The relaxed Dirichlet energy of manifold constrained mappings

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Abstract. *The Dirichlet energy of Sobolev mappings between Riemannian manifolds is studied. After giving an explicit formula of the polyconvex extension of the energy for currents between manifolds, we prove a strong density result. As a consequence, we give an explicit formula for the relaxed energy. The fractional space of traces of $W^{1,2}$-mappings is also treated.*

Let $\mathcal{X} = \mathcal{X}^n$ and $\mathcal{Y} = \mathcal{Y}^m$ be two smooth compact connected oriented Riemannian manifolds of dimension n and m, respectively, where $\mathcal{Y}$ is boundaryless and $\mathcal{X}$ possibly with a non-empty boundary $\partial\mathcal{X}$. We assume $\mathcal{X}$ and $\mathcal{Y}$ equipped with metric tensors $(g_{\alpha\beta})$ and (γ_{ij}), respectively, in some local coordinate charts $x = (x_1, \dots, x_n)$ and $U = (U^1, \dots U^m)$ on $\mathcal{X}$ and $\mathcal{Y}$, respectively. The *Dirichlet energy*, or *action* in physics, of a smooth map $U : \mathcal{X} \to \mathcal{Y}$ is defined as the integral of the square of the derivatives dU. More precisely, the *energy density* of U is

$$e(x, U) := \frac{1}{2}|dU_x|^2 = \frac{1}{2}\mathrm{tr}[(dU_x)^* dU_x] \tag{0.1}$$

and the Dirichlet energy of U is

$$\mathbf{D}_g(U, \mathcal{X}) := \int_{\mathcal{X}} e(x, U)\, d\mathrm{vol}_{\mathcal{X}} = \frac{1}{2}\int_{\mathcal{X}} |dU_x|^2\, d\mathrm{vol}_{\mathcal{X}}\,. \tag{0.2}$$

In local coordinates $(\frac{\partial}{\partial x^i})_{i=1}^n$ in $T_x\mathcal{X}$ and $(\frac{\partial}{\partial y^j})_{j=1}^m$ in $T_{U(x)}\mathcal{Y}$, one computes

$$2e(x, U)(x) = g^{\alpha\beta}(x)\gamma_{ij}(U)\frac{\partial U^i}{\partial x^\alpha}\frac{\partial U^j}{\partial x^\beta}\,,$$

where $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, and therefore, since

$$d\mathrm{vol}_{\mathcal{X}} = \sqrt{\det g}\, dx\,,$$

one concludes that the Dirichlet energy in local coordinates for continuous maps U takes the form

$$\frac{1}{2}\int g^{\alpha\beta}(x)\gamma_{ij}(U)\frac{\partial U^i}{\partial x^\alpha}\frac{\partial U^j}{\partial x^\beta}\sqrt{\det g(x)}\, dx\,. \tag{0.3}$$

This generalizes the classical Dirichlet's energy for maps between the flat manifolds $\mathbb{R}^n$ and $\mathbb{R}^m$.

By Nash embedding theorem, we may and will assume, without loss of generality, that $\mathcal{Y}$ is *isometrically embedded*, as a submanifold, in some Euclidean space $\mathbb{R}^N$ with induced Riemannian metric. This means that the inner product of two tangent vectors to $\mathcal{Y}$ at a point $y \in \mathcal{Y}$ is simply their Euclidean inner product, i.e., $\gamma_{ij} = \delta_{ij}$, the Kronecker symbols. We then consider maps $u : \mathcal{X} \to \mathbb{R}^N$ that are constrained to take values into a smooth, boundaryless, compact submanifold $\mathcal{Y}$ of $\mathbb{R}^N$.

Therefore, in local coordinates, the energy density (0.1) agrees with $e_g(x, Du)$, where

$$e_g(x, G) := \frac{1}{2}\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N} g^{\alpha\beta}(x)\delta_{ij}\, G^i_\alpha\, G^j_\beta\sqrt{\det g(x)} \tag{0.4}$$

for every $x \in \mathcal{X}$, every $y \in \mathcal{Y}$, and every $(N \times n)$-matrix G, say $G \in M(N, n)$, with $\mathrm{im}\, G$ in $T_y\mathcal{Y}$, the tangent space to $\mathcal{Y}$ at y. Therefore, for every local parameterization $\phi : \Omega \to \mathcal{X}$, the Dirichlet energy in Ω

of a map $U\in W^{1,2}(\mathcal{X},\mathcal{Y})$ on $\phi(\Omega)$ agrees with

$$\mathbf{D}_g(u,\Omega):=\int_\Omega e_g(x,Du(x))\,dx\,,\qquad u=U\circ\phi\,. \tag{0.5}$$

Assume that *the integral* 2*-homology group* $H_2(\mathcal{Y}):=H_2(\mathcal{Y};\mathbb{Z})$ *has no torsion.* We recall from [9] [19], see Sec. 2 below, that the class of *Cartesian currents* $\mathrm{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ arises as weak $\mathcal{D}_{n,2}$-limits of sequences of currents G_{u_k} carried by the graphs of smooth maps $u_k:\mathcal{X}\to\mathcal{Y}$ with equibounded $W^{1,2}$-energies,

$$\sup_k \|u_k\|_{W^{1,2}(\mathcal{X},\mathcal{Y})}<\infty\,,$$

the weak $\mathcal{D}_{n,2}$-convergence being given, by duality, by testing with forms in the class $\mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y})$, i.e., with smooth, compactly supported n-forms in $\mathcal{X}\times\mathcal{Y}$ with *at most two vertical differentials* in the $\mathcal{Y}$-directions. We refer to [7] and [11, Vol. I] for general definitions of currents on Riemannian manifolds.

Every weak limit current $T\in\mathrm{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ satisfies the *null boundary condition* (2.3) and can be decomposed as

$$T=G_{u_T}+\sum_{s=1}^{\widetilde{s}}\mathbb{L}_s(T)\times\gamma_s+S_{T,sing}\,. \tag{0.6}$$

G_{u_T} is the current integration of forms in $\mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y})$ over the rectifiable graph of u_T, see Example 2.1, where $u_T\in W^{1,2}(\mathcal{X},\mathcal{Y})$ is the weak $W^{1,2}$-limit of the u_k's. The γ_i's are integral cycles in $\mathcal{Z}_2(\mathcal{Y})$ such that $\{[\gamma_i]\}_{i=1}^{\widetilde{s}}$ generates the *spherical subgroup* $H_2^{sph}(\mathcal{Y})$ of $H_2(\mathcal{Y})$, see (2.1). The $\mathbb{L}_s(T)$'s are integer multiplicity (say i.m.) rectifiable current in $\mathcal{R}_{n-2}(\mathcal{X})$. Finally, $S_{T,sing}$, though completely vertical and homologically trivial, in general is non zero only possibly on forms $\omega\in\mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y})$ for which $d_y\omega^{(2)}\neq 0$, where $\omega^{(2)}$ is the component of ω with exactly two vertical differentials. Moreover, as shown in [10], in principle $S_{T,sing}$ may be any measure.

In Sec. 1, we shall consider the *parametric polyconvex lower semicontinuous envelop* of the integrand e_g, defined for every $x\in\mathcal{X}$ and every n-vector ξ in $\mathbb{R}^{n+N}$, say $\xi\in\Lambda_n\mathbb{R}^{n+N}$, by

$$\begin{aligned}F_g(x,\xi)=\sup\{\phi(\xi)\mid \phi:\Lambda_n\mathbb{R}^{n+N}\to\overline{\mathbb{R}}_+\,,\ \phi \text{ linear},\\ \phi(M(G))\le e_g(x,G)\quad\forall\, G\in M(N,n)\}\,,\end{aligned} \tag{0.7}$$

where $M(G)$, see (1.1), is the n-vector in $\Lambda_n\mathbb{R}^{n+N}$ orienting the graph of G. Since $(x,G)\mapsto e_g(x,G)$ is continuous, it turns out that $F_g(x,\xi)$ *is l.s.c. in all variables and convex in* ξ *for any* x. Using (0.7), the *polyconvex parametric extension* $\mathbf{D}_g(T)$ of the Dirichlet energy (0.5) turns out to be well-defined on currents T in $\mathcal{D}_{n,2}(\mathcal{X}\times\mathcal{Y})$ with finite $\mathbf{D}$*-norm*, see (2.5), and is *lower semicontinuous* with respect to the weak $\mathcal{D}_{n,2}$-convergence.

In Sec. 2 we shall give an explicit formula of the *Dirichlet energy* $\mathbf{D}_g(T)$ for currents T in $\mathrm{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$. More precisely, if $S_{T,sing}=0$ in the decomposition formula (0.6) of T, we shall prove that

$$\mathbf{D}_g(T)=\mathbf{D}_g(u_T)+\sum_{s=1}^{\widetilde{s}}\mathbf{M}_g(\mathbb{L}_s(T))\cdot\mathbf{M}(\gamma_s)\,, \tag{0.8}$$

where $\mathbf{M}_g(\mathbb{L}_s(T))$ denotes the *g-mass* of $\mathbb{L}_s(T)$, see (2.8). Writing $\mathbb{L}_s(T)$ as $\mathbb{L}_s(T)=\tau(\mathcal{L}_s,\theta_s,\tau_s)$ for some $(n-2)$-rectifiable set $\mathcal{L}_s$, integer multiplicity θ_s, and unit orienting $(n-2)$-vector τ_s, we have

$$\mathbf{M}_g(\mathbb{L}_s(T))=\int_{\mathcal{L}_s}\theta_s(x)\,d\mathcal{H}^{n-2}\,,$$

if we choose τ_s with unit *g-norm*, $|\tau_s|_g=1$.

In Secs. 3 and 4 we will then prove a strong density result for the Dirichlet energy in $\mathrm{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$. For that, we recall, following Hang-Lin [21], that $\mathcal{X}$ satisfies the *d-extension property with respect to* $\mathcal{Y}$ if for any given CW-complex K on $\mathcal{X}$, denoting by K^d its d-dimensional skeleton, any continuous map $f:K^{d+1}\to\mathcal{Y}$ is such that its restriction to K^d can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$.

In [21] it is shown that if $\mathcal{X}$ satisfies the 1-extension property with respect to $\mathcal{Y}$, and $\pi_2(\mathcal{Y})=0$, every Sobolev map in $W^{1,2}(\mathcal{X},\mathcal{Y})$ is the strong limit in $W^{1,2}$ of a sequence of smooth maps from $\mathcal{X}$ to $\mathcal{Y}$. We notice that, if $\pi_1(\mathcal{X}) = 0$ and $\pi_2(\mathcal{Y}) = 0$, then the 1-extension property is automatically satisfied. In the case $\mathcal{X} = B^n$, the unit ball in $\mathbb{R}^n$, the problem of strong density of smooth maps in the Sobolev classes $W^{1,p}(B^n, \mathcal{Y})$ was solved by Bethuel [2].

In this paper, we shall assume that $\mathcal{X}$ satisfies the 1-extension property with respect to $\mathcal{Y}$, and that for any base point $y_0 \in \mathcal{Y}$ the *Hurewicz homomorphism* from the second homotopy group $\pi_2(\mathcal{Y}; y_0)$ onto the second real homology group $H_2(\mathcal{Y}; \mathbb{R})$ is injective (notice that by the Hurewicz theorem this last condition holds true if $\mathcal{Y}$ is 1-connected, i.e., if $\pi_1(\mathcal{Y}) = 0$). Moreover, we assume that *for every* $x \in \mathcal{X}$ *the metric* g *is equivalent to the Euclidean metric,* see (2.7), and that $x \mapsto g(x)$ *is continuous in* $\mathcal{X}$. Then in Sec. 3 we will prove the following density result.

Theorem 0.1 *Let* T *in* $\text{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$ *be such that* $S_{T,sing} = 0$ *in* (0.6). *There exists a sequence of smooth maps* $u_k : \mathcal{X} \to \mathcal{Y}$ *such that* $G_{u_k} \rightharpoonup T$ *weakly in* $\mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ *and*

$$\lim_{k\to\infty} \mathbf{D}_g(u_k) = \mathbf{D}_g(T)\,.$$

Remark 0.2 Notice that, as shown in [9], compare [19, Sec. 4.9], in every *vertical homology class* of currents in $\text{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$ there exists a minimizer of the Dirichlet energy that satisfies the condition $S_{T,sing} = 0$. On the other hand, a part from the regular case $n = 2$, it is not clear how to find an explicit formula for the Dirichlet energy if $S_{T,sing}$ in (0.6) is non-zero.

As an application of Theorem 0.1, in Sec. 5 we shall obtain a representation formula of the *relaxed Dirichlet energy* for $W^{1,2}$-maps in the weak $W^{1,2}$-topology. For every $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$ and every open set $\Omega \subset \mathcal{X}$ we let

$$\begin{aligned}\widetilde{\mathbf{D}}_g(u,\Omega) := \inf\big\{\liminf_{k\to\infty} \mathbf{D}_g(u_k,\Omega) \mid \{u_k\} \subset C^1(\mathcal{X},\mathcal{Y})\,,\\ u_k \rightharpoonup u \quad \text{weakly in } W^{1,2}(\mathcal{X},\mathcal{Y})\big\}\end{aligned} \tag{0.9}$$

where $\mathbf{D}_g(u, \Omega)$ is defined by (0.5). By Schoen-Uhlenbeck density theorem [25], in case of dimension $n = 2$ we clearly have

$$\widetilde{\mathbf{D}}_g(u,\Omega) = \mathbf{D}_g(u,\Omega) \qquad \forall\, u \in W^{1,2}(\mathcal{X},\mathcal{Y})\,.$$

In any dimension $n \geq 3$, the following weak sequential density result was proved by Pakzad-Rivière [24], see [19, Sec. 5.6] for a proof in the easier case $\mathcal{Y} = \mathbb{S}^2$, the unit 2-sphere in $\mathbb{R}^3$.

Theorem 0.3 *For every* $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$ *there exists a sequence of smooth maps* $\{u_k\} \subset C^1(\mathcal{X}, \mathcal{Y})$ *such that* $u_k \rightharpoonup u$ *weakly in* $W^{1,2}(\mathcal{X}, \mathcal{Y})$.

This clearly yields that for every open set $\Omega \subset \mathcal{X}$

$$\widetilde{\mathbf{D}}_g(u,\Omega) < \infty \qquad \forall\, u \in W^{1,2}(\mathcal{X},\mathcal{Y})\,.$$

By Theorem 0.1 we then obtain that for every $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$

$$\widetilde{\mathbf{D}}_g(u,\Omega) = \inf\{\mathbf{D}_g(T,\Omega\times\mathcal{Y}) \mid T \in \mathcal{T}^{2,1}_u\}.$$

In this formula, $\mathcal{T}^{2,1}_u$ denotes the family of *vertical equivalence classes* of currents in $\text{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$, denoted by $\text{CART}^{2,1}(\mathcal{X} \times \mathcal{Y})$, such that the underlying $W^{1,2}$-maps u_T in (0.6) are equal to u, see (5.4). We recall that every element in $\text{CART}^{2,1}(\mathcal{X} \times \mathcal{Y})$ has a representative of the type (0.6) with $S_{T,sing} = 0$, see Remark 0.2. Moreover, the i.m. rectifiable currents $\mathbb{L}_s(T) \in \mathcal{R}_{n-2}(\mathcal{X})$ in the decomposition (0.6) do not depend on the choice of the representative in a class of $\text{CART}^{2,1}(\mathcal{X} \times \mathcal{Y})$. Therefore, the Dirichlet energy $\mathbf{D}_g(T)$ of $T \in \text{CART}^{2,1}(\mathcal{X} \times \mathcal{Y})$ is well defined by the right-hand side of the formula (0.8), by taking γ_s as the *mass minimizing* integral chain of $\mathcal{Z}_2(\mathcal{Y})$ in the homology class $[\gamma_s]$.

As a consequence, we deduce that for every $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$

$$\widetilde{\mathbf{D}}_g(u,\Omega) = \mathbf{D}_g(u,\Omega) + \inf\bigg\{\sum_{s=1}^{\widetilde{s}} M_s \cdot \mathbf{M}_g(\mathbb{L}_s(T) \llcorner \Omega) \mid T \in \mathcal{T}^{2,1}_u\bigg\}\,,$$

where

$$M_s := \inf\{\mathbf{M}(\gamma) \mid \gamma \in \mathcal{Z}_2(\mathcal{Y})\,,\ \gamma \in [\gamma_s]\}\,.$$

Therefore, the gap between $\mathbf{D}_g(u,\Omega)$ and the relaxed energy $\widetilde{\mathbf{D}}_g(u,\Omega)$ is related to the minimum value of the g-mass among all the $(n-2)$-dimensional i.m. rectifiable currents Γ in Ω that bound the *singular set* of u. The above formula reads as

$$\widetilde{\mathbf{D}}_g(u) = \mathbf{D}_g(u) + \sum_{s=1}^{\widetilde{s}} M_s \cdot \mathbf{M}_g(L_s)\,,$$

where $L_s \in \mathcal{R}_{n-2}(\Omega)$, for $s = 1,\dots,\widetilde{s}$, is an *integral minimal connection for the g-mass* of the singular set $\mathbb{P}_s(u)$ *allowing connections to the boundary* of Ω, see Definitions 5.8 and 5.11. In the case $\mathcal{X} = B^n$ or $\mathbb{S}^n$, and for the standard Dirichlet integral, this formula was obtained in [26], in the case $\mathcal{Y} = \mathbb{S}^2$, and in [14], for more general target manifolds $\mathcal{Y}$ as above.

Remark 0.4 From another point of view, one may be interested in studying the quadratic energy

$$\int_{B^n} f(x, Du)\, dx \tag{0.10}$$

of mappings $u : B^n \to \mathcal{Y} \subset \mathbb{R}^N$, where the quadratic integrand $f : B^n \times M(N,n) \to \mathbb{R}^+$ is defined by

$$f(x,G) := \frac{1}{2}\operatorname{tr}(G\,A(x)\,G^T)\,, \qquad x \in B^n\,, \quad G \in M(N,n)\,, \tag{0.11}$$

$x \mapsto A(x)$ being a continuous map from B^n to the space of positive definite matrices in $M(n,n)$. Setting

$$g := (\det A)^{1/(n-2)}\, A^{-1} \iff A_{\alpha\beta}(x) := \sqrt{\det(g_{\alpha\beta}(x))}\, g^{\alpha\beta}(x)\,, \quad x \in B^n, \tag{0.12}$$

it turns out that the quadratic energy (0.10) agrees with the Dirichlet energy (0.5) of mappings $u : \mathcal{X} \to \mathcal{Y} \subset \mathbb{R}^N$, where $(\mathcal{X}, g) = (B^n, g)$, i.e., we have

$$f(x,G) = e_g(x,G) \qquad \forall\,(x,G) \in B^n \times M(N,n)\,. \tag{0.13}$$

In the case $n = 3$, since $|\tau|_g^2 = \tau^T g\, \tau$, we have

$$|\tau|_g^2 = \tau^T (\operatorname{cof} A)\, \tau\,.$$

Using the same techniques, in Sec. 6 we briefly discuss some analogous features for the fractional Sobolev class $W^{1/2}(B^n,\mathcal{Y})$, given by the L^2-mappings $u : B^n \to \mathbb{R}^N$ such that $u(x) \in \mathcal{Y}$ a.e. on B^n and that are the traces on $B^n \simeq B^n \times \{0\}$ of some Sobolev map U in $W^{1,2}(B^n \times]0,1[, \mathbb{R}^N)$, equipped with the seminorm

$$|u|_{1/2} := \inf\Big\{ \int_{B^n}\int_0^1 e_g(x, DU)\, dt\, dx \;\mid\; U \in W^{1,2}(B^n \times]0,1[, \mathbb{R}^N)\,,\; U = u \text{ on } B^n \times \{0\} \Big\}\,,$$

compare [23] for the case of $W^{1/2}$-maps from B^2 into $\mathbb{S}^1$, the unit circle in $\mathbb{R}^2$.

1 The parametric envelop of the Dirichlet energy density

NOTATION ON MULTIVECTORS. Denote by $I(k,m)$ the class of ordered multi-indices α in $\{1,\dots,m\}$ of length k, i.e., $\alpha = (\alpha_1,\dots,\alpha_k)$ where $1 \le \alpha_1 < \cdots < \alpha_k \le m$, and, for convenience, $I(0,m) := \{0\}$. Moreover, denote by $|\alpha|$ the length of α, by $\overline{\alpha}$ the element in $I(m-k,m)$ which complements α, and by $\sigma(\alpha,\overline{\alpha})$ the sign of the permutation that reorders the multi-index $(\alpha,\overline{\alpha})$ in the natural way.

Let $e_1,\dots,e_n$ and $\varepsilon_1,\dots,\varepsilon_N$ be the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, and denote by e_α and ε_β, for $\alpha \in I(k,n)$ and $\beta \in I(h,N)$, the unit simple multi-vectors $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}$ and $\varepsilon_\beta := \varepsilon_{\beta_1} \wedge \cdots \wedge \varepsilon_{\beta_h}$.

If $G : \mathbb{R}^n \to \mathbb{R}^N$ is a linear map we let G also denote the $(N \times n)$-matrix in $M(N,n)$ associated to G with respect to the standard bases. For multi-indices α and β with length respectively $|\alpha| = n-k$

and $|\beta| = k$, we shall denote by $G^{\beta}_{\overline{\alpha}}$ the $(k\times k)$-submatrix of G with rows $\beta = (\beta_1,\dots,\beta_k)$ and columns $\overline{\alpha} = (\overline{\alpha}_1,\dots,\overline{\alpha}_k)$, and by

$$M^{\beta}_{\overline{\alpha}}(G) := \det G^{\beta}_{\overline{\alpha}}$$

the determinant of $G^{\beta}_{\overline{\alpha}}$, where by definition

$$M^0_0(G) := 1\,.$$

Let $\Lambda_n\mathbb{R}^{n+N}$ denote the space of n-vectors in $\mathbb{R}^{n+N}$. Every n-vector ξ in $\Lambda_n\mathbb{R}^{n+N}$ can be written as

$$\xi = \sum_{|\alpha|+|\beta|=n} \xi^{\alpha\beta} e_\alpha \wedge \varepsilon_\beta\,, \qquad \xi^{\alpha\beta}\in\mathbb{R}\,,$$

we refer to $\xi^{\bar{0}0}$ as the *first component* of ξ, or as $\xi = \sum_{k=0}^{n} \xi_{(k)}$, where

$$\xi_{(k)} := \sum_{\substack{|\alpha|+|\beta|=n\\ |\beta|=k}} \xi^{\alpha\beta} e_\alpha\wedge\varepsilon_\beta\,, \qquad k=0,\dots,\underline{n} := \min\{n,N\}\,,$$

so that $\xi_{(0)} = \xi^{\bar{0}0} e_1\wedge\cdots\wedge e_n$. We also denote by Σ the class of *simple* n-vectors in $\Lambda_n\mathbb{R}^{n+N}$ and set

$$\begin{aligned}\Sigma_1 &:= \{\xi\in\Sigma \mid \xi^{\bar{0}0} = 1\}\,, \quad \Lambda_1 := \{\xi\in\Lambda_n\mathbb{R}^{n+N} \mid \xi^{\bar{0}0}=1\}\,,\\ \Sigma_+ &:= \{\xi\in\Sigma \mid \xi^{\bar{0}0} > 0\}\,, \quad \Lambda_+ := \{\xi\in\Lambda_n\mathbb{R}^{n+N} \mid \xi^{\bar{0}0}>0\}\,.\end{aligned}$$

For $G\in M(N,n)$, the vectors $e_i + Ge_i \in\mathbb{R}^{n+N}$, $i=1,\dots,n$, yield a basis of the tangent n-plane to the graph of G in $\mathbb{R}^{n+N}$ that agrees with the graph of G. Letting

$$M(G) := (e_1+Ge_1)\wedge\cdots\wedge(e_n+Ge_n)\in\Lambda_n\mathbb{R}^{n+N}, \tag{1.1}$$

we find that the unit simple n-vector

$$\xi_G := \frac{M(G)}{|M(G)|}\,,$$

called the *tangent n-vector to the graph of G*, identifies the n-plane graph of G, and in fact orients such an n-plane. The map $G\mapsto M(G)$ from $M(N,n)$ to $\Lambda_n\mathbb{R}^{n+N}$ is injective, as

$$M(G) = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha,\overline{\alpha})\, M^{\beta}_{\overline{\alpha}}(G)\, e_\alpha\wedge\varepsilon_\beta \in \Lambda_n\mathbb{R}^{n+N}\,. \tag{1.2}$$

Moreover, if $M_{(k)}(G) := M(G)_{(k)}$, for every $G\in M(N,n)$ we have $M_{(0)}(G) = e_1\wedge\cdots\wedge e_n$ and

$$M_{(1)}(G) = \sum_{j=1}^{N}\sum_{i=1}^{n} (-1)^{n-i}\, G^j_i\, \widehat{e_i}\wedge\varepsilon_j\,, \qquad G = (G^j_i)^{N,n}_{j,i=1}\,.$$

Conversely, to every $\xi\in\Lambda_+$ we associate the matrix $G_\xi\in M(N,n)$ defined by

$$G_\xi := {M_{(1)}}^{-1}\left(\frac{\xi_{(1)}}{\xi^{\bar{0}0}}\right).$$

For $\xi\in\Lambda_+$ we have $G_\xi = 0$ if and only if $\xi_{(1)}=0$, whereas $G_{\lambda\xi} = G_\xi$ for every $\forall\,\lambda>0$. Most importantly, $G_\xi = M^{-1}(\xi)$ if and only if $\xi\in\Sigma_1$, i.e.

$$\begin{cases} G_{M(G)} = G \quad \forall\, G\in M(N,n)\,,\\ \xi = M(G_\xi) \quad \Longleftrightarrow \quad \xi\in\Sigma_1 \end{cases} \tag{1.3}$$

and finally

$$\xi\in\Lambda_+ \quad \text{is simple if and only if} \quad \frac{\xi}{\xi^{\bar{0}0}} = M(G_\xi)\,,$$

whereas Λ_1 agrees with the convex envelop of the set of n-vector $M(G)$,

$$\text{co}\,(\{M(G)\mid G\in M(N,n)\})=\Lambda_1\,. \tag{1.4}$$

We refer to [11] for background material concerning this section.

LINEAR MAPPINGS ON n-VECTORS. If $L:V\to W$ is a linear map between finite dimensional vector spaces V and W, and $\Lambda_k L:\Lambda_k V\to\Lambda_k W$ is the *induced linear transformation*, defined on simple k-vectors by

$$\Lambda_k L(v_1\wedge\cdots\wedge v_k):=Lv_1\wedge\cdots\wedge Lv_k\,,$$

we have

$$M(G)=\Lambda_n(\text{Id}\bowtie G)(e_1\wedge\cdots\wedge e_n)\qquad\forall\,G\in M(N,n)\,,$$

where $(\text{Id}\bowtie G):\mathbb{R}^n\to\mathbb{R}^{n+N}$ is given by $(\text{Id}\bowtie G)(x):=(x,Gx)$. Moreover, the following Laplace's formulas hold:

Lemma 1.1 *Let $L:\mathbb{R}^n\to\mathbb{R}^n$ be a non-singular linear map. Then*

$$\sigma(\gamma,\overline{\gamma})\,\sigma(\alpha,\overline{\alpha})\,M^{\overline{\alpha}}_{\overline{\gamma}}(L)=(\det L)\,M^{\gamma}_{\alpha}(L^{-1})$$

for any $0\le|\alpha|=|\gamma|\le n$.

PROOF: Let $(e_1,\dots e_n)$ and $(\epsilon_1,\dots\epsilon_n)$ be two orthonormal bases in the domain and in the target space, respectively. From

$$(\text{Id}\bowtie L)=(L^{-1}\bowtie\text{Id})\circ L$$

we get

$$\Lambda_n(\text{Id}\bowtie L)=\Lambda_n(L^{-1}\bowtie\text{Id})\circ\Lambda_n L$$

so that, as $\Lambda_n(e_1\wedge\cdots\wedge e_n)=(\det L)(\epsilon_1\wedge\cdots\wedge\epsilon_n)$, we get

$$\Lambda_n(\text{Id}\bowtie L)(e_1\wedge\cdots\wedge e_n)=(\det L)\Lambda_n(L^{-1}\bowtie\text{Id})(\epsilon_1\wedge\cdots\wedge\epsilon_n)\,,$$

which in components reads as the above Laplace's formulas. □

Definition 1.2 *For any square matrix $L\in M(n,n)$, let $\mathcal{L}_L:\Lambda_n\mathbb{R}^{n+N}\to\Lambda_n\mathbb{R}^{n+N}$ be the linear map defined by*

$$\mathcal{L}_L(\xi):=\sum_{|\alpha|+|\beta|=n}\sigma(\alpha,\overline{\alpha})\,\xi_L^{\alpha\beta}e_\alpha\wedge\varepsilon_\beta\,,\qquad\xi_L^{\alpha\beta}:=\sum_{|\gamma|=|\alpha|}\sigma(\gamma,\overline{\gamma})\,\xi^{\gamma\beta}\,M^{\overline{\gamma}}_{\overline{\alpha}}(L)\,,$$

if $\xi=\displaystyle\sum_{|\gamma|+|\beta|=n}\xi^{\gamma\beta}e_\gamma\wedge\varepsilon_\beta\in\Lambda_n\mathbb{R}^{n+N}$.

Lemma 1.3 *We have*

$$\mathcal{L}_L(M(G))=M(GL)\qquad\forall\,G\in M(N,n)\,.$$

Moreover, if $\det L\ne0$, then $\mathcal{L}_L$ is bijective and ${\mathcal{L}_L}^{-1}=\mathcal{L}_{L^{-1}}$.

PROOF: Since by the Binet's formulas $M^{\beta}_{\overline{\alpha}}(GL)=\displaystyle\sum_{|\gamma|=|\alpha|}M^{\beta}_{\overline{\gamma}}(G)\,M^{\overline{\gamma}}_{\overline{\alpha}}(L)$, by (1.2) we compute

$$M(GL)=\sum_{|\alpha|+|\beta|=n}\sigma(\alpha,\overline{\alpha})\Big(\sum_{|\gamma|=|\alpha|}M^{\beta}_{\overline{\gamma}}(G)\,M^{\overline{\gamma}}_{\overline{\alpha}}(L)\Big)e_\alpha\wedge\varepsilon_\beta\,,$$

that proves the first assertion. If $\det L\ne0$, we trivially have

$$\mathcal{L}_{L^{-1}}\circ\mathcal{L}_L(M(G))=\mathcal{L}_{L^{-1}}(M(GL))=M(GLL^{-1})=M(G)\,.$$

Using (1.4), by linearity and continuity we obtain the second assertion. □

THE PARAMETRIC POLYCONVEX L.S.C. ENVELOP. Let $e_g : B^n \times M(N,n) \to \mathbb{R}$ be the Dirichlet energy density (0.4), and let $F_g : B^n \times \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ be its *parametric polyconvex lower semicontinuous envelop* given by (0.7). Since e_g is continuous, it turns out that $F_g(x,\xi)$ *is l.s.c. in all variables and convex in* ξ *for any* x. In fact, if $\underline{e}_g : B^n \times \Sigma_1 \to \overline{\mathbb{R}}_+$ is defined, according to (1.3), by $\underline{e}_g(x,\xi) := e_g(x, G_\xi)$, taking x as a parameter, the map $\xi \mapsto F_g(x,\xi)$ agrees with the *convex l.s.c. envelop* of $\xi \mapsto \overline{e}_g(x,\xi)$,

$$F_g(x,\cdot) := \Gamma C \overline{e}_g(x,\cdot)\,,$$

where for every $x \in B^n$ we set

$$\overline{e}_g(x,\xi) := \begin{cases} \xi^{\bar{0}0}\underline{e}_g(x,\xi/\xi^{\bar{0}0}) = \xi^{\bar{0}0} e_g(x,G_\xi) & \text{if} \quad \xi \in \Sigma_+, \\ +\infty & \text{otherwise}. \end{cases}$$

For future use, we shall denote by $F : \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ the parametric polyconvex l.s.c. envelop of the standard Dirichlet integrand $G \mapsto \frac{1}{2}|G|^2$, i.e., $F = F_g$ with $g = \delta_{\alpha\beta}$, so that F does not depend on x and

$$\begin{aligned} F(\xi) = \sup\Big\{ & \phi(\xi) \mid \phi : \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+\,,\ \phi \text{ linear}, \\ & \phi(M(G)) \le \frac{1}{2}\,|G|^2 \quad \forall\, G \in M(N,n) \Big\}\,. \end{aligned} \tag{1.5}$$

Proposition 1.4 *For every* $x \in B^n$ *we have*

$$F_g(x,\xi) = F(\mathcal{L}_L(\xi)) \qquad \forall\, \xi \in \Lambda_n\mathbb{R}^{n+N}\,,$$

where $L = L(x)$ *is the unique symmetric positive definite square matrix in* $M(n,n)$ *satisfying*

$$L(x)L(x)^T = \sqrt{\det g(x)}\, g(x)^{-1}\,, \tag{1.6}$$

and $\mathcal{L}_L$ *is given by Definition* 1.2.

PROOF: If $A = A(x) \in M(n,n)$ is the positive definite symmetric square matrix given by (0.12), we actually have $LL^T = A$, i.e., $L := \sqrt{A}$ in (1.6). Therefore, by (0.11) and (0.13) we infer that

$$2\, e_g(x,G) = \mathrm{tr}(GAG^T) = \mathrm{tr}((GL)(GL)^T) = |GL|^2 \qquad \forall\, G \in M(N,n)\,. \tag{1.7}$$

Because of (0.7), this yields that for every $x \in B^n$ and $\xi \in \Lambda_n\mathbb{R}^{n+N}$

$$\begin{aligned} F_g(x,\xi) = \sup\Big\{ & \phi(\xi) \mid \phi : \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+\,,\ \phi \text{ linear}, \\ & \phi(M(G)) \le \frac{1}{2}\,|GL(x)|^2 \quad \forall\, G \in M(N,n) \Big\}\,. \end{aligned} \tag{1.8}$$

Since the matrix $L(x)$ in (1.6) is invertible, by (1.8), Lemma 1.3 and (1.5) we get

$$\begin{aligned} F_g(x,\xi) &= \sup\Big\{ \phi(\xi) \mid \phi \text{ linear},\ \phi(M(GL^{-1})) \le \frac{1}{2}|G|^2\ \ \forall\, G \in M(N,n) \Big\} \\ &= \sup\Big\{ \phi(\xi) \mid \phi \text{ linear},\ \phi \circ \mathcal{L}_{L^{-1}}(M(G)) \le \frac{1}{2}|G|^2\ \ \forall\, G \in M(N,n) \Big\} \\ &= \sup\Big\{ \phi \circ \mathcal{L}_{L^{-1}}(\mathcal{L}_L\xi) \mid \phi \text{ linear},\ \phi \circ \mathcal{L}_{L^{-1}}(M(G)) \le \frac{1}{2}|G|^2\ \ \forall\, G \in M(N,n) \Big\} \\ &= \sup\Big\{ \widetilde{\phi}(\mathcal{L}_L\xi) \mid \widetilde{\phi} \text{ linear},\ \widetilde{\phi}(M(G)) \le \frac{1}{2}|G|^2\ \ \forall\, G \in M(N,n) \Big\} = F(\mathcal{L}_L(\xi))\,, \end{aligned}$$

as required. □

An explicit formula. We are interested in writing more explicitly the polyconvex extension of the energy density e_g on simple n-vectors ξ in $\Lambda_{n-2}\mathbb{R}^n \otimes \Lambda_2\mathbb{R}^N$. We will see that for every $x \in B^n$ it agrees with the length of ξ in the metric on $\mathbb{R}^{n+N}$ given by the product of the metric $g(x)$ on $\mathbb{R}^n$ and of the Euclidean metric on $\mathbb{R}^N$. To this purpose, we recall that a metric g on $\mathbb{R}^n$ induces a metric on the whole exterior algebra. In particular, we have

$$\langle \tau, \eta \rangle_g := \langle \Lambda_k(g)\tau, \eta \rangle \qquad \forall\, \tau,\, \eta \in \Lambda_k\mathbb{R}^n\,,$$

so that

$$|\tau|_g = |\Lambda_k(g^{1/2})(\tau)| \qquad \forall\, \tau \in \Lambda_k\mathbb{R}^n\,, \tag{1.9}$$

where $g^{1/2} := \sqrt{g}$ is the unique symmetric positive definite square matrix $\widetilde{g}$ such that $\widetilde{g}^2 = g$.

Proposition 1.5 *If $\xi = \tau \wedge \eta \in \Lambda_{n-2}\mathbb{R}^n \otimes \Lambda_2\mathbb{R}^N$ and $L \in M(n,n)$ is non-singular, then*

$$\mathcal{L}_L(\tau \wedge \nu) = (\det L)(\Lambda_{n-2}L^{-1}(\tau) \wedge \eta) = \Lambda_{n-2}(g^{1/2})(\tau) \wedge \eta\,.$$

Proof: For any $\alpha \in I(n-2,n)$ and $\beta \in I(2,n)$, by Definition 1.2 we have

$$\mathcal{L}_L(e_\alpha \wedge \epsilon_\beta) = \sum_{|\gamma|=|\alpha|} \sigma(\gamma,\overline{\gamma})\, \sigma(\alpha,\overline{\alpha})\, M_{\overline{\gamma}}^{\overline{\alpha}}(L)\, e_\gamma \wedge \varepsilon_\beta\,,$$

whereas

$$\Lambda_{n-2}L^{-1}(e_\alpha) = \sum_{|\gamma|=|\alpha|} M_\alpha^\gamma(L^{-1})\, e_\gamma\,.$$

By Lemma 1.1 we thus obtain

$$\mathcal{L}_L(e_\alpha \wedge \varepsilon_\beta) = (\det L) \sum_{|\gamma|=|\alpha|} M_\alpha^\gamma(L^{-1})\, e_\gamma \wedge \varepsilon_\beta = (\det L)\,(\Lambda_{n-2}L^{-1}(e_\alpha) \wedge \varepsilon_\beta)\,.$$

The first equality follows by using an argument by linearity on the two factors $\Lambda_{n-2}\mathbb{R}^n$ and $\Lambda_2\mathbb{R}^N$. Moreover, by (1.6) we have

$$\det L = ((\det g)^{n/2}\, g^{-1})^{1/2} = (\det g)^{(n-2)/4}$$

and

$$L^{-1} = (\det g)^{-1/4}\, g^{1/2}.$$

This yields

$$(\det L)\, \Lambda_{n-2}L^{-1} = \Lambda_{n-2}(g^{1/2})$$

and hence the second equality. □

We recall from [11, Vol. II, Sec. 5.4.4], see also [19, Sec. 4.8], that *if $\xi = \tau \wedge \eta \in \Lambda_{n-2}\mathbb{R}^n \otimes \Lambda_2\mathbb{R}^N$ is simple,* and F is given by (1.5), we have

$$F(\tau \wedge \eta) = |\tau| \cdot |\eta|\,.$$

As a consequence of Propositions 1.4 and 1.5, on account of (1.9) we immediately obtain:

Theorem 1.6 *Let $\xi = \tau \wedge \eta \in \Lambda_{n-2}\mathbb{R}^n \otimes \Lambda_2\mathbb{R}^N$ be a* simple *n-vector, and let F_g be given by* (1.8). *For every $x \in B^n$ we have*

$$F_g(x, \tau \wedge \eta) = F(\Lambda_{n-2}(g^{1/2})(\tau) \wedge \eta) = |\Lambda_{n-2}(g^{1/2})(\tau)| \cdot |\eta| = |\tau|_g \cdot |\eta|\,.$$

Manifold constrained mappings. In the sequel we shall deal with mappings that are constrained to take values into a smooth manifold $\mathcal{Y}$ isometrically embedded in $\mathbb{R}^N$. To this purpose, we notice that in fact the energy density (0.1) is given by the integrand $\widehat{e}_g : B^n \times \mathbb{R}^N \times M(N,n) \to \overline{\mathbb{R}}_+$ defined by

$$\widehat{e}_g(x,u,G) := \begin{cases} e_g(x,G) & \text{if} \quad u \in \mathcal{Y} \quad \text{and} \quad G \in S_u \\ +\infty & \text{otherwise}\,, \end{cases}$$

where

$$S_u := \{G \in M(N,n) \mid G \in T_u\mathcal{Y}\}\,, \qquad u \in \mathcal{Y}\,,$$

$T_u\mathcal{Y}$ being the tangent space to $\mathcal{Y}$ at u. We denote by $\widehat{F}_g(x,u,\xi) : B^n \times \mathbb{R}^N \times \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ the parametric polyconvex l.s.c. extension of the integrand $\widehat{e}_g$. The n-vector $M(G)$ corresponding to matrices $G \in S_u$ belongs to the subspace $\Lambda_n(\mathbb{R}^N \times T_u\mathcal{Y})$. This implies the following property, compare [11, Vol. II, Sec. 1.2.4] or [19, Sec. 4.8].

Proposition 1.7 *For every $x \in B^n$ we have:*

$$\widehat{F}_g(x,u,\xi) := \begin{cases} F_g(x,\xi) & \text{if } \ u \in \mathcal{Y},\ \xi \in \Lambda_n(\mathbb{R}^n \times T_u\mathcal{Y}) \\ +\infty & \text{otherwise}\,, \end{cases} \tag{1.10}$$

where $F_g(x,\xi)$ is given by (1.8) *and $T_u\mathcal{Y}$ is the tangent space to $\mathcal{Y}$ at u.*

Since e_g is a local representation of the energy density (0.1), actually $\widehat{F}_g$ defines locally pointwise a map $\widehat{F}_g(x,u,\xi) : \mathcal{X} \times \mathbb{R}^N \times \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$

2 Cartesian currents and Dirichlet energy

Cartesian currents. Following [11] [19], we recall that an integral 2-cycle in $\mathcal{Z}_2(\mathcal{Y})$ is said to be *of spherical type* if its homology class contains a Lipschitz image of the 2-sphere $\mathbb{S}^2$. We denote by

$$H_2^{sph}(\mathcal{Y}) := \{[\gamma] \in H_2(\mathcal{Y}) \mid \exists\, \phi \in \mathrm{Lip}(\mathbb{S}^2, \mathcal{Y}) \,:\, \phi_\# [\![\, \mathbb{S}^2 \,]\!] \in [\gamma]\} \tag{2.1}$$

the spherical subgroup of $H_2(\mathcal{Y})$, and we shall also assume that $H_2(\mathcal{Y})/H_2^{sph}(\mathcal{Y})$ has no torsion. Therefore, there are generators $[\gamma_1], \dots, [\gamma_{\overline{s}}]$, i.e. integral cycles $\gamma_1, \dots, \gamma_{\overline{s}}$ in $\mathcal{Z}_2(\mathcal{Y})$, such that

$$H_2(\mathcal{Y}) = \left\{ \sum_{s=1}^{\overline{s}} n_s\,[\gamma_s] \mid n_s \in \mathbb{Z} \right\}\,,$$

see e.g. [11], Vol. I, Sec. 5.4.1, and we may and do choose the γ_s's in such a way that $[\gamma_1], \dots, [\gamma_{\widetilde{s}}]$ generate the spherical homology classes in $H_2^{sph}(\mathcal{Y})$ for some $\widetilde{s} \le \overline{s}$. By de Rham's theorem, we may and do choose a dual basis $[\sigma^1], \dots, [\sigma^{\overline{s}}]$ in $H^2_{dR}(\mathcal{Y})$ so that $\gamma_s(\sigma^r) = \delta_{sr}$, the Kronecker symbol. Also, we may and do assume that σ^s is the harmonic form in its cohomology class.

We denote by $\mathcal{D}^{k,p}(\mathcal{X} \times \mathcal{Y})$ the subspace of $\mathcal{D}^k(\mathcal{X} \times \mathcal{Y})$ of compactly supported smooth k-forms in $\mathcal{X} \times \mathcal{Y}$ of the type $\omega = \sum_{j=0}^p \omega^{(j)}$, where $\omega^{(j)}$ is the component of ω that contains exactly j differentials in the vertical $\mathcal{Y}$ variables. Also, $\mathcal{D}_{k,p}(\mathcal{X} \times \mathcal{Y})$ denotes the dual space of $\mathcal{D}^{k,p}(\mathcal{X} \times \mathcal{Y})$.

We also remark that if $T \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ the *boundary* current ∂T makes sense only as an element of the dual space of $\mathcal{Z}^{n-1,2}(\mathcal{X} \times \mathcal{Y})$, where

$$\mathcal{Z}^{k,p}(\mathcal{X} \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{k,p}(\mathcal{X} \times \mathcal{Y}) \mid d_y\omega^{(p)} = 0\} \tag{2.2}$$

and $d = d_x + d_y$ is the natural splitting of the exterior differential into a horizontal and a vertical differential.

Example 2.1 If $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$, the current G_u carried by the "graph" of u is well-defined in an approximate sense, see [11], by

$$G_u(\omega) := \int_{\mathcal{X}} (\mathrm{Id} \bowtie u)^\# \omega\,, \qquad \omega \in \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})\,,$$

where $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$, and hence $G_u \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$.

Cartesian currents in $\mathrm{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$, see Definition 2.2 below, arise as weak limit points of sequences of graphs G_{u_k} of smooth maps $u_k : \mathcal{X} \to \mathcal{Y}$ with equibounded $W^{1,2}$-norms

$$\sup_k \|u_k\|_{W^{1,2}(\mathcal{X},\mathcal{Y})} < \infty\,.$$

It turns out that every such weak limit point satisfies the *null-boundary condition*

$$\partial T(\omega)=0 \qquad \forall\, \omega\in \mathcal{Z}^{n-1,2}(\mathcal{X}\times\mathcal{Y}) \tag{2.3}$$

and decomposes as

$$T=G_{u_T}+S_T\,,\qquad S_T=\sum_{s=1}^{\widetilde{s}}\mathbb{L}_s(T)\times\gamma_s \qquad \text{on}\quad \mathcal{Z}^{n,2}(\mathcal{X}\times\mathcal{Y})\,, \tag{2.4}$$

where $u_T\in W^{1,2}(\mathcal{X},\mathcal{Y})$ and $\mathbb{L}_s(T)$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\mathcal{X})$, for every s. Setting

$$S_{T,sing}:=T-(G_{u_T}+S_T)$$

though completely vertical and homologically trivial, i.e., $S_{T,sing}(\omega)=0$ if $\omega^{(2)}=0$ or $\omega\in\mathcal{Z}^{n,2}(\mathcal{X}\times\mathcal{Y})$, in general $S_{T,sing}$ is non zero only possibly on forms $\omega\in\mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y})$ for which $d_y\omega^{(2)}\neq 0$. Moreover, even if T is the weak limit of a sequence of smooth graphs with equibounded Dirichlet energies, in principle $S_{T,sing}$ may be any measure, compare [10].

By lower semicontinuity, it turns out that every such weak limit point T has *finite* **D**-*norm*,

$$\|T\|_{\mathbf{D}}(\mathcal{X})<\infty\,,$$

where we define for any open set $\Omega\subset\mathcal{X}$

$$\begin{aligned}\|T\|_{\mathbf{D}}(\Omega) &:= \sup\Big\{T(\omega)\mid \omega\in\mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y}),\ \|\omega\|_{\mathbf{D}}\le 1,\ \operatorname{spt}\omega\subset\Omega\times\mathcal{Y}\Big\}\\ \|\omega\|_{\mathbf{D}} &:= \max\Big\{\sup_{x,y}\frac{|\omega^{(0)}(x,y)|}{1+|y|^2}\,,\int_{\mathcal{X}}\sup_y|\omega^{(1)}(x,y)|^2\,d_{vol}x\,,\int_{\mathcal{X}}\sup_y|\omega^{(2)}(x,y)|\,d_{vol}x\Big\}\,.\end{aligned} \tag{2.5}$$

Definition 2.2 *The class* $\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ *is the class of the currents* T *in* $\mathcal{D}_{n,2}(\mathcal{X}\times\mathcal{Y})$ *that satisfy the null-boundary condition* (2.3)*, have finite* **D**-*norm, and decompose as in* (2.4) *for some* $u_T\in W^{1,2}(\mathcal{X},\mathcal{Y})$ *and some i.m. rectifiable current* $\mathbb{L}_s(T)\in\mathcal{R}_{n-2}(\mathcal{X})$*, for* $s=1,\dots,\widetilde{s}$.

Remark 2.3 By the structure theorem, see e.g. [19, Thm. 4.66], the class $\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ agrees with the one considered in [9] [19].

Example 2.4 If $u\in W^{1,2}(\mathcal{X},\mathcal{Y})$, the norms $\|u\|_{W^{1,2}}$ and $\|G_u\|_{\mathbf{D}}$ are equivalent. Therefore, the current G_u belongs to $\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ if and only if

$$\partial G_u=0 \qquad \text{on}\quad \mathcal{Z}^{n-1,2}(\mathcal{X}\times\mathcal{Y}) \tag{2.6}$$

or, equivalently,

$$G_u(d\omega):=\int_{\mathcal{X}}(\mathrm{Id}\bowtie u)^{\#}d\omega=0 \qquad \forall\,\omega\in\mathcal{Z}^{n-1,2}(\mathcal{X}\times\mathcal{Y})\,.$$

Thanks to Schoen-Uhlenbeck density theorem [25], condition (2.6) is always satisfied in dimension $n=2$. However, if $n\ge 3$, for maps $u\in W^{1,2}(\mathcal{X},\mathcal{Y})$ in general (2.6) is violated. For example, if $n=3$, $\mathcal{X}=B^3$, $\mathcal{Y}=\mathbb{S}^2$, and $u(x):=x/|x|$, we have, compare [11, Vol. I, Sec. 3.2.2],

$$\partial G_u=-\delta_0\times[\![\,\mathbb{S}^2\,]\!] \qquad \text{on}\ \mathcal{D}^2(B^3\times\mathbb{S}^2)\,.$$

The Dirichlet energy on currents. Every current $T\in\mathcal{D}_{n,2}(\mathcal{X}\times\mathcal{Y})$ can be identified, in terms of its components, with the $\mathbb{R}^{c(n,N)}$-valued linear functional $T:=\big(T^{\bar{0}0},(T^{\bar{i}j}),(T^{\overline{\alpha}\beta})\big)$, where for every $\phi\in C_0^\infty(\mathcal{X}\times\mathcal{Y})$ we set $T^{\bar{0}0}(\phi):=T(\phi\,dx)$,

$$T^{\bar{i}j}(\phi):=T(\phi\,\widehat{dx^i}\wedge dy^j)\,,\qquad i=1,\dots n\,,\quad j=1,\dots N\,,$$

$$T^{\overline{\alpha}\beta}(\phi):=T(\phi\,dx^{\overline{\alpha}}\wedge dy^\beta)\,,\qquad |\alpha|=|\beta|=2\,,$$

and $c(n,N):=1+Nn+\binom{n}{2}\binom{N}{2}$. If $\|T\|_{\mathbf{D}}<\infty$, we can decompose $T=\|T\|_{\mathbf{D}}\llcorner\overrightarrow{T}$, where $\overrightarrow{T}$ is the *Radon-Nikodym derivative* of T with respect to $\|T\|_{\mathbf{D}}$.

Definition 2.5 *The Dirichlet integral* (0.2) *is extended to currents* T *in* $\mathcal{D}_{n,2}(\mathcal{X}\times\mathcal{Y})$ *with finite* **D**-*norm,* $\|T\|_{\mathbf{D}}<\infty$*, by letting*

$$\mathbf{D}_g(T):=\int \widehat{F}_g(x,u,\overrightarrow{T})\,d\|T\|_{\mathbf{D}}\,,$$

where $\widehat{F}_g(x,u,\xi)$ *is the parametric polyconvex l.s.c. extension given in local coordinates by* (1.10)*. For any measurable set* $B\subset\mathcal{X}$ *we define*

$$\mathbf{D}_g(T,B\times\mathcal{Y}):=\mathbf{D}_g(T\llcorner(B\times\mathcal{Y}))\,.$$

Remark 2.6 Since our considerations are all local, a part from the proof of Theorem 3.1 and the covering argument in the proof of Theorem 3.4, it looks convenient to look at $\mathcal{X}$ as (B^n,g). We shall then denote by $\mathbf{D}(T)$ the Dirichlet energy of T in the case $g\equiv\delta_{\alpha\beta}$, the Euclidean metric, i.e., when $e_g(G)\equiv\frac{1}{2}|G|^2$. Finally, for every map $u\in W^{1,2}(\mathcal{X},\mathcal{Y})$ we set

$$\mathbf{D}_g(u,B):=\int_B e_g(x,Du(x))\,dx\,,\qquad \mathbf{D}(u,B):=\frac{1}{2}\int_B |Du(x)|^2\,dx\,.$$

PROPERTIES. From now on we shall *assume that there exists an absolute constant* $C>0$ *such that for every* $x\in\mathcal{X}$ *and* $\tau\in\mathbb{R}^n$

$$C|\tau|^2\le|\tau|^2_{g(x)}\le\frac{1}{C}\,|\tau|^2\,,\qquad |\tau|^2_{g(x)}:=\tau^T g(x)\,\tau. \tag{2.7}$$

This clearly yields that for some absolute constant $\widetilde{C}>0$ we have

$$\widetilde{C}\,\mathbf{D}(T)\le\mathbf{D}_g(T)\le\frac{1}{\widetilde{C}}\,\mathbf{D}(T)\qquad\forall\, T\in\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})\,.$$

As a consequence of the closure theorem in [9], we readily obtain the following properties:

i) $\mathbf{D}_g(T)<\infty$ for every $T\in\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$;

ii) the functional $T\mapsto\mathbf{D}_g(T)$ is lower semicontinuous in $\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ with respect to the weak $\mathcal{D}_{n,2}$-convergence;

iii) the class $\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ is closed in the weak $\mathcal{D}_{n,2}$-convergence along sequences with equibounded $\mathbf{D}_g$-energies;

iv) $\mathbf{D}_g$-bounded sequences in $\operatorname{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ are relatively compact in the $\mathcal{D}_{n,2}$-topology.

THE g-MASS. The *g-comass* $\|\omega\|_g$ of a k-form $\omega\in\mathcal{D}^k(\mathcal{X})$ is defined by

$$\|\omega(x)\|_{g(x)}:=\sup\{\langle\omega(x),\xi\rangle\mid\xi\in\Lambda^k(T_x\mathcal{X})\text{ simple, }|\xi|_{g(x)}\le 1\}\,,\qquad x\in\mathcal{X}\,,$$

where $T_x\mathcal{X}$ is the tangent n-space to $\mathcal{X}$ at x, and the g-mass of a current $\Gamma\in\mathcal{D}_k(\mathcal{X})$ by

$$\mathbf{M}_g(\Gamma):=\sup\{\Gamma(\omega)\mid\omega\in\mathcal{D}^k(\mathcal{X})\,,\ \|\omega(x)\|_{g(x)}\le 1\ \forall\, x\in\mathcal{X}\}\,. \tag{2.8}$$

If $g(x)\equiv\delta_{\alpha\beta}$, they agree with the standard comass and mass, respectively. Moreover, if Γ is an i.m. rectifiable current in $\mathcal{R}_k(\mathcal{X})$, writing $\Gamma=\tau(\mathcal{G},\theta,\xi)$, where $\mathcal{G}$ is k-rectifiable in $\mathcal{X}$, $\theta(x)$ is an integer-valued multiplicity function on $\mathcal{G}$ and $\xi(x)$ is a simple k-vector in $\Lambda_k(T_x\mathcal{X})$, with $|\xi(x)|_{g(x)}=1$, orienting $\mathcal{G}$ at x, we have

$$\begin{aligned}\mathbf{M}_g(\Gamma)&=\sup\Big\{\int_{\mathcal{G}}\theta(x)\,\langle\omega(x),\xi(x)\rangle\,d\mathcal{H}^k\mid\omega\in\mathcal{D}^k(\mathcal{X})\,,\ \|\omega(x)\|_{g(x)}\le 1\ \forall\, x\in\mathcal{X}\Big\}\\&=\int_{\mathcal{G}}\theta(x)\,d\mathcal{H}^k(x)\,.\end{aligned}$$

Remark 2.7 For future use, we point out that in local coordinates, e.g. when $\mathcal{X}$ is equal to (B^n, g), the g-mass of a current $\Gamma = \tau(\mathcal{G}, \theta, \xi)$, where $|\xi| \equiv 1$ in the Euclidean metric, agrees with

$$\mathbf{M}_g(\Gamma) = \int_{\mathcal{G}} \theta(x)\, |\xi(x)|_{g(x)}\, d\mathcal{H}^k(x)\,.$$

AN EXPLICIT FORMULA. Assume now that $T \in \text{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$ can be decomposed as in (2.4) on the whole of $\mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})$, where $u_T \in W^{1,2}(\mathcal{X}, \mathcal{Y})$ and $\mathbb{L}_s(T) \in \mathcal{R}_{n-2}(\mathcal{X})$. Write $\mathbb{L}_s(T) = \tau(\mathcal{L}_s, \theta_s, \tau_s)$, where $\mathcal{L}_s$ is $(n-2)$-rectifiable in $\mathcal{X}$, $\theta_s(x)$ is an integer-valued multiplicity function on $\mathcal{L}_s$ and $\tau_s(x)$ is a simple $(n-2)$-vector in $\Lambda_{n-2}\mathbb{R}^n$ orienting $\mathcal{L}_s$ at x, with $|\tau_s(x)|_{g(x)} = 1$. In this case, for every Borel set $B \subset \mathcal{X}$ we have

$$\mathbf{M}_g(\mathbb{L}_s(T) \llcorner B) = \int_{\mathcal{L}_s \cap B} \theta_s(x)\, d\mathcal{H}^{n-2}(x)\,. \tag{2.9}$$

Arguing as for the standard Dirichlet integral $\mathbf{D}(T)$, we then compute explicitly:

Proposition 2.8 *For every Borel set $B \subset \mathcal{X}$ we have*

$$\mathbf{D}_g(T, B \times \mathcal{Y}) = \mathbf{D}_g(u_T, B) + \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\gamma_s) \cdot \mathbf{M}_g(\mathbb{L}_s(T) \llcorner B)\,. \tag{2.10}$$

PROOF: If $\eta_s \in \Lambda_2\mathbb{R}^N$ yields an orientation to γ_s at $u \in \mathcal{Y}$, and $|\eta_s| = 1$, the simple n-vector $\tau_s \wedge \eta_s$ yields an orientation to $\mathbb{L}_s(T) \times \gamma_s$ at (x, u). By Theorem 1.6 and Proposition 1.7 we have

$$\widehat{F}_g(x, u, \tau_s \wedge \eta_s) = |\tau_s|_{g(x)} \cdot |\eta| = 1\,.$$

Due to Definition 2.5, using the same argument as for the standard Dirichlet integral, compare [11, Vol. II, Sec. 5.4.4] or [19, Sec. 4.9], we obtain

$$\mathbf{D}_g(T, B \times \mathcal{Y}) = \int_B e_g(x, Du_T)\, dx + \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\gamma_s) \cdot \int_{\mathcal{L}_s \cap B} \theta_s(x)\, d\mathcal{H}^{n-2}(x)\,.$$

The assertion follows from (2.9). □

THE CASE OF CONSTANT METRICS. Assume now that the metric g is constant, so that $e_g(x, G) \equiv e_g(G)$ in (0.5). Equivalently, compare Remark 0.4, assume that $A(x) \equiv A$ is a constant positive definite symmetric matrix in $M(n, n)$. If $g \equiv \delta_{\alpha\beta}$, the Euclidean metric, i.e., if A is the identity matrix, then

$$e_g(G) \equiv \frac{1}{2}\, |G|^2 \qquad \forall\, G \in M(N, n)\,.$$

Therefore, the energy $\mathbf{D}_g(T)$ agrees with the standard Dirichlet energy $\mathbf{D}(T)$ and for every Borel set $B \subset \mathcal{X}$ we clearly have

$$\mathbf{D}(T, B \times \mathcal{Y}) = \frac{1}{2} \int_B |Du_T|^2\, dx + \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\gamma_s) \cdot \mathbf{M}(\mathbb{L}_s(T) \llcorner B)\,.$$

In the case $(\mathcal{X}, g_{\alpha\beta}) = (B^n, \delta_{\alpha\beta})$, the following density result holds true, compare [17] [19, Sec. 5.4]:

Theorem 2.9 *For every $T \in \text{cart}^{2,1}(B^n \times \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ and $\frac{1}{2}\int_{B^n} |Du_k|^2\, dx \to \mathbf{D}(T)$ as $k \to \infty$.*

Remark 2.10 In dimension $n \geq 3$, the hypothesis on the Hurewicz maps is a necessary condition to strong approximability by smooth sequences. In fact, if the Hurewicz homomorphism $\pi_2(\mathcal{Y}; y_0) \to H_2(\mathcal{Y}; \mathbb{R})$ is not injective, there are maps in $W^{1,2}(B^3, \mathcal{Y})$ that are smooth outside the origin, i.e., with only one point singularity, which cannot be approximated weakly with the **D**-energy by graphs of smooth maps, even if G_u satisfies the null-boundary condition (2.6), i.e., $G_u \in \text{cart}^{2,1}(B^3, \mathcal{Y})$, compare [19, Sec. 5.3].

In the case of constant metrics g, the following link with the standard Dirichlet energy clearly holds true. For every $T \in \text{cart}^{2,1}(B^n \times \mathcal{Y})$, we will denote by $T_L := (L^{-1} \bowtie \text{Id}_{\mathbb{R}^N})_\# T$ the Cartesian current in $\text{cart}^{2,1}(L^{-1}(B^n) \times \mathcal{Y})$ given by the push forward of T by means of the linear map $(L^{-1} \bowtie \text{Id}_{\mathbb{R}^N})(x,y) := (L^{-1}x, y)$, where L is given by (1.6), i.e.,

$$T_L(\widetilde{\omega}) := T((L^{-1} \bowtie \text{Id}_{\mathbb{R}^N})^\# \widetilde{\omega}), \qquad \widetilde{\omega} \in \mathcal{D}^{n,2}(L^{-1}(B^n) \times \mathcal{Y}).$$

Notice that if $T = G_{u_T}$ for some Sobolev map $u_T \in W^{1,2}(B^n, \mathcal{Y})$, then

$$(L^{-1} \bowtie \text{Id}_{\mathbb{R}^N})_\# G_{u_T} = G_{v_T},$$

where $v_T : L^{-1}(B^n) \to \mathcal{Y}$ is given by $v_T(\widetilde{x}) := u_T(L\widetilde{x})$. This yields that the function v_T corresponding to T_L agrees with $u_T \circ L$.

Proposition 2.11 *Assume that the metric g is constant on B^n. Let $T \in \text{cart}^{2,1}(B^n \times \mathcal{Y})$ be such that* (2.4) *holds in the whole of $\mathcal{D}^{n,2}(B^n \times \mathcal{Y})$. For every Borel set $B \subset B^n$ we have*

$$\mathbf{D}_g(T, B \times \mathcal{Y}) = (\det L) \cdot \mathbf{D}(T_L, L^{-1}(B) \times \mathcal{Y}),$$

where L is given by (1.6)*. In particular, if $T = G_{u_T}$ for some $u_T \in W^{1,2}(B^n, \mathcal{Y})$, then*

$$\int_{B^n} e_g(Du_T(x))\, dx = (\det L) \cdot \frac{1}{2} \int_{L^{-1}(B^n)} |Dv_T(\widetilde{x})|^2\, d\widetilde{x}, \qquad v_T(\widetilde{x}) := u_T(L\widetilde{x}).$$

PROOF: The isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ induces an isomorphism $L^\# : W^{1,2}(B^n, \mathcal{Y}) \to W^{1,2}(L^{-1}(B^n), \mathcal{Y})$ and an isomorphism map $(L^{-1} \bowtie \text{Id}_{\mathbb{R}^N})_\#$ between $\mathcal{D}^{n,2}(B^n \times \mathcal{Y})$ onto $\mathcal{D}^{n,2}(L^{-1}(B^n) \times \mathcal{Y})$. Moreover, it is easily seen that $T \in \text{cart}^{2,1}(B^n \times \mathcal{Y})$ if and only if $T_L := (L^{-1} \bowtie \text{Id}_{\mathbb{R}^N})_\# T$ belongs to $\text{cart}^{2,1}(L^{-1}(B^n) \times \mathcal{Y})$ and

$$e_g(G) = (\det L) \cdot \frac{1}{2} |G \circ L|^2 \qquad \forall\, G \in M(N,n).$$

This yields the assertions. □

3 A density result for the Dirichlet energy

In this section and in the next one we shall prove a density result for the Dirichlet energy. As before, *we assume that for any $y_0 \in \mathcal{Y}$ the Hurewicz homomorphism from $\pi_2(\mathcal{Y}; y_0)$ onto $H_2(\mathcal{Y}; \mathbb{R})$ is injective.* Moreover, we assume that the metric $g(x)$ is continuous in $\mathcal{X}$ and satisfies the bound (2.7). Finally, *we assume that $\mathcal{X}$ satisfies the 1-extension property with respect to $\mathcal{Y}$*. Alternatively, we may assume that $\mathcal{X}$ *is* 1*-connected*, i.e., that $\pi_1(\mathcal{X}) = 0$.

Theorem 3.1 *Let $T \in \text{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$ be such that*

$$T = G_{u_T} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T) \times \gamma_s \qquad \text{on} \quad \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y}), \tag{3.1}$$

where $u_T \in W^{1,2}(\mathcal{X}, \mathcal{Y})$ and $\mathbb{L}_s(T)$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\mathcal{X})$, for every s. There exists a sequence of smooth maps $\{u_k\} \subset C^1(\mathcal{X}, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ as $k \to \infty$ and

$$\lim_{k \to \infty} \int_{\mathcal{X}} e_g(x, Du_k(x))\, dx = \mathbf{D}_g(T).$$

Remark 3.2 Since the metric function $x \mapsto g(x)$ is continuous in $\mathcal{X}$, whereas

$$G \mapsto \frac{e_g(x,G) - e_g(x_0,G)}{|G|^2}$$

is positively homogeneous of degree zero, it turns out that there exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\omega(t) \to 0$ if $t \to 0$, such that for every $x, x_0 \in \mathcal{X}$ and every $G \in M(N,n)$

$$|e_g(x,G) - e_g(x_0,G)| \leq \omega(|x - x_0|) \cdot |G|^2. \tag{3.2}$$

Remark 3.3 If $(\mathcal{X}, g_{\alpha\beta}) = (B^n, g_{\alpha\beta})$, and the metric g is constant on B^n, we immediately deduce Theorem 3.1. In fact, setting $T_L := (L^{-1} \bowtie \mathrm{Id}_{\mathbb{R}^N})_\# T$, by Theorem 2.9 we find a sequence $\{v_k\} \subset C^1(L^{-1}(B^n), \mathcal{Y})$ such that $G_{v_k} \rightharpoonup T_L$ weakly in $\mathcal{D}_{n,2}$ and $\frac{1}{2}\int_{L^{-1}(B^n)} |Dv_k|^2 \, d\widetilde{x} \to \mathbf{D}(T_L, L^{-1}(B^n) \times \mathcal{Y})$ as $k \to \infty$. It then suffices to apply Proposition 2.11, by taking $u_k := v_k \circ L^{-1}$.

In the case of dimension $n = 2$, the proof of Theorem 3.1 is an easy adaptation of the one from [13], by using the continuity of the metric g and Proposition 4.2 below, so we omit to write it. In the case of higher dimension $n \geq 3$, we shall use arguments taken from the density theorem in [18] [20], see also [19], and we shall adapt the dipole construction to our situation, Theorem 3.8.

To this purpose, for every current $T \in \mathrm{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$ satisfying (3.1), with $\mathbb{L}_s(T) \in \mathcal{R}_{n-2}(\mathcal{X})$ for every s, we will denote by μ_T the finite Radon measure on $\mathcal{X}$ given for every Borel set $B \subset \mathcal{X}$ by

$$\mu_T(B) := \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\gamma_s) \cdot \mathbf{M}_g(\mathbb{L}_s(T) \cap B)\,, \tag{3.3}$$

so that we have

$$\mathbf{D}_g(T, B \times \mathcal{Y}) = \mathbf{D}_g(u_T, B) + \mu_T(B)\,.$$

For any T as above, we will also denote by $\mathbf{F}(T)$ the *flat norm*

$$\mathbf{F}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})\,,\ \mathbf{F}(\omega) \leq 1\}\,,$$

where for every $\omega \in \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})$

$$\mathbf{F}(\omega) := \max\Big\{ \sup_{z \in \mathcal{X} \times \mathcal{Y}} \|\omega(z)\|\,,\ \sup_{z \in \mathcal{X} \times \mathcal{Y}} \|d\omega(z)\| \Big\}\,.$$

As $|T(\omega)| \leq \mathbf{F}(T)\,\mathbf{F}(\omega)$, we infer that $T_k \rightharpoonup T$ weakly in $\mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ provided that $\mathbf{F}(T_k - T) \to 0$.

Moreover, following [9], see also [19, Sec. 4.6], if $T \in \mathrm{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$ decomposes as in Theorem 3.1, we can write

$$T = G_u + S_T\,, \quad S_T := \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q \qquad \text{on} \quad \mathcal{D}^{n,2}(\mathcal{X} \times \mathcal{Y})\,.$$

Here, $u = u_T \in W^{1,2}(\mathcal{X}, \mathcal{Y})$ and every $\mathbb{L}_q$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\mathcal{X})$ with multiplicity one such that, writing $\mathbb{L}_q = \tau(\mathcal{L}_q, 1, \tau_q)$, the $(n-2)$-rectifiable sets $\mathcal{L}_q := \mathrm{set}(\mathbb{L}_q)$ are pairwise disjoint and $|\tau_q(x)|_{g(x)} = 1$ for all $x \in \mathcal{L}_q$. Moreover, $R_q \in \mathcal{Z}_2(\mathcal{Y})$ is an integral 2-cycle of spherical type in the homology class q. As a consequence, on account of (2.10), the Dirichlet energy of T can be equivalently written for every Borel set $B \subset \mathcal{X}$ as

$$\mathbf{D}_g(T, B) := \mathbf{D}_g(T, B \times \mathcal{Y}) = \mathbf{D}_g(u, B) + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbf{M}(R_q) \cdot \mathbf{M}_g(\mathbb{L}_q \llcorner B)\,. \tag{3.4}$$

By (3.3) and (3.4) we then infer that the rectifiable measure μ_T satisfies

$$\mu_T = \theta_T\, \mathcal{H}^{n-2} \llcorner \mathcal{L}_T\,,$$

where $\mathcal{L}_T$ is the $(n-2)$-rectifiable set $\mathcal{L}_T := \bigcup_{q \in H_2^{sph}(\mathcal{Y})} \mathrm{set}(\mathbb{L}_q)$, so that $\mathcal{H}^{n-2}(\mathcal{L}_T) < \infty$, and the density $\theta_T : \mathcal{L}_T \to [0, +\infty)$ is the non-negative $\mathcal{H}^{n-2} \llcorner \mathcal{L}_T$-measurable function on $\mathcal{L}_T$ given by

$$\theta_T(x) := \mathbf{M}(R_q) \qquad \text{if} \quad x \in \mathrm{set}(\mathbb{L}_q)\,.$$

Finally, by the smoothness and compactness of $\mathcal{Y}$ we infer that the function $x \mapsto \theta_T(x)$ is uniformly bounded on $\mathcal{L}_T$, and by an *isoperimetric theorem*, see e.g. [19, Thm. 1.101], there exist an absolute constant C_1, depending on $\mathcal{Y}$ and μ_T, such that

$$0 < C_1 \leq \theta_T(x) \leq C_2 < \infty \qquad \forall\, x \in \mathcal{L}_T\,. \tag{3.5}$$

Similarly to [19, Sec. 5.4], the proof of Theorem 3.1 is based on the following

Theorem 3.4 *Let $T \in \text{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ be as in Theorem 3.1. Let $\varepsilon \in (0,1/2)$ and $k\in\mathbb{N}$. We can find a current $\widetilde{T}\in \text{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ of the type in (3.1), with $\mathbb{L}_s(\widetilde{T})\in\mathcal{R}_{n-2}(\mathcal{X})$ for every s, such that*

$$\mathbf{D}_g(\widetilde{T}) \le \mathbf{D}_g(T)+\varepsilon^k\,,$$
$$\mathbf{F}(\widetilde{T}-T)\le\varepsilon^k \qquad \textit{and} \qquad \mu_{\widetilde{T}}(\mathcal{X})\le \frac{1}{2}\cdot\mu_T(\mathcal{X})\,.$$

Proof of Theorem 3.1: By Theorem 3.4, using a diagonal argument, we find a sequence $\{T_k\}\subset \text{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ that weakly converges to T in $\mathcal{D}_{n,2}$ with $\mathbf{D}_g(T_k)\to\mathbf{D}_g(T)$ as $k\to\infty$ and such that $\mu_{T_k}(\mathcal{X})=0$. Therefore, T_k agrees with the current G_{u_k} given by the integration of forms in $\mathcal{D}^{n,2}(\mathcal{X}\times\mathcal{Y})$ over the "graph" of some $u_k\in W^{1,2}(\mathcal{X},\mathcal{Y})$, see Example 2.4, and hence $\mathbf{D}_g(T_k)=\mathbf{D}_g(u_k)$.

If $(\mathcal{X},g_{\alpha\beta})=(B^n,g_{\alpha\beta})$, by means of Bethuel's density theorem [2], for every k we find a smooth sequence $\{u_h^{(k)}\}_h\subset C^1(B^n,\mathcal{Y})$ that strongly converges to u_k in the $W^{1,2}$-sense as $h\to\infty$. In fact, even if the second homotopy group $\pi_2(\mathcal{Y})$ is non-trivial, the injectivity hypothesis on the Hurewicz homomorphisms from $\pi_2(\mathcal{Y};y_0)$ onto $H_2(\mathcal{Y};\mathbb{R})$, in conjunction with the null-boundary condition (2.6) for u_k, and the bound (2.7) for the energy, allows us to remove the $(n-2)$-dimensional singularities, compare [4] and e.g. [19, Sec. 5.3]. Lower dimensional singularities are removed as in [2]. By the dominated convergence theorem, we infer that the strong convergence yields $G_{u_h^{(k)}}\rightharpoonup G_{u_k}$ with $\mathbf{D}_g(u_h^{(k)})\to\mathbf{D}_g(u_k)$. Theorem 3.1 then follows by means of a diagonal argument.

More generally, if $\mathcal{X}$ satisfies the 1-extension property with respect to $\mathcal{Y}$, or if $\pi_1(\mathcal{X})=0$, using arguments from [21], the hypothesis on the Hurewicz homomorphisms, in conjunction with the null-boundary condition (2.6) for u_k, plays the role of the triviality of $\pi_2(\mathcal{Y})$, and we find again a smooth sequence $\{u_h^{(k)}\}_h\subset C^1(\mathcal{X},\mathcal{Y})$ that strongly converges to u_k in the $W^{1,2}$-sense as $h\to\infty$. More precisely, if the Hurewicz homomorphisms from $\pi_2(\mathcal{Y};y_0)$ onto $H_2(\mathcal{Y};\mathbb{R})$ are injective, it turns out that $\mathcal{X}$ has the 1-extension property with respect to $\mathcal{Y}$ if and only if $\mathcal{X}$ has the 2-extension property with respect to $\mathcal{Y}$, compare [21, Lemma 6.4] for the case of $\partial\mathcal{X}=\emptyset$, and [22] if $\partial\mathcal{X}\neq\emptyset$. □

In order to prove Theorem 3.4, we need the following.

Slicing properties. Let $T\in\text{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$ be as in Theorem 3.4. Similarly to the case of normal currents, for every point $x_0\in\mathcal{X}$ and for a.e. radius $r\in(0,r_0)$, where $r_0=r_0(x)>0$ is sufficiently small, in dependence of x, the *sliced current*

$$\langle T,d_{x_0},r\rangle = \langle G_{u_T},d_{x_0},r\rangle + \langle S_T,d_{x_0},r\rangle\,,$$

where $d_{x_0}(x,y)=\delta_{x_0}(x):=\text{dist}_{\mathcal{X}}(x_0,x)$, is a well-defined Cartesian current in $\text{cart}^{2,1}(\partial B_r(x_0)\times\mathcal{Y})$, where $B_r(x_0)$ denotes the *geodesic ball* of radius r centered at x_0, and $\partial B_r(x_0)$ its boundary. More precisely, see Example 2.1, we have

$$\langle G_{u_T},d_{x_0},r\rangle(\omega)=\int_{\partial B_r(x_0)}(\text{Id}\bowtie u_{|\partial B_r(x_0)})^{\#}\omega\,,\qquad \omega\in\mathcal{D}^{n-1,2}(\partial B_r(x_0)\times\mathcal{Y})\,,$$

where $u_{|\partial B_r(x_0)}$ is the restriction of u to $\partial B_r(x_0)$, which is a function in $W^{1,2}(\partial B_r(x_0),\mathcal{Y})$. Also,

$$\langle S_T,d_{x_0},r\rangle=\sum_{q\in H_2^{sph}(\mathcal{Y})}\langle\mathbb{L}_q,\delta_{x_0},r\rangle\times R_q \qquad\text{on}\quad \mathcal{D}^{n-1,2}(\partial B_r(x_0)\times\mathcal{Y})\,.$$

As a consequence, we infer that for every Borel set $B\subset\mathcal{X}$ the Dirichlet energy of $\langle T,d_{x_0},r\rangle$ on $B\times\mathcal{Y}$ is given by

$$\mathbf{D}_g(\langle T,d_{x_0},r\rangle,B\times\mathcal{Y})=\mathbf{D}_g(u_{|\partial B_r(x_0)},B)+\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbf{M}(R_q)\cdot\mathbf{M}_g(\langle\mathbb{L}_q,\delta_{x_0},r\rangle\llcorner B) \tag{3.6}$$

where $\mathbf{D}_g(u_{|\partial B_r(x_0)},B)$ can be written in local coordinates in a way similar to (0.4) (0.5), by using the distributional derivative D_τ w.r.t. an orthonormal frame τ tangential to $\partial B_r(x_0)$. For example, in the case $g_{\alpha\beta} = \delta_{\alpha\beta}$, we clearly have

$$\mathbf{D}_g(u_{|\partial B_r(x_0)}, B) = \frac{1}{2} \int_{\partial B_r(x_0)\cap B} |D_\tau u_{(r,x_0)}|^2 \, d\mathcal{H}^{n-1} \,.$$

We also let

$$\mathbf{D}_g(\langle T, d_{x_0}, r\rangle) := \mathbf{D}_g(\langle T, d_{x_0}, r\rangle, \partial B_r(x_0) \times \mathcal{Y}) \,.$$

Remark 3.5 For future use, we denote by

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \operatorname{dist}(y, \mathcal{Y}) \leq \varepsilon\}$$

the ε-neighborhood of $\mathcal{Y}$ and we observe that, since $\mathcal{Y}$ is smooth and compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the *nearest point projection* Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $L_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$. For $y \in \mathcal{Y}$ and $0 < \varepsilon < \varepsilon_0$ we denote by

$$B_{\mathcal{Y}}(y, \varepsilon) := \overline{B}^N(y, \varepsilon) \cap \mathcal{Y}$$

the intersection of $\mathcal{Y}$ with the closed N-ball of radius ε centered at y, so that $\Pi_\varepsilon(\overline{B}^N(y, \varepsilon)) = B_{\mathcal{Y}}(y, \varepsilon)$. Moreover, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^N \to B_{\mathcal{Y}}(y, \varepsilon)$ be the retraction map given by $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if} \quad z \in \overline{B}^N(y, \varepsilon) \\ \varepsilon \dfrac{z - y}{|z - y|} & \text{if} \quad z \in \mathbb{R}^N \setminus \overline{B}^N(y, \varepsilon) \end{cases} \tag{3.7}$$

so that $\Psi_{(y,\varepsilon)}$ is a Lipschitz continuous function with $\operatorname{Lip} \Psi_{(y,\varepsilon)} = \operatorname{Lip} \Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

The proof of Theorem 3.4 is based on the following local arguments. First, Proposition 3.6, we show how to "deform" a current T, satisfying suitable energy estimates on the boundary of a ball, into a current satisfying a bound on the oscillation. Secondly, Proposition 3.7 and Theorem 3.8, we use a local approximation argument. In the sequel we will denote by $c > 0$ an absolute constant, possibly varying from line to line.

Projecting the image of a current. For $n \geq 3$, we set

$$B^n_\rho := B^n(0, \rho) \,, \qquad x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^{n-2} \times \mathbb{R}^2 \,, \qquad D_\rho := B^{n-2}(0_{\mathbb{R}^{n-2}}, \rho) \,.$$

Proposition 3.6 *Let* $0 < R < d < 1$ *and* $T \in \operatorname{cart}^{2,1}(B^n_d \times \mathcal{Y})$ *be such that*

$$\begin{aligned} &\mathbf{D}_g(\langle T, d_0, R\rangle, \partial B^n_R \setminus (\overline{D}_R \times \{0\})) \leq c\, \sigma\, \theta_T(0) R^{n-3} \,, \\ &\mathbf{D}_g(\langle T, d_0, R\rangle) \leq c\, \theta_T(0)\, R^{n-3} \,, \\ &\int_{\partial B_R} |u_T(x) - y|^2 \, d\mathcal{H}^{n-1} \leq c\, \sigma\, R^{n-1} \end{aligned} \tag{3.8}$$

for some $y \in \mathcal{Y}$ *and for* $\sigma > 0$ *small enough. Then there exists an absolute constant* $c > 0$ *such that, if* $q \in \mathbb{N}^+$ *is the integer part of* $c\, \sigma^{\alpha(n)}$*, where* $\alpha(n) := 1/(6(2 - n)) < 0$*, we can find a Cartesian current* $\widetilde{T} \in \operatorname{cart}^{2,1}((B^n_R \setminus \overline{B}^n_r) \times \mathcal{Y})$*, where* $r = R(1 - 1/q)$*, such that the following facts hold:*

(*a*) $\langle \widetilde{T}, d_0, R\rangle = \langle T, d_0, R\rangle$ *and* $\langle \widetilde{T}, d_0, r\rangle = (\psi_{R,r} \bowtie \Psi_{(y,\varepsilon_\sigma)})_\# \langle T, d_0, R\rangle$*, where* $\varepsilon_\sigma := c \cdot \sigma^{2/3}$*,* $\psi_{R,r}(x) := rx/R$*, and* $\Psi_{(y,\varepsilon_\sigma)}(z) := \Pi_{\varepsilon_\sigma} \circ \xi_{(y,\varepsilon_\sigma)}$*, see* (3.7)*, so that* $\operatorname{spt}\langle \widetilde{T}, d_0, r\rangle \subset \partial B^n_r \times B_{\mathcal{Y}}(y, \varepsilon_\sigma)$*;*

(*b*) $\widetilde{T}$ *has small energy on* $B^n_R \setminus B^n_r$*, i.e.,*

$$\mathbf{D}_g(\widetilde{T}, B^n_R \setminus B^n_r) \leq c\, \frac{R}{q}\, \mathbf{D}_g(\langle T, d_0, R\rangle) \,; \tag{3.9}$$

(*c*) *we finally have*

$$\mathbf{F}((\widetilde{T} - G_y) \llcorner (B^n_R \setminus \overline{B}^n_r) \times \mathcal{Y}) \leq c \cdot \frac{\sigma}{q} \cdot R^n \leq c \cdot \sigma \cdot d^{n-1} \,. \tag{3.10}$$

Proof: We use an argument very similar to the one in Step 3 of [20]. Roughly speaking, using the first inequality in (3.8), we can find a suitable subdivision of ∂B^n_R in a grid made of small $(n-1)$-dimensional "cubes" of side R/q. Denoting by Σ^k_R the union of the k-faces of the grid that do not intersect $\overline{D}_R \times \{0\}$, we may and do estimate the energy of the restriction of $\langle T, d_0, R\rangle$ to $\Sigma^k_R \times \mathcal{Y}$ by $c\,\sigma\,\theta_T(0)\,R^{k-2}$, for every $k = 1,\dots,n-2$. In particular, if $\sigma^{1/3} < 1/C_2$, see (3.5), the energy of the restriction of $\langle T, d_0, R\rangle$ to $\Sigma^2_R \times \mathcal{Y}$ is smaller than $c\,\sigma^{2/3}$. Now, by the above cited isoperimetric theorem, the infimum of the mass of the nontrivial spherical cycles R_q in $H^{sph}_2(\mathcal{Y})$ is bounded from below by a positive constant. Therefore, taking $\sigma > 0$ small, it turns out that the restriction of $\langle T, d_0, R\rangle$ to $\Sigma^2_R \times \mathcal{Y}$ has no vertical part, hence agrees with the current carried by the graph of a $W^{1,2}$-map w with values into $\mathcal{Y}$. As a consequence, by using the bound (2.7) we obtain that

$$\int_{\Sigma^1_R} |Dw_{|\Sigma^1_R}|^2 \, d\mathcal{H}^1 \le c\,\sigma^{2/3}\,\frac{1}{R}\,.$$

The grid of ∂B^n_R being made of q^{n-1} cubes of side R/q, we have $\mathcal{H}^1(\Sigma^1_R) \le c\,R\,q^{n-2}$ and hence, by the Hölder inequality,

$$\int_{\Sigma^1_R} |Dw_{|\Sigma^1_R}| \, d\mathcal{H}^1 \le c\,q^{(n-2)/2}\,\sigma^{1/3} \le c\,\sigma^{1/4}\,,$$

provided that $q \in \mathbb{N}^+$ is chosen as in the thesis. By using the third inequality in (3.8) and the above formula, we infer that we may and do assume that the oscillation of $w_{|\Sigma^1_R}$ is smaller than $c\,\sigma^{1/4}$ and that the image $w(\Sigma^1_R)$ is contained in the geodesic ball $B_{\mathcal{Y}}(y, \varepsilon_\sigma)$.

Therefore, using the argument of Step 3 of [20], we may and do define the current $\widetilde{T}$ satisfying the above properties. However, since we deal with currents acting on k-forms in $\mathcal{D}^{k,2}$, i.e., with two vertical differentials in the $\mathcal{Y}$-direction, when extending $\widetilde{T}$ from the 2-skeleton to the 3-skeleton of a partition of $B^n_R \setminus B^n_r$ in "cubes", it turns out that $\widetilde{T}$ has a non-zero boundary of the type $\delta_{x_l} \times S_l$ for each 3-face F_l of such a cubeulation, where x_l is the barycenter of F_l and S_l is the integral 2-cycle in $\mathcal{Y}$ given by $w_\# [\![I_l]\!] - \Psi_{(y,\varepsilon_\sigma)} \circ w_\# [\![I_l]\!]$, where I_l is the 2-face of F_l that intersects the boundary ∂B^n_R. By the rectangular inequality, it turns out that the mass of S_l is lower than twice the Dirichlet energy of $w_{|I_l}$, which is small with σ, by construction. Therefore, using again the cited isoperimetric theorem, we infer that for $\sigma > 0$ small the 2-cycle S_l is homologically trivial in $\mathcal{Y}$, and hence we can find an integral 3-current R_l in $\mathcal{Y}$ such that $\partial R_l = S_l$ and $\mathbf{M}(R_l) \le c_n\,\mathbf{M}(S_l)^{3/2}$. As a consequence, in case of dimension $n = 3$, by adding the terms $-\delta_{x_l} \times R_l$ for each 3-cube F_l, we may and do define the current T_l with no interior boundary, by paying an amount of energy that is bounded by the energy of the restriction of $\langle T, d_0, R\rangle$ to the 2-skeleton Σ^2_R. In dimension $n \ge 4$, for $k = 4,\dots,n$, no extra-boundary is produced when extending $\widetilde{T}$ from the $(k-1)$-skeleton to the k-skeleton of the cubes of the partition of $B^n_R \setminus B^n_r$, as $(k-1)$-currents of the type $\delta_x \times R$, where $R \in \mathcal{D}_{k-1}(\mathcal{Y})$, are always zero when tested on forms in $\mathcal{D}^{k-1,2}$.

Arguing this way on the cubes of $B^n_R \setminus B^n_r$ that do not intersect $\overline{D}_R \times \{0\}$ yields a bound of the energy of $\widetilde{T}$ in terms of

$$c\,\frac{R}{q}\,\mathbf{D}_g(\langle T, d_0, R\rangle, \partial B^n_R \setminus (\overline{D}_R \times \{0\}))$$

and hence in terms of the right-hand side of (3.9).

Using a slightly different argument when defining $\widetilde{T}$ on the cubes of $B^n_R \setminus B^n_r$ that intersect $\overline{D}_R \times \{0\}$, by the second inequality in (3.8) we obtain an extra term in the estimate of the energy of $\widetilde{T}$ given by the right-hand side of (3.9).

By the third inequality in (3.8), by the construction of $\widetilde{T}$, and since $0 < R < d < 1$, we obtain the bound (3.10) of the flat distance, whereas property (a) follows by using the argument of Step 3 of [20]. □

Approximation on a ball. Let $y(\widetilde{x}) := (r - |\widetilde{x}|)$ denote the distance of $\widetilde{x}$ from the boundary of the $(n-2)$-disk D_r and

$$\phi_\delta(x) := (\widetilde{x}, \varphi_\delta(y(\widetilde{x}))\,\widehat{x})\,, \quad x \in D_r \times \overline{B}^2\,, \quad \varphi_\delta(y) := \min\{y, \delta\}\,, \tag{3.11}$$

so that $\Omega_\delta := \phi_\delta(D_r \times \overline{B}^2)$ is a small neighborhood of the interior of the disk $D_r \times \{0_{\mathbb{R}^2}\}$ in B^n_R. Also, let

$$\widetilde{\Omega}_\delta := \phi_\delta(D_r \times \overline{B}^2_{1/2}) = \{(\widetilde{x}, \widehat{x}) \mid \widetilde{x} \in D_r\,,\ \rho \le \varphi_\delta(y(\widetilde{x}))/2\}\,, \tag{3.12}$$

where in the sequel $\rho := |\widehat{x}| = \sqrt{x_{n-1}{}^2 + x_n{}^2}$, and

$$\Omega_{(r,\delta)} := \Omega_\delta \setminus (D_r \times \{0_{\mathbb{R}^2}\}) .$$

In the proof of Theorem 3.4 we shall make use of the following

Proposition 3.7 *Let* $T \in \text{cart}^{2,1}(B^n_r \times \mathcal{Y})$ *be such that* $T = G_u + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q$. *Assume that* $\text{spt}\, T \subset \overline{B}^n_r \times B_{\mathcal{Y}}(y, \varepsilon_\sigma)$, *where* $y \in \mathcal{Y}$ *and* $\varepsilon_\sigma = c \cdot \sigma^{2/3}$, *with* $\sigma > 0$ *small, and that* $D_r \times \{0_{\mathbb{R}^2}\} \subset \text{set}(\mathbb{L}_{q_0})$ *for some* $q_0 \in H_2^{sph}(\mathcal{Y})$. *For* $\delta > 0$ *small enough, we can find a current* $\widetilde{T} \in \text{cart}^{2,1}((B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y})$ *satisfying the following properties:*

i) $\partial(\widetilde{T} \llcorner (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}) = \partial(T \llcorner B^n_r \times \mathcal{Y}) - [\![\widetilde{\Omega}_\delta]\!] \times \delta_y - [\![\partial D_r \times \{0_{\mathbb{R}^2}\}]\!] \times R_{q_0}$;

ii) $\mathbf{D}_g(\widetilde{T}, (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}) \le \mathbf{D}_g(u, (B^n_r \setminus \Omega_\delta)) + c\,\sigma\, r^{n-2} + c\,\mu_T(\Omega_{(r,\delta)})$;

iii) $\mathbf{F}((\widetilde{T} - T) \llcorner (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}) \le c\,\sigma\, r^{n-2}$.

Proof: Let $\psi_\delta : \Omega_\delta \setminus \widetilde{\Omega}_\delta \to \Omega_{(r,\delta)}$ be the bijective map

$$\psi_\delta(\widetilde{x}, \widehat{x}) := \left(\widetilde{x}, \left(2 - \frac{\varphi_\delta(y(\widetilde{x}))}{\rho}\right) \widehat{x}\right).$$

Similarly to [19, Sec. 5.5], we infer that the current

$$\overline{T} := ((\psi_\delta)^{-1} \bowtie \text{Id}_{\mathbb{R}^N})_\#(T \llcorner (\text{int}(\Omega_{(r,\delta)}) \times \mathcal{Y}))$$

belongs to $\text{cart}^{2,1}(\text{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y})$, its underlying $W^{1,2}$-function is $v := u_T \circ \psi_\delta : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to B_{\mathcal{Y}}(y, \varepsilon_\sigma)$, where $u_T : B^n_r \to B_{\mathcal{Y}}(y, \varepsilon_\sigma)$ is the $W^{1,2}$-function corresponding to T, and

$$\mu_{\overline{T}}(\text{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \le \mu_T(\text{int}(\Omega_{(r,\delta)})) .$$

Setting then $w : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to \mathbb{R}^N$ by

$$w(x) := \left(\frac{2\rho}{\varphi_\delta(y(\widetilde{x}))} - 1\right) \cdot v(x) + \left(2 - \frac{2\rho}{\varphi_\delta(y(\widetilde{x}))}\right) \cdot y ,$$

by using the bound (2.7) and the fact that the oscillation of v is small with $\sigma > 0$, we infer that the energy $\mathbf{D}_g(w, \Omega_\delta \setminus \widetilde{\Omega}_\delta)$ is small if δ and σ are small. Moreover, by projecting w into the manifold $\mathcal{Y}$, we may and will assume that w belongs to $W^{1,2}(\Omega_\delta \setminus \widetilde{\Omega}_\delta, \mathcal{Y})$.

We then may and do define a current $\widehat{T} \in \text{cart}^{2,1}(\text{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y})$, with underlying $W^{1,2}$-function w, that satisfies the boundary condition

$$\partial\widehat{T} = \partial(T \llcorner \Omega_\delta \times \mathcal{Y}) - [\![\partial\widetilde{\Omega}_\delta]\!] \times \delta_y - [\![\partial D_r \times \{0_{\mathbb{R}^2}\}]\!] \times R_{q_0}$$

and, taking δ small, the energy estimate

$$\mathbf{D}_g(\widehat{T}, \text{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y}) \le c\,\sigma\, r^{n-2} + c\,\mu_T(\Omega_{(r,\delta)}) .$$

We finally set

$$\widetilde{T} := T \llcorner (B^n_r \setminus \text{int}(\Omega_\delta)) \times \mathcal{Y} + \widehat{T} \llcorner (\text{int}(\Omega_\delta) \setminus \widetilde{\Omega}_\delta) \times \mathcal{Y} .$$

Property iii) readily follows, for $\delta > 0$ small. □

The dipole construction. We shall finally make use of the following theorem, the proof of which is postponed to the next section.

Theorem 3.8 *Let $C \in \mathcal{Z}_2(\mathcal{Y})$ be an integral 2-cycle of spherical type and $y \in \mathcal{Y}$ be a given point. For every $\sigma > 0$ there exists a function $v_\sigma \in W^{1,2}(\widetilde{\Omega}_\delta, \mathcal{Y})$, with $\delta > 0$ sufficiently small, such that $G_{v_\sigma} \in$* $\text{cart}^{2,1}(\text{int}(\widetilde{\Omega}_\delta) \times \mathcal{Y})$,

$$\int_{\widetilde{\Omega}_\delta} e_g(0, Dv_\sigma)\, dx \le \sigma\, r^{n-2} + |\tau|_{g(0)} \cdot \mathcal{H}^{n-2}(D_r) \cdot \mathbf{M}(C)\,, \tag{3.13}$$

where $\tau := e_1 \wedge \cdots \wedge e_{n-2} \in \Lambda_{n-2}\mathbb{R}^n$, and

$$\partial G_{v_\sigma} = \partial [\![\, \widetilde{\Omega}_\delta \,]\!] \times \delta_y + [\![\, \partial D_r \times \{0_{\mathbb{R}^2}\} \,]\!] \times C\,. \tag{3.14}$$

Moreover, $v_{\sigma\#}[\![\, \widetilde{\Omega}_\delta \,]\!] \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$, as $\sigma \to 0^+$.

We are now ready to give the

Proof of Theorem 3.4: Applying arguments as for instance in the proof of Federer [7, 4.2.19], by [7, 3.2.29] there exists a countable family $\mathcal{G}$ of $(n-2)$-dimensional C^1-submanifolds $\mathcal{M}_j$ of $\mathcal{X}$ such that μ_T-almost all of $\mathcal{X}$ is covered by $\mathcal{G}$.

Let $\sigma \in (0,1)$ to be fixed. By the Vitali-Besicovitch theorem, and by the properties of the class $\text{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$, we can find a number $t = t_\sigma \in (1/2, 1)$, a countable disjoint family of closed geodesic balls $B_j := \overline{B}(p_j, r_j)$, contained in $\mathcal{X}$ and centered at points p_j in $\mathcal{L}_T$, satisfying the properties listed below. In the sequel we will denote by $c > 0$ an absolute constant, possibly varying from line to line, which is independent of σ and of the radii r_j of the balls B_j.

i) $\mu_T(\mathcal{X} \setminus \bigcup_j B_j) = 0$.

ii) For every j there is a manifold $\mathcal{M}_j$ of $\mathcal{G}$ such that the center p_j of B_j belongs to $\mathcal{M}_j$.

iii) Since $\mathcal{H}^{n-2}(\mathcal{L}_T) < \infty$, then
$$\sum_{j=1}^{\infty} {r_j}^{n-2} \le c \cdot \mathcal{H}^{n-2}(\mathcal{L}_T) < \infty\,. \tag{3.15}$$

iv) We have
$$\mu_T(B(p_j, r_j) \setminus (B(p_j, tr_j) \cap \mathcal{M}_j)) \le \sigma \cdot \mu_T(B(p_j, r_j)) \qquad \forall\, j\,. \tag{3.16}$$

v) We have $\mathcal{M}_j \subset \text{set}(\mathbb{L}_q)$ for some $q = q_j \in H_2^{sph}(\mathcal{Y})$.

vi) All the p_j's are Lebesgue points of $u = u_T$ and of Du, with Lebesgue values $u(p_j) = z_j$, and by a slicing argument
$$\int_{\partial B(p_j, tr_j)} |u(x) - z_j|^2\, d\mathcal{H}^{n-1} \le c \cdot \sigma\, {r_j}^{n-1}\,. \tag{3.17}$$

vii) Using a blow-up argument at p_j in the x-variables, we may and do assume that the current $S_j := [\![\, B_j \,]\!] \times \delta_{z_j} + [\![\, \mathcal{M}_j \,]\!] \times R_{q_j}$ has small flat distance from T on $B_j \times \mathcal{Y}$, i.e.
$$\mathbf{F}((S_j - T) \llcorner B_j \times \mathcal{Y}) \le c \cdot \sigma \cdot {r_j}^{n-2}\,. \tag{3.18}$$

viii) By a slicing argument, we may and will assume that for some $R \in (tr_j, 2tr_j)$ the current $\langle T, d_{p_j}, tr_j \rangle$ belongs to $\text{cart}^{2,1}$ and satisfies
$$\mathbf{D}_g(\langle T, d_{p_j}, tr_j \rangle, \partial B(p_j, tr_j) \setminus \mathcal{M}_j) \le \frac{c}{r_j} \cdot \mathbf{D}_g(T, B(p_j, R) \setminus \mathcal{M}_j)\,.$$

Moreover, by the construction, and by the bound (2.7), we may assume that both (3.16) and
$$\mu_T(B(p_j, \rho)) \le c\, \theta_T(p_j)\, \rho^{n-2}\,, \qquad \mathbf{D}_g(u, B(p_j, \rho)) \le c\, |Du(p_j)|^2\, \rho^n \tag{3.19}$$
hold true for any $0 < \rho < 2r_j$. Therefore, taking r_j small so that $|Du(p_j)|^2\, {r_j}^2 \le \sigma\, \theta_T(p_j)$, we readily obtain that
$$\mathbf{D}_g(\langle T, d_{p_j}, tr_j \rangle, \partial B(p_j, tr_j) \setminus \mathcal{M}_j) \le c\, \sigma\, \theta_T(p_j)\, {r_j}^{n-3}\,. \tag{3.20}$$

ix) Using a similar slicing argument and (3.19), we also may and do assume that

$$\mathbf{D}_g(\langle T, d_{p_j}, tr_j\rangle) \le c\,\theta_T(p_j)\, {r_j}^{n-3}\,. \tag{3.21}$$

x) By the continuity property (3.2), we may take the radii r_j sufficiently small so that for every $x \in B_j$

$$|e_g(x,G) - e_g(p_j,G)| \le \sigma\,|G|^2 \qquad \forall\, G \in M(N,n)\,. \tag{3.22}$$

xi) Since $\theta_T(p_j)$ is the $(n-2)$-dimensional density of μ_T at p_j, and $p_j \in \text{set}(\mathbb{L}_q)$, we also may and will assume that

$$|\mu_T(B_j) - \mathbf{M}(R_q)\cdot \omega_{n-2}\, {r_j}^{n-2}| \le \sigma\cdot\omega_{n-2}\, {r_j}^{n-2}\,. \tag{3.23}$$

xii) There exists a bilipschitz homeomorphism ψ_σ from $\mathcal{X}$ onto itself, with $\text{Lip}\,\psi_\sigma \le 2$ and $\text{Lip}\,\psi_\sigma^{-1} \le 2$, such that ψ_σ maps bijectively B_j onto B_j, with $\psi_{\sigma|\partial B_j} = \text{Id}_{|\partial B_j}$, for all j, and ψ_σ is equal to the identity outside the union of the balls B_j.

xiii) For every j, $\psi_\sigma(B(p_j,t_\sigma r_j)\cap\mathcal{M}_j) = B(p_j,\rho_j)\cap(p_j+\text{Tan}(\mathcal{M}_j,p_j))$ and $\psi_\sigma(\partial B(p_j,t_\sigma r_j)) = \partial B(p_j,\rho_j)$, where $\rho_j \in (r_j/2, r_j)$ and $\text{Tan}(\mathcal{M}_j,p_j)$ is the $(n-2)$-dimensional tangent space to $\mathcal{M}_j$ at p_j.

Setting now for any j

$$T_j^\sigma := (\psi_\sigma \bowtie \text{Id}_{\mathbb{R}^N})_\# T \llcorner \text{int}(B_j)\times\mathcal{Y}\,,$$

T_j^σ belongs to $\text{cart}^{2,1}(\text{int}(B_j)\times\mathcal{Y})$, with underlying function $u_j^\sigma := (u_T\circ\psi_\sigma^{-1})_{|\,\text{int}(B_j)}$ in $W^{1,2}(\text{int}(B_j),\mathcal{Y})$, and $\mu_{T_j^\sigma} = \psi_{\sigma\#}(\mu_T\llcorner\text{int}(B_j))$. Moreover, by (3.20), (3.21), and (3.17) we readily infer that T_j^σ satisfies (3.8), where $y = z_j \in \mathcal{Y}$ is the Lebesgue value of u_T at p_j, with $p_j = 0$, $d = r_j$ and $R = \rho_j$, i.e.,

$$B_j = \overline{B}^n_d\,,\quad B(p_j,\rho_j) = B^n_R\,,\quad B(p_j,\rho_j)\cap(p_j+\text{Tan}(\mathcal{M}_j,p_j)) = D_R\times\{0\}\subset\mathbb{R}^{n-2}\times\mathbb{R}^2\,.$$

Proposition 3.6 yields a Cartesian current $\widetilde{T}_j \in \text{cart}^{2,1}((B(p_j,\rho_j)\setminus\overline{B}(p_j,\delta_j))\times\mathcal{Y})$, where $\delta_j := \rho_j(1-1/q)$. Set now $\beta(n) := 1/(12(n-2)) > 0$. Since $1/q \le c\,\sigma^{1/(6(n-2))}$, by (3.9), (3.8), and (3.21), taking $\sigma > 0$ small so that $\sigma^{\beta(n)} < 1/C_2$, see (3.5), we readily obtain that

$$\mathbf{D}_g(\widetilde{T}_j, B(p_j,\rho_j)\setminus\overline{B}(p_j,\delta_j)) \le c\ \sigma^{\beta(n)}\ {\rho_j}^{n-2}\,, \tag{3.24}$$

whereas by (3.10)

$$\mathbf{F}((\widetilde{T}_j - G_{z_j})\llcorner(B(p_j,\rho_j)\setminus\overline{B}(p_j,\delta_j))\times\mathcal{Y}) \le c\cdot\sigma\cdot {r_j}^{n-1}\,. \tag{3.25}$$

Setting now

$$\breve{T}_j^\sigma := (\psi_j \bowtie \Psi_{(z_j,\varepsilon_\sigma)})_\#(T_j^\sigma\llcorner\overline{B}(p_j,\rho_j)\times\mathcal{Y})\,,$$

where $\psi_j(x) := p_j + \dfrac{\delta_j}{\rho_j}(x-p_j)$, we have $\text{spt}\,\breve{T}_j^\sigma \subset \overline{B}(p_j,\delta_j)\times B_{\mathcal{Y}}(z_j,\varepsilon_\sigma)$, whence $\breve{T}_j^\sigma$ satisfies the hypotheses of Proposition 3.7, with $B(p_j,\delta_j)$ instead of B^n_r, $y = z_j$, and $q_0 = q_j$, that yields a current $\widehat{T}_j^\sigma \in \text{cart}^{2,1}((B(p_j,\delta_j)\setminus\widetilde{\Omega}^j_\delta)\times\mathcal{Y})$, where $\widetilde{\Omega}^j_\delta$ is defined similarly to (3.12), but in correspondence of $B(p_j,\delta_j)$.

Moreover, by applying Theorem 3.8, with $B(p_j,\delta_j)$ instead of B^n_r and $C = R_{q_j}$, we find a suitable function $v_j^\sigma \in W^{1,2}(\widetilde{\Omega}^j_\delta,\mathcal{Y})$. Setting then

$$\overline{T}_j^\sigma := \widehat{T}_j^\sigma + G_{v_j^\sigma}\,,$$

(3.14) and i) in Proposition 3.7 yield that $\overline{T}_j^\sigma \in \text{cart}^{2,1}(B(p_j,\delta_j)\times\mathcal{Y})$ and that

$$\partial(\overline{T}_j^\sigma\llcorner B(p_j,\delta_j)\times\mathcal{Y}) = \partial(\breve{T}_j^\sigma\llcorner B(p_j,\delta_j)\times\mathcal{Y})\,. \tag{3.26}$$

Moreover, according to Remark 2.7, by (3.13) we have

$$\int_{\widetilde{\Omega}_\delta} e_g(p_j, Dv_j^\sigma)\,dx \le \sigma\,{\delta_j}^{n-2} + \mathcal{H}^{n-2}(D_{r_j})\cdot\mathbf{M}(R_q)\,.$$

Therefore, since $\delta_j \in (r_j/2, r_j)$, by (3.23) we obtain that

$$\int_{\widetilde{\Omega}_\delta} e_g(p_j, Dv_j^\sigma)\, dx \le c\,\sigma\, {r_j}^{n-2} + \mu_T(B_j)\,. \tag{3.27}$$

On the other hand, as $0 < \sigma < 1$, by (3.22), (2.7), and (3.27) we obtain

$$\begin{aligned} &\left| \int_{\widetilde{\Omega}_\delta} e_g(x, Dv_j^\sigma)\, dx - \int_{\widetilde{\Omega}_\delta} e_g(p_j, Dv_j^\sigma)\, dx \right| \le \\ &\le \sigma \int_{\widetilde{\Omega}_\delta} |Dv_j^\sigma|^2\, dx \le c\,\sigma \int_{\widetilde{\Omega}_\delta} e_g(p_j, Dv_j^\sigma)\, dx \\ &\le c\,\sigma\,(\mu_T(B_j) + {r_j}^{n-2}) \end{aligned}$$

where $c > 0$ is an absolute constant. Therefore, if $\delta > 0$ is small, (3.27) yields

$$\int_{\widetilde{\Omega}_\delta} e_g(x, Dv_j^\sigma)\, dx \le c\,\sigma\, {r_j}^{n-2} + (1 + c\,\sigma)\, \mu_T(B_j)\,.$$

Finally, using (3.16) to estimate the last term in the right-hand side of ii) in Proposition 3.7, we obtain

$$\mathbf{D}_g(\overline{T}_j^\sigma, B(p_j, \delta_j) \times \mathcal{Y}) \le \mathbf{D}_g(u_j^\sigma, B(p_j, \delta_j)) + c\,\sigma\, {r_j}^{n-2} + (1 + c\,\sigma)\, \mu_T(B(p_j, \delta_j))\,. \tag{3.28}$$

We now set

$$\widetilde{T}_j^\sigma := \overline{T}_j^\sigma + \widetilde{T}_j + T_j^\sigma \llcorner (B(p_j, r_j) \setminus B(p_j, \rho_j)) \times \mathcal{Y}\,.$$

Property (a) in Proposition 3.6, the definition of $\breve{T}_j^\sigma$, and (3.26) yield that $\widetilde{T}_j^\sigma$ belongs to $\mathrm{cart}^{2,1}(\mathrm{int}(B_j) \times \mathcal{Y})$. Moreover, by (3.24) and (3.28) we obtain that

$$\mathbf{D}_g(\widetilde{T}_j^\sigma, B_j \times \mathcal{Y}) \le \mathbf{D}_g(T_j^\sigma, B_j \times \mathcal{Y}) + c\,\sigma^{\beta(n)}\, {r_j}^{n-2} + c\,\sigma \mu_{T_j^\sigma}(B_j)\,. \tag{3.29}$$

Finally, arguing as in [19, Sec. 5.5, Step 3], by (3.10), property iii) in Proposition 3.7, and by the dipole construction, Theorem 3.8, we obtain that for ε, δ small enough

$$\mathbf{F}((\widetilde{T}_j^\sigma - T_j^\sigma) \llcorner B_j \times \mathcal{Y}) \le c \cdot \sigma \cdot {r_j}^{n-2}\,.$$

Setting now

$$T_j^{(\sigma)} := (\psi_\sigma^{-1} \bowtie \mathrm{Id}_{\mathbb{R}^N})_\# \widetilde{T}_j^{(\sigma)} \llcorner \mathrm{int}(B_j) \times \mathcal{Y}\,,$$

by (3.29) we infer that for every j

$$\begin{aligned} \mathbf{D}_g(T_j^{(\sigma)}, \mathrm{int}(B_j) \times \mathcal{Y}) \;\le\;& \mathbf{D}_g(u_T, B_j) \\ +\;& (1 + c\,\sigma)\, \mu_T(B_j) + c\,\sigma^{\beta(n)}\, {r_j}^{n-2} \end{aligned} \tag{3.30}$$

whereas

$$\mathbf{F}((T_j^{(\sigma)} - T) \llcorner B_j \times \mathcal{Y}) \le c \cdot \sigma \cdot {r_j}^{n-2}\,. \tag{3.31}$$

In conclusion, setting $T^\sigma \in \mathcal{D}_{n,2}(\mathcal{X} \times \mathcal{Y})$ by

$$T^\sigma := \sum_{j=1}^\infty T_j^{(\sigma)} + T \llcorner \big(\mathcal{X} \setminus \bigcup_{j=1}^\infty \mathrm{int}(B_j)\big) \times \mathcal{Y}\,,$$

we have $T^\sigma \in \mathrm{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$. By (3.30) and the hypothesis $\sum_{j=1}^\infty {r_j}^{n-2} \le c \cdot \mathcal{H}^{n-2}(\mathcal{L}_T)$ we obtain that

$$\mathbf{D}_g(T^\sigma) \le \mathbf{D}_g(u_T) + (1 + c\,\sigma)\, \mu_T(\mathcal{X}) + c\,\sigma^{\beta(n)}\, \mathcal{H}^{n-2}(\mathcal{L}_T)\,,$$

so that if $\sigma = \sigma(\varepsilon, k, \mathcal{L}_T, \mu_T) > 0$ is small, we have

$$\mathbf{D}_g(T^\sigma) \le \mathbf{D}_g(T) + \varepsilon^k\,.$$

Moreover, by (3.16), taking σ small, the above construction yields that

$$\begin{aligned}\mu_{T^\sigma}(\mathcal{X}) &\leq c\sum_{j=1}^{\infty}\mu_T(B(p_j,r_j)\setminus(B(p_j,tr_j)\cap\mathcal{M}_j))+\mu_T(\mathcal{X}\setminus\mathcal{L}_T)\\ &\leq c\,\sigma\,\mu_T(\mathcal{X})<\frac{1}{2}\cdot\mu_T(\mathcal{X})\,.\end{aligned}$$

Also, by (3.31) and (3.15) we have

$$\begin{aligned}\mathbf{F}(T^\sigma-T) &\leq \sum_{j=1}^{\infty}\mathbf{F}((T_j^{(\sigma)}-T)\llcorner B_j\times\mathcal{Y})\\ &\leq c\cdot\sigma\sum_{j=1}^{\infty}{r_j}^{n-2}<\varepsilon^k\,,\end{aligned}$$

if $\sigma=\sigma(\varepsilon,k,\mathcal{L}_T,\mu_T)>0$ is small. Taking $\widetilde{T}=T^\sigma$ for $\sigma>0$ small, the proof is complete. $\square$

4 The dipole construction

In this section we shall prove Theorem 3.8.

Set $\Omega:=D_r\times B^2_{1/2}$, and assume that $u\in W^{1,2}(\Omega,\mathcal{Y})$ only depends on the last two variables,

$$u=u(\widehat{x})\,,\qquad x=(\widetilde{x},\widehat{x})\in\mathbb{R}^{n-2}\times\mathbb{R}^2\,.$$

By Fubini's theorem, for every $0<\rho<r$ we have

$$\int_{D_\rho\times B^2_{1/2}}e_g(0,Du(x))\,dx=\mathcal{H}^{n-2}(D_\rho)\cdot\int_{B^2_{1/2}}e_g(0,Du(\widehat{x}))\,d\widehat{x}\,.$$

Now, writing $u:=\widetilde{u}\circ L^{-1}$, $L=L(0)$, by (1.7) we have

$$e_g(0,Du(\widehat{x}))=\frac{1}{2}|D\widetilde{u}(z)|^2\,,\qquad z:=L^{-1}x\,.$$

Let $\{v_1,\dots,v_n\}\subset\mathbb{R}^n$ be a $g(0)$-orthogonal basis given by eigenvectors of the matrix $g(0)$, and let $S\in M(n,n)$ be given by $S^i_j:=v^i_j$, where $v_j:=(v^1_j,\dots,v^n_j)$. Since τ orients the $(n-2)$-disk D_r, it turns out that $v\in W^{1,2}(L^{-1}(\Omega),\mathcal{Y})$ only depends on the orthogonal directions to $S^T\tau$. Setting $\widetilde{e}_i:=S^Te_i$, this means that

$$\widetilde{u}(z)=F(z^{n-1},z^n)\,,\qquad z=\sum_{i=1}^{n}z^i\,\widetilde{e}_i \tag{4.1}$$

for some function $F\in W^{1,2}(\widetilde{D},\mathcal{Y})$, where $\widetilde{D}:=L^{-1}(\{0\}\times B^2_{1/2})$. On the other hand, since $\widehat{x}=\widehat{L}z$, where $\widehat{L}\in M(2,n)$ is the matrix of the last two rows of L, by a change of variable we find that

$$\int_{B^2_{1/2}}e_g(0,Du(\widehat{x}))\,d\widehat{x}=|M_{(2)}\widehat{L}|\cdot\frac{1}{2}\int_{\widetilde{D}}|DF|^2\,d\mathcal{H}^2\,, \tag{4.2}$$

where $|M_{(2)}\widehat{L}|$ is the 2-*dimensional Jacobian* of $\widehat{L}$. In addition, we obtain:

Lemma 4.1 *We have* $|M_{(2)}\widehat{L}|=|\tau|_g$, *where* $g=g(0)$.

Proof: Setting $\alpha_0:=(1,\dots,n-2)\in I(n-2,n)$, we have

$$|M_{(2)}\widehat{L}|^2=\sum_{|\gamma|=n-2}M^{\overline{\alpha}_0}_{\overline{\gamma}}(L)^2\,,$$

whereas by (1.9) and Proposition 1.5

$$|\tau|_g = (\det L)\, |\Lambda_{n-2} L^{-1}(\tau)| \,, \qquad L = L(0)\,, \quad g = g(0)\,.$$

Since $\Lambda_{n-2} L^{-1}(\tau) = L^{-1}e_1 \wedge \cdots \wedge L^{-1}e_{n-2}$, we compute

$$\Lambda_{n-2} L^{-1}(\tau) = \sum_{|\gamma|=n-2} M^{\gamma}_{\alpha_0}(L^{-1})\, e_\gamma \,.$$

Moreover, Lemma 1.1 yields

$$(\det L)\, M^{\gamma}_{\alpha_0}(L^{-1}) = \sigma(\gamma, \overline{\gamma})\, \sigma(\alpha_0, \overline{\alpha}_0)\, M^{\overline{\alpha}_0}_{\overline{\gamma}}(L)\,,$$

so that we obtain

$$|\tau|_g^2 = \sum_{|\gamma|=n-2} (\det L)^2\, M^{\gamma}_{\alpha_0}(L^{-1})^2 = \sum_{|\gamma|=n-2} M^{\overline{\alpha}_0}_{\overline{\gamma}}(L)^2$$

and hence the assertion. □

We now make use of following proposition, that was essentially proved in [13], see also [19, Sec. 5.1]. As before, we let $\widetilde{D} := L^{-1}(\{0\} \times B^2_{1/2})$.

Proposition 4.2 *Let $C \in \mathcal{Z}_2(\mathcal{Y})$ be an integral 2-cycle of spherical type and $y \in \mathcal{Y}$ be a given point. There exists a family of Lipschitz functions $F^y_\varepsilon : \widetilde{D} \to \mathcal{Y}$ such that ${F^y_\varepsilon}_{|\partial \widetilde{D}} \equiv y$ and*

$$\frac{1}{2} \int_{\widetilde{D}} |DF^y_\varepsilon|^2 \, d\mathcal{H}^2 \le \mathbf{M}(C) + \varepsilon \,.$$

Moreover, the 2-cycle $C_\varepsilon := F^y_{\varepsilon\#} [\![\widetilde{D}]\!]$ in $\mathcal{Z}_2(\mathcal{Y})$ does not depend on the choice of $y \in \mathcal{Y}$, and $C_\varepsilon \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$ with $\mathbf{M}(C_\varepsilon) \to \mathbf{M}(C)$, as $\varepsilon \to 0$.

As a consequence, taking $F = F^y_\varepsilon$ in (4.1), by (4.2) and Lemma 4.1 we obtain $u_\varepsilon \in W^{1,2}(\Omega, \mathcal{Y})$ such that for every $\rho \in (0, r]$

$$\int_{D_\rho \times B^2_{1/2}} e_g(0, Du_\varepsilon)\, dx \le \mathcal{H}^{n-2}(D_\rho) \cdot |\tau|_{g(0)} \cdot (\mathbf{M}(C) + \varepsilon)\,. \tag{4.3}$$

The following lemma is proved in a way similar e.g. to the one in [19, Sec. 5.5], by using the bound (2.7).

Lemma 4.3 *Let $0 < \delta < 1$ and $u^\varepsilon_\delta := u_\varepsilon \circ \phi_\delta^{-1} : \widetilde{\Omega}_\delta \to \mathcal{Y}$, where ϕ_δ is given by (3.11). Then we have*

$$\int_{\widetilde{\Omega}_\delta} e_g(0, Du^\varepsilon_\delta)\, dx \le \int_{D_r \times B^2_{1/2}} e_g(0, Du_\varepsilon)\, dx + c \int_{(D_r \setminus D_{r-\delta}) \times B^2_{1/2}} e_g(0, Du_\varepsilon)\, dx\,,$$

where $c > 0$ is an absolute constant.

Proof of Theorem 3.8: On account of (4.3), we obtain the energy estimate

$$\int_{\widetilde{\Omega}_\delta} e_g(0, Du^\varepsilon_\delta)\, dx \le (\mathcal{H}^{n-2}(D_r) + c\, \mathcal{H}^{n-2}(D_r \setminus D_{r-\delta})) \cdot |\tau|_{g(0)} \cdot (\mathbf{M}(C) + \varepsilon)$$

and hence, setting $v_\sigma := u^\varepsilon_\delta$ for $\varepsilon > 0$ sufficiently small, and for δ sufficiently small in dependence of ε and of the Lipschitz constant of F^y_ε, we get (3.13), whereas (3.14) and the last assertion in Theorem 3.8 trivially follow. □

5 The relaxed Dirichlet energy

In this section, as an application of the density theorem from Sec. 3, we give a representation formula for the relaxed energy (0.9), Propositions 5.5 and 5.6. Of course, we shall assume that the manifolds $\mathcal{X}$ and $\mathcal{Y}$ satisfy the hypotheses of Theorem 3.1.

To our purpose, we may and do consider *equivalence classes* of Cartesian currents. More precisely, if $T, \widetilde{T} \in \mathrm{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$, see Definition 2.2, we say that

$$T \sim \widetilde{T} \iff T(\omega) = \widetilde{T}(\omega) \quad \forall\, \omega \in \mathcal{Z}^{n,2}(\mathcal{X}\times\mathcal{Y})\,, \tag{5.1}$$

the forms in $\mathcal{Z}^{n,2}(\mathcal{X}\times\mathcal{Y})$ being defined as in (2.2). We also say that $T_k \rightharpoonup T$ *weakly in* $\mathcal{Z}_{n,2}$ if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{Z}^{n,2}(\mathcal{X}\times\mathcal{Y})$. It is easily checked that equivalent currents have the same underlying $W^{1,2}$-function, i.e.,

$$T \sim \widetilde{T} \implies u_T = u_{\widetilde{T}} \in W^{1,2}(\mathcal{X},\mathcal{Y})\,. \tag{5.2}$$

Moreover, if T and $\widetilde{T}$ are decomposed as in (2.4), then

$$T \sim \widetilde{T} \implies \mathbb{L}_s(T) = \mathbb{L}_s(\widetilde{T}) \in \mathcal{R}_{n-2}(\mathcal{X}) \quad \forall\, s = 1,\ldots,\widetilde{s}\,.$$

Definition 5.1 *Denote by* $\mathrm{CART}^{2,1}(\mathcal{X}\times\mathcal{Y})$ *the family of all the equivalence classes of Cartesian currents in* $\mathrm{cart}^{2,1}(\mathcal{X}\times\mathcal{Y})$*, where the equivalence relation is given by* (5.1)*.*

On account of (2.10), we also set:

Definition 5.2 *Let* $T \in \mathrm{CART}^{2,1}(\mathcal{X}\times\mathcal{Y})$*, one of its representatives being decomposed as in* (2.4)*, where* $\mathbb{L}_s(T) \in \mathcal{R}_{n-2}(\mathcal{X})$*. For every open set* $\Omega \subset \mathcal{X}$ *we define the* Dirichlet energy *of* T *by*

$$\mathbf{D}_g(T, \Omega\times\mathcal{Y}) := \mathbf{D}_g(u_T, \Omega) + \sum_{s=1}^{\widetilde{s}} \mathbf{M}_g(\mathbb{L}_s(T) \llcorner \Omega)\cdot M_s\,,$$

where M_s *is the mass of the mass minimizing integral spherical* 2*-cycle in the homology class* $[\gamma_s]$*, i.e.,*

$$M_s := \min\{\mathbf{M}(C) \mid C \in \mathcal{Z}_2(\mathcal{Y})\,, \quad C \in [\gamma_s]\}\,. \tag{5.3}$$

Remark 5.3 For the sake of simplicity, in this section we denote by T an equivalence class of currents. We also notice that the weak convergence $T_k \rightharpoonup T$ in $\mathcal{Z}_{n,2}$ is well-defined for Cartesian currents in $\mathrm{CART}^{2,1}(\mathcal{X}\times\mathcal{Y})$ as the weak $\mathcal{Z}_{n,2}$-convergence of any representative of T_k to any representative of T.

One checks:

i) the class $\mathrm{CART}^{2,1}$ is closed under the weak convergence in $\mathcal{Z}_{n,2}$ with equibounded Dirichlet energies;

ii) the Dirichlet energy is lower semicontinuous with respect to the weak $\mathcal{Z}_{n,2}$-convergence in $\mathrm{CART}^{2,1}$;

iii) if $\{T_k\} \subset \mathrm{CART}^{2,1}$ satisfies $\sup_k \mathbf{D}_g(T) < \infty$, possibly passing to a subsequence, T_k weakly converges in $\mathcal{Z}_{n,2}$ to some current T in $\mathrm{CART}^{2,1}$;

iv) if $\mathcal{Y} = \mathbb{S}^2$ or, more generally, $\mathcal{Y}$ has dimension $m = 2$, we have $\mathrm{CART}^{2,1} = \mathrm{cart}^{2,1}$.

A REPRESENTATION FORMULA. Arguing as in Theorem 3.1, we readily prove the following.

Theorem 5.4 *For every* $T \in \mathrm{CART}^{2,1}(\mathcal{X}\times\mathcal{Y})$ *there exists a sequence of smooth maps* $\{u_k\} \subset C^1(\mathcal{X},\mathcal{Y})$ *such that* $G_{u_k} \rightharpoonup T$ *weakly in* $\mathcal{Z}_{n,2}$ *and* $\mathbf{D}_g(u_k) \to \mathbf{D}_g(T)$ *as* $k \to \infty$*.*

For every $u \in W^{1,2}(\mathcal{X},\mathcal{Y})$, we now denote by

$$\mathcal{T}^{2,1}_u := \{T \in \mathrm{CART}^{2,1}(\mathcal{X}\times\mathcal{Y}) \mid u_T = u\} \tag{5.4}$$

the class of Cartesian current in $\mathrm{CART}^{2,1}(\mathcal{X}\times\mathcal{Y})$ such that the underlying $W^{1,2}$-function u_T in the decomposition (2.4) is equal to u, compare (5.2). As a consequence of Theorems 0.3 and 5.4, we obtain:

Proposition 5.5 *For every* $u \in W^{1,2}(\mathcal{X}\times\mathcal{Y})$ *the class* $\mathcal{T}^{2,1}_u$ *is non-empty. Moreover, for every open set* $\Omega \subset \mathcal{X}$ *we have*

$$\widetilde{\mathbf{D}}_g(u,\Omega) = \inf\{\mathbf{D}_g(T,\Omega\times\mathcal{Y}) \mid T \in \mathcal{T}^{2,1}_u\} < \infty\,. \tag{5.5}$$

Proof: Let $\{u_k\}\subset C^1(\mathcal{X},\mathcal{Y})$ be such that $u_k \rightharpoonup u$ weakly in $W^{1,2}$, see Theorem 0.3. Since by (2.7)

$$\widetilde{C}\int_{\mathcal{X}} |Du_k|^2\,dx \le \mathbf{D}_g(u_k) \le \frac{1}{\widetilde{C}}\int_{\mathcal{X}} |Du_k|^2\,dx\,,$$

the relaxed energy $\widetilde{\mathbf{D}}_g(u)$ is always finite. By closure-compactness, possibly passing to a subsequence $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,2}$ to some current $T \in \mathrm{CART}^{2,1}(\mathcal{X}\times\mathcal{Y})$ such that $u_T = u$, whence $\mathcal{T}^{2,1}_u$ is non-empty. As to the second assertion, since $\widetilde{\mathbf{D}}_g(u)<\infty$, by closure-compactness, for every $\varepsilon>0$ we find a sequence $\{u_k\}\subset C^1(\mathcal{X},\mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in $W^{1,2}$, with energies $\mathbf{D}_g(u_k)\le \widetilde{\mathbf{D}}_g(u)+\varepsilon$ for every k, such that the graphs G_{u_k} weakly converge in $\mathcal{Z}_{n,2}$ to a current $T\in\mathcal{T}^{2,1}_u$. Since by lower semicontinuity

$$\mathbf{D}_g(T,\Omega\times\mathcal{Y}) \le \liminf_{k\to\infty}\mathbf{D}_g(u_k,\Omega)\,,\qquad \mathbf{D}_g(u_k,\Omega)=\mathbf{D}_g(G_{u_k},\Omega\times\mathcal{Y})\,,$$

we readily obtain the inequality "$\ge$" in (5.5). Moreover, by applying Theorem 5.4, for every $T\in\mathcal{T}^{2,1}_u$ we find a sequence $\{u_k\}\subset C^1(\mathcal{X},\mathcal{Y})$ such that $G_{u_k}\rightharpoonup T$ weakly in $\mathcal{Z}_{n,2}$ and $\mathbf{D}_g(u_k)\to\mathbf{D}_g(T)$ as $k\to\infty$. Since the weak convergence $G_{u_k}\rightharpoonup T$ yields the convergence $u_k\rightharpoonup u_T$ weakly in $W^{1,2}$-sense, and $u_T=u$, we find that $\widetilde{\mathbf{D}}_g(u,\Omega)\le\mathbf{D}_g(T,\Omega\times\mathcal{Y})$, which yields the inequality "$\le$" in (5.6), by the arbitrariness of $T\in\mathcal{T}^{2,1}_u$. □

As a consequence, on account of Definition 5.2 we immediately obtain the following

Proposition 5.6 *For every* $u\in W^{1,2}(\mathcal{X},\mathcal{Y})$ *and every open set* $\Omega\subset\mathcal{X}$ *we have*

$$\begin{aligned}\widetilde{\mathbf{D}}_g(u,\Omega) &= \mathbf{D}_g(u,\Omega)+\inf\Big\{\sum_{s=1}^{\widetilde{s}} M_s\cdot\mathbf{M}_g(\mathbb{L}_s(T)\llcorner\Omega)\mid T\in\mathcal{T}^{2,1}_u\Big\}\\ &= \int_\Omega e_g(x,Du)\,dx+\inf\Big\{\sum_{s=1}^{\widetilde{s}} M_s\cdot\int_{\mathcal{L}_s\cap\Omega}\theta_s(x)\,|\tau_s(x)|_g\,d\mathcal{H}^{n-2}\mid T\in\mathcal{T}^{2,1}_u\Big\}\,,\end{aligned} \tag{5.6}$$

where M_s *is given by* (5.3) *and* $\mathbb{L}_s(T)=\tau(\mathcal{L}_s,\theta_s,\tau_s)\in\mathcal{R}_{n-2}(\mathcal{X})$ *in the decomposition* (2.4) *of* T*, for* $s=1,\dots,\widetilde{s}$.

Remark 5.7 If the second homology group $H_2(\mathcal{Y})$ is trivial, e.g., if $\mathcal{Y}$ is 2-connected, from (5.6) we readily infer that in any dimension n

$$\widetilde{\mathbf{D}}_g(u,\Omega)=\int_\Omega e_g(x,Du)\,dx\qquad\forall\,u\in W^{1,2}(\mathcal{X}\times\mathcal{Y})\,.$$

The singular set. To write more explicitly the formula (5.6), we recall the following facts from [11, Sec. 5.4.2] or [19, Sec. 4.3]. In the sequel we shall denote by $\pi:\mathcal{X}\times\mathcal{Y}\to\mathcal{X}$ and $\widehat{\pi}:\mathcal{X}\times\mathcal{Y}\to\mathcal{Y}$ the orthogonal projections onto the first and second factor, respectively. Following Sec. 2, we set

Definition 5.8 *Let* $u\in W^{1,2}(\mathcal{X},\mathcal{Y})$*. For every* $s=1,\dots,\overline{s}$*, we set* $\mathbb{P}_s(u):=\pi_\#((\partial G_u)\llcorner\widehat{\pi}^\#\sigma^s)\in\mathcal{D}_{n-3}(\mathcal{X})$*, and* $\mathbb{D}_s(u):=\pi_\#(G_u\llcorner\widehat{\pi}^\#\sigma^s)\in\mathcal{D}_{n-2}(\mathcal{X})$*, so that, in local coordinates*

$$\begin{aligned}\mathbb{P}_s(u)(\phi) &= \partial G_u(\widehat{\pi}^\#\sigma^s\wedge\pi^\#\phi)\\ &= G_u(\widehat{\pi}^\#\sigma^s\wedge\pi^\# d\phi)=\int_{\mathcal{X}} u^\#\sigma^s\wedge d\phi\end{aligned}$$

for every $\phi\in\mathcal{D}^{n-3}(\mathcal{X})$*, and similarly*

$$\mathbb{D}_s(u)(\gamma)=G_u(\widehat{\pi}^\#\sigma^s\wedge\pi^\#\gamma)=\int_{\mathcal{X}} u^\#\sigma^s\wedge\gamma\qquad\forall\,\gamma\in\mathcal{D}^{n-2}(\mathcal{X})\,.$$

It turns out that $\mathbb{P}_s(u)$ *does not depend on the representative in the cohomology class* $[\sigma^s]$, and that for every open set $\Omega\subset\mathcal{X}$

$$\mathbb{P}_s(u)\llcorner\Omega=(\partial\,\mathbb{D}_s(u))\llcorner\Omega\qquad\forall\,s=1,\dots,\overline{s}\,.$$

Moreover, since by Bethuel's theorem [2] the so called class $R_2^\infty(\mathcal{X},\mathcal{Y})$ is strongly dense in $W^{1,2}(\mathcal{X},\mathcal{Y})$, it turns out that

$$\mathbb{P}_s(u)=0\qquad\forall\,s=\widetilde{s}+1,\dots,\overline{s}\,.$$

Therefore, the *homological singular set* of u is well-defined by the current $\mathbb{P}(u)\in\mathcal{D}_{n-3}(\mathcal{X};H_2^{sph}(\mathcal{Y};\mathbb{R}))$, where $H_2^{sph}(\mathcal{Y};\mathbb{R}):=H_2^{sph}(\mathcal{Y})\otimes\mathbb{R}$, given by

$$\mathbb{P}(u):=\sum_{s=1}^{\widetilde{s}}\mathbb{P}_s(u)\otimes[\gamma_s]\,,$$

and it satisfies $\partial\,\mathbb{P}(u)=0$. In general, $\mathbb{P}(u)\neq 0$.

Example 5.9 If $\mathcal{Y}=\mathbb{S}^2$, we let $\omega_{\mathbb{S}^2}$ denote the *volume* 2*-form on* $\mathbb{S}^2$,

$$\omega_{\mathbb{S}^2}:=y^1dy^2\wedge dy^3+y^2dy^3\wedge dy^1+y^3dy^1\wedge dy^2\,,$$

so that

$$[\![\,\mathbb{S}^2\,]\!](\omega_{\mathbb{S}^2})=\int_{\mathbb{S}^2}\omega_{\mathbb{S}^2}=4\pi\,.$$

The current $\mathbb{P}_s(u)$ simply reduces to the $(n-3)$-current $4\pi\ \mathbb{P}(u):=\pi_\#((\partial G_u)\llcorner\widehat{\pi}^\#\omega_{\mathbb{S}^2})\in\mathcal{D}_{n-3}(\mathcal{X})$, i.e.,

$$\mathbb{P}(u)(\phi):=\frac{1}{4\pi}\partial G_u(\widehat{\pi}^\#\omega_{\mathbb{S}^2}\wedge\pi^\#\phi)=\frac{1}{4\pi}\int_{\mathcal{X}}u^\#\omega_{\mathbb{S}^2}\wedge d\phi$$

for every $\phi\in\mathcal{D}^{n-3}(\mathcal{X})$, and $\mathbb{D}_s(u)$ to the $(n-2)$-current $4\pi\ \mathbb{D}(u):=\pi_\#(G_u\llcorner\widehat{\pi}^\#\omega_{\mathbb{S}^2})\in\mathcal{D}_{n-2}(\mathcal{X})$, i.e.,

$$\mathbb{D}(u)(\gamma):=\frac{1}{4\pi}G_u(\widehat{\pi}^\#\omega_{\mathbb{S}^2}\wedge\pi^\#\gamma)=\frac{1}{4\pi}\int_{\mathcal{X}}u^\#\omega_{\mathbb{S}^2}\wedge\gamma$$

for every $\gamma\in\mathcal{D}^{n-2}(\mathcal{X})$, so that for every open set $\Omega\subset\mathcal{X}$

$$\mathbb{P}(u)\llcorner\Omega=(\partial\,\mathbb{D}(u))\llcorner\Omega\,.\tag{5.7}$$

In the particular case $n=3$, the above can be stated in terms of the so called *D-field* of Brezis-Coron-Lieb, see [6], defined by

$$D(u):=(u\cdot u_{x_2}\times u_{x_3},\,u\cdot u_{x_3}\times u_{x_1},\,u\cdot u_{x_1}\times u_{x_2})\,,$$

where

$$u\cdot u_{x_i}\times u_{x_j}:=\det\begin{pmatrix}u^1&u^2&u^3\\u^1_{x_i}&u^2_{x_i}&u^3_{x_i}\\u^1_{x_j}&u^2_{x_j}&u^3_{x_j}\end{pmatrix}\,.$$

It turns out that the vector $D(u)(x)$ is tangent to the naturally oriented level lines $\{z\in\mathcal{X}\mid u(z)=u(x)\}$, if u is smooth. More precisely, when normalized, the vector $D(u)(x)$ orients the slices of $[\![\,\mathcal{X}\,]\!]$ by the map u at $u(x)\in\mathbb{S}^2$. Moreover, by (5.7) we have

$$\mathbb{P}(u)=0\qquad\Longleftrightarrow\qquad\mathrm{div}D(u)=0\qquad\text{on }\mathcal{X}\,.$$

In higher dimension $n\geq 4$, the smooth $(n-2)$-vector field $D(u)$ can be defined as the dual to $u^\#\omega_{\mathbb{S}^2}$,

$$\langle\eta,D(u)(x)\rangle\,dx:=u^\#\omega_{\mathbb{S}^2}(x)\wedge\eta\qquad\forall\,\eta\in\Lambda^{n-2}(\mathbb{R}^n)\,.$$

More precisely, $D(u)$ may be identified with $\star\,u^\#\omega_{\mathbb{S}^2}$, where $\star$ is the *Hodge operator*. Of course, we have

$$\mathbb{D}(u)(\gamma)=\frac{1}{4\pi}\int_{\mathcal{X}}\langle\gamma,D(u)\rangle\,dx\qquad\forall\,\gamma\in\mathcal{D}^{n-2}(\mathcal{X})\,.$$

Moreover, the $(n-2)$-vector $D(u)(x)$ is tangent to the naturally oriented level $(n-2)$-surfaces $\{z\in\mathcal{X}\mid u(z)=u(x)\}$, if u is smooth. Finally, we have:

Proposition 5.10 *For every* $u \in W^{1,2}(\mathcal{X}, \mathbb{S}^2)$ *and every open set* $\Omega \subset \mathcal{X}$

$$(\mathbb{P}(u) \llcorner \Omega) \times [\![\mathbb{S}^2]\!] = ((\partial \mathbb{D}(u)) \llcorner \Omega) \times [\![\mathbb{S}^2]\!] = (\partial G_u) \llcorner \Omega \times \mathbb{S}^2 .$$

Also, defining the differential $du^{\#}\omega_{\mathbb{S}^2}$ *in the weak sense, we have*

$$\begin{aligned} \mathbb{P}(u) \llcorner \Omega = 0 \quad &\iff \quad (\partial \mathbb{D}(u)) \llcorner \Omega = 0 \\ &\iff \quad (\partial G_u) \llcorner \Omega \times \mathbb{S}^2 = 0 \quad \iff \quad du^{\#}\omega_{\mathbb{S}^2} = 0 \quad \text{in } \Omega . \end{aligned}$$

Minimal connections. Let $\Omega \subset \mathcal{X}$ be an open set. Extending the well-known definition for the standard mass, we set:

Definition 5.11 *Let* $0 \le k \le n-2$*. For every* $\Gamma \in \mathcal{D}_k(\Omega)$ *we denote by*

$$m^g_{i,\Omega}(\Gamma) := \inf\{\mathbf{M}_g(L) \mid L \in \mathcal{R}_{k+1}(\Omega) , \quad (\partial L) \llcorner \Omega = \Gamma\}$$

the integral g-mass *of* Γ relative to Ω*. In case* $m^g_{i,\Omega}(\Gamma) < \infty$*, an i.m. rectifiable current* $L \in \mathcal{R}_{k+1}(\Omega)$ *is an* integral minimal connection for the g-mass *of* Γ allowing connections to the boundary *of* Ω *if* $(\partial L) \llcorner \Omega = \Gamma$ *and* $\mathbf{M}_g(L) = m^g_{i,\Omega}(\Gamma)$*. Finally, we denote by* $m_{i,\Omega}(\Gamma)$ *the standard integral mass of* Γ *relative to* Ω*.*

Now, if $T \in \text{cart}^{2,1}(\mathcal{X} \times \mathcal{Y})$ is decomposed as in (2.4), the null-boundary condition (2.3) reads as

$$(\partial \mathbb{L}_s(T)) \llcorner \Omega = - \mathbb{P}_s(u) \llcorner \Omega \qquad \forall\, s = 1, \ldots, \widetilde{s} ,$$

i.e., $\mathbb{L}_s(T)$ yields (up to the sign) an integral connection of $\mathbb{P}_s(u_T)$. As a consequence, we infer that for every $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$

$$\mathcal{T}^{2,1}_u = \left\{ G_u + \sum_{s=1}^{\widetilde{s}} L_s \times \gamma_s \mid L_s \in \mathcal{R}_{n-2}(\mathcal{X}) , \quad (\partial L_s) \llcorner \Omega = - \mathbb{P}_s(u) \llcorner \Omega \quad \forall\, s ,\ \forall\, \Omega \subset \mathcal{X} \text{ open} \right\} . \tag{5.8}$$

In particular, as $\mathcal{T}^{2,1}_u$ is non-empty, see Proposition 5.5, it turns out that for every $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$

$$m^g_{i,\Omega}(\mathbb{P}_s(u)) < \infty \qquad \forall\, s = 1, \ldots, \widetilde{s}, \quad \forall\, \Omega \subset \mathcal{X} \text{ open} .$$

On account of (5.8), by Proposition 5.6 we readily obtain the following formula, already obtained in [14] in the case of the standard Dirichlet integrand $e_g(G) := \frac{1}{2} |G|^2$, i.e., when g is the Euclidean metric.

Corollary 5.12 *For every* $u \in W^{1,2}(\mathcal{X}, \mathcal{Y})$ *and every open set* $\Omega \subset \mathcal{X}$ *we have*

$$\widetilde{\mathbf{D}}_g(u, \Omega) = \mathbf{D}_g(u, \Omega) + \sum_{s=1}^{\widetilde{s}} M_s \cdot m^g_{i,\Omega}(\mathbb{P}_s(u)) ,$$

where M_s *is given by* (5.3) *and* $m^g_{i,\Omega}(\mathbb{P}_s(u))$ *is the integral g-mass of the singular set* $\mathbb{P}_s(u)$ *allowing connections to the boundary of* Ω*, see Definitions* 5.8 *and* 5.11*.*

The case of constant metrics. Assume now that the metric g is constant on $\mathcal{X}$. Arguing as in the proof of Proposition 2.11, it turns out that an integral minimal connection L_s for the g-mass of $\mathbb{P}_s(u)$, allowing connections to the boundary of Ω, is also an integral minimal connection for the mass of $\mathbb{P}_s(u)$, and

$$m^g_{i,\Omega}(\mathbb{P}_s(u)) = C(g) \cdot m_{i,\Omega}(\mathbb{P}_s(u)) \qquad \forall\, s = 1, \ldots, \widetilde{s} ,$$

where the constant $C(g) > 0$ is given by the formula

$$\mathbf{M}_g(L) = C(g)\, \mathbf{M}(L) \qquad \forall L \in \mathcal{R}_{n-3}(\mathcal{X}) . \tag{5.9}$$

Boundary data. Let $\Omega, \widetilde{\Omega}$ be open sets in $\mathcal{X}$ such that $\Omega \subset\subset \widetilde{\Omega}$ and $\varphi : \widetilde{\Omega} \to \mathcal{Y}$ be a given smooth $W^{1,2}$-function. For $X = W^{1,2}$, C^1, let

$$X_\varphi(\widetilde{\Omega}, \mathcal{Y}) := \{u \in X(\widetilde{\Omega}, \mathcal{Y}) \mid u = \varphi \quad \text{on} \quad \widetilde{\Omega} \setminus \overline{\Omega}\} .$$

For $u\in W^{1,2}_\varphi(\widetilde\Omega,\mathcal{Y})$, let

$$\widetilde{\mathbf{D}}_{g,\varphi}(u,\widetilde\Omega):=\inf\{\liminf_{k\to\infty}\mathbf{D}_g(u_k,\widetilde\Omega)\mid\{u_k\}\subset C^1_\varphi(\widetilde\Omega,\mathcal{Y})\,,\\ u_k\rightharpoonup u\quad\text{weakly in } W^{1,2}(\widetilde\Omega,\mathcal{Y})\}$$

denote the relaxed energy functional *with prescribed boundary data*. Moreover, let

$$\mathrm{CART}^{2,1}_\varphi(\widetilde\Omega\times\mathcal{Y})\ :=\ \{T\in\mathrm{CART}^{2,1}(\widetilde\Omega\times\mathcal{Y})\mid\\ (T-G_\varphi)\llcorner(\widetilde\Omega\setminus\overline\Omega)\times\mathcal{Y}=0\}\,.$$

Similarly to Theorem 5.4, it can be shown that *for every $T\in\mathrm{CART}^{2,1}_\varphi(\widetilde\Omega\times\mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\}\subset C^1_\varphi(\widetilde\Omega,\mathcal{Y})$ such that $G_{u_k}\rightharpoonup T$ weakly in $\mathcal{Z}_{n,2}$ and $\mathbf{D}_g(u_k,\widetilde\Omega)\to\mathbf{D}_g(T,\widetilde\Omega\times\mathcal{Y})$ as $k\to\infty$.* As a consequence, setting

$$\mathcal{T}^{2,1}_{u,\varphi}:=\{T\in\mathrm{CART}^{2,1}_\varphi(\widetilde\Omega\times\mathcal{Y})\mid u_T=u\}\,,$$

arguing as in Proposition 5.5 we obtain that for every $u\in W^{1,2}_\varphi(\widetilde\Omega,\mathcal{Y})$

$$\widetilde{\mathbf{D}}_{g,\varphi}(u,\widetilde\Omega)=\inf\{\mathbf{D}_g(T,\widetilde\Omega\times\mathcal{Y})\mid T\in\mathcal{T}^{2,1}_{u,\varphi}\}\,.$$

Since every Cartesian current $T\in\mathcal{T}^{2,1}_{u,\varphi}$ can be written as in (2.4), this time on forms in $\mathcal{Z}^{n,2}(\widetilde\Omega\times\mathcal{Y})$, where $u_T=u$ and $\mathbb{L}_s(T)\in\mathcal{R}_{n-2}(\widetilde\Omega)$, similarly to Proposition 5.6 we obtain that for every $u\in W^{1,2}_\varphi(\widetilde\Omega,\mathcal{Y})$

$$\begin{aligned}\widetilde{\mathbf{D}}_{g,\varphi}(u,\widetilde\Omega)\ &=\ \mathbf{D}_g(u,\widetilde\Omega)+\inf\Big\{\sum_{s=1}^{\widetilde s}M_s\cdot\mathbf{M}_g(\mathbb{L}_s(T))\mid T\in\mathcal{T}^{2,1}_{u,\varphi}\Big\}\\ &=\ \int_{\widetilde\Omega}e_g(x,Du(x))\,dx+\inf\Big\{\sum_{s=1}^{\widetilde s}M_s\cdot\int_{\mathcal{L}_s}\theta_s(x)\,|\tau_s(x)|_g\,d\mathcal{H}^{n-2}\mid T\in\mathcal{T}^{2,1}_{u,\varphi}\Big\}\end{aligned}$$

where M_s is given by (5.3) and $\mathbb{L}_s(T)=\tau(\mathcal{L}_s,\theta_s,\tau_s)\in\mathcal{R}_{n-2}(\Omega)$, for $s=1,\dots,\widetilde s$.

For $0\le k\le n-2$ and $\Gamma\in\mathcal{D}_k(\widetilde\Omega)$ with $\operatorname{spt}\Gamma\subset\overline\Omega$, we let

$$m^g_i(\Gamma):=\inf\{\mathbf{M}_g(L)\mid L\in\mathcal{R}_{k+1}(\widetilde\Omega)\,,\quad\operatorname{spt}L\subset\overline\Omega\,,\quad\partial L=\Gamma\}$$

denote the *integral g-mass* of Γ. Also, in case $m^g_i(\Gamma)<\infty$, an i.m. rectifiable current $L\in\mathcal{R}_{k+1}(\widetilde\Omega)$ is said to be an *integral minimal connection for the g-mass* of Γ if $\operatorname{spt}L\subset\overline\Omega$, $\partial L=\Gamma$, and $\mathbf{M}_g(L)=m^g_i(\Gamma)$. Finally, we denote by $m_i(\Gamma)$ the standard integral mass of Γ, i.e., when g is the Euclidean metric or, equivalently, $\mathbf{M}_g(L)=\mathbf{M}(L)$.

For every $u\in W^{1,2}_\varphi(\widetilde\Omega,\mathcal{Y})$ and $s=1,\dots,\overline s$, setting $\mathbb{P}_s(u):=\pi_\#((\partial G_u)\llcorner\widehat\pi^\#\sigma^s)\in\mathcal{D}_{n-3}(\widetilde\Omega)$, i.e., by Definition 5.8, with $\widetilde\Omega$ instead of $\mathcal{X}$, we infer that $\operatorname{spt}\mathbb{P}_s(u)\subset\overline\Omega$, as $u=\varphi$ on $\widetilde\Omega\setminus\overline\Omega$, with $\mathbb{P}_s(u)=0$ for $s=\widetilde s+1,\dots,\overline s$. Moreover, if $T\in\mathcal{T}^{2,1}_{u,\varphi}$ the i.m. rectifiable currents $\mathbb{L}_s(T)\in\mathcal{R}_{n-2}(\widetilde\Omega)$ have support $\operatorname{spt}\mathbb{L}_s(T)\subset\overline\Omega$ and boundary $\partial\,\mathbb{L}_s(T)=-\,\mathbb{P}_s(u_T)$. Therefore, similarly to Corollary 5.12 we find that

$$\widetilde{\mathbf{D}}_{g,\varphi}(u,\widetilde\Omega)=\mathbf{D}_g(u,\widetilde\Omega)+\sum_{s=1}^{\widetilde s}M_s\cdot m^g_i(\mathbb{P}_s(u))\qquad\forall\,u\in W^{1,2}_\varphi(\widetilde\Omega,\mathcal{Y})\,.$$

Moreover, if the metric g is constant on $\widetilde\Omega$, again we have that an integral minimal connection L_s for the g-mass is also an integral minimal connection for the mass, and

$$m^g_i(\mathbb{P}_s(u))=C(g)\cdot m_i(\mathbb{P}_s(u))\qquad\forall\,s=1,\dots,\widetilde s\,,$$

where the constant $C(g)>0$ is given by the formula (5.9).

THE CASE $n = 3$ AND $\mathcal{Y} = \mathbb{S}^2$. Let $\Gamma \in \mathcal{D}_k(\mathcal{X})$ be such that $\Gamma = (\partial D) \llcorner \Omega$ for some current $D \in \mathcal{D}_{k+1}(\Omega)$ with finite g-mass; moreover, let $\widetilde{\Gamma} \in \mathcal{D}_k(\widetilde{\Omega})$, with support in $\overline{\Omega}$, be such that $\widetilde{\Gamma} = \partial \widetilde{D}$ for some $\widetilde{D} \in \mathcal{D}_{k+1}(\widetilde{\Omega})$ with $\operatorname{spt} \widetilde{D} \subset \overline{\Omega}$. By Federer's theorem [8], if $k = 0$ we have

$$m^g_{i,\Omega}(\Gamma) = m^g_{r,\Omega}(\Gamma)\,, \qquad m^g_i(\widetilde{\Gamma}) = m^g_r(\widetilde{\Gamma})\,,$$

where

$$\begin{aligned} m^g_{r,\Omega}(\Gamma) &:= \inf\{\mathbf{M}_g(D) \mid D \in \mathcal{D}_{k+1}(\Omega)\,, \quad (\partial D) \llcorner \Omega = \Gamma\} \\ m^g_r(\widetilde{\Gamma}) &:= \inf\{\mathbf{M}_g(\widetilde{D}) \mid \widetilde{D} \in \mathcal{D}_{k+1}(\widetilde{\Omega})\,, \quad \operatorname{spt} \widetilde{D} \subset \overline{\Omega}\,, \quad \partial \widetilde{D} = \Gamma\} \end{aligned}$$

denote the *real g-mass* of Γ *relative to* Ω and the *real g-mass* of Γ, respectively. Moreover, for every k we have

$$m^g_{r,\Omega}(\Gamma) = F^g_\Omega(\Gamma)\,, \qquad m^g_r(\widetilde{\Gamma}) = F^g_{\overline{\Omega}}(\widetilde{\Gamma})\,,$$

where $F^g_\Omega(\Gamma)$ is the *flat g-norm* of Γ *relative to* Ω

$$F^g_\Omega(\Gamma) := \sup\{\Gamma(\xi) \mid \xi \in \mathcal{D}^k(\Omega)\,,\ \|d\xi(x)\|_{g(x)} \le 1\ \forall\, x \in \Omega\}$$

and $F^g_{\overline{\Omega}}(\widetilde{\Gamma})$ is the *flat g-norm* of $\widetilde{\Gamma}$

$$F^g_{\overline{\Omega}}(\widetilde{\Gamma}) := \sup\{\widetilde{\Gamma}(\xi) \mid \xi \in \mathcal{D}^k(\Omega)\,,\ \max\{\|\xi(x)\|_{g(x)}, \|d\xi(x)\|_{g(x)}\} \le 1\ \forall\, x \in \overline{\Omega}\}\,.$$

Assume now that $n = 3$ and $\mathcal{Y} = \mathbb{S}^2$. Taking $\Gamma = \mathbb{P}(u) \llcorner \Omega$ for some $u \in W^{1,2}(\mathcal{X}, \mathbb{S}^2)$, by Example 5.9 we infer that *the integral g-mass* $m^g_{i,\Omega}(\mathbb{P}(u))$ *of* $\mathbb{P}(u)$ *relative to* Ω *agrees with*

$$L_g(u) := \frac{1}{4\pi} \sup\Big\{\int_\Omega D(u) \cdot D\phi\, dx \mid \phi \in C^\infty_c(\Omega)\,,\ \|d\phi(x)\|_{g(x)} \le 1\ \forall\, x \in \Omega\Big\}\,.$$

Similarly, taking $\widetilde{\Gamma} = \mathbb{P}(u)$ for some $u \in W^{1,2}_\varphi(\widetilde{\Omega}, \mathcal{Y})$, if the boundary $\partial\Omega$ is smooth we infer that *the integral g-mass* $m^g_i(\mathbb{P}(u))$ *agrees with*

$$\widetilde{L}_g(u) := \frac{1}{4\pi} \sup_{\phi \in \widetilde{\mathcal{D}}^0(\widetilde{\Omega})} \Big\{\int_\Omega D(u) \cdot D\phi\, dx - \int_{\partial\Omega} D(\varphi) \cdot \nu\, \phi\, d\mathcal{H}^2\Big\}\,,$$

where ν is the outward unit normal to $\partial\Omega$ and

$$\widetilde{\mathcal{D}}^0(\widetilde{\Omega}) := \{\phi \in C^\infty_0(\widetilde{\Omega}) \mid \max\{\|\phi(x)\|_{g(x)}, \|d\phi(x)\|_{g(x)}\} \le 1\ \forall\, x \in \overline{\Omega}\}\,.$$

This was proved in [6] in the case of the Euclidean metric and $\mathcal{X} = B^3$ or $\mathbb{S}^3$, where $L_g(u)$ is called the *length of the minimal connection of the singularities of* u.

Remark 5.13 We finally notice that for any $u \in W^{1,2}_\varphi(\widetilde{\Omega}, \mathcal{Y})$ we clearly have

$$\widetilde{\mathbf{D}}_g(u, \widetilde{\Omega}) \le \widetilde{\mathbf{D}}_{g,\varphi}(u, \widetilde{\Omega})\,,$$

and that the strict inequality may occur, in general. For example, in the case of a constant metric g, the strict inequality holds if for some $s = 1, \ldots, \widetilde{s}$ we have

$$m_{i,\widetilde{\Omega}}(\mathbb{P}_s(u)) < m_i(\mathbb{P}_s(u))\,,$$

i.e., if the mass of an integral minimal connection $L \in \mathcal{R}_{n-2}(\widetilde{\Omega})$ of $\mathbb{P}_s(u)$ allowing connections to the boundary of $\widetilde{\Omega}$, i.e., such that $(\partial L) \llcorner \widetilde{\Omega} = \mathbb{P}(u)$, see Definition 5.11, is strictly lower than the mass of an integral minimal connection $\widetilde{L} \in \mathcal{R}_{n-2}(\widetilde{\Omega})$ of $\mathbb{P}_s(u)$, i.e., such that $\operatorname{spt} \widetilde{L} \subset \overline{\Omega}$ and $\partial \widetilde{L} = \mathbb{P}_s(u)$. This happen e.g. if $\Omega \subset \mathbb{R}^3$ and $\mathbb{P}_s(u) = \delta_{a_+} - \delta_{a_-}$ for some points $a_\pm \in \Omega$ such that the line segment connecting them is not contained in Ω.

6 The case of $W^{1/2}$-maps

In this section we shall briefly consider the analogous problem for manifold constrained $W^{1/2}$-maps. We refer to [12] [16] [19, Ch. 6] for details on the definitions and properties involved.

For the sake of simplicity, *in the sequel we let* $\mathcal{X}^n = B^n$ *or* $\mathbb{S}^n$, the unit sphere in $\mathbb{R}^{n+1}$. Moreover, $\mathcal{Y} = \mathcal{Y}^m$ is a smooth compact boundaryless connected oriented Riemannian submanifold of $\mathbb{R}^N$. We shall also assume that *the first homology group* $H_1(\mathcal{Y})$ *is torsion-free.* Setting

$$W^{1/2}(\mathcal{X},\mathcal{Y}) := \{u \in W^{1/2}(\mathcal{X},\mathbb{R}^N) \mid u(x) \in \mathcal{Y} \quad \text{for a.e. } x \in \mathcal{X}\},$$

see [1], every map $u \in W^{1/2}(\mathcal{X},\mathcal{Y})$ is the *trace* on $\mathcal{X} \times \{0\}$ of Sobolev functions U in $W^{1,2}(\mathcal{C}^{n+1},\mathbb{R}^N)$, say $\mathbf{T}(U) = u$, where $\mathcal{C}^{n+1}$ is the cylinder

$$\mathcal{C}^{n+1} := \mathcal{X} \times [0,1].$$

Moreover, the classical norm $\|u\|^2_{L^2(\mathcal{X})} + |u|_{1/2,\mathcal{X}}$ is equivalent to the standard Dirichlet integral $\mathbf{D}(U) := \mathbf{D}(U,\mathcal{C}^{n+1})$ of the *extension* $U = U(x,t) := \text{Ext}(u)$ of u, i.e., of the harmonic function that minimizes $\mathbf{D}(U,\mathcal{C}^{n+1})$ among all functions in $W^{1,2}(\mathcal{C}^{n+1},\mathbb{R}^N)$ such that $\mathbf{T}(U) = u$.

THE ENERGY ON MAPS. As above, we assume that for every $x \in \mathcal{X}$ the metric $g(x)$ on $\mathcal{X}$ satisfies the bound (2.7) and is continuous in $\mathcal{X}$. We equip $\mathcal{C}^{n+1}$ with the metric $\widehat{g}$ given by the product of the metric g on $\mathcal{X}$ times the Euclidean metric on $[0,1]$. This yields that $\widehat{g}_{\alpha\beta} = \widehat{g}_{\beta\alpha}$ and

$$\widehat{g}_{\alpha\beta} = g_{\alpha\beta}, \quad \widehat{g}_{(n+1)\beta} = 0, \quad \widehat{g}_{(n+1)(n+1)} = 1, \qquad \alpha,\beta = 1,\dots n.$$

The Dirichlet energy of a map $U \in W^{1,2}(\mathcal{C}^{n+1},\mathbb{R}^N)$ is then given by

$$\mathbf{D}_g(U) := \int_{\mathcal{C}^{n+1}} e_g(x, DU(x,t))\,dx\,dt, \tag{6.1}$$

where this time the quadratic integrand $e_g : \mathcal{X} \times M(N,n+1) \to \mathbb{R}^+$ is defined by

$$2e_g(x,G) := \widehat{g}^{\alpha\beta}(x)\delta_{ij}\,G^i_\alpha G^j_\beta\,\sqrt{\det \widehat{g}(x)}, \qquad x \in \mathcal{X}, \quad G \in M(N,n+1).$$

Therefore, if g is the Euclidean metric on $\mathcal{X}$, the energy (6.1) agrees with the standard Dirichlet integral $\mathbf{D}(U)$.

GRAPHS OF $W^{1/2}$-MAPS. To any map $u \in W^{1/2}(\mathcal{X},\mathcal{Y})$ we can associate an $(n,1)$-current G_u in $\mathcal{D}_{n,1}(\mathcal{X}\times\mathcal{Y})$, compare Sec. 2. If u is "smooth", G_u agrees with the current carried by the graph of u. Moreover, if $U := \text{Ext}(u)$, by Stokes' theorem, and by a density argument, we infer that

$$(-1)^{n-1}\partial G_U = G_u \qquad \text{on} \quad \mathcal{D}^{n,1}(\mathcal{X}\times\{0\}\times\mathcal{Y}). \tag{6.2}$$

Definition 6.1 *We say that an i.m.* 1*-cycle* $C \in \mathcal{Z}_1(\mathcal{Y})$ *is an* integral flat cycle *if there exists an i.m. rectifiable current* $R \in \mathcal{R}_2(\mathbb{R}^N)$ *such that* $\partial R = C$.

It turns out that an element q in $H_1(\mathcal{Y},\emptyset;\mathbb{Z})$, the *relative integral homology*, see [8], is an equivalence class of integral flat 1-cycles of $\mathcal{Y}$, where

$$C \sim Z \quad \iff \quad \exists\, W \in \mathcal{R}_2(\mathcal{Y}) : C - Z = \partial W.$$

In each homology class q in $H_1(\mathcal{Y},\emptyset;\mathbb{Z})$ there exists a homological mass minimizer, i.e., an integral flat cycle $\widetilde{C} \in \mathcal{Z}_1(\mathcal{Y})$ with finite mass such that

$$\mathbf{M}(\widetilde{C}) = \inf\{\mathbf{M}(C) \mid C \in Z_1(\mathcal{Y},\emptyset;\mathbb{Z}),\ [C] = \gamma\} < \infty.$$

Moreover, $H_1(\mathcal{Y},\emptyset;\mathbb{Z})$ is isomorphic to $H_1(\mathcal{Y})$, that is assumed to be torsion-free. Therefore, we may and will denote by $[\widetilde{\gamma}_1],\dots,[\widetilde{\gamma}_{\overline{s}}]$ a family of generators of $H_1(\mathcal{Y},\emptyset;\mathbb{Z})$, i.e., the $\widetilde{\gamma}_s$'s are integral flat cycles, and by $[\widetilde{\sigma}^1],\dots,[\widetilde{\sigma}^{\overline{s}}]$ a dual basis in $H^1_{dR}(\mathcal{Y})$ so that $\widetilde{\gamma}_s(\widetilde{\sigma}^r) = \delta_{sr}$. We will then denote by R_s the i.m. rectifiable current of least mass among all currents in $\mathcal{R}_2(\mathbb{R}^N)$ such that ∂R_s is in the homology class $\widetilde{\gamma}_s$. Notice that a priori the mass of ∂R_s is not finite. Moreover, for $s = 1,\dots,\overline{s}$, we set

$$\widetilde{M}_s := \mathbf{M}(R_s) = \inf\{\mathbf{M}(R) \mid R \in \mathcal{R}_2(\mathbb{R}^N),\ \partial R \in [\widetilde{\gamma}_s]\} < \infty. \tag{6.3}$$

Definition 6.2 *Let* $T \in \mathcal{D}_{n,1}(\mathcal{X} \times \mathcal{Y})$*. We say that* T *is in* $\mathcal{E}_{1/2}$-graph$(\mathcal{X} \times \mathcal{Y})$ *if*

$$\partial T = 0 \quad on \quad \mathcal{Z}^{n-1,1}(\mathcal{X} \times \mathcal{Y}) \tag{6.4}$$

and T *can be decomposed as*

$$T = G_{u_T} + S_T\,, \;\; S_T := \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \widetilde{\gamma}_s\,, \;\; on \;\; \mathcal{Z}^{n,1}(\mathcal{X} \times \mathcal{Y}) \tag{6.5}$$

where $u_T \in W^{1/2}(\mathcal{X}, \mathcal{Y})$ *and the* $\mathbb{L}_s(T)$*'s are i.m. rectifiable current in* $\mathcal{R}_{n-1}(\mathcal{X})$*.*

Remark 6.3 Currents in $\mathcal{E}_{1/2}$-graph$(\mathcal{X} \times \mathcal{Y})$ are defined in a homological sense, compare Remark. 5.3, as the decomposition (6.5) does not depend on the choice of the representative $\widetilde{\gamma}_s$ in the homology class $[\widetilde{\gamma}_s]$.

Definition 6.4 *Let* $T \in \mathcal{E}_{1/2}$-graph$(\mathcal{X} \times \mathcal{Y})$ *be such that* (6.5) *holds. We define its* extension $\widetilde{T} := \mathrm{Ext}(T)$ *in* $\mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ *by*

$$\widetilde{T} = (-1)^{n-1}\Big(G_{U_T} + \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times R_s\Big)\,, \qquad U_T := \mathrm{Ext}(u_T)\,. \tag{6.6}$$

Remark 6.5 From Definition 6.4 and (6.2) we infer that the *boundary* of $\widetilde{T}$ over $\mathcal{X} \times \{0\} \times \mathcal{Y}$ is equal to T on $\mathcal{Z}^{n,1}(\mathcal{X} \times \mathcal{Y})$.

THE $\mathcal{E}_g$-ENERGY. If $\widetilde{T} \in \mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ satisfies (6.6), where $\mathbb{L}_s(T) = \tau(\mathcal{L}_s, \theta_s, \tau_s)$, arguing as in Sec. 2 we infer that its Dirichlet energy is given by

$$\mathbf{D}_g(\widetilde{T}) = \int_{\mathcal{C}^{n+1}} e_g(x, DU_T(x,t))\, dx\, dt + \sum_{s=1}^{\bar{s}} \widetilde{M}_s \cdot \mathbf{M}_g(\mathbb{L}_s(T))\,, \tag{6.7}$$

where $\widetilde{M}_s$ is given by (6.3), with $\mathbf{D}_g(G_U) = \mathbf{D}_g(U)$ if $\widetilde{T} = G_U$ for some $U \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^N)$. In fact, since $\mathcal{L}_s \subset \mathcal{X} \times \{0\}$, the orienting $(n-1)$-vector $\tau_s \in \Lambda_{n-1}\mathbb{R}^{n+1}$ does not depend on the t-direction, whence $|\tau_s(x)|_{\widehat{g}(x)} = |\tau_s(x)|_{g(x)}$, and the $\widehat{g}$-mass of $\mathbb{L}_s(T)$ agrees with its g-mass,

$$\mathbf{M}_{\widehat{g}}(\mathbb{L}_s(T)) = \int_{\mathcal{L}_s} |\tau_s(x)|_{g(x)}\, d\mathcal{H}^{n-1} = \mathbf{M}_g(\mathbb{L}_s(T))\,.$$

Remark 6.6 If g is the Euclidean metric, the energy of $\widetilde{T}$ agrees with the Dirichlet energy

$$\mathbf{D}(\widetilde{T}) = \frac{1}{2}\int_{\mathcal{C}^{n+1}} |DU_T|^2\, dx\, dt + \sum_{s=1}^{\bar{s}} \widetilde{M}_s \cdot \mathbf{M}(\mathbb{L}_s(T))\,.$$

Moreover, see Remark 0.4, in the simple case $n = 2$ the g-norm of the tangent vector $\tau_s(x)$ is given by

$$|\tau_s(x)|_g^2 = \tau_s(x)^T (\mathrm{cof}\, A(x)) \tau_s(x)\,,$$

where $A \in M(n,n)$ is given by (0.12).

Definition 6.7 *Let* T *be in* $\mathcal{E}_{1/2}$-graph$(\mathcal{X} \times \mathcal{Y})$*, so that* (6.5) *holds. The* $\mathcal{E}_g$-energy $\mathcal{E}_g(T)$ *of* T *is defined as the Dirichlet energy* $\mathbf{D}_g(\widetilde{T})$ *of the extension* $\widetilde{T} := \mathrm{Ext}(T)$*, see* (6.6) *and* (6.7)*.*

Therefore, if $e_g(x,G) = \frac{1}{2}|G|^2$, the $\mathcal{E}_g$-energy $\mathcal{E}_g(T)$ reduces to the $\mathcal{E}_{1/2}$-energy studied in [16]. Moreover, if $T = G_u$ for some $u \in W^{1/2}(\mathcal{X}, \mathcal{Y})$ and $U = \mathrm{Ext}(u)$, we let

$$\mathrm{Ext}(G_u) := (-1)^{n-1} G_U\,, \quad \mathcal{E}_g(u) := \mathcal{E}_g(G_u) = \mathbf{D}_g(G_U) = \mathbf{D}_g(U)\,,$$

see (6.1), and by the bound (2.7) we have

$$\mathbf{D}_g(U) \simeq \mathbf{D}(U) \simeq |u|_{1/2}\,.$$

Finally, for every open set $\Omega \subset \mathcal{X}$ we let

$$\begin{array}{rcll} \mathcal{E}_g(T,\Omega\times\mathcal{Y}) & := & \mathbf{D}_g(\widetilde{T},\Omega\times[0,1]\times\mathbb{R}^N)\,, & \widetilde{T}:=\mathrm{Ext}(T) \\ \mathcal{E}_g(u,\Omega) & := & \mathbf{D}_g(U,\Omega\times[0,1])\,, & U:=\mathrm{Ext}(u)\,. \end{array}$$

Definition 6.8 *A current $T \in \mathcal{D}_{n,1}(\mathcal{X}\times\mathcal{Y})$ is said to be in* $\mathrm{cart}^{1/2}(\mathcal{X}\times\mathcal{Y})$ *if T belongs to the class* $\mathcal{E}_{1/2}$-graph$(\mathcal{X}\times\mathcal{Y})$ *and the $\mathcal{E}_{1/2}$-energy $\mathcal{E}_{1/2}(T)$ of T is finite, see Definitions* 6.2 *and* 6.7.

We also say that $T_k \rightharpoonup T$ *weakly in* $\mathcal{Z}_{n,1}$ if $T_k(\omega)\to T(\omega)$ for every $\omega\in\mathcal{Z}^{n,1}(\mathcal{X}\times\mathcal{Y})$.

Density results. It is well-known that if $n=1$ maps in $C^1(\mathcal{X}^1,\mathcal{Y})$ are dense in $W^{1/2}(\mathcal{X}^1,\mathcal{Y})$, compare e.g. [5]. For $n\geq 2$, let $R^\infty_{1/2}(\mathcal{X},\mathcal{Y})$ be the set of all maps $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ which are smooth except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u)=\bigcup_{i=1}^r \Sigma_i\,,\qquad r\in\mathbb{N}\,,$$

where Σ_i is a smooth $(n-2)$-dimensional subset of $\mathcal{X}$ with smooth boundary, if $n\geq 3$, and Σ_i is a point if $n=2$. In [15] we proved that *in any dimension $n\geq 2$ the class $R^\infty_{1/2}(\mathcal{X},\mathcal{Y})$ is dense in $W^{1/2}(\mathcal{X},\mathcal{Y})$*. On account of the dominated convergence theorem, we then obtain:

Proposition 6.9 *For every $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ there exists a sequence $\{u_k\}\subset R^\infty_{1/2}(\mathcal{X},\mathcal{Y})$ such that $u_k\rightharpoonup u$ weakly in $W^{1/2}$ and $\mathcal{E}_g(u_k)\to\mathcal{E}_g(u)$ as $k\to\infty$.*

We recall that if the first homotopy group of the target manifold is nontrivial, $\pi_1(\mathcal{Y})\neq 0$, there exist functions $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$, for $n\geq 2$, which cannot be approximated strongly in $W^{1/2}$ by smooth maps in $W^{1/2}(\mathcal{X},\mathcal{Y})$. In [15] we showed that the converse holds true. As a consequence, by the dominated convergence theorem, in any dimension $n\geq 2$ we obtained the following.

Proposition 6.10 *Let $\pi_1(\mathcal{Y})=0$. For every $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\}\subset C^1(\mathcal{X},\mathcal{Y})$ such that $u_k\rightharpoonup u$ weakly in $W^{1/2}$ and $\mathcal{E}_g(u_k)\to\mathcal{E}_g(u)$ as $k\to\infty$.*

Finally, we have:

Theorem 6.11 *Let $n\geq 1$ and let $\pi_1(\mathcal{Y})$ be commutative. For every $T\in\mathrm{cart}^{1/2}(\mathcal{X}\times\mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\}\subset C^\infty(\mathcal{X},\mathcal{Y})$ such that $G_{u_k}\rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}$ and*

$$\lim_{k\to\infty}\mathcal{E}_g(u_k)=\mathcal{E}_g(T)\,.$$

This theorem was proved in [16] in the case of the $\mathcal{E}_{1/2}$-energy, i.e., when $e_g(x,G)=\frac{1}{2}|G|^2$. In the case of dimension $n=1$, the commutativity hypothesis on the first homotopy group can be removed, compare [19, Sec. 6.6]. However, even in the case of dimension $n=2$, and $e_g(x,G)=\frac{1}{2}|G|^2$, if $\pi_1(\mathcal{Y})$ is non-commutative there exist currents T in $\mathrm{cart}^{1/2}(B^2\times\mathcal{Y})$ of the type $T=G_u$ which cannot be approximated weakly in $\mathcal{Z}_{n,1}$ by graphs of smooth maps $u_k:B^2\to\mathcal{Y}$ such that $\mathcal{E}_{1/2}(G_{u_k})\to\mathcal{E}_{1/2}(G_u)$, compare [16].

Proof of Theorem 6.11: Since the metric $\widehat{g}$ is continuous in $\mathcal{C}^{n+1}$, we infer that (3.2) holds true, this time for every $G\in M(N,n+1)$. The proof can be obtained by an adaptation of the one given for the $\mathcal{E}_{1/2}$-energy in [16], by using arguments similar to the one in the proof of Theorem 3.8 for the dipole construction. For this reason, we omit any further comment. ☐

The relaxed $\mathcal{E}_g$-energy. We now introduce the *relaxed $\mathcal{E}_g$-energy* with respect to the weak $W^{1/2}$-convergence, defined for every $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ and every open set $\Omega\subset\mathcal{X}$ by

$$\begin{aligned}\widetilde{\mathcal{E}}_g(u,\Omega):=\inf\big\{\liminf_{k\to\infty}\mathcal{E}_g(u_k,\Omega)\mid\{u_k\}\subset C^1(\mathcal{X},\mathcal{Y})\,,\\ u_k\rightharpoonup u\quad\text{weakly in } W^{1/2}(\mathcal{X},\mathcal{Y})\big\}\,.\end{aligned}$$

Moreover, for every $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ we denote by

$$\mathcal{T}^{1/2}_u := \{T\in \operatorname{cart}^{1/2}(\mathcal{X}\times\mathcal{Y}) \mid u_T = u\}$$

the class of Cartesian current in $\operatorname{cart}^{1/2}(\mathcal{X}\times\mathcal{Y})$ such that the underlying $W^{1/2}$-function u_T in the decomposition (6.5) is equal to u.

By the strong density of smooth maps, in case of dimension $n=1$ we clearly have

$$\widetilde{\mathcal{E}}_g(u,\Omega)=\mathcal{E}_g(u,\Omega)=\int_{\Omega\times[0,1]} e_g(x,DU(x,t))\,dx\,dt \qquad \forall\, u\in W^{1/2}(\mathcal{X}^1,\mathcal{Y})\,,$$

where $U=\operatorname{Ext}(u)$. In dimension $n\geq 2$, as a consequence of Theorem 6.11, using Theorem 0.3 and the closure-compactness of the class $\operatorname{cart}^{1/2}(\mathcal{X}\times\mathcal{Y})$, and arguing as in the proof of Proposition 5.5, we obtain the following

Proposition 6.12 *Under the hypotheses of Theorem* 6.11*, for every* $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ *and every open set* $\Omega\subset\mathcal{X}$ *we have*

$$\widetilde{\mathcal{E}}_g(u,\Omega)=\inf\{\mathcal{E}_g(T,\Omega\times\mathcal{Y}) \mid T\in\mathcal{T}^{1/2}_u\}<\infty\,.$$

By Definition 6.7 we then obtain:

Proposition 6.13 *For every* $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ *and every open set* $\Omega\subset\mathcal{X}$ *we have*

$$\widetilde{\mathcal{E}}_g(u,\Omega)=\int_{\Omega\times[0,1]} e_g(x,DU(x,t))\,dx\,dt+\inf\Big\{\sum_{s=1}^{\overline{s}}\widetilde{M}_s\cdot\mathbf{M}_g(\mathbb{L}_s(T)\llcorner\Omega) \mid T\in\mathcal{T}^{1/2}_u\Big\}\,,$$

where $U:=\operatorname{Ext}(u)$*,* $\widetilde{M}_s$ *is given by* (6.3)*, and* $\mathbb{L}_s(T)\in\mathcal{R}_{n-1}(\mathcal{X})$ *is given by the decomposition* (6.5) *of* T*, for* $s=1,\dots,\overline{s}$*.*

Remark 6.14 If the first homotopy group $\pi_1(\mathcal{Y})$ is trivial, e.g., if $\mathcal{Y}=\mathbb{S}^p$ for some $p\geq 2$, by the Hurewicz theorem we have $H_1(\mathcal{Y})=0$. As a consequence, we readily infer that in any dimension n

$$\widetilde{\mathcal{E}}_g(u,\Omega)=\mathcal{E}_g(u,\Omega) \qquad \forall\, u\in W^{1/2}(\mathcal{X},\mathcal{Y})\,.$$

In order to write more explicitly the relaxed energy, for every $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ and every s we set $\mathbb{P}_s(u):=-\pi_{\#}((\partial G_u)\llcorner\widehat{\pi}^{\#}\widetilde{\sigma}^s)\in\mathcal{D}_{n-2}(\mathcal{X})$, so that

$$\mathbb{P}_s(u)(\phi)=\int_{\mathcal{X}} u^{\#}\widetilde{\sigma}^s\wedge d\phi\,, \qquad \phi\in\mathcal{D}^{n-2}(\mathcal{X})\,. \tag{6.8}$$

By the null-boundary condition (6.4), we infer that for every $T\in\mathcal{T}^{1/2}_u$ and every open set $\Omega\subset\mathcal{X}$ we have

$$\partial(\mathbb{L}_s(T))\llcorner\Omega=(-1)^n\ \mathbb{P}_s(u)\llcorner\Omega \qquad \forall\, s=1,\dots,\overline{s}\,.$$

Remark 6.15 Notice that $m^g_{i,\Omega}(\mathbb{P}_s(u))<\infty$ for every map $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ and every open set $\Omega\subset\mathcal{X}$, see Definition 5.11. Moreover, by the definition of the metric $\widehat{g}$ it turns out that when minimizing the $\widehat{g}$-mass $\mathbf{M}_{\widehat{g}}(L)$ among all i.m. rectifiable currents L in $\mathcal{R}_{n-1}(\mathcal{C}^{n+1})$ such that $(\partial L_s)\llcorner(\Omega\times\{0\})=\mathbb{P}_s(u)\llcorner\Omega$, there exists a solution L_s such that $\operatorname{spt} L_s\subset\mathcal{X}\times\{0\}$, so that $\mathbf{M}_{\widehat{g}}(L_s)=\mathbf{M}_g(L_s)$.

Similarly to Corollary 5.12, using Definition 5.11 with $k=n-2$ we then obtain the following formula, that goes back to [16] in the case of the Euclidean metric g, i.e., when $\mathcal{E}_g=\mathcal{E}_{1/2}$ and $m^g_{i,\Omega}=m_{i,\Omega}$.

Corollary 6.16 *For every* $u\in W^{1/2}(\mathcal{X},\mathcal{Y})$ *and every open set* $\Omega\subset\mathcal{X}$ *we have*

$$\widetilde{\mathcal{E}}_g(u,\Omega)=\mathcal{E}_g(u,\Omega)+\sum_{s=1}^{\overline{s}}\widetilde{M}_s\cdot m^g_{i,\Omega}(\mathbb{P}_s(u))$$

where $\widetilde{M}_s$ *is given by* (6.3) *and* $\mathbb{P}_s(u)$ *by* (6.8)*.*

We finally mention that the case with prescribed boundary data can be treated in a similar way.

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