nabla\phi,\nabla\Delta_{\mathrm{g}}\,\phi\rangle_{\mathrm{g}}-\tfrac{1}{2}\Delta_{\mathrm{g}}\left(|\nabla\phi|_{\mathrm{g}}^2\right)=-|\mathrm{Hess}\,\phi|_{\mathrm{g}}^2-\mathrm{Ric}(\nabla\phi,\nabla\phi),\tag{4.15}$$

we get

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{\mathbb{M}}|\nabla\tilde\phi_t^s|_{\mathrm{g}}^2\,\tilde\rho_t^s\,\mathrm{dV}+\int_{\mathbb{M}}\left\langle\nabla U(\tilde\rho_t^s),\nabla\tilde\phi_t^s\right\rangle_{\mathrm{g}}\mathrm{dV}=s\tilde D_t^s.\tag{4.16}$$

Now we observe that the second term in the right-hand side of (4.16) is the derivative of the functional (4.3) along the curve $s\mapsto\tilde\rho_t^s V\in\mathscr{P}_2^{ar}(\mathbb{M})$:

$$\frac{\partial}{\partial s}\mathscr{E}(\tilde\mu_t^s)=\int_{\mathbb{M}}e'(\tilde\rho_t^s)\,\tfrac{\partial}{\partial s}\tilde\rho_t^s\,\mathrm{dV}=-\int_{\mathbb{M}}e'(\tilde\rho_t^s)\nabla\cdot(\tilde\rho_t^s\nabla\tilde\phi_t^s)\,\mathrm{dV}=\int_{\mathbb{M}}\nabla U\left(\tilde\rho_t^s\right)\cdot\nabla\tilde\phi_t^s\,\mathrm{dV}\tag{4.17}$$

and we eventually obtain (4.8).

Finally, when $\mathrm{Ric}(\mathbb{M})\ge 0$, using the inequality $(\Delta_{\mathrm{g}}\,\phi)^2\le n|\mathrm{Hess}\,\phi|_{\mathrm{g}}^2$ and (1.7) we easily get $\tilde D_t^s\le 0$ and (4.9). □

Proof of Theorem 4.2. We argue as in the proof of Theorem 2.2: we fix $\varepsilon>0$ and we choose a smooth curve $(\rho,\phi)\in\mathscr{C}(\nu,\mu)$ such that

$$\int_0^1\tilde A_0^s\,\mathrm{d}s=\int_0^1\int_{\mathbb{M}}|\nabla\phi^s|_{\mathrm{g}}^2\,\rho^s\,\mathrm{dVd}s\le W_2^2(\nu,\mu)+\varepsilon.\tag{4.18}$$

Let $(\tilde\rho,\tilde\phi)$ a smooth variation defined as in Lemma 4.3; since $\tilde\rho_t^0\mathrm{V}=\rho^0\mathrm{V}=\nu$ and $\tilde\rho_t^1\mathrm{V}=\mu_t$, for every $t>0$ we have $(\tilde\rho_t^s,\tilde\phi_t^s)\in\mathscr{C}(\nu,\mu_t)$ and therefore

$$W_2^2(\nu,\mu_t)\le\int_0^1\int_{\mathbb{M}}|\nabla\tilde\phi_t^s|_{\mathrm{g}}^2\,\tilde\rho_t^s\,\mathrm{dV}\,\mathrm{d}s=\int_0^1\tilde A_t^s\,\mathrm{d}s.\tag{4.19}$$

Integrating (4.9) for $s\in[0,1]$ and $t\in[0,\tau]$ and recalling that $t\mapsto\mathscr{E}(\mu_t)$ is not increasing, we get

$$\frac{1}{2}\int_0^1\tilde A_\tau^s\,\mathrm{d}s-\frac{1}{2}\int_0^1\tilde A_0^s\,\mathrm{d}s\le\tau\Big(\mathscr{E}(\nu)-\mathscr{E}(\mu_\tau)\Big).\tag{4.20}$$

Combining (4.20) with (4.19) and (4.18) we get

$$\frac{1}{2}W_2^2(\nu,\mu_\tau)-\frac{1}{2}W_2^2(\nu,\mu)\le\tau\Big(\mathscr{E}(\nu)-\mathscr{E}(\mu_\tau)\Big)+\varepsilon,\tag{4.21}$$

and, as ε is arbitrary,

$$\frac{1}{2}W_2^2(\nu,\mu_\tau)-\frac{1}{2}W_2^2(\nu,\mu)\le\tau\Big(\mathscr{E}(\nu)-\mathscr{E}(\mu_\tau)\Big).\tag{4.22}$$

Since the semigroup associated to (4.1) is translation invariant, (4.22) is the integral formulation (3.3) of (4.4). □

Remark 4.4 Taking into account Theorem 2.3, (4.8) perfectly fits with the calculation performed by [19, Lemma 4.4], which provides the same expression for $\tilde D_t^s$.

Applying now Theorem 3.3, with the choices $\mathrm{X}:=\mathscr{P}_2(\mathbb{M})$, $\mathrm{X}_0:=\mathscr{P}_2^{ar}(\mathbb{M})$, $F:=\mathscr{E}$ (which satisfies the approximation condition (3.19), see [3]) we can prove the first part of Theorem 1.1.

Corollary 4.5 *Let $\mathscr{E}:\mathscr{P}_2(\mathbb{M})\to(-\infty,+\infty]$ be the functional defined in* (1.1). *If e satisfies* McCann *conditions* (1.7) *and $Ric(\mathbb{M})\ge 0$, then $\mathscr{E}$ is (strongly) displacement convex along* every *geodesic $\mu:s\in[0,1]\mapsto\mu^s\in\mathscr{P}_2(\mathbb{M})$, i.e.*

$$\mathscr{E}(\mu^s)\le(1-s)\mathscr{E}(\mu^0)+s\mathscr{E}(\mu^1)\quad\forall\,s\in[0,1].\tag{4.23}$$

5 The Heat equation and the displacement λ-convexity of the logarithmic Entropy

In this last section we prove the second part of Theorem 1.1: we thus assume that the Riemannian manifold $\mathbb{M}$ satisfies the lower Ricci curvature bound

$$\mathrm{Ric}(\mathbb{M}) \geq \lambda \quad \text{i.e.} \quad \mathrm{Ric}_x(\xi,\xi) \geq \lambda\,|\xi|^2_{\mathsf{g}} \quad \forall\,\xi \in T_x\,\mathbb{M}, \tag{5.1}$$

and we consider the logarithmic entropy functional

$$\mathscr{E}(\mu) = \int_{\mathbb{M}} \rho\log\rho\,\mathrm{dV}, \quad \rho = \frac{d\mu}{\mathrm{dV}}, \tag{5.2}$$

corresponding to $e(\rho) := \rho\log\rho$. Since $U(\rho) = \rho$, the Wasserstein gradient flow associated to $\mathscr{E}$ is the Heat equation

$$\frac{\partial}{\partial t}\rho_t - \Delta_{\mathsf{g}}\,\rho_t = 0 \quad \text{in } \mathbb{M}, \qquad \rho_{|t=0} = \rho_0. \tag{5.3}$$

The main result of this section is the following:

Theorem 5.1 *The semigroup $\mathcal{S}_t : \mu_0 = \rho_0\mathrm{V} \mapsto \mu_t = \rho_t\mathrm{V}$, generated by the solution of the Heat equation* (5.3) *is a λ-flow in $\mathscr{P}^{ar}_2(\mathbb{M})$ for the logarithmic entropy functional, i.e. μ_t satisfies the inequality*

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\nu,\mu_t) + \frac{\lambda}{2}W_2^2(\nu,\mu_t) \leq \mathscr{E}(\nu) - \mathscr{E}(\mu_t) \quad \forall\,t \in [0,+\infty),\ \nu \in \mathscr{P}^{ar}_2(\mathbb{M}). \tag{5.4}$$

In particular, the logarithmic entropy functional (5.2) *is (strongly) displacement λ-convex, i.e. for every geodesic $\mu^s : [0,1] \to \mathscr{P}_2(\mathbb{M})$ between μ^0 and μ^1, we have*

$$\mathscr{E}(\mu^s) \leq (1-s)\mathscr{E}(\mu^0) + s\mathscr{E}(\mu^1) - \frac{\lambda}{2}s(1-s)W_2^2(\mu^0,\mu^1), \quad \forall\,s \in [0,1]. \tag{5.5}$$

Proof. By Theorem 3.3, if $\mathcal{S}$ is a λ-flow for the functional (5.2) in $\mathscr{P}^{ar}_2(\mathbb{M})$ then $\mathscr{E}$ is (strongly) displacement λ-convex. In order to prove that $\mathcal{S}$ is a λ-flow, since (3.2a,b) are immediate, we check that $\mathcal{S}$ satisfies the E.V.I. (3.2c) and we argue as in the proof of Theorem 4.2 and Theorem 2.2. We thus fix $\varepsilon > 0$ and we choose a smooth curve $(\rho,\phi) \in \mathscr{C}(\nu,\mu)$

$$\int_0^1 \tilde{A}^s_0\,\mathrm{d}s = \int_0^1\int_{\mathbb{M}} |\nabla\phi^s|^2_{\mathsf{g}}\,\rho^s\,\mathrm{dVd}s \leq W_2^2(\nu,\mu) + \varepsilon^2. \tag{5.6}$$

By a standard re-parametrization technique (see next Lemma 5.2), we can also assume that

$$W_2(\mu^{s_0},\mu^{s_1}) \leq L|s_0 - s_1|, \quad L^2 := W_2^2(\nu,\mu) + \varepsilon^2 \qquad \forall\,s_0,s_1 \in [0,1]; \qquad \mu^s := \rho^s\,\mathrm{V}. \tag{5.7}$$

We keep the same notation of Theorem 4.2 and Lemma 4.3, i.e.

$$\tilde{\mu}^s_t = \tilde{\rho}^s_t\,\mathrm{V} := \mathcal{S}_{st}(\mu^s), \quad \tilde{A}^s_t := \int_{\mathbb{M}} |\nabla\tilde{\phi}^s_t|^2_{\mathsf{g}}\,\tilde{\rho}^s_t\,\mathrm{dV}, \quad \tilde{F}^s_t = \mathscr{E}(\tilde{\mu}^s_t) \tag{5.8}$$

where $\tilde{\phi}^s_t$ is family of potentials associated to $\tilde{\rho}^s_t$ as in (4.6). Since $U(\rho) = \rho$ the term $\rho U'(\rho) - U(\rho)$ in the definition of $\tilde{D}^s_t$ vanishes, so that in the present case

$$\tilde{D}^s_t = -\int_{\mathbb{M}} \Big(|\mathrm{Hess}\,\tilde{\phi}^s_t|^2_{\mathsf{g}} + \mathrm{Ric}\,(\nabla\tilde{\phi}^s_t,\nabla\tilde{\phi}^s_t)\Big)\,\tilde{\rho}^s_t\,\mathrm{dV} \overset{(5.1)}{\leq} -\lambda\int_{\mathbb{M}} |\nabla\tilde{\phi}^s_t|^2_{\mathsf{g}}\tilde{\rho}^s_t\,\mathrm{dV} = -\lambda\tilde{A}^s_t, \tag{5.9}$$

(4.8) yields the differential inequality

$$\frac{1}{2}\frac{\partial}{\partial t}\tilde{A}^s_t + \lambda s\tilde{A}^s_t + \frac{\partial}{\partial s}\tilde{F}^s_t \leq 0 \qquad \forall\,s \in [0,1], \quad \forall\,t > 0. \tag{5.10}$$

Multiplying inequality (5.10) by $e^{2\lambda st} > 0$ we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(e^{2\lambda st}\tilde A_t^s\Big) + \frac{\partial}{\partial s}\Big(e^{2\lambda st}\tilde F_t^s\Big) \le 2\lambda t\, e^{2\lambda st}\,\tilde F_t^s. \tag{5.11}$$

Integrating with respect to s from 0 to 1 we get

$$\frac{\mathrm d}{\mathrm dt}\Big(\frac{1}{2}\int_0^1 e^{2\lambda st}\tilde A_t^s\,\mathrm ds\Big) + e^{2\lambda t}\tilde F_t^1 - \tilde F_t^0 \le \int_0^1 2\lambda\, t\, e^{2\lambda st}\tilde F_t^s\,\mathrm ds, \tag{5.12}$$

and a further integration with respect to t yields

$$\frac{1}{2}\int_0^1 e^{2\lambda st}\tilde A_t^s\,\mathrm ds - \frac{1}{2}\int_0^1 A_0^s\,\mathrm ds + \mathsf E_{2\lambda}(t)\mathscr E(\mu_t) - t\mathscr E(\nu) \le \int_0^t\int_0^1 2\lambda\, r\, e^{2\lambda sr}\,\tilde F_r^s\,\mathrm ds\,\mathrm dr. \tag{5.13}$$

Applying the next Lemma 5.2, since for $\lambda \neq 0$ $\int_0^1 \frac{1}{e^{2\lambda st}}\,\mathrm ds = \frac{1-e^{-2\lambda t}}{2\lambda t} = \frac{1}{e^{\lambda t}\mathsf s(\lambda t)}$, $\quad \mathsf s(t) := \frac{t}{\sinh(t)}$, we get

$$\frac{e^{\lambda t}\mathsf s(\lambda t)}{2}W_2^2(\mu_t,\nu) - \frac{1}{2}W_2^2(\mu,\nu) + \mathsf E_{2\lambda}(t)\mathscr E(\mu_t) - t\mathscr E(\nu) \le \int_0^t\int_0^1 2\lambda r e^{2\lambda sr}\tilde F_r^s\,\mathrm ds\mathrm dr + \frac{\varepsilon^2}{2}. \tag{5.14}$$

Let us first consider the case $\lambda < 0$: being $\mathscr E$ nonnegative, the right hand side in (5.14) is less or equal than ε; since $\varepsilon > 0$ is arbitrary, we obtain the same inequality with 0 in the right-hand side. Since $t^{-1}\mathsf E_{2\lambda}(t) \to 1$ as $t \downarrow 0$ and $\mathsf s(0) = 1$, we thus obtain

$$\frac{1}{2}\frac{\mathrm d^+}{\mathrm dt}\Big(e^{\lambda t}\mathsf s(\lambda t)W_2^2(\mu_t,\nu)\Big)\Big|_{t=0} + \mathscr E(\mu) \le \mathscr E(\nu). \tag{5.15}$$

Being $\mathsf s'(0) = 0$ it is then easy to check that

$$\frac{\mathrm d^+}{\mathrm dt}\Big(e^{\lambda t}\mathsf s(\lambda t)W_2^2(\mu_t,\nu)\Big)\Big|_{t=0} = \frac{\mathrm d^+}{\mathrm dt}\Big(W_2^2(\mu_t,\nu)\Big)\Big|_{t=0} + \lambda\, W_2^2(\mu,\nu),$$

which yields (5.4).
Let us now consider the case $\lambda > 0$. By (5.7) we can apply the estimate (3.11) obtaining

$$\begin{aligned} r\tilde F_r^s = r\mathscr E(\mathcal S_{rs}(\mu^s)) &\overset{(3.11)}{\le} r\Big((1-s)\mathscr E(\mu^0) + s\mathscr E(\mu^1) - \frac{\lambda}{2}s(1-s)W_2^2(\mu^0,\mu^1) + \frac{\varepsilon^2}{2\mathsf E_\lambda(rs)}s(1-s)\Big) \\ &\le r\Big(\mathscr E(\mu^0) + \mathscr E(\mu^1)\Big) + \varepsilon^2, \end{aligned}$$

since $s \in [0,1]$ and $rs/\mathsf E_\lambda(rs) \le 1$. We thus get

$$\int_0^t\int_0^1 2\lambda\, r\, e^{2\lambda sr}\tilde F_r^s\,\mathrm ds\,\mathrm dr \le 2\lambda t e^{2\lambda t}\Big(t\big(\mathscr E(\mu_0) + \mathscr E(\mu_1)\big) + \varepsilon^2\Big); \tag{5.16}$$

inserting this bound in (5.14) and passing to the limit as $\varepsilon \downarrow 0$ we find

$$\frac{e^{\lambda t}\mathsf s(\lambda t)}{2}W_2^2(\mu_t,\nu) - \frac{1}{2}W_2^2(\mu,\nu) + \mathsf E_{2\lambda}(t)\mathscr E(\mu_t) - t\mathscr E(\nu) \le 2\lambda t^2 e^{2\lambda t}\Big(\mathscr E(\mu_0) + \mathscr E(\mu_1)\Big). \tag{5.17}$$

Dividing by t and letting t tend to 0 the second term vanishes, so we obtain the EVI also in the case in which $\lambda > 0$. $\square$

Lemma 5.2 *Let $\nu,\mu \in \mathscr P_2^{ar}(\mathbb M)$ and let $(\rho,\phi) \in \mathscr C(\nu,\mu)$ be a smooth solution of the continuity equation*

$$\frac{\partial}{\partial s}\rho^s + \nabla\cdot(\rho^s\,\nabla\phi^s) = 0 \quad \textit{in } [0,1]\times\mathbb M \quad \textit{with} \quad \rho^0\mathrm V = \nu,\ \rho^1\mathrm V = \mu \quad \textit{and} \quad A^s := \int_{\mathbb M}|\nabla\phi^s|_{\mathsf g}^2\,\rho^s\,\mathrm dV.$$

For every positive function $f \in C^\infty[0,1]$

$$W_2^2(\nu,\mu) \le L_f \int_0^1 f(s) A^s\,ds, \quad \textit{where} \quad L_f := \int_0^1 \frac{1}{f(s)}\,ds. \tag{5.18}$$

Moreover, for every $\varepsilon > 0$ *there exists a smooth rescaling* $\mathsf{s}_\varepsilon : [0,1] \to [0,1]$ *so that the re-parametrized families*

$$\bar\rho^r := \rho^{\mathsf{s}_\varepsilon(r)}, \quad \bar\phi^r := \mathsf{s}_\varepsilon'(r)\phi^{\mathsf{s}_\varepsilon(r)}, \quad \bar\mu^r := \bar\rho^r\,\mathrm{V} \tag{5.19}$$

satisfy

$$(\bar\rho,\bar\phi) \in \mathscr{C}(\nu,\mu), \quad W_2(\bar\mu^{r_0},\bar\mu^{r_1}) \le L|r_0 - r_1|, \quad L^2 \le \int_0^1 A^s\,\mathrm{d}s + \varepsilon^2. \tag{5.20}$$

Proof. Let us consider the smooth increasing map $\mathsf{r} : [0,1] \to [0,1]$

$$\mathsf{r}(s) := L_f^{-1} \int_0^s \frac{1}{f(s)}\,ds \quad \text{and its inverse } \mathsf{s} := \mathsf{r}^{-1} \quad \text{with} \quad \mathsf{s}'(\mathsf{r}(s)) = L_f f(s).$$

It is immediate to check that the smooth (reparametrized) curve

$$\bar\rho^r(x) := \rho^{\mathsf{s}(r)}(x), \quad \bar\phi^r(x) := \mathsf{s}'(r)\phi^{\mathsf{s}(r)}(x) \tag{5.21}$$

belongs to $\mathscr{C}(\nu,\mu)$. It follows that

$$W_2^2(\nu,\mu) \le \int_0^1 \bar A^r\,\mathrm{d}r, \quad \text{where} \quad \bar A^r := \int_{\mathbb{M}} |\nabla\bar\phi^r|_{\mathsf{g}}^2\,\bar\rho^r\,\mathrm{dV} \overset{(5.21)}{=} \big(\mathsf{s}'(r)\big)^2 A^{\mathsf{s}(r)},$$

so that

$$\int_0^1 \bar A^r \mathrm{d}r = \int_0^1 A^{\mathsf{s}(r)}\big(\mathsf{s}'(r)\big)^2\,\mathrm{d}r = \int_0^1 A^s \mathsf{s}'(\mathsf{r}(s))\,\mathrm{d}s = L_f \int_0^1 f(s)A^s\,\mathrm{d}s.$$

Choosing now the re-parametrization s_ε corresponding to the choice

$$f_\varepsilon(s) := \frac{1}{\sqrt{\varepsilon^2 + A^s}}, \quad L_{f_\varepsilon} := \int_0^1 \sqrt{\varepsilon^2 + A^s}\,\mathrm{d}s, \quad L_{f_\varepsilon}^2 \le \varepsilon^2 + \int_0^1 A^s\,\mathrm{d}s, \tag{5.22}$$

we get

$$W^2(\bar\mu^{r_0},\bar\mu^{r_1}) \le |r_1 - r_0| \int_{r_0}^{r_1} \bar A^r\,\mathrm{d}r = |r_1 - r_0| L_{f_\varepsilon}^2 \int_{r_0}^{r_1} A^{\mathsf{s}(r)} f_\varepsilon^2(\mathsf{s}(r))\,\mathrm{d}r \le (r_1 - r_0)^2 L_{f_\varepsilon}^2,$$

which yields (5.20). □

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