

A YOUNG MEASURE APPROACH TO QUASISTATIC EVOLUTION FOR A CLASS OF MATERIAL MODELS WITH NONCONVEX ELASTIC ENERGIES

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ABSTRACT. Rate-independent evolution for material models with nonconvex elastic energies is studied without any spatial regularization of the inner variable; due to lack of convexity, the model is developed in the framework of Young measures. An existence result for the quasistatic evolution is obtained in terms of compatible systems of Young measures. We also show as this result can be equivalently reformulated with probabilistic language and leads to the description of the quasistatic evolution in terms of stochastic processes on a suitable probability space.

Keywords: quasistatic evolution, rate-independent processes, elastic materials, incremental problems, Young measures

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1. INTRODUCTION

The energetic formulation of many rate-independent evolution processes for elastic materials involve an energy functional depending on the elastic deformation of the body and on an internal variable; an important role is played by the dissipation distance, which depends just on the internal variable (see e.g. [9], [13] [14]).

The standard method to solve this kind of problems is via time-discretization and resolution of incremental minimum problems (see [15], [11], [10], [12]); to apply this method,

lower semicontinuity and coerciveness of the energy functional are needed in the space where the problem is formulated.

As in [7], we consider the case in which the energy functional \mathcal{W} and the dissipation distance \mathcal{H} take the form

$$\mathcal{W}(z, v) = \int_D W(z(x), \nabla v(x)) \, dx, \quad (1.1)$$

$$\mathcal{H}(z_1 - z_0) = \int_D H(z_1(x) - z_0(x)) \, dx, \quad (1.2)$$

where $D \subset \mathbb{R}^d$ is the reference configuration.

Since for the mechanical applications considered in [7] it is not natural to assume \mathcal{W} convex with respect to the internal variable z , there is no function space where the energy functional can be simultaneously lower semicontinuous and coercive. The way proposed in [7] to overcome this difficulty is to assume quasiconvexity of W with respect to its second argument and to add a spatial regularizing term of the form

$$\eta \|\nabla z\|_2^2, \quad (1.3)$$

with $\eta > 0$: thanks to the hypotheses made on W and H , the regularized functional is coercive in suitable Sobolev spaces, and classical results (see [2]) guarantees its lower semicontinuity in the same spaces; however the regularizing term (1.3) is not always physically justified.

In this paper we propose an approach to this problem which does not require any assumption on the convexity/quasiconvexity of the energy functional and does not need any regularizing term: lower semicontinuity and coerciveness are obtained by formulating the problem in a suitable space of Young measures, as proposed in [5] for a different problem.

Without entering in the details of the hypotheses, we just mention that we will assume that the body is subjected to an external load $\mathbf{l}(t)$ depending on time and to a time-dependent prescribed boundary deformation $\varphi(t)$; moreover we will ask that $W: \mathbb{R}^m \times \mathbb{R}^{N \times d} \rightarrow [0, +\infty)$ has quadratic growth and that $H: \mathbb{R}^m \rightarrow [0, +\infty)$ is convex and has linear growth.

Given an initial value of the variables (z_0, v_0) and a partition of the time interval $[0, T]$ in which we study the evolution

$$0 = t^0 < t^1 < \dots < t^k = T,$$

the approximate solution should be defined inductively by solving the following incremental minimum problem:

$$\inf \{ \mathcal{W}(z, v) - \langle \mathbf{l}(t^i), v \rangle + \mathcal{H}(z - z(t^{i-1})) \} \quad (1.4)$$

among all (z, v) which make the energy finite and satisfy the boundary condition at time t^i . As anticipated, these problems are not well-posed in Sobolev spaces and we will present an explicit example in which (1.4) has actually no solution (see Remark 7.3).

To obtain lower semicontinuity and coerciveness of the energy functional we place the problem in a suitable space of Young measures and solve the incremental minimum problems in this extended setting. Differently from [5], the quadratic growth hypothesis on W allows us to use Young measures with finite second moments instead of generalized Young measures; to this aim we specialize to the more regular setting the definition and properties of the space of compatible systems introduced in [4].

The next step is the study of the convergence of the approximate solutions as the time step $t^i - t^{i-1}$ tends to 0. Up to careful choices of subsequences, we can obtain the convergence of the approximations to a pair (ν, μ) , with ν a time-dependent family of Young measures with finite second moments and values in $\mathbb{R}^m \times \mathbb{R}^{N \times d}$, and μ a compatible system of Young measures with finite second moments and values in \mathbb{R}^m , connected to ν by a suitable projection property.

The main result is Theorem 6.15, which shows that this pair satisfies a global stability condition and an energy inequality, suitably reformulated in Young measures language (see Definition 6.14); therefore it can be considered as a solution of the quasistatic evolution problem in the framework of Young measures.

Since we do not need to use generalized Young measures, we can rephrase the definition of compatible systems introduced in [5] using a probabilistic language: in Theorem 3.3, with a modified version of Kolmogorov Theorem, we prove that we can associate to any compatible system of Young measures with finite second moments a suitable stochastic process on a probability space of the form $(D \times \Omega, P)$.

The advantage of the probabilistic formulation is that it motivates, in some sense, and clarifies the notions related to compatible systems of Young measures: in particular the notion of variation of a compatible system in a time interval $[a, b]$ agrees with the usual variation on $[a, b]$ of the corresponding stochastic process $(\mathbf{Z}_t)_{t \in [0, T]}$ considered as function from $[0, T]$ into $L^1(D \times \Omega; \mathbb{R}^m)$. Moreover, thanks to Theorem 3.5, the pair (ν, μ) representing the solution can be described by a unique stochastic process with values in $\mathbb{R}^m \times \mathbb{R}^{N \times d}$.

In the last section we give an alternative proof of the main theorem under the special assumption that the stored energy density is quasiconvex with respect to its second argument; we use the result in [7, Section 4] to obtain solutions of spatially regularized problems, and prove that we can pass to the limit as the regularization parameter vanishes, to obtain a globally stable quasistatic evolution in terms of Young measures in the sense of Definition 6.14.

2. MATHEMATICAL PRELIMINARIES

Functions and measures. We denote by \mathcal{L}^d the Lebesgue measure on \mathbb{R}^d , $d \geq 1$, and by \mathcal{H}^k the k -dimensional Hausdorff measure. $\|\cdot\|_p$ is the usual norm on L^p . $H^1(D; \mathbb{R}^N)$ denotes the usual Sobolev space of all functions from an open domain $D \subseteq \mathbb{R}^d$ into \mathbb{R}^N lying in L^2 with their first derivatives. The symbol $\langle \cdot, \cdot \rangle$ will denote a duality pairing depending on the context.

Given a finite dimensional Hilbert space Ξ , $f: D \times \Xi \rightarrow \mathbb{R}$ is said to be a Carathéodory function if $f(x, \cdot)$ is continuous for a.e. $x \in D$ and $f(\cdot, \xi)$ is measurable for every $\xi \in \Xi$.

Given D an open bounded domain in \mathbb{R}^d and Ξ a finite dimensional Hilbert space, we denote by $M_b(D \times \Xi)$ the space of all bounded Radon measures on $D \times \Xi$; this space can be identified with the dual of the Banach space $\mathcal{C}_0(D \times \Xi)$ of all continuous functions $\phi: D \times \Xi \rightarrow \mathbb{R}$ such that $|\phi| \geq \varepsilon$ is compact for every $\varepsilon > 0$. We will consider on $M_b(D \times \Xi)$ the weak* topology deriving from this duality.

Let ν and μ be nonnegative measures in $M_b(D)$ and $M_b(D \times \Xi)$, respectively, such that $\pi_D(\mu) = \nu$. The Disintegration Theorem (see, e.g., [17, Appendix A2]) guarantees the existence of a measurable family $(\mu^x)_{x \in D}$ of probability measures on Ξ , such that

$$\int_{D \times \Xi} f(x, \xi) d\mu(x, \xi) = \int_D \left(\int_{\Xi} f(x, \xi) d\mu^x(\xi) \right) d\nu(x),$$

for every bounded Borel function $f: D \times \Xi \rightarrow \mathbb{R}$. The measures μ^x are uniquely determined for a.e. $x \in D$ and we will write

$$\mu = \int_D \mu^x d\nu(x).$$

In the whole paper we will use the following notation: π_D and π_{Ξ} will denote the usual projections of the product space $D \times \Xi$ on D and Ξ respectively; in the case $\Xi = \Xi_1 \times \Xi_2$, $\tilde{\pi}_{\Xi_i}$ will denote the projection of $D \times \Xi_1 \times \Xi_2$ on $D \times \Xi_i$ and π_{Ξ_i} the projection of $\Xi_1 \times \Xi_2$ on Ξ_i , for $i = 1, 2$.

Young measures. Now we will recall the definition and the main properties of Young measures.

In the whole paper D will denote a bounded open domain in \mathbb{R}^d and Ξ, Ξ_1, Ξ_2 will denote finite dimensional Hilbert spaces; we will assume, without loss of generality, that

$$\mathcal{L}^d(D) = 1. \quad (2.1)$$

The space $Y(D; \Xi)$ of the Young measures on D with values in Ξ is the space of all nonnegative measures $\mu \in M_b(D \times \Xi)$ such that

$$\pi_D(\mu) = \mathcal{L}^d. \quad (2.2)$$

Applying the Disintegration Theorem to $\mu \in Y(D; \Xi)$, we deduce the existence of a measurable family of probability measures on Ξ , $(\mu^x)_{x \in D}$, with

$$\mu = \int_D \mu^x dx.$$

$Y^2(D; \Xi)$ denotes the space of all $\mu \in Y(D; \Xi)$, whose second moment

$$\int_{D \times \Xi} |\xi|^2 d\mu(x, \xi) = \int_D \left(\int_{\Xi} |\xi|^2 d\mu^x(\xi) \right) dx$$

is finite.

To every $u \in L^2(D; \Xi)$ we can associate a Young measure with finite second moment denoted by δ_u and defined by

$$\int_{D \times \Xi} f(x, \xi) d\delta_u(x, \xi) := \int_D f(x, u(x)) dx, \quad (2.3)$$

for every bounded Borel function $f: D \times \Xi \rightarrow \mathbb{R}$. In particular for every $\bar{\xi} \in \Xi$, we will denote the Young measure associated to the function $x \mapsto \bar{\xi}$ simply by $\delta_{\bar{\xi}}$.

3. YOUNG MEASURES WITH PROBABILISTIC LANGUAGE

In this section we want to point out that Young measures can be presented using a probabilistic language, and precisely the notion of random variable. While for a single Young measure this probabilistic presentation does not introduce relevant simplifications, it will be very useful in the case of families of time-dependent Young measures.

Probability spaces of the form $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$, where Ω is a measurable space, $\mathcal{B}(D)$ denotes the Borel σ -algebra on D , \mathcal{F} is a σ -algebra on Ω and P a probability measure with the property $\pi_D(P) = \mathcal{L}^d$, will be called (D, \mathcal{L}^d) -probability spaces.

We can associate to every Young measure μ on D with values in Ξ a random variable Y defined on a (D, \mathcal{L}^d) -probability space with values in Ξ , in such a way that

$$\int_{D \times \Xi} f(x, \xi) d\mu(x, \xi) = \int_{D \times \Omega} f(x, Y(x, \omega)) dP(x, \omega), \quad (3.1)$$

for every bounded Borel function $f: D \times \Xi \rightarrow \mathbb{R}$. Indeed it is enough to take as Ω the space Ξ itself, as Y the projection on Ξ , and as P the measure μ itself, which is a probability measure thanks to (2.1) and (2.2). Conversely, given any Ξ -valued random variable Y on a (D, \mathcal{L}^d) -probability space, formula (3.1) defines a Young measure μ , which will be denoted by $(\pi_D, Y)(P)$, since it coincides with the image of the measure P under the map $(\pi_D, Y): D \times \Omega \rightarrow D \times \Xi$.

We say that a random variable Y on a (D, \mathcal{L}^d) -probability space $(D \times \Omega, P)$ has finite second moment if

$$\int_{D \times \Omega} |Y(x, \omega)|^2 dP(x, \omega) < \infty.$$

Hence a Young measure has finite second moment if and only if the associated random variable does.

In the particular case of $\mu = \delta_u \in Y^2(D; \Xi)$ with $u \in L^2(D; \Xi)$, for every (D, \mathcal{L}^d) -probability space we can associate to δ_u the random variable $Y: D \times \Omega \rightarrow \Xi$, defined by $Y(x, \omega) := u(x)$ for \mathcal{L}^d -a.e. $x \in D$ and for every $\omega \in \Omega$; we will denote this random variable simply by u .

3.1. Stochastic processes and compatible systems of Young measures. As we have seen, if we deal with a single random variable the association to a Young measure is immediate; more complicated is the case of a stochastic process $(\mathbf{Y}_t)_{t \in T}$ on a (D, \mathcal{L}^d) -probability space in a time set T : indeed the family of measures $((\pi_D, \mathbf{Y}_t)(P))_{t \in T}$ gives an insufficient information on the stochastic process, since in general we cannot go back to $(\pi_D, \mathbf{Y}_{t_1}, \dots, \mathbf{Y}_{t_n})(P)$, for an increasing sequence of time instants $t_1 < \dots < t_n$, just using $(\pi_D, \mathbf{Y}_{t_i})(P)$, $i = 1, \dots, n$.

In this section, using a modification of Kolmogorov Theorem (see [8, pg. 29]), we want to show that the correct correspondence is between stochastic processes and compatible systems of Young measures, which have been introduced in [4] in the more general context of generalized Young measures.

Given a stochastic process $(\mathbf{Y}_t)_{t \in T}$ on a (D, \mathcal{L}^d) -probability space, with

$$\mathbf{Y}_t \in L^2(D \times \Omega; \Xi_t),$$

we can define a family of Young measures on D , indexed by the nonempty finite subsets F of T , as

$$\boldsymbol{\mu}_F := (\pi_D, (\mathbf{Y}_t)_{t \in F})(P). \quad (3.2)$$

It is immediate to see that every $\boldsymbol{\mu}_F$ has finite second moment and that this family satisfies the following compatibility condition

$$\tilde{\pi}_G^F(\boldsymbol{\mu}_F) = \boldsymbol{\mu}_G, \quad (3.3)$$

for every nonempty finite subsets $G \subset F$ of T , where $\tilde{\pi}_G^F: D \times \prod_{t \in F} \Xi_t \rightarrow D \times \prod_{s \in G} \Xi_s$ maps $(x, (\xi_t)_{t \in F})$ in $(x, (\xi_s)_{s \in G})$.

According to [4], we define a *compatible system of Young measures with finite second moments* on D with time set T and values in $\prod_{t \in T} \Xi_t$ as a family $\boldsymbol{\mu} = (\boldsymbol{\mu}_F)$ of Young measures $\boldsymbol{\mu}_F \in Y^2(D; \prod_{t \in F} \Xi_t)$, with F varying among all nonempty finite subsets of T , such that the compatibility condition (3.3) is satisfied.

The space of all such systems is denoted by $SY^2(D; \prod_{t \in T} \Xi_t)$; in the special case of $\Xi_t \equiv \Xi$, for every $t \in T$ we will use the notation $SY^2(T, D; \Xi)$.

The two following remarks are technical and will be used to prove the correspondence between compatible systems and stochastic processes.

Remark 3.1. If $\boldsymbol{\mu}$ satisfies the compatibility condition, for every nonempty finite subsets $G \subset F$ of T there exists a set N_G^F of D with $\mathcal{L}^d(N_G^F) = 0$, such that

$$\pi_G^F(\boldsymbol{\mu}_F^x) = \boldsymbol{\mu}_G^x \quad \text{for every } x \in D \setminus N_G^F. \quad (3.4)$$

Conversely, if (3.4) holds for a.e. $x \in D$, then $\boldsymbol{\mu}$ satisfies the compatibility condition (3.3).

Hence, up to subsets of D with zero measure, the compatibility condition commutes in some sense with the disintegration.

Remark 3.2. Let $Y^2(D; \Xi)^T$ denote the set of all families of Young measures with finite second moments on D with values in Ξ , indexed on the set T . Given $(\mu_t)_{t \in T} \in Y^2(D; \Xi)^T$, we can always construct a compatible system $\boldsymbol{\mu} \in SY^2(T, D; \Xi)$ satisfying $\boldsymbol{\mu}_t = \mu_t$ for every $t \in T$. Indeed it is enough to define

$$\boldsymbol{\mu}_F := \int_D \left(\bigotimes_{t \in F} \mu_t^x \right) dx,$$

for every nonempty finite subset F of T .

In the next Theorem we will show that to every compatible system of Young measures with finite second moments we can associate a stochastic process on a suitable (D, \mathcal{L}^d) -probability space.

Theorem 3.3. *Given a set of indices T and a compatible system $\boldsymbol{\mu} \in SY^2(D; \prod_{t \in T} \Xi_t)$, there exist a (D, \mathcal{L}^d) -probability space and a stochastic process $(\mathbf{X}_t)_{t \in T}$ with*

$$\mathbf{X}_t \in L^2(D \times \Omega; \Xi_t), \quad (3.5)$$

for every $t \in T$, such that

$$(\pi_D, (\mathbf{X}_t)_{t \in F})(P) = \boldsymbol{\mu}_F, \quad (3.6)$$

for every nonempty finite subset F of T .

Proof. Let t_∞ be an index such that $t_\infty \notin T$. Set $\hat{T} := \{t_\infty\} \cup T$ and let $\Xi_{t_\infty} := D$.

We want to construct a system of finite dimensional distributions indexed on \hat{T} , satisfying the hypotheses of Kolmogorov Theorem.

Let F be a nonempty finite subset of \hat{T} ; we distinguish three cases. If $t_\infty \in F$ and $F \setminus \{t_\infty\} \neq \emptyset$ we set

$$\boldsymbol{\nu}_F = \boldsymbol{\mu}_{F \setminus \{t_\infty\}}. \quad (3.7)$$

If $t_\infty \notin F$, we set

$$\boldsymbol{\nu}_F := \pi_{\prod_{\tau \in F} \Xi_\tau}(\boldsymbol{\mu}_F), \quad (3.8)$$

where $\pi_{\prod_{\tau \in F} \Xi_\tau} : D \times \prod_{\tau \in F} \Xi_\tau \rightarrow \prod_{\tau \in F} \Xi_\tau$ denotes the usual projection. Finally, if $F = \{t_\infty\}$, we set

$$\boldsymbol{\nu}_F := \mathcal{L}^d. \quad (3.9)$$

The system of finite dimensional distributions $\boldsymbol{\nu}$ satisfies the following compatibility condition: for every nonempty finite subsets $G \subset F$ of \hat{T} , we have

$$\pi_G^F(\boldsymbol{\nu}_F) = \boldsymbol{\nu}_G. \quad (3.10)$$

Indeed, in the case $t_\infty \in F$ and $F \setminus \{t_\infty\} \neq \emptyset$, it follows from compatibility condition for $\boldsymbol{\mu}$ if $t_\infty \in G$ and $G \setminus \{t_\infty\} \neq \emptyset$, it comes from (2.2) if $G = \{t_\infty\}$, and it easily follows from (3.8) if $t_\infty \notin G$. In case $t_\infty \notin F$, (3.10) can be proved using the construction in (3.8) and the analysis of the previous case.

By (3.10), $\boldsymbol{\nu}$ satisfies the hypotheses of Kolmogorov Theorem; therefore it is enough to choose $\Omega := \prod_{t \in T} \Xi_t$, \mathcal{F} the product of the Borel σ -algebras of Ξ_t , for $t \in T$, $\mathbf{X}_t : \Omega \rightarrow \Xi_t$, for $t \in T$, the usual projections, and Kolmogorov Theorem guarantees the existence of a probability measure P on $(D \times \Omega, \mathcal{B}(D) \times \mathcal{F})$ with $\pi_D(P) = \mathcal{L}^d$ which satisfies (3.6). \square

Using the previous result we will prove that we can associate to a pair of two compatible systems, connected by a further compatibility condition, a pair of stochastic processes on the same probability space.

Lemma 3.4. *Let V and W finite dimensional Hilbert spaces, $\mu \in Y^2(D; V)$ and $\nu \in Y^2(D; V \times W)$ be such that*

$$\tilde{\pi}_V(\nu) = \mu. \quad (3.11)$$

Then for \mathcal{L}^d -a.e. $x \in D$ we have

$$(\nu^x)^v = \nu^{(x,v)},$$

for μ^x -a.e. $v \in V$, where $(\nu^x)^v$ is the disintegration of ν^x with respect to μ^x and $\nu^{(x,v)}$ the disintegration of ν with respect to μ .

Proof. It is easy to see, as for Remark 3.1, that there exists a set $N \subseteq D$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in D \setminus N$

$$\pi_V(\nu^x) = \mu^x. \quad (3.12)$$

Hence for every bounded Borel function $f: D \times V \times W \rightarrow \mathbb{R}$,

$$\begin{aligned} \int_{D \times V \times W} f(x, v, w) d\nu(x, v, w) &= \int_D \left(\int_{V \times W} f(x, v, w) d\nu^x(v, w) \right) dx = \\ &= \int_D \left(\int_V \left(\int_W f(x, v, w) d(\nu^x)^v(w) \right) d\mu^x(v) \right) dx. \end{aligned}$$

On the other hand, thanks to (3.11),

$$\begin{aligned} \int_{D \times V \times W} f(x, v, w) d\nu(x, v, w) &= \int_{D \times V} \left(\int_W f(x, v, w) d\nu^{(x,v)}(w) \right) d\mu(x, v) = \\ &= \int_D \left(\int_V \left(\int_W f(x, v, w) d\nu^{(x,v)}(w) \right) d\mu^x(v) \right) dx. \end{aligned}$$

This concludes the proof. \square

Theorem 3.5. *Let T a set of indices, V_t and W_t finite dimensional Hilbert spaces, for every $t \in T$. Let $\mu \in SY^2(D; (V_t)_{t \in T})$ and $\nu \in SY^2(D; (V_t \times W_t)_{t \in T})$. Assume that*

$$\tilde{\pi}_{V_t}(\nu_t) = \mu_t,$$

for every $t \in T$. Then there exist a (D, \mathcal{L}^d) -probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$ and a stochastic process $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in T}$ with

$$\begin{aligned} \mathbf{Z}_t &\in L^2(D \times \Omega; V_t), \\ \mathbf{Y}_t &\in L^2(D \times \Omega; W_t), \end{aligned}$$

for every $t \in T$, such that

$$(\pi_D, (\mathbf{Z}_t)_{t \in F})(P) = \mu_F, \quad (3.13)$$

for every nonempty finite subset F of T , and

$$(\pi_D, \mathbf{Z}_t, \mathbf{Y}_t)(P) = \nu_t, \quad (3.14)$$

for every $t \in T$.

Proof. We want to construct from μ and ν a unique compatible system and to apply to it Theorem 3.3.

Fix a nonempty finite subset F of T . Denote by $(\mu_F^x)_{x \in D}$, $(\nu_t^x)_{x \in D}$ the disintegrations with respect to \mathcal{L}^d of μ_F , and ν_t , $t \in F$, respectively. As observed in (3.12), $\pi_{V_t}(\nu_t^x) = \mu_t^x$, for a.e. $x \in D$, for every $t \in F$. Hence, for a.e. $x \in D$, we can write

$$\nu_t^x = \int_{V_t} (\nu_t^x)^{v_t} d\mu_t^x(v_t).$$

Using the fact that the disintegration is a measurable family, a Dynkin class argument, and Lemma 3.4, we can deduce that

$$(x, (v_t)_{t \in F}) \mapsto \left(\otimes_{t \in F} (\nu_t^x)^{v_t} \right)(B) \quad (3.15)$$

is a Borel measurable function, for every Borel subset B of $\prod_{t \in F} W_t$. In particular for a.e. $x \in D$ the function $(v_t)_{t \in F} \mapsto \left(\otimes_{t \in F} (\nu_t^x)^{v_t} \right)(B)$ is Borel measurable; hence, for every Borel sets $A \subseteq \prod_{t \in F} V_t$ and $B \subseteq \prod_{t \in F} W_t$, we can define a measure $\tilde{\nu}_F^x$ on $\prod_{t \in F} (V_t \times W_t)$ by

$$\tilde{\nu}_F^x(A \times B) := \int_A \left(\otimes_{t \in F} (\nu_t^x)^{v_t} \right)(B) d\mu_F^x((v_t)_{t \in F}), \quad (3.16)$$

for a.e. $x \in D$.

By construction, for a.e. $x \in D$, $\tilde{\nu}_F^x$ is a probability measure with the properties

$$\pi_{\prod_{t \in F} V_t}(\tilde{\nu}_F^x) = \mu_F^x, \quad (3.17)$$

$$\pi_{V_t \times W_t}(\tilde{\nu}_F^x) = \nu_t^x, \quad (3.18)$$

for every $t \in F$.

The Borel measurability of the function in (3.15) guarantees that $(\tilde{\nu}_F^x)_{x \in D}$ is a measurable family of probability measures on $\prod_{t \in F} (V_t \times W_t)$ and we can define

$$\tilde{\nu}_F := \int_D \tilde{\nu}_F^x dx.$$

It is easy to check that, for F running over all nonempty finite subsets of T , $\tilde{\nu}_F \in Y^2(D; \prod_{t \in F} (V_t \times W_t))$ and satisfies the compatibility condition; hence we have $\tilde{\nu} \in SY^2(D; (V_t \times W_t)_{t \in T})$. Moreover, thanks to (3.17) and (3.18),

$$\tilde{\pi}_{\prod_{t \in F} V_t}(\tilde{\nu}_F) = \mu_F, \quad (3.19)$$

for every nonempty finite subset F of T and

$$\tilde{\pi}_{V_t \times W_t}(\tilde{\nu}_t) = \nu_t, \quad (3.20)$$

for every $t \in T$.

Applying Theorem 3.3 to $\tilde{\nu}$, we obtain a (D, \mathcal{L}^d) -probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$ and a stochastic process $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in T}$, with $(\mathbf{Z}_t, \mathbf{Y}_t): D \times \Omega \rightarrow V_t \times W_t$, such that

$$(\pi_D, (\mathbf{Y}_t, \mathbf{Z}_t)_{t \in F})(P) = \tilde{\nu}_F, \quad (3.21)$$

for every nonempty finite subset F of T . Using the construction of $(\mathbf{Z}_t, \mathbf{Y}_t)$ and (3.19), (3.20), we can obtain the thesis. \square

If the time set is an interval $[0, T] \subset \mathbb{R}$ and $\Xi_t \equiv \Xi$ for every $t \in [0, T]$, the notion of variation of a stochastic process $(\mathbf{Y}_t)_{t \in [0, T]}$ on a (D, \mathcal{L}^d) -probability space, with $\mathbf{Y}_t \in L^1(D \times \Omega; \Xi)$, is defined in the usual way: for every time interval $[a, b] \subseteq [0, T]$ we set

$$\begin{aligned} \text{Var}(\mathbf{Y}, P; a, b) &:= \sup \sum_{i=1}^k \int_{D \times \Omega} |\mathbf{Y}_{t_i}(x, \omega) - \mathbf{Y}_{t_{i-1}}(x, \omega)| dP(x, \omega) = \\ &= \sup \sum_{i=1}^k \|\mathbf{Y}_{t_i} - \mathbf{Y}_{t_{i-1}}\|_1, \end{aligned}$$

where the supremum is taken over all finite partitions $a = t^0 < \dots < t^k = b$ of the interval $[a, b]$ (with the convention $\text{Var}(\mathbf{Y}, P; a, b) = 0$, if $a = b$).

Using the correspondence between stochastic processes and compatible systems we define the *variation* of $\mu \in SY^2([0, T], D; \Xi)$ on $[a, b] \subseteq [0, T]$ by

$$\text{Var}(\mu; a, b) := \sup \sum_{i=1}^k \int_{D \times \Xi^{k+1}} |\xi_i - \xi_{i-1}| d\mu_{t_0 \dots t_k}(x, \xi_0, \dots, \xi_k)$$

where the supremum is taken over all finite partitions $a = t_0 < \dots < t_k = b$ of the interval $[a, b]$ (with the convention $\text{Var}(\mu; a, b) = 0$, if $a = b$).

From these definitions, if μ is the compatible system associated to the stochastic process \mathbf{Y} we have $\text{Var}(\mathbf{Y}, P; a, b) = \text{Var}(\mu; a, b)$.

4. SOME TECHNICAL TOOLS

For technical reasons it is more convenient to deal with Young measures and compatible systems than with random variables and stochastic processes; therefore we want to point out some results about measures which will be needed in the sequel.

We say that a sequence μ_k in $Y(D; \Xi)$ weakly* converges if it converges in the weak* topology of $M_b(D \times \Xi)$.

Remark 4.1. Since the total variation of a Young measure μ is $|\mu|(D \times \Xi) = \mathcal{L}^d(D) = 1$, $Y(D; \Xi)$ is contained in a bounded subset of the dual of a separable Banach space, therefore it is metrizable with respect to the weak* topology.

We will say that $\mu_k \rightharpoonup \mu$ 2-weakly* if the second moments of μ_k are equibounded and $\mu_k \rightharpoonup \mu$ weakly*.

We recall that $Y(D; \Xi)$ is not closed with respect to the weak* convergence, differently from $Y^2(D; \Xi)$, which is closed under 2-weakly* convergence, as can be deduced from the following remarks.

Remark 4.2. If μ_k is a sequence in $Y(D; \Xi)$ and μ_k weakly* converges to some $\mu \in M_b(D \times \Xi)$, then for every positive Carathéodory function f on $D \times \Xi$ we have

$$\int_{D \times \Xi} f(x, \xi) d\mu(x, \xi) \leq \liminf_{k \rightarrow \infty} \int_{D \times \Xi} f(x, \xi) d\mu_k(x, \xi), \quad (4.1)$$

(see [17, Theorem 4]). Moreover, for every bounded Carathéodory function f with compact support on $D \times \Xi$, we have

$$\int_{D \times \Xi} f(x, \xi) d\mu_k(x, \xi) \rightarrow \int_{D \times \Xi} f(x, \xi) d\mu(x, \xi).$$

Therefore, if a sequence $(\mu_k)_k \subset Y^2(D; \Xi)$ 2-weakly* converges to a measure $\mu \in M_b(D \times \Xi)$, then $\mu \in Y^2(D; \Xi)$; moreover from the previous remark we can deduce that a sequence in $Y^2(D; \Xi)$ with equibounded second moments has always a subsequence which converges 2-weakly*.

The following remark is a slight modification of [16, Proposition 6.5].

Remark 4.3. If $\mu_k \rightharpoonup \mu$ 2-weakly*, for every Carathéodory function $f: D \times \Xi \rightarrow \mathbb{R}$, with $|f(x, \xi)| \leq a(x) + b(x)|\xi|$, for every $x \in D$, $\xi \in \Xi$, for suitable $b \in L^2(D)$ and $a \in L^1(D)$, it holds

$$\int_{D \times \Xi} f(x, \xi) d\mu_k(x, \xi) \longrightarrow \int_{D \times \Xi} f(x, \xi) d\mu(x, \xi).$$

This allows to prove that if $\mu_k \rightharpoonup \mu$ 2-weakly*, $\mu_k, \mu \in Y^2(D; \Xi_1 \times \Xi_2)$, then $\tilde{\pi}_i(\mu_k) \rightharpoonup \tilde{\pi}_i(\mu)$ in $Y^2(D; \Xi_i)$, as $k \rightarrow \infty$, for $i = 1, 2$.

Moreover we have the following result, where for every $g \in L^2(X; \Xi)$ the translation map \mathcal{T}_g from $D \times \Xi$ into itself is defined by $\mathcal{T}_g(x, \xi) := (x, \xi + g(x))$.

Lemma 4.4. Let $(\mu_k)_k$ be a sequence in $Y^2(D; \Xi)$, such that $\mu_k \rightharpoonup \mu$ 2-weakly*. Then $\mathcal{T}_g(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ 2-weakly*, for every $g \in L^2(X; \Xi)$.

More in general

Lemma 4.5. Let $\mu_k \rightharpoonup \mu$ 2-weakly* and g_k be a bounded sequence in $L^2(D; \Xi)$ such that $g_k \rightarrow g$ strongly in $L^1(D; \Xi)$. Then $\mathcal{T}_{g_k}(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ 2-weakly*.

Proof. Since $(g_k)_k$ is bounded in $L^2(D; \Xi)$, the second moments of $\mathcal{T}_{g_k}(\mu_k)$ are equibounded. We now prove that $\mathcal{T}_{g_k}(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ weakly*. Since $\mathcal{C}_0(D \times \Xi)$ is the closure of $\mathcal{C}_0^\infty(D \times \Xi)$

with respect to the norm $\|\cdot\|_\infty$, thanks to (2.2), to prove that $\mathcal{T}_{g_k}(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ weakly* it is enough to show that

$$\int_{D \times \Xi} f(x, \xi + g_k(x)) d\mu_k(x, \xi) \rightarrow \int_{D \times \Xi} f(x, \xi + g(x)) d\mu(x, \xi),$$

for every $f \in \mathcal{C}_0^\infty(D \times \Xi)$.

Let $f \in \mathcal{C}_0^\infty(D \times \Xi)$, we have

$$\begin{aligned} & \left| \int_{D \times \Xi} f(x, \xi + g(x)) d\mu(x, \xi) - \int_{D \times \Xi} f(x, \xi + g_k(x)) d\mu_k(x, \xi) \right| \leq \\ & \leq \left| \int_{D \times \Xi} f(x, \xi + g(x)) d\mu(x, \xi) - \int_{D \times \Xi} f(x, \xi + g(x)) d\mu_k(x, \xi) \right| + \\ & \quad + \int_{D \times \Xi} |f(x, \xi + g(x)) - f(x, \xi + g_k(x))| d\mu_k(x, \xi). \end{aligned}$$

By the Lipschitz continuity of f and (2.2), the last line can be estimated by $c\|g - g_k\|_1$ for a positive constant c ; Lemma 4.4 implies now the thesis. \square

If we deal with Young measures generated by gradients, the following Lemma, in the version of Fonseca, Müller, and Pedregal ([6, Lemma 1.2]), can be very useful.

Lemma 4.6. (Decomposition Lemma) *Let $(v_j)_j$ be a bounded sequence in $H^1(D; \Xi)$. Then there exists a subsequence $(v_{j_k})_k$ of $(v_j)_j$, and another sequence $(w_k)_k$ bounded in $H^1(D; \Xi)$, such that*

$$\mathcal{L}^d(\{v_{j_k} \neq w_k \text{ or } \nabla v_{j_k} \neq \nabla w_k\}) \rightarrow 0, \quad (4.2)$$

as $k \rightarrow \infty$, and $(|\nabla w_k|^2)_k$ is equiintegrable.

Note that condition (4.2) implies that both sequences $(\nabla v_{j_k})_k$ and $(\nabla w_k)_k$ generate the same Young measure, i.e., $\delta_{\nabla v_{j_k}}$ and $\delta_{\nabla w_k}$ converge to the same Young measure.

Using part of the arguments of [6] and a more careful diagonalization argument, it can be proved the following Lemma.

Lemma 4.7. *Let $(v_j)_j$ be a bounded sequence in $L^2(D; \Xi)$ such that there exists a Young measure $\mu \in Y^2(D; \Xi)$ with $\delta_{v_j} \rightharpoonup \mu$ weakly*. Then there exists another sequence $(w_j)_j$, bounded in $L^2(D; \Xi)$, such that*

$$\mathcal{L}^d(\{v_j \neq w_j\}) \rightarrow 0, \quad (4.3)$$

as $j \rightarrow \infty$, and $(|w_j|^2)_j$ is equiintegrable.

The following theorem (see [1]) gives an important convergence result in case we deal with equiintegrable sequences.

Theorem 4.8. (Fundamental Theorem for Young measures) *Given a Young measure μ , generated by a sequence of functions $(u_j)_j$, and a function $f \in \mathcal{C}(\Xi; \mathbb{R})$ such that the sequence $(f(u_j))_j$ is weakly sequentially relatively compact in $L^1(D)$, then*

$$f(u_j) \rightharpoonup f_\mu \text{ weakly in } L^1(D),$$

where the function $f_\mu \in L^1(D)$ is defined by $f_\mu(x) := \int_\Xi f(\xi) d\mu^x(\xi)$ for a.e. $x \in D$. In particular

$$\int_D f(u_j(x)) dx \rightarrow \int_{D \times \Xi} f(\xi) d\mu(x, \xi).$$

The *barycentre* of a Young measure $\mu \in Y^2(D; \Xi)$ is the function $\text{bar}(\mu) \in L^2(D; \Xi)$ defined as

$$\text{bar}(\mu)(x) := \int_\Xi \xi d\mu^x(\xi),$$

for a.e. $x \in D$.

Lemma 4.9. *Let μ_k be a sequence in $Y^2(D; \Xi)$, such that $\mu_k \rightharpoonup \mu$ 2-weakly*. Assume that there exists a sequence of functions $v_k \in H_0^1(D; \Xi)$ such that $\nabla v_k = \text{bar}(\mu_k)$ for every k . Then there exists a function $v \in H_0^1(D; \Xi)$ such that $v_k \rightharpoonup v$ weakly in H^1 and $\nabla v = \text{bar}(\mu)$.*

Proof. Since $\|\nabla v_k\|_2^2 \leq \int_{D \times \Xi} |\xi|^2 d\mu_k(x, \xi)$ which is bounded uniformly with respect to k by hypothesis, using Poincaré inequality we can deduce that there exists a subsequence v_{k_h} and a function $v \in H_0^1(D; \Xi)$, such that $v_{k_h} \rightharpoonup v$ weakly in H^1 . Using the definition of barycentre and (4.3), we deduce that $\nabla v = \text{bar}(\mu)$ and hence $\nabla v_{k_h} \rightharpoonup \text{bar}(\mu)$ weakly in $L^2(D; \Xi)$; together with Poincaré inequality, this implies that the whole sequence v_k converges to v weakly in H^1 . \square

The space $SY^2(D; \prod_{t \in T} \Xi_t)$ will be equipped with the weakest topology for which the maps $\mu \mapsto \mu_F$ from $SY^2(D; \prod_{t \in T} \Xi_t)$ into $Y^2(D; \prod_{t \in F} \Xi_t)$, endowed with the weak* topology of $M_b(D \times \prod_{t \in F} \Xi_t)$, are continuous for every nonempty finite subset F of T . We will refer to this topology as the weak* topology of $SY^2(D; \prod_{t \in T} \Xi_t)$.

From now on we consider the case in which the time set is an interval $[0, T]$. A compatible system of Young measures with finite second moments μ is said to be *left continuous* if for every finite sequence t_1, \dots, t_m in $[0, T]$, with $t_1 < \dots < t_m$, the following property holds:

$$\mu_{s_1 \dots s_m} \rightharpoonup \mu_{t_1 \dots t_m} \text{ weakly* in } Y^2(D; \Xi^m),$$

as $s_i \rightarrow t_i$, with $s_i \in [0, T]$ and $s_i \leq t_i$. We will denote the space of all left continuous compatible systems by $SY_-^2([0, T], D; \Xi)$.

The following theorem can be considered as a version of Helly's Theorem.

Theorem 4.10. *Let μ^k be a sequence in $SY^2([0, T], D; \Xi)$ such that*

$$\sup_k \text{Var}(\mu^k; 0, T) \leq C, \quad (4.4)$$

$$\sup_{t \in [0, T]} \sup_k \int_{D \times \Xi} |\xi|^2 d\mu_t^k(x, \xi) \leq C^*, \quad (4.5)$$

for finite constants C and C^* . Then there exist a subsequence, still denoted by μ^k , a set $\Theta \subset [0, T]$, containing 0 and such that $[0, T] \setminus \Theta$ is at most countable, and $\mu \in SY_-^2([0, T], D; \Xi)$ with

$$\text{Var}(\mu; 0, T) \leq C, \quad (4.6)$$

$$\int_{D \times \Xi} |\xi|^2 d\mu_t(x, \xi) \leq C^* \quad \text{for every } t \in [0, T], \quad (4.7)$$

such that, for every nonempty finite subset F of Θ , we have

$$\mu_F^k \rightharpoonup \mu_F \text{ 2-weakly*}. \quad (4.8)$$

The proof of this Theorem follows easily from Theorem 8.10 of [4], since every Young measure can be seen as a generalized Young measure and our more restrictive hypotheses force the limit to be an element of $SY^2([0, T], D; \Xi)$.

If $H: \Xi \rightarrow [0, +\infty)$ is positively homogeneous of degree one and satisfies the triangle inequality, we can define the H -variation of $\mu \in SY^2([0, T], D; \Xi)$ on the time interval $[a, b] \subseteq [0, T]$ as

$$\text{Var}_H(\mu; a, b) := \sup \sum_{i=1}^k \int_{D \times \Xi^{k+1}} H(\xi_i - \xi_{i-1}) d\mu_{t_0 \dots t_k}(x, \xi_0, \dots, \xi_k), \quad (4.9)$$

where the supremum is taken over all finite partitions $a = t_0 < \dots < t_k = b$ of the interval $[a, b]$ (with the convention $\text{Var}_H(\mu; a, b) = 0$, if $a = b$).

Adapting the argument in [4] (Theorem 8.11), it can be proved the following Lemma.

Lemma 4.11. *Let μ^k be a sequence in $SY^2([0, T], D; \Xi)$. Suppose that there exist a dense set $\Theta \subset [0, T]$ containing 0 and $\mu \in SY^2_-([0, T], D; \Xi)$, such that*

$$\mu_F^k \rightharpoonup \mu_F \quad 2\text{-weakly}^*$$

for every nonempty finite subset F in Θ ; then

$$\text{Var}_H(\mu; 0, T) \leq \liminf_{k \rightarrow \infty} \text{Var}_H(\mu^k; 0, T)$$

for every positively one homogeneous function $H: \Xi \rightarrow [0, +\infty)$ satisfying the triangle inequality.

Definition 4.12. Fix a finite sequence $0 = t_1 < \dots < t_m = T$ in $[0, T]$. For every $\mu \in Y^2(D; \Xi^m)$, it is possible to define the *piecewise constant interpolation* $\mu^{pwc} \in SY^2([0, T], D; \Xi)$ in the following way. For every finite sequence $\tau_1 < \dots < \tau_n$ of elements of $[0, T]$ let $\rho_{\tau_1 \dots \tau_n}: D \times \Xi^m \times \mathbb{R} \rightarrow D \times \Xi^n \times \mathbb{R}$ be defined by

$$\rho_{\tau_1 \dots \tau_n}(x, \xi_{t_1}, \dots, \xi_{t_m}) := (x, \xi_{\tau_1}, \dots, \xi_{\tau_n}),$$

with $\xi_{\tau_i} = \xi_{t_j}$, where j is the largest index such that $t_j \leq \tau_i$. The compatible system of Young measures with finite second moments μ^{pwc} is then defined by

$$\mu_{\tau_1 \dots \tau_n}^{pwc} := \rho_{\tau_1 \dots \tau_n}(\mu).$$

Lemma 4.13. *Let $(\mu^n)_n$ and $(\nu^n)_n$ be sequences in $SY^2([0, T], D; \Xi_1)$ and $SY^2([0, T], D; \Xi_1 \times \Xi_2)$, respectively, satisfying $\tilde{\pi}_{\Xi_1}(\nu_t^n) = \mu_t^n$, for every $t \in [0, T]$. Assume that*

$$\sup_{t \in [0, T]} \sup_n \int_{D \times \Xi_1 \times \Xi_2} |(\xi_1, \xi_2)|^2 d\nu_t^n(x, \xi_1, \xi_2) \leq C, \quad (4.10)$$

for a positive constant C , and that there exist a subset Θ of $[0, T]$, containing 0, with $\mathcal{L}^1([0, T] \setminus \Theta) = 0$, and $\mu \in SY^2([0, T], D; \Xi_1)$ such that

$$\mu_{t_1 \dots t_m}^n \rightharpoonup \mu_{t_1 \dots t_m}, \quad 2\text{-weakly}^*,$$

for every $t_1 < \dots < t_m$ in Θ . For every $t \in \Theta$ let $(n_k^t)_k$ be an increasing sequence of integers; then there exists $\nu \in SY^2([0, T], D; \Xi_1 \times \Xi_2)$, such that $\tilde{\pi}_{\Xi_1}(\nu_t) = \mu_t$, for every $t \in [0, T]$ and satisfying the following properties:

- (1) for every $t \in \Theta$, there exists a subsequence $\nu_t^{n_{k,i}^t}$ of $\nu_t^{n_k^t}$ such that

$$\nu_t^{n_{k,i}^t} \rightharpoonup \nu_t \quad 2\text{-weakly}^*; \quad (4.11)$$

- (2) for every $t \in [0, T] \setminus \Theta$, there exists a sequence s^j in Θ , converging to t , with $s^j \leq t$, such that

$$\nu_{s^j} \rightharpoonup \nu_t \quad 2\text{-weakly}^*. \quad (4.12)$$

The proof of this Lemma was contained in a preliminary version of [5].

Proof. Fix $t \in \Theta$; thanks to (4.10), we can deduce that there exist $\nu_t^0 \in Y^2(D; \Xi_1 \times \Xi_2)$ and a subsequence $(\nu_t^{n_{k,i}^t})_i$ of $(\nu_t^{n_k^t})_k$, satisfying (4.11). Thanks to Remark 4.2 and Remark 4.3, for every $t \in \Theta$ we have

$$\int_{D \times \Xi_1 \times \Xi_2} |(\xi_1, \xi_2)|^2 d\nu_t^0(x, \xi_1, \xi_2) \leq C \quad (4.13)$$

and $\tilde{\pi}_{\Xi_1}(\nu_t^0) = \mu_t$.

Consider now the sets B_t defined in the following way:

- if $t \in \Theta$, B_t denotes the collection of all $\nu \in SY^2([0, T], D; \Xi_1 \times \Xi_2)$ such that the second moments of ν_s are bounded by the constant C appearing in (4.10), for every $s \in [0, T]$, and satisfying $\nu_t = \nu_t^0$;

- if $t \notin \Theta$, B_t is the collection of all $\nu \in SY^2([0, T], D; \Xi_1 \times \Xi_2)$, such that the second moments of ν_s are bounded by the constant C appearing in (4.10), for every $s \in [0, T]$, and for which there exists a sequence s^j in Θ , converging to t with $s^j \leq t$, such that $\nu_{s^j}^0 \rightharpoonup \nu_t$, weakly*.

For every $t \in [0, T]$, $B_t \neq \emptyset$: indeed, if $t \in \Theta$ it comes immediately from Remark 3.2 applied to $(\nu_s)_{s \in [0, T]} \in Y^2(D; \Xi)^{[0, T]}$ defined by $\nu_s := \nu_t^0$ for every $s \in [0, T]$; if $t \notin \Theta$, thanks to (4.13), there exist s^j , with $s^j \in \Theta$, $s^j \rightarrow t$ and $s^j \leq t$, and $\mu \in Y^2(D; \Xi_1 \times \Xi_2)$, such that $\nu_{s^j}^0 \rightharpoonup \mu$ 2-weakly*, hence the second moment of μ is bounded by C and we can apply Remark 3.2 to $(\nu_s)_{s \in [0, T]} \in Y^2(D; \Xi)^{[0, T]}$ defined by $\nu_s := \mu$, for every $s \in [0, T]$, and find an element of B_t . Using Remark 4.1 we can see that the set of all Young measures μ for which there exists a sequence $s^j \rightarrow t$ with $s^j \leq t$ and $\nu_{s^j}^0 \rightharpoonup \mu$ weakly* is sequentially closed with respect to the weak* topology (thanks to (4.13)); moreover we observe that the set of all $\nu \in SY^2([0, T], D; \Xi)$ with the second moments equibounded by the constant C is closed in the weak* topology of $SY^2([0, T], D; \Xi)$, therefore, using again Remark 4.1, we can conclude that, for every $t \in [0, T]$, B_t are closed subsets of $SY^2([0, T], D; \Xi_1 \times \Xi_2)$, endowed with the weak* topology. Moreover the family has the finite intersection property (for every finite sequence $t_1 < \dots < t_m$ in $[0, T]$, using Definition 4.12, we can find an element belonging to $B_{t_1} \cap \dots \cap B_{t_m}$) and is contained in the set of all $\nu \in SY^2([0, T], D; \Xi_1 \times \Xi_2)$ for which the second moments of ν_t are uniformly bounded by the constant C appearing in (4.10); since, thanks to Tychonoff's Theorem, this is a compact subset of $SY^2([0, T], D; \Xi_1 \times \Xi_2)$, endowed with the weak* topology, we can conclude that there exists ν belonging to B_t , for every $t \in [0, T]$. By construction ν satisfies (1) and (2) and from the left continuity of μ we can deduce that $\tilde{\pi}_{\Xi_1}(\nu_t) = \mu_t$, for every $t \in [0, T]$, as required. \square

5. MECHANICAL MODEL

The *reference configuration* D is a bounded connected open subset of \mathbb{R}^d with Lipschitz boundary $\partial D = \Gamma_0 \cup \Gamma_1$, where Γ_0 is assumed to be a nonempty closed subset of ∂D with $\mathcal{H}^{d-1}(\Gamma_0) \neq 0$, and $\Gamma_1 = \partial D \setminus \Gamma_0$. Without loss of generality, we also assume for simplicity that $\mathcal{L}^d(D) = 1$.

We will indicate the deformation by v and the internal variable by z . We will denote the *stored energy density* by $W: \mathbb{R}^m \times \mathbb{R}^{N \times d} \rightarrow [0, +\infty)$ and the *dissipation rate density* by $H: \mathbb{R}^m \rightarrow [0, +\infty)$. For every $\theta, \tilde{\theta} \in \mathbb{R}^m$ and $F \in \mathbb{R}^{N \times d}$, we will make the following assumptions:

(W.1) there exist positive constants c, C such that

$$c(|\theta|^2 + |F|^2) - C \leq W(\theta, F) \leq C(1 + |\theta|^2 + |F|^2);$$

(W.2) $W(\theta, \cdot)$ is of class \mathcal{C}^1 ,

$$\left| \frac{\partial W}{\partial F}(\theta, F) \right| \leq C(1 + |\theta| + |F|),$$

and

$$|W(\theta + \tilde{\theta}, F) - W(\theta, F)| \leq C|\tilde{\theta}|(1 + |\theta| + |\tilde{\theta}| + |F|);$$

(H.1) H is positively homogeneous of degree one and convex;

(H.2) there exists a positive constant λ , such that $\frac{1}{\lambda}|\theta| \leq H(\theta) \leq \lambda|\theta|$.

Let \mathcal{W} be the functional $\mathcal{W}(z, v) := \int_D W(z(x), \nabla v(x)) dx$, for every $z \in L^2(D; \mathbb{R}^m)$ and every $v \in H^1(D; \mathbb{R}^N)$, and \mathcal{H} the functional $\mathcal{H}(z) := \int_D H(z(x)) dx$, for every $z \in L^1(D; \mathbb{R}^m)$.

Given two distinct times $s < t$, the *global dissipation* of a possibly discontinuous function $\mathbf{z}: [0, T] \rightarrow L^2(D; \mathbb{R}^m)$ in the interval $[s, t]$ will be

$$\text{Var}_H(\mathbf{z}; s, t) := \sup \sum_{i=1}^k \mathcal{H}(\mathbf{z}(\tau_i) - \mathbf{z}(\tau_{i-1})),$$

where the supremum will be taken among all finite partitions $s = \tau_0 < \tau_1 < \dots < \tau_k = t$.

The external load at time t and the prescribed boundary datum on Γ_0 at time t are denoted by $\mathbf{l}(t)$ and $\boldsymbol{\varphi}(t)$, respectively; we assume $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$ and $\boldsymbol{\varphi} \in AC([0, T]; H^1(D; \mathbb{R}^N))$.

The kinematically admissible values at time t for z and v are those which make the total energy finite and satisfy the boundary condition, i.e., $v = \boldsymbol{\varphi}(t)$ on Γ_0 \mathcal{H}^{d-1} -a.e. (in the sense of traces). From the previous assumption it follows that the kinematically admissible values at time t are contained in $L^2(D; \mathbb{R}^m) \times \mathcal{A}(t)$, where $\mathcal{A}(t) = H_{\Gamma_0}^1(\boldsymbol{\varphi}(t)) := \{v \in H^1(D; \mathbb{R}^N) : v = \boldsymbol{\varphi}(t) \text{ } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_0\}$.

6. GLOBALLY STABLE QUASISTATIC EVOLUTION FOR YOUNG MEASURES

6.1. Admissible set in terms of stochastic processes. Now we describe the set of admissible stochastic processes in which we look for a solution of our quasistatic evolution problem: the definition takes into account approximation properties with functions which satisfy the boundary condition.

Definition 6.1. Given $A \subset \mathbb{R}$ and $\mathbf{w}: A \rightarrow H^1(D; \mathbb{R}^N)$, we define $AY_{sp}(A, \mathbf{w})$ as the set of all stochastic processes $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in A}$ on a (D, \mathcal{L}^d) -probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$ with

$$\begin{aligned} \mathbf{Z}_t &\in L^2(D \times \Omega; \mathbb{R}^m), \\ \mathbf{Y}_t &\in L^2(D \times \Omega; \mathbb{R}^{N \times d}), \end{aligned}$$

satisfying the following property: for every finite sequence $t_1 < \dots < t_n$ in A there exist sequences $(z_i^k)_k \subset L^2(D; \mathbb{R}^m)$, $(v_i^k)_k \subset H_{\Gamma_0}^1(\mathbf{w}(t_i))$, for $i = 1, \dots, n$ such that

(ap_{sp}1) we have

$$(\pi_D, z_1^k, \dots, z_n^k)(P) \rightharpoonup (\pi_D, \mathbf{Z}_{t_1}, \dots, \mathbf{Z}_{t_n})(P)$$

2-weakly* as $k \rightarrow \infty$;

(ap_{sp}2) for every $i = 1, \dots, n$, there exist a subsequence, possibly depending on i , $(z_i^{k_j}, v_i^{k_j})_j$, such that

$$(\pi_D, z_i^{k_j}, \nabla v_i^{k_j})(P) \rightharpoonup (\pi_D, \mathbf{Z}_{t_i}, \mathbf{Y}_{t_i})(P)$$

2-weakly* as $j \rightarrow \infty$.

6.2. Admissible set in terms of Young measures. The notion of admissible set is now presented in terms of Young measures.

Definition 6.2. Given $A \subset \mathbb{R}$ and $\mathbf{w}: A \rightarrow H^1(D; \mathbb{R}^N)$, we define $AY(A, \mathbf{w})$ as the set of all pairs $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^A \times SY^2(A, D; \mathbb{R}^m)$ satisfying the following property: for every finite sequence $t_1 < \dots < t_n$ in A there exist sequences $(z_i^k)_k \subset L^2(D; \mathbb{R}^m)$, $(v_i^k)_k \subset H_{\Gamma_0}^1(\mathbf{w}(t_i))$, for $i = 1, \dots, n$ such that

(ap1) we have

$$\delta_{(z_1^k, \dots, z_n^k)} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_n}, \quad (6.1)$$

2-weakly* as $k \rightarrow \infty$;

(ap2) for every $i = 1, \dots, n$, there exists a subsequence, possibly depending on i , $(z_i^{k_j}, v_i^{k_j})_j$, such that

$$\delta_{(z_i^{k_j}, \nabla v_i^{k_j})} \rightharpoonup \nu_{t_i} \quad (6.2)$$

2-weakly* as $j \rightarrow \infty$.

Remark 6.3. If $(\nu, \mu) \in AY(A, \mathbf{w})$, then $\tilde{\pi}_{\mathbb{R}^m}(\nu_t) = \mu_t$, for every $t \in A$. Indeed, fixed $t \in A$, by definition there exist $(z^k)_k \subset L^2(D; \mathbb{R}^m)$, $(v^k)_k \subset H_{\Gamma_0}^1(\mathbf{w}(t))$ such that $\delta_{(z^k, \nabla v^k)} \rightharpoonup \nu_t$ 2-weakly* and $\delta_{z^k} \rightharpoonup \mu_t$ 2-weakly*; in particular $\tilde{\pi}_{\mathbb{R}^m}(\delta_{(z^k, \nabla v^k)}) \rightharpoonup \tilde{\pi}_{\mathbb{R}^m}(\nu_t)$ 2-weakly* and this prove the claim.

Remark 6.4. If $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in A} \in AY_{sp}(A, \mathbf{w})$, we can define $(\nu, \mu) \in AY(A, \mathbf{w})$ as

$$\nu_t := (\pi_D, \mathbf{Z}_t, \mathbf{Y}_t)(P) \quad \text{for every } t \in A$$

$$\mu_{t_1 \dots t_n} := (\pi_D, \mathbf{Z}_{t_1}, \dots, \mathbf{Z}_{t_n})(P) \quad \text{for every finite sequence } t_1 < \dots < t_n \text{ in } A.$$

On the other side, thanks to Remark 6.3 and Theorem 3.5, for every $(\nu, \mu) \in AY(A, \mathbf{w})$ there exists a stochastic process $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in A} \in AY_{sp}(A, \mathbf{w})$ such that, for every finite sequence $t_1 < \dots < t_n$ in A ,

$$\begin{aligned} (\pi_D, \mathbf{Z}_{t_1}, \dots, \mathbf{Z}_{t_n})(P) &= \mu_{t_1 \dots t_n} \\ (\pi_D, \mathbf{Z}_{t_i}, \mathbf{Y}_{t_i})(P) &= \nu_{t_i} \quad \text{for every } i = 1, \dots, n. \end{aligned}$$

Remark 6.5. Thanks to Decomposition Lemma 4.6 and Lemma 4.7, given $(\nu, \mu) \in AY(A, \mathbf{w})$ and a finite sequence $t_1 < \dots < t_m$ in A , we can always choose $z_i^k \in L^2(D; \mathbb{R}^m)$ and $v_i^k \in H_{\Gamma_0}^1(\mathbf{w}(t_i))$, for $i = 1, \dots, m$, in such a way that $|z_i^k|^2$ are equiintegrable, satisfy (6.1), and for every i there exists a subsequence $(z_i^{k_j}, v_i^{k_j})_j$ satisfying (6.2), such that $|\nabla v_i^{k_j}|^2$ are equiintegrable with respect to j . Hence, by Theorem 4.8, we can always assume that

$$\|(z_1^k, \dots, z_n^k)\|_2^2 \rightarrow \int_{D \times (\mathbb{R}^m)^n} |(\theta_1, \dots, \theta_n)|^2 d\mu_{t_1 \dots t_n}(x, \theta_1, \dots, \theta_n), \quad (6.3)$$

$$\|(z_i^{k_j}, \nabla v_i^{k_j})\|_2^2 \rightarrow \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d\nu_{t_i}(x, \theta, \xi), \quad (6.4)$$

as $j \rightarrow \infty$, for $i = 1, \dots, m$. This allows us to assume, without loss of generality, that

$$\sup_k \|z_i^k\|_2^2 \leq C_1 + 1, \quad (6.5)$$

$$\sup_j \|(z_i^{k_j}, \nabla v_i^{k_j})\|_2^2 \leq C_2 + 1, \quad (6.6)$$

with

$$\begin{aligned} C_1 &:= \sup_{i=1, \dots, n} \int_{D \times \mathbb{R}^m} |\theta|^2 d\mu_{t_i}(x, \theta), \\ C_2 &:= \sup_{i=1, \dots, n} \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d\nu_{t_i}(x, \theta, F). \end{aligned}$$

In the following two Lemmas we want to point out some closure properties of $AY(A, \mathbf{w})$.

Lemma 6.6. Let $(\nu, \mu) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^A \times SY^2(A, D; \mathbb{R}^m)$, and assume that for every finite sequence $t_1 < \dots < t_n$ in A there exists a sequence $(\nu^j, \mu^j)_j$ in $AY(\{t_1, \dots, t_n\}, \mathbf{w})$, such that

$$\mu_{t_1 \dots t_n}^j \rightharpoonup \mu_{t_1 \dots t_n} \quad \text{2-weakly*}, \quad (6.7)$$

as $j \rightarrow \infty$, and such that for every i there exists a subsequence, possibly depending on i , $(\nu^{j_h})_h$, satisfying

$$(\nu^{j_h})_{t_i} \rightharpoonup \nu_{t_i}, \quad \text{2-weakly*}, \quad (6.8)$$

as $h \rightarrow \infty$. Then $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$.

Proof. Fix a finite sequence $t_1 < \dots < t_n$ in A . By definition of $AY(\{t_1, \dots, t_n\}, \boldsymbol{w})$, for every j and every $i = 1, \dots, n$, there exist $(z_i^{j,k})_k \in L^2(D; \mathbb{R}^m)$ and $(v_i^{j,k})_k \in H_{\Gamma_0}^1(\boldsymbol{w}(t_i))$ satisfying (6.1) for $\boldsymbol{\mu}_j$ and such that for every i and j there exists an increasing sequence of integers $(k_l^{i,j})_l$ for which $(z_i^{j,k_l^{i,j}}, v_i^{j,k_l^{i,j}})_l$ satisfies (6.2) for $\boldsymbol{\nu}^j$; thanks to Remark 6.5 we can assume, without loss of generality, that

$$\|z_i^{j,k}\|_2^2 \leq \int_{D \times \mathbb{R}^m} |\theta|^2 d\boldsymbol{\mu}_{t_i}^j(x, \theta) + 1, \quad \text{for every } k,$$

and

$$\|\nabla v_i^{j,k_l^{i,j}}\|_2^2 \leq \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d\boldsymbol{\nu}_{t_i}^j(x, \theta, F) + 1, \quad \text{for every } l;$$

hence thanks to (6.7) and (6.8) there exists a positive constant C such that

$$\|z_i^{j,k}\|_2^2 \leq C + 1, \quad (6.9)$$

for every i, j, k , and

$$\sup_h \sup_l \|\nabla v_i^{j,k_l^{i,j}}\|_2^2 \leq C + 1, \quad (6.10)$$

for every i, h, l . Thanks to Remark 4.1, we can find a metric d_1 on $Y(D; (\mathbb{R}^m)^n)$ and a metric d_2 on $Y(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})$ which induce the weak* topologies of $Y(D; (\mathbb{R}^m)^n)$ and $Y(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})$, respectively; therefore, for every j we can find an integer $\kappa(j)$ such that, for every $k \geq \kappa(j)$ it holds

$$d_1(\delta_{(z_1^{j,k}, \dots, z_n^{j,k}), \boldsymbol{\mu}_{t_1 \dots t_n}^j}) < \frac{1}{j}; \quad (6.11)$$

analogously, for every $i = 1, \dots, n$, there exists an integer $\kappa_i(j)$ such that,

$$d_2(\delta_{(z_i^{j,k_l^{i,j}}, \nabla v_i^{j,k_l^{i,j}}), \boldsymbol{\nu}_{t_i}^j}) < \frac{1}{j} \quad (6.12)$$

whenever $k_l^{i,j} \geq \kappa_i(j)$.

By taking, if needed, a larger value of $\kappa(j)$, we may assume that (6.11) and (6.12) are satisfied whenever $k \geq \kappa(j)$ and $k_l^{i,j} \geq \kappa(j)$, respectively. Another slight modification allows us to assume that, for every $i = 1, \dots, n$ and for every j , there exists $k_{i,j}^{i,j}$ with

$$\kappa(j) < k_{i,j}^{i,j} \leq \kappa(j+1). \quad (6.13)$$

Let $(\alpha(k))_{k > \kappa(1)}$ be the unique sequence such that $\kappa(\alpha(k)) < k \leq \kappa(\alpha(k)+1)$, for every $k > \kappa(1)$. This implies

$$d_1(\delta_{(z_1^{\alpha(k),k}, \dots, z_n^{\alpha(k),k}), \boldsymbol{\mu}_{t_1 \dots t_n}^{\alpha(k)}}) \leq \frac{1}{\alpha(k)}, \quad (6.14)$$

which, together with (6.7) and (6.9), implies that

$$\delta_{(z_1^{\alpha(k),k}, \dots, z_n^{\alpha(k),k})} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_n}, \quad (6.15)$$

2-weakly* as $k \rightarrow \infty$.

Now, for every $i = 1, \dots, n$ we can choose an integer $\beta_i(j)$ in such a way that

$$\beta_i(j) = k_{i,j}^{i,j}, \quad (6.16)$$

for every j , so that we have $\kappa(j) < \beta_i(j) \leq \kappa(j+1)$, for every j , by (6.13). This implies that $\alpha(\beta_i(j)) = j$ and $\beta_i(j) > \kappa(j)$ so that, by (6.12)

$$d_2(\delta_{(z_i^{\beta_i(j)}, \nabla v_i^{\beta_i(j)})}, \boldsymbol{\nu}_{t_i}^j) < \frac{1}{j}.$$

Therefore, thanks to (6.8) and (6.10) we can conclude that

$$\delta_{(z_i^{\alpha(\beta_i(j_h^i)), \beta_i(j_h^i)}, \nabla v_i^{\alpha(\beta_i(j_h^i)), \beta_i(j_h^i)})} \rightharpoonup \nu_{t_i}, \quad (6.17)$$

as $h \rightarrow \infty$, for every $i = 1, \dots, n$. Since for every i the sequence $(\beta_i(j_h^i))_h$ is increasing, (6.15) and (6.17) show that conditions (ap1) and (ap2) in Definition 6.2 are satisfied. \square

The following lemma consider the case of varying boundary conditions.

Lemma 6.7. *Let \mathbf{w}^j be a sequence of functions from A into $H^1(D, \mathbb{R}^m)$, such that $\mathbf{w}^j(t) \rightarrow \mathbf{w}(t)$ strongly in H^1 , for every $t \in A$ and let $(\nu, \mu) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^A \times SY^2(A, D; \mathbb{R}^m)$. Assume that for every finite sequence $t_1 < \dots < t_n$ in A there exists a sequence $(\nu^j, \mu^j) \in AY(\{t_1, \dots, t_n\}, \mathbf{w}^j)$ such that*

$$\mu_{t_1 \dots t_n}^j \rightharpoonup \mu_{t_1 \dots t_n} \quad 2\text{-weakly}^*, \quad (6.18)$$

as $j \rightarrow \infty$, and such that for every i there exists a subsequence, possibly depending on i , $(\nu^{j_h^i})_h$, satisfying

$$(\nu^{j_h^i})_{t_i} \rightharpoonup \nu_{t_i}, \quad 2\text{-weakly}^*, \quad (6.19)$$

as $h \rightarrow \infty$. Then $(\nu, \mu) \in AY(A, \mathbf{w})$.

Proof. Fixed $t_1 < \dots < t_n$ in $[0, T]$, thanks to Lemma 4.5 from (6.19) we can deduce that for every $i = 1, \dots, n$

$$\tilde{T}_{\nabla \mathbf{w}(t_i) - \nabla \mathbf{w}^{j_h^i}(t_i)}^2 (\nu^{j_h^i})_{t_i} \rightharpoonup \nu_{t_i} \quad 2\text{-weakly}^*$$

as $h \rightarrow \infty$, where $\tilde{T}_{\nabla \mathbf{w}(t_i) - \nabla \mathbf{w}^{j_h^i}(t_i)}^2$ is the map defined by $\tilde{T}_{\nabla \mathbf{w}(t_i) - \nabla \mathbf{w}^{j_h^i}(t_i)}^2(x, \theta, F) := (x, \theta, F + \nabla \mathbf{w}(t_i) - \nabla \mathbf{w}^{j_h^i}(t_i))$. Thanks to Lemma 4.4 it is easy to see that the hypotheses of Lemma 6.6 are satisfied with $(\nu^{j_h^i})_{t_i}$ replaced by $\tilde{T}_{\nabla \mathbf{w}(t_i) - \nabla \mathbf{w}^{j_h^i}(t_i)}^2((\nu^{j_h^i})_{t_i})$. \square

Remark 6.8. If $(\nu, \mu) \in AY(A, \mathbf{w})$, for every $t \in A$ there exists a function $\mathbf{v}(t) \in H_{\Gamma_0}^1(\mathbf{w}(t))$ such that $\nabla \mathbf{v}(t) = \text{bar}(\tilde{\pi}_{\mathbb{R}^{N \times d}}(\nu_t))$. Indeed, by definition of $AY(A, \mathbf{w})$, for every $t \in A$ there exists a sequence $v^k \in H_{\Gamma_0}^1(\mathbf{w}(t))$ such that $\delta_{\nabla v^k} \rightharpoonup \tilde{\pi}_{\mathbb{R}^{N \times d}}(\nu_t)$, 2-weakly*; thanks to a variant of Lemma 4.9 with H_0^1 replaced by $H_{\Gamma_0}^1(\mathbf{w}(t))$, there exists a function $\mathbf{v}(t) \in H^1(D; \mathbb{R})$, such that $v^k \rightharpoonup \mathbf{v}(t)$ weakly in H^1 and $\nabla \mathbf{v}(t) = \text{bar}(\tilde{\pi}_{\mathbb{R}^{N \times d}}(\nu_t))$; since $H_{\Gamma_0}^1(\mathbf{w}(t))$ is closed with respect to the weak convergence, we can conclude that $\mathbf{v}(t) \in H_{\Gamma_0}^1(\mathbf{w}(t))$.

Translating the previous remark in terms of stochastic processes we obtain the following

Remark 6.9. If $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in [0, T]} \in AY_{sp}([0, T], \varphi)$, for every $t \in A$ there exists a function $\mathbf{v}(t) \in H_{\Gamma_0}^1(\mathbf{w}(t))$ such that $\nabla \mathbf{v}(t) = \text{bar}((\pi_D, \mathbf{Y}_t)(P))$.

Remark 6.10. If $(\nu, \mu) \in AY(A, \mathbf{w})$, for every $t \in A$ we define

$$\sigma(t, x) := \int_{\mathbb{R}^m \times \mathbb{R}^{N \times d}} \frac{\partial W}{\partial F}(\theta, F) d\nu_t^x(\theta, F), \quad (6.20)$$

for a.e. $x \in D$. For every $t \in A$ we have that $\sigma(t) \in L^2(D; \mathbb{R}^{N \times d})$: this comes immediately from (W.2), (2.2), and from the fact that $\nu_t \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})$. In the language of stochastic processes $\sigma(t)$ can be characterized as the unique element of $L^2(D; \mathbb{R}^{N \times d})$ such that

$$\int_D \sigma(t, x) g(x) dx = \int_{D \times \Omega} \frac{\partial W}{\partial F}(\mathbf{Z}_t(x, \omega), \mathbf{Y}_t(x, \omega)) g(x) dP(x, \omega), \quad (6.21)$$

for every $g \in L^2(D; \mathbb{R}^{N \times d})$, where $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in [0, T]}$ is the stochastic process corresponding to (ν, μ) .

Remark 6.11. Since $\varphi \in AC([0, T]; H^1(D; \mathbb{R}^N))$ and $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, the time derivative $\dot{\varphi}$ and $\dot{\mathbf{l}}$ are well defined for a.e. $t \in [0, T]$ and belong to the space $L^1([0, T]; H^1(D; \mathbb{R}^N))$ and $L^1([0, T]; H^1(D; \mathbb{R}^N)^*)$, respectively. Moreover the fundamental Theorem of Calculus holds (see, e.g., [3, Appendice]).

6.3. Main result. We are now in the position to define the notion of globally stable quasistatic evolution of stochastic processes.

Definition 6.12. Given $\varphi \in AC([0, T]; H^1(D; \mathbb{R}^N))$, $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, $z_0 \in L^2(D; \mathbb{R}^m)$, $v_0 \in \mathcal{A}(0)$, and $T > 0$, a *globally stable quasistatic evolution of stochastic processes* with boundary datum φ , external load \mathbf{l} , and initial condition (z_0, v_0) , in the time interval $[0, T]$, is a stochastic process $(\mathbf{Z}_t, \mathbf{Y}_t)_{t \in [0, T]} \in AY_{sp}([0, T], \varphi)$, such that for every finite sequence $t_1 < \dots < t_n$ in $[0, T]$ we have

$$(\pi_D, \mathbf{Z}_{s_1^j}, \dots, \mathbf{Z}_{s_n^j})(P) \rightharpoonup (\pi_D, \mathbf{Z}_{t_1}, \dots, \mathbf{Z}_{t_n})(P)$$

2-weakly*, as $s_i^j \rightarrow t_i$ with $s_i^j \leq t_i$, and satisfying the following conditions:

(ev0) *initial condition:* $(\mathbf{Z}_0, \mathbf{Y}_0) = (z_0, \nabla v_0)$;

(ev1) *global stability:* for every $t \in [0, T]$, we have

$$\begin{aligned} & \int_{D \times \Omega} W(\mathbf{Z}_t(x, \omega), \mathbf{Y}_t(x, \omega)) dP(x, \omega) \leq \\ & \leq \int_{D \times \Omega} W(\mathbf{Z}_t(x, \omega) + \tilde{z}(x), \mathbf{Y}_t(x, \omega) + \nabla \tilde{u}(x)) dP(x, \omega) - \langle \mathbf{l}(t), \tilde{u} \rangle + \mathcal{H}(\tilde{z}), \end{aligned}$$

for every $\tilde{z} \in L^2(D; \mathbb{R}^m)$ and every $\tilde{u} \in H_{\Gamma_0}^1(0)$;

(ev2) *energy inequality:* for every $t \in [0, T]$ we have

$$\text{Var}_H(\mathbf{Z}, P; 0, t) := \sup \sum_{i=1}^k \int_{D \times \Omega} H(\mathbf{Z}_{t_i}(x, \omega) - \mathbf{Z}_{t_{i-1}}(x, \omega)) dP(x, \omega) < \infty,$$

where the supremum is taken over all finite partitions $0 = t_0 < \dots < t_k = t$, and the map

$$t \mapsto [\langle \boldsymbol{\sigma}(t), \nabla \dot{\varphi}(t) \rangle - \langle \dot{\mathbf{l}}(t), \mathbf{v}(t) \rangle]$$

is measurable on $[0, T]$, where $\boldsymbol{\sigma}(t)$ is the function defined in (6.21) and $\mathbf{v}(t)$ that one defined in Remark 6.9; moreover

$$\begin{aligned} & \int_{D \times \Omega} W(\mathbf{Z}_t(x, \omega), \mathbf{Y}_t(x, \omega)) dP(x, \omega) - \langle \mathbf{l}(t), \mathbf{v}(t) \rangle + \text{Var}_H(\mathbf{Z}, P; 0, t) \leq \\ & \leq \mathcal{W}(z_0, v_0) - \langle \mathbf{l}(0), v_0 \rangle + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla \dot{\varphi}(s) \rangle ds + \\ & \quad - \int_0^t [\langle \mathbf{l}(s), \dot{\varphi}(s) \rangle + \langle \dot{\mathbf{l}}(s), \mathbf{v}(s) \rangle] ds. \end{aligned}$$

We now state the main existence theorem in terms of stochastic processes.

Theorem 6.13. *Let $\varphi \in AC([0, T]; H^1(D; \mathbb{R}^N))$, $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, $T > 0$, $z_0 \in L^2(D; \mathbb{R}^m)$ and $v_0 \in \mathcal{A}(0)$ be such that*

$$\mathcal{W}(z_0, v_0) \leq \mathcal{W}(z_0 + \tilde{z}, v_0 + \tilde{u}) - \langle \mathbf{l}(0), \tilde{u} \rangle + \mathcal{H}(\tilde{z}), \quad (6.22)$$

for every $\tilde{z} \in L^2(D; \mathbb{R}^m)$ and every $\tilde{u} \in H_{\Gamma_0}^1(0)$. Then there exists a globally stable quasistatic evolution for stochastic processes with boundary datum φ , external load \mathbf{l} , and initial condition (z_0, v_0) , in the time interval $[0, T]$.

Thanks to Remark 6.4, we can translate the definition of globally stable quasistatic evolution of stochastic processes in terms of globally stable quasistatic evolution of Young measures.

Definition 6.14. Given $\varphi \in AC([0, T]; H^1(D; \mathbb{R}^N))$, $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, $z_0 \in L^2(D; \mathbb{R}^m)$, $v_0 \in \mathcal{A}(0)$, and $T > 0$, a *globally stable quasistatic evolution of Young measures* with boundary datum φ , external load \mathbf{l} and initial condition (z_0, v_0) , in the time interval $[0, T]$, is a pair $(\nu, \mu) \in AY([0, T], \varphi)$, with $\mu \in SY_-^2([0, T], D; \mathbb{R}^m)$, satisfying the following conditions:

- (ev0) *initial condition*: $\nu_0 = \delta_{(z_0, \nabla v_0)}$;
- (ev1) *global stability*: for every $t \in [0, T]$, we have

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d\nu_t(x, \theta, F) \leq \\ & \leq \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) d\nu_t(x, \theta, F) - \langle \mathbf{l}(t), \tilde{u} \rangle + \mathcal{H}(\tilde{z}), \end{aligned}$$

for every $\tilde{z} \in L^2(D; \mathbb{R}^m)$ and every $\tilde{u} \in H_{\Gamma_0}^1(0)$;

- (ev2) *energy inequality*: for every $t \in [0, T]$ we have that $\text{Var}_H(\mu; 0, t) < \infty$, (see (4.9)), and the map

$$t \mapsto [\langle \sigma(t), \nabla \dot{\varphi}(t) \rangle - \langle \dot{\mathbf{l}}(t), \mathbf{v}(t) \rangle] \quad (6.23)$$

is measurable on $[0, T]$, where $\sigma(t)$ is the function defined in (6.20) and $\mathbf{v}(t)$ that one defined in Remark 6.8; moreover

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d\nu_t(x, \theta, F) - \langle \mathbf{l}(t), \mathbf{v}(t) \rangle + \text{Var}_H(\mu; 0, t) \leq \\ & \leq \mathcal{W}(z_0, v_0) - \langle \mathbf{l}(0), v_0 \rangle + \int_0^t \langle \sigma(s), \nabla \dot{\varphi}(s) \rangle ds + \\ & \quad - \int_0^t [\langle \mathbf{l}(s), \dot{\varphi}(s) \rangle + \langle \dot{\mathbf{l}}(s), \mathbf{v}(s) \rangle] ds. \end{aligned}$$

Thanks to Theorem 3.5, to obtain the main theorem it is enough to prove the following version for Young measures.

Theorem 6.15. *Let $\varphi \in AC([0, T]; H^1(D; \mathbb{R}^N))$, $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, $T > 0$, $z_0 \in L^2(D; \mathbb{R}^m)$, and $v_0 \in \mathcal{A}(0)$ be such that*

$$\mathcal{W}(z_0, v_0) \leq \mathcal{W}(z_0 + \tilde{z}, v_0 + \tilde{u}) - \langle \mathbf{l}(0), \tilde{u} \rangle + \mathcal{H}(\tilde{z}), \quad (6.24)$$

for every $\tilde{z} \in L^2(D; \mathbb{R}^m)$ and every $\tilde{u} \in H_{\Gamma_0}^1(0)$. Then there exists a globally stable quasistatic evolution for Young measures with boundary datum φ , external load \mathbf{l} , and initial condition (z_0, v_0) , in the time interval $[0, T]$.

Remark 6.16. In the proof of Theorem 6.15 we will obtain, in particular, a globally stable quasistatic evolution (ν, μ) and a subset Θ of $[0, T]$ which satisfy the following property:

- (appr) for every $t \in [0, T]$ there exists a sequence s^j in Θ with $s^j \leq t$ and $s^j \rightarrow t$, such that

$$\nu_{s^j} \rightharpoonup \nu_t \quad \text{2-weakly*}.$$

7. PROOF OF THE MAIN THEOREM

The proof is obtained via time discretization, resolution of incremental minimum problems, and passing to the limit as the discretization step tends to 0.

7.1. The incremental minimum problem. Let us fix a sequence of subdivisions of $[0, T]$, $0 = t_n^0 < t_n^1 < \dots < t_n^{k(n)} = T$, such that $\sup_{i=1, \dots, k(n)} \tau_n^i \rightarrow 0$, as $n \rightarrow \infty$, where $\tau_n^i := t_n^i - t_n^{i-1}$, for every $i = 1, \dots, k(n)$.

For every $i = 0, 1, \dots, k(n)$ we set $\mathbf{l}_n^i := \mathbf{l}(t_n^i)$ and $\varphi_n^i := \varphi(t_n^i)$.

We will define $(\boldsymbol{\nu}_n^i, \boldsymbol{\mu}_n^i) \in AY(\{t_n^0, \dots, t_n^i\}, \varphi)$ by induction on i : set $(\boldsymbol{\nu}_n^0, \boldsymbol{\mu}_n^0) := \delta_{(z_0, \nabla v_0)}$, and for $i > 0$ we define $(\boldsymbol{\nu}_n^i, \boldsymbol{\mu}_n^i)$ as a minimizer (see Lemma 7.2 below) of the functional

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d\boldsymbol{\nu}_{t_n^i}^i(x, \theta, F) - \langle \mathbf{l}_n^i, \mathbf{v}(t_n^i) \rangle + \\ & + \int_{D \times (\mathbb{R}^m)^2} H(\theta_i - \theta_{i-1}) d\boldsymbol{\mu}_{t_n^{i-1} t_n^i}^i(x, \theta_{i-1}, \theta_i), \end{aligned} \quad (7.1)$$

in the set A_n^i of all $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(\{t_n^0, \dots, t_n^i\}, \varphi)$, satisfying

$$\boldsymbol{\mu}_{t_n^0 \dots t_n^{i-1}}^i = (\boldsymbol{\mu}_n^{i-1})_{t_n^0 \dots t_n^{i-1}} \quad (7.2)$$

$$\boldsymbol{\nu}_{t_n^j}^i = (\boldsymbol{\nu}_n^{i-1})_{t_n^j}, \quad \text{for every } j < i, \quad (7.3)$$

where the function $\mathbf{v}(t_n^i)$ appearing in (7.1) is that one defined in Remark 6.8.

Lemma 7.1. *The set A_n^i is nonempty, for every $i > 1$.*

Proof. Fixed $(\boldsymbol{\nu}_n^{i-1}, \boldsymbol{\mu}_n^{i-1})$, we consider the map $\tilde{\mathcal{T}}_{\nabla \varphi_n^i - \nabla \varphi_n^{i-1}}^2 : (x, \theta, F) \mapsto (x, \theta, F + \nabla \varphi_n^i(x) - \nabla \varphi_n^{i-1}(x))$, and the map $\pi^{(i)} : D \times (\mathbb{R}^m)^i \rightarrow D \times (\mathbb{R}^m)^{i+1}$ defined by $\pi^{(i)}(x, \theta_1, \dots, \theta_{i-1}) := (x, \theta_1, \dots, \theta_{i-1}, \theta_{i-1})$; let $\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^m)^{\{t_n^0, \dots, t_n^i\}}$ be defined by $\boldsymbol{\nu}_{t_n^j}^i := (\boldsymbol{\nu}_n^{i-1})_{t_n^j}$, for $j < i$ and $\boldsymbol{\nu}_{t_n^i}^i := \tilde{\mathcal{T}}_{\nabla \varphi_n^i - \nabla \varphi_n^{i-1}}^2((\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}})$ and $\boldsymbol{\mu}$ the unique element of $SY^2(\{t_n^0, \dots, t_n^i\}, D; \mathbb{R}^m)$ satisfying $\boldsymbol{\mu}_{t_n^0 \dots t_n^i}^i = \pi^{(i)}((\boldsymbol{\mu}_n^{i-1})_{t_n^0 \dots t_n^{i-1}})$. It is evident that $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ so defined satisfy the projection properties (7.2) and (7.3). Moreover it is easy to prove that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(\{t_n^0, \dots, t_n^i\}, \varphi)$: since $\pi^{(i)}(\delta_{(z^0, \dots, z^{i-1})}) = \delta_{(z^0, \dots, z^{i-1}, z^{i-1})}$ and $\tilde{\mathcal{T}}_{\nabla \varphi_n^i - \nabla \varphi_n^{i-1}}^2(\delta_{(z, \nabla v)}) = \delta_{(z, \nabla v + \nabla \varphi_n^i - \nabla \varphi_n^{i-1})}$, with $v + \varphi_n^i - \varphi_n^{i-1} \in \mathcal{A}(t_n^i)$ whenever $v \in \mathcal{A}(t_n^{i-1})$, applying Lemma 4.4 we can obtain the approximation properties (6.1) and (6.2). \square

Lemma 7.2. *For every i the functional (7.1) has a minimizer over A_n^i .*

Proof. Let $(\boldsymbol{\nu}^h, \boldsymbol{\mu}^h)_h \subset A_n^i$ be a minimizing sequence. By the bounds on W and the assumption on \mathbf{l} , using Poincaré inequality we have

$$\begin{aligned} & c \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [|\theta|^2 + |F|^2] d\boldsymbol{\nu}_{t_n^i}^h(x, \theta, F) - C'(1 + \|\nabla \mathbf{v}^h(t_n^i)\|_2) \leq \\ & \leq \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d\boldsymbol{\nu}_{t_n^i}^h(x, \theta, F) - \langle \mathbf{l}_n^i, \mathbf{v}^h(t_n^i) \rangle \leq C', \end{aligned}$$

for every h , for positive constants c, C' . Since by Remark 6.8

$$\|\nabla \mathbf{v}^h(t_n^i)\|_2 \leq \left(\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |F|^2 d\boldsymbol{\nu}_{t_n^i}^h(x, \theta, F) \right)^{1/2}, \quad (7.4)$$

we can deduce that

$$\sup_h \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d\boldsymbol{\nu}_{t_n^i}^h(x, \theta, F) \leq C'. \quad (7.5)$$

Since, thanks to Remark 6.3

$$\begin{aligned} & \int_{D \times (\mathbb{R}^m)^{i+1}} |(\theta_0, \dots, \theta_i)|^2 d\boldsymbol{\mu}_{t_n^0 \dots t_n^i}^h(x, \theta_0, \dots, \theta_i) = \sum_{j=0}^{i+1} \int_{D \times \mathbb{R}^m} |\theta_j|^2 d\boldsymbol{\mu}_{t_n^j}^h(x, \theta_j) = \\ & = \sum_{j=0}^{i+1} \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |\theta_j|^2 d\boldsymbol{\nu}_{t_n^j}^h(x, F, \theta_j), \end{aligned}$$

the projection property (7.3) and (7.5) imply that the second moments of $\mu_{t_n^0 \dots t_n^i}^h$ are bounded uniformly with respect to h . From this and from (7.5) we can deduce that, up to a subsequence, there exist $\bar{\mu} \in Y^2(D; (\mathbb{R}^m)^{i+1})$ and $\bar{\nu} \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})$ such that

$$\mu_{t_n^0 \dots t_n^i}^h \rightharpoonup \bar{\mu} \quad \text{2-weakly}^*, \quad (7.6)$$

$$\nu_{t_n^i}^h \rightharpoonup \bar{\nu} \quad \text{2-weakly}^*. \quad (7.7)$$

Hence defining $\bar{\nu}_{t_n^j} := (\nu_n^{i-1})_{t_n^j}$, for every $j < i$, $\bar{\nu}_{t_n^i} := \bar{\nu}$, and $\bar{\mu}$ as the unique element of $SY^2(\{t_n^0, \dots, t_n^i\}, D; \mathbb{R}^m)$ such that $\bar{\mu}_{t_n^0 \dots t_n^i} = \bar{\mu}$, we obtain

$$(\bar{\nu}, \bar{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{\{t_n^0, \dots, t_n^i\}} \times SY^2(\{t_n^0, \dots, t_n^i\}, D; \mathbb{R}^m).$$

Since the hypotheses of Lemma 6.6 are satisfied, we deduce that $(\bar{\nu}, \bar{\mu}) \in AY(\{t_n^0, \dots, t_n^i\}, \varphi)$. Moreover, by construction, $(\bar{\nu}, \bar{\mu})$ satisfies also the projection properties (7.2) and (7.3), so we can conclude that $(\bar{\nu}, \bar{\mu}) \in A_n^i$.

By (4.1) the terms of (7.1) containing W and H are lower semicontinuous with respect to the 2-weak* convergence; on the other hand a variant of Lemma 4.9, with H_0^1 replaced by $\mathcal{A}(t_n^i)$, shows that the term of (7.1) containing \mathbf{l}_n^i is continuous with respect to the 2-weak* convergence, therefore the functional (7.1) is 2-weakly* lower semicontinuous and this implies that $(\bar{\nu}, \bar{\mu})$ is a minimizer for it in A_n^i . \square

Remark 7.3. Even if $W(\theta, \cdot): \mathbb{R}^{N \times d} \rightarrow [0, +\infty)$ is convex for every $\theta \in \mathbb{R}^m$, it may happen that the incremental minimum problems have no solutions representable by functions.

We give an example in which this happens even for the first time step. By definition of A_n^1 , there exists a solution of the first incremental minimum problem representable by function if and only if there exist $z_1 \in L^2(D; \mathbb{R}^m)$ and $v_1 \in \mathcal{A}(t_n^1)$ such that $((\delta_{(z_0, \nabla v_0)}, \delta_{(z_1, \nabla v_1)}), \delta_{(z_0, z_1)})$ realizes the minimum of the functional (7.1) on A_n^1 . Consider the following case: $D = (0, 1)^2$ and $N = m = 1$, $T = 1$, $\mathbf{l} \equiv 0$ and $\varphi(t, (x_1, x_2)) := (1-t)x_1$, for every $t \in [0, 1]$ and every $(x_1, x_2) \in (0, 1)^2$. We consider

$$W(\theta, (F_1, F_2)) := |F_1 - a(\theta)|^2 + |F_2|^2 + b(\theta) + c, \quad (7.8)$$

where a is a \mathcal{C}^1 function satisfying $a(0) = a(1) = 1$ and $a(-1) = -1$, while b a \mathcal{C}^1 function such that $\tilde{b}(\theta) := b(\theta) + |\theta|$ is positive and vanishes only at 0, 1, and -1 , and $c := -\inf b$. It can be easily verified that (W.1) and (W.2) are satisfied by suitable choices of a and b compatible with the requirements above. Now choose $H(\theta) := |\theta|$, $z_0 \equiv 0$ and $v_0(x_1, x_2) := x_1$. It is immediate to check that (z_0, v_0) satisfy the boundary conditions and (6.24). Moreover, by standard arguments, it can be easily shown that the infimum of functional in (7.1), for $i = 1$, is c and cannot be attained by functions which satisfy the boundary conditions. A minimizer of (7.1) on A_n^1 is defined by

$$\begin{aligned} (\nu_n^1)_{t_n^1} &:= \frac{t_n^1}{2} \delta_{(-1, (-1, 0))} + (1 - \frac{t_n^1}{2}) \delta_{(1, (1, 0))}, \\ (\mu_n^1)_{t_n^0, t_n^1} &:= \frac{t_n^1}{2} \delta_{(0, -1)} + (1 - \frac{t_n^1}{2}) \delta_{(0, 1)}. \end{aligned}$$

Set $\tau^n(s) := t_n^i$, whenever $t_n^i \leq s < t_n^{i+1}$.

For every i and n we set

$$\sigma_n^i(x) := \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} \frac{\partial W}{\partial F}(\theta, F) d(\nu_n^i)_{t_n^i}^x,$$

and define

$$\sigma_n(t, x) := \sigma_n^i(x), \quad (7.9)$$

for a.e. $x \in D$, whenever $t_n^i \leq t < t_n^{i+1}$.

We define $\nu_n \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{[0, T]}$ by

$$(\nu_n)_s := (\nu_n^i)_{t_n^i}, \quad (7.10)$$

whenever $t_n^i = \tau^n(s)$, for every $s \in [0, T]$; we define also $\boldsymbol{\mu}_n \in SY^2([0, T], D; \mathbb{R}^m)$ as the piecewise constant interpolation of $\boldsymbol{\mu}_n^{k(n)}$, as in Definition 4.12.

Note that $(\boldsymbol{\nu}_n, \boldsymbol{\mu}_n) \in AY([0, T], \boldsymbol{\varphi}(\tau^n(\cdot)))$ by construction.

7.2. A priori estimates. First of all we want to deduce a discrete version of the energy inequality for $(\boldsymbol{\nu}_n, \boldsymbol{\mu}_n)$.

Using the competitor defined in the proof of Lemma 7.1 and the fact that

$$\text{bar}(\tilde{\pi}_{\mathbb{R}^N \times d}(\tilde{T}_{\nabla \boldsymbol{\varphi}_n^i - \nabla \boldsymbol{\varphi}_n^{i-1}}^2((\boldsymbol{\nu}_n^i)_{t_n^i}))) = \text{bar}(\tilde{\pi}_{\mathbb{R}^N \times d}((\boldsymbol{\nu}_n^i)_{t_n^i})) + \nabla \boldsymbol{\varphi}_n^i - \nabla \boldsymbol{\varphi}_n^{i-1},$$

for every i and n , we have

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d(\boldsymbol{\nu}_n^i)_{t_n^i}(x, \theta, F) - \langle \boldsymbol{l}_n^i, \boldsymbol{v}_n^i(t_n^i) \rangle + \\ & \quad + \int_{D \times (\mathbb{R}^m)^2} H(\theta_i - \theta_{i-1}) d(\boldsymbol{\mu}_n^i)_{t_n^{i-1} t_n^i}(x, \theta_{i-1}, \theta_i) \leq \\ & \leq \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F + \nabla \boldsymbol{\varphi}_n^i(x) - \nabla \boldsymbol{\varphi}_n^{i-1}(x)) d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x, \theta, F) + \\ & \quad - \langle \boldsymbol{l}_n^i, \boldsymbol{v}_n^i(t_n^{i-1}) \rangle + \boldsymbol{\varphi}_n^i - \boldsymbol{\varphi}_n^{i-1} + \\ & \quad + \int_{D \times (\mathbb{R}^m)^{i+1}} H(\theta_i - \theta_{i-1}) d(\pi^{(i)}((\boldsymbol{\mu}_n^{i-1})_{t_n^0 \dots t_n^{i-1}}))(x, \theta_0, \dots, \theta_{i-1}, \theta_i). \end{aligned}$$

We deduce that

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d(\boldsymbol{\nu}_n^i)_{t_n^i}(x, \theta, F) - \langle \boldsymbol{l}_n^i, \boldsymbol{v}_n^i(t_n^i) \rangle + \\ & \quad + \int_{D \times (\mathbb{R}^m)^2} H(\theta_i - \theta_{i-1}) d(\boldsymbol{\mu}_n^i)_{t_n^{i-1} t_n^i}(x, \theta_{i-1}, \theta_i) \leq \\ & \leq \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x, \theta, F) - \langle \boldsymbol{l}_n^{i-1}, \boldsymbol{v}_n^{i-1}(t_n^{i-1}) \rangle + \\ & \quad + \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F + \nabla \boldsymbol{\varphi}_n^i(x) - \nabla \boldsymbol{\varphi}_n^{i-1}(x)) - W(\theta, F)] d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x, \theta, F) + \\ & \quad - \langle \boldsymbol{l}_n^i, \boldsymbol{v}_n^{i-1}(t_n^{i-1}) \rangle + \boldsymbol{\varphi}_n^i - \boldsymbol{\varphi}_n^{i-1} + \langle \boldsymbol{l}_n^{i-1}, \boldsymbol{v}_n^{i-1}(t_n^{i-1}) \rangle. \end{aligned}$$

Let us fix $t \in [0, T]$ such that $t_n^j \leq t < t_n^{j+1}$; using

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F + \nabla \boldsymbol{\varphi}_n^i(x) - \nabla \boldsymbol{\varphi}_n^{i-1}(x)) - W(\theta, F)] d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x, \theta, F) = \\ & = \int_{t_n^{i-1}}^{t_n^i} \left(\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} \frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) \nabla \dot{\boldsymbol{\varphi}}(s, x) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right) ds, \end{aligned}$$

where $\varepsilon^n(s, x) := \nabla \boldsymbol{\varphi}(s, x) - \nabla \boldsymbol{\varphi}(\tau^n(s), x)$, for every $s \in [0, T]$ and every $x \in D$, and

$$\begin{aligned} & \langle \boldsymbol{l}_n^i, \boldsymbol{v}_n^{i-1}(t_n^{i-1}) \rangle + \boldsymbol{\varphi}_n^i - \boldsymbol{\varphi}_n^{i-1} - \langle \boldsymbol{l}_n^{i-1}, \boldsymbol{v}_n^{i-1}(t_n^{i-1}) \rangle = \\ & = \int_{t_n^{i-1}}^{t_n^i} [\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle + \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_n^{i-1}(t_n^{i-1}) - \boldsymbol{\varphi}(\tau^n(s)) + \boldsymbol{\varphi}(s) \rangle] ds, \end{aligned}$$

and iterating from 0 to j , we obtain

$$\begin{aligned}
& \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d(\boldsymbol{\nu}_n)_t(x, \theta, F) - \langle \mathbf{l}(\tau^n(t)), \mathbf{v}_n(t) \rangle + \text{Var}_H(\boldsymbol{\mu}_n; 0, t) \leq \\
& \leq \mathcal{W}(z_0, v_0) - \langle \mathbf{l}(0), v_0 \rangle + \int_0^{\tau^n(t)} \langle \boldsymbol{\sigma}_n(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle ds + \\
& \quad - \int_0^{\tau^n(t)} [\langle \dot{\mathbf{i}}(s), \mathbf{v}_n(s) \rangle + \langle \mathbf{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle] ds + \\
& + \int_0^{\tau^n(t)} \left(\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} \left[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) - \frac{\partial W}{\partial F}(\theta, F) \right] \nabla \dot{\boldsymbol{\varphi}}(s) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right) ds + \\
& \quad + \int_0^{\tau^n(t)} \langle \dot{\mathbf{i}}(s), \boldsymbol{\varphi}(\tau^n(s)) - \boldsymbol{\varphi}(s) \rangle ds.
\end{aligned} \tag{7.11}$$

From (7.11), we can deduce the following a priori estimates on $(\boldsymbol{\nu}_n, \boldsymbol{\mu}_n)$.

Lemma 7.4. *There exists a positive constant C , such that*

$$\sup_n \sup_{t \in [0, T]} \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d(\boldsymbol{\nu}_n)_t(x, \theta, F) \leq C, \tag{7.12}$$

$$\sup_n \text{Var}_H(\boldsymbol{\mu}_n; 0, T) \leq C. \tag{7.13}$$

Proof. Using the fact that $\sup_{t \in [0, T]} \|\mathbf{l}(t)\|_{(H^1)^*}$, $\int_0^T \|\dot{\mathbf{i}}(t)\|_{(H^1)^*} dt$, and $\int_0^T \|\dot{\boldsymbol{\varphi}}(t)\|_{H^1} dt$ are finite, the hypotheses on W and the inequality

$$\sup_{s \in [0, T]} \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d(\boldsymbol{\nu}_n)_s(x, \theta, F) < \infty,$$

(since $\boldsymbol{\nu}_n$ are piecewise constant interpolations of Young measures with finite second moments), we can deduce from (7.11) that, for n sufficiently large,

$$\begin{aligned}
& \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d(\boldsymbol{\nu}_n)_t(x, \theta, F) \leq \\
& \leq \tilde{C} + \tilde{C} \sup_{s \in [0, T]} \left(1 + \tilde{c} \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right)^{1/2},
\end{aligned} \tag{7.14}$$

for suitable positive constants \tilde{C} and \tilde{c} independent of t and n (to estimate the terms in the third line in (7.11) we use (7.4), while the term in the fourth line of (7.11) can be treated using $\pi_D((\boldsymbol{\nu}_n)_s) = \mathcal{L}^d$ and Hölder inequality).

Since this can be repeated for every $t \in [0, T]$, we deduce (7.12). Inequality (7.13) comes now from (7.12) and (7.11). \square

We can also deduce the following energy inequality for $(\boldsymbol{\nu}_n, \boldsymbol{\mu}_n)$.

Lemma 7.5. *For every $t \in [0, T]$ we have*

$$\begin{aligned}
& \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d(\boldsymbol{\nu}_n)_t(x, \theta, F) - \langle \mathbf{l}(\tau^n(t)), \mathbf{v}_n(t) \rangle + \text{Var}_H(\boldsymbol{\mu}_n; 0, t) \leq \\
& \leq \mathcal{W}(z_0, v_0) - \langle \mathbf{l}(0), v_0 \rangle + \int_0^{\tau^n(t)} \langle \boldsymbol{\sigma}_n(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle ds + \\
& \quad - \int_0^{\tau^n(t)} [\langle \dot{\mathbf{i}}(s), \mathbf{v}_n(s) \rangle + \langle \mathbf{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle] ds + \rho_n,
\end{aligned} \tag{7.15}$$

where $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Thanks to (7.11) it is enough to prove that

$$\rho_n^1 := \int_0^{\tau^n(t)} \left(\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} \left[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) - \frac{\partial W}{\partial F}(\theta, F) \right] \nabla \dot{\varphi}(s, x) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right) ds$$

and

$$\rho_n^2 := \int_0^{\tau^n(t)} \langle \dot{\mathbf{i}}(s), \varphi(\tau^n(s)) - \varphi(s) \rangle ds$$

tend to 0 as $n \rightarrow \infty$. Since φ is uniformly continuous on $[0, T]$ with values in $H^1(D; \mathbb{R}^N)$, it is immediate to see that $\rho_n^2 \rightarrow 0$ as $n \rightarrow \infty$. It remains to prove that, fixed $\delta > 0$, $\rho_n^1 < \delta$ for n sufficiently large.

We recall that, since $\nabla \dot{\varphi} \in L^1([0, T]; L^2(D; \mathbb{R}^{N \times d}))$, we can find a sequence $v_j \in \mathcal{C}^0([0, T]; \mathcal{C}^0(\bar{D}; \mathbb{R}^N))$ such that $\int_0^T \|v_j(t) - \nabla \dot{\varphi}(t)\|_2 dt \rightarrow 0$, as $j \rightarrow \infty$.

Since $\pi_D((\boldsymbol{\nu}_n)_s) = \mathcal{L}^d$, using (W.2), we can deduce for every $M > 1$ and every $s \in [0, T]$

$$\begin{aligned} & \int_{\{(x, \theta, F): |\theta| + |F| > M\}} \left[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) - \frac{\partial W}{\partial F}(\theta, F) \right] \nabla \dot{\varphi}(s) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \leq \\ & \leq C \int_{\{(x, \theta, F): |\theta| + |F| > M\}} 2(1 + |\theta| + |F|) |\nabla \dot{\varphi}(s, x) - v_j(s, x)| d(\boldsymbol{\nu}_n)_s(x, \theta, F) + \\ & + C \int_{\{(x, \theta, F): |\theta| + |F| > M\}} 2(1 + |\theta| + |F|) |v_j(s, x)| d(\boldsymbol{\nu}_n)_s(x, \theta, F) + \\ & + C \|\varepsilon^n(s)\|_2 \|\nabla \dot{\varphi}(s)\|_2; \end{aligned}$$

therefore, thanks to Lemma 7.4, for every j we have

$$\begin{aligned} & \int_0^{\tau^n(t)} \left(\int_{\{(x, \theta, F): |\theta| + |F| > M\}} \left[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) - \frac{\partial W}{\partial F}(\theta, F) \right] \nabla \dot{\varphi}(s) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right) ds \leq \\ & \leq 8C \left[\sup_{s \in [0, T]} \sup_n \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right]^{1/2} \int_0^t \|\nabla \dot{\varphi}(s) - v_j(s)\|_2 ds + \\ & + 4T \sup_{s \in [0, T]} \sup_n \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 d(\boldsymbol{\nu}_n)_s(x, \theta, F) \frac{\|v_j\|_\infty}{M} + \\ & + C \sup_{s \in [0, T]} \|\varepsilon^n(s)\|_2 \int_0^T \|\nabla \dot{\varphi}(s)\|_2 ds. \end{aligned}$$

Since $s \mapsto \varphi(s)$ is continuous from $[0, T]$ into $H^1(D; \mathbb{R}^N)$, the term in the last line tends to 0 as $n \rightarrow \infty$. Therefore there exist \bar{j} and \bar{M} such that

$$\begin{aligned} & \int_0^{\tau^n(t)} \left(\int_{\{(x, \theta, F): |\theta| + |F| > \bar{M}\}} \left[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) - \frac{\partial W}{\partial F}(\theta, F) \right] \nabla \dot{\varphi}(s) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right) ds \leq \\ & \leq \frac{\delta}{2}, \end{aligned} \tag{7.16}$$

for n sufficiently large.

We now consider the contribution of the integral on $\{(x, \theta, F) : |\theta| + |F| \leq \bar{M}\}$.

For every $M > 0$ and $r > 0$, define

$$\omega_M(r) := \sup_{|(\theta, F)| \leq M, |(\theta', F')| \leq r} \left| \frac{\partial W}{\partial F}(\theta + \theta', F + F') - \frac{\partial W}{\partial F}(\theta, F) \right|.$$

Thanks to the continuity properties of $\frac{\partial W}{\partial F}$, for every M we have $\omega_M(r) \rightarrow 0$ as r tends to 0; moreover it is $\omega_M(r) \leq 2C(M+1) + Cr$.

In particular, for \bar{M} chosen before, it is immediate that

$$\int_0^{\tau^n(t)} \left(\int_{\{(x,\theta,F):|\theta|+|F|\leq\bar{M}\}} \left[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) - \frac{\partial W}{\partial F}(\theta, F) \right] \nabla \dot{\varphi}(s) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right) ds \leq \int_0^t \left(\int_D \omega_{\bar{M}}(|\varepsilon^n(s, x)|) |\nabla \dot{\varphi}(s, x)| dx \right) ds;$$

by the Dominated Convergence Theorem, we have

$$\|\omega_{\bar{M}}(|\varepsilon^n(s)|)\|_2 \rightarrow 0,$$

for a.e. $s \in [0, T]$, as n tends to ∞ ; since we have the estimate

$$\|\omega_{\bar{M}}(|\varepsilon^n(s)|)\|_2 \|\nabla \dot{\varphi}(s)\|_2 \leq C \|\nabla \dot{\varphi}(s)\|_2 (2\bar{M} + 2 + \|\varepsilon^n(s)\|_2),$$

for every $s \in [0, T]$ and $\sup_{s \in [0, T]} \|\varepsilon^n(s)\|_2 \rightarrow 0$ as $n \rightarrow \infty$, we can apply again the Dominated Convergence Theorem and obtain

$$\begin{aligned} \int_0^{\tau^n(t)} \left(\int_{\{(x,\theta,F):|\theta|+|F|\leq\bar{M}\}} \left[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s, x)) - \frac{\partial W}{\partial F}(\theta, F) \right] \nabla \dot{\varphi}(s) d(\boldsymbol{\nu}_n)_s(x, \theta, F) \right) ds &\leq \\ &\leq \frac{\delta}{2}, \end{aligned} \quad (7.17)$$

for n sufficiently large. Therefore (7.16) and (7.17) give the thesis. \square

7.3. Passage to the limit. Thanks to (7.12), (7.13), and hypothesis (H.2), we can apply our version of Helly Theorem (Theorem 4.10) to the sequence $\boldsymbol{\mu}_n$ and obtain a subsequence, still indicated by $\boldsymbol{\mu}_n$, a subset Θ of $[0, T]$, containing 0, with $\mathcal{L}^1([0, T] \setminus \Theta) = 0$, and $\boldsymbol{\mu} \in SY_-^2([0, T], D; \mathbb{R}^m)$, such that, for every finite sequence $t_1 < \dots < t_l$ in Θ , we have

$$(\boldsymbol{\mu}_n)_{t_1 \dots t_l} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_l}, \quad 2\text{-weakly}^*. \quad (7.18)$$

For every $t \in \Theta$ we choose an increasing sequence of integers n_k^t , possibly depending on t , such that

$$\limsup_n \langle \boldsymbol{\sigma}_n(t), \nabla \dot{\varphi}(t) \rangle + \langle \dot{\mathbf{i}}(t), \mathbf{v}_n(t) \rangle = \lim_k \langle \boldsymbol{\sigma}_{n_k^t}(t), \nabla \dot{\varphi}(t) \rangle + \langle \dot{\mathbf{i}}(t), \mathbf{v}_{n_k^t}(t) \rangle, \quad (7.19)$$

where $\mathbf{v}_n(t)$ is defined as in Remark (6.8). Thanks to (7.12) and Lemma 4.13, there exists $\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{[0, T]}$, such that $\tilde{\pi}_{\mathbb{R}^m}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$ for every $t \in [0, T]$, and satisfying the following properties:

(conv) for every $t \in \Theta$, there exists a subsequence of $(\boldsymbol{\nu}_{n_k^t})_k$, still denoted by $(\boldsymbol{\nu}_{n_k^t})_k$, such that

$$(\boldsymbol{\nu}_{n_k^t})_t \rightharpoonup \boldsymbol{\nu}_t, \quad 2\text{-weakly}^*; \quad (7.20)$$

(appr) for every $t \in [0, T] \setminus \Theta$, there exists a sequence $s^j \in \Theta$, converging to t , with $s^j \leq t$, such that

$$\boldsymbol{\nu}_{s^j} \rightharpoonup \boldsymbol{\nu}_t, \quad 2\text{-weakly}^*. \quad (7.21)$$

Note that the map (6.23) is measurable on $[0, T]$ since

$$\langle \boldsymbol{\sigma}(t), \nabla \dot{\varphi}(t) \rangle - \langle \dot{\mathbf{i}}(t), \mathbf{v}(t) \rangle = \limsup_n \langle \boldsymbol{\sigma}_n(t), \nabla \dot{\varphi}(t) \rangle - \langle \dot{\mathbf{i}}(t), \mathbf{v}_n(t) \rangle, \quad (7.22)$$

for every $t \in \Theta$, thanks to (7.19), (W.2), Remark 4.3, and Remark 6.8.

It can be shown that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY([0, T], \boldsymbol{\varphi})$. Indeed, thanks to (7.18) and (7.20), we can apply Lemma 6.7 to get

$$(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(\Theta, \boldsymbol{\varphi}). \quad (7.23)$$

Let now $t_1 < \dots < t_l$ be a finite sequence in $[0, T]$. Thanks to left continuity of $\boldsymbol{\mu}$ and to (appr), for every $i = 1, \dots, l$, there exist a sequence $s_i^j \rightarrow t_i$ as $j \rightarrow \infty$, with $s_i^j \leq t_i$, such that

$$\boldsymbol{\mu}_{s_1^j \dots s_n^j} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_n}, \quad 2\text{-weakly}^*, \quad (7.24)$$

$$\boldsymbol{\nu}_{s_i^j} \rightharpoonup \boldsymbol{\nu}_{t_i}, \quad 2\text{-weakly}^*, \quad \text{for every } i = 1, \dots, n \quad (7.25)$$

as $j \rightarrow \infty$. If we define $\boldsymbol{\varphi}^j: \{t_1, \dots, t_l\} \rightarrow H^1(D; \mathbb{R}^N)$ by $\boldsymbol{\varphi}^j(t_i) := \boldsymbol{\varphi}(s_i^j)$, for every $i = 1, \dots, n$, we have that $\boldsymbol{\varphi}^j(t_i) \rightarrow \boldsymbol{\varphi}(t_i)$ strongly in $H^1(D; \mathbb{R}^N)$, for every $i = 1, \dots, n$; if we define $(\tilde{\boldsymbol{\nu}}^j, \tilde{\boldsymbol{\mu}}^j) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{\{t_1, \dots, t_l\}} \times SY^2(\{t_1, \dots, t_l\}, D; \mathbb{R}^m)$ by

$$\tilde{\boldsymbol{\nu}}_{t_i}^j := \boldsymbol{\nu}_{s_i^j}, \quad \text{for every } i = 1, \dots, n,$$

$$\tilde{\boldsymbol{\mu}}_{t_1 \dots t_l}^j := \boldsymbol{\mu}_{s_1^j \dots s_l^j},$$

thanks to (7.23), we have that $(\tilde{\boldsymbol{\nu}}^j, \tilde{\boldsymbol{\mu}}^j) \in AY(\{t_1, \dots, t_n\}, \boldsymbol{\varphi}^j)$ and

$$\begin{aligned} (\tilde{\boldsymbol{\mu}}^j)_{t_1 \dots t_l} &\rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_l}, \quad 2\text{-weakly}^* \\ (\tilde{\boldsymbol{\nu}}^j)_{t_i} &\rightharpoonup \boldsymbol{\nu}_{t_i}, \quad 2\text{-weakly}^* \text{ for every } i = 1, \dots, l, \end{aligned}$$

as $j \rightarrow \infty$. Hence we are again in the hypotheses of Lemma 6.7 and we can conclude that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY([0, T], \boldsymbol{\varphi})$.

By construction, $(\boldsymbol{\nu}, \boldsymbol{\mu})$ satisfies (ev0).

Now we want to prove that $(\boldsymbol{\nu}, \boldsymbol{\mu})$ satisfies (ev1).

Let $\tilde{z} \in L^2(D; \mathbb{R}^m)$, $\tilde{u} \in H_{\Gamma_0}^1(0)$; for every n and for every $i = 1, \dots, k(n)$, let consider the pair $(\hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\mu}})$, where $\hat{\boldsymbol{\nu}} := ((\boldsymbol{\nu}_n^i)_{t_n^0}, \dots, (\boldsymbol{\nu}_n^i)_{t_n^{i-1}}, \mathcal{T}_{(\tilde{z}, \nabla \tilde{u})}((\boldsymbol{\nu}_n^i)_{t_n^i}))$ and $\hat{\boldsymbol{\mu}}$ is the unique compatible system in $SY^2(\{t_n^0, \dots, t_n^i\}, D; \mathbb{R}^m)$ satisfying $\hat{\boldsymbol{\mu}}_{t_n^0 \dots t_n^i} = \tilde{\mathcal{T}}_{\tilde{z}}^{i+1}((\boldsymbol{\mu}_n^i)_{t_n^0 \dots t_n^i})$, with

$$\mathcal{T}_{(\tilde{z}, \nabla \tilde{u})}(x, \theta, F) = (x, \theta + \tilde{z}(x), F + \nabla \tilde{u}(x))$$

and

$$\tilde{\mathcal{T}}_{\tilde{z}}^{i+1}(x, \theta_0, \dots, \theta_i) = (x, \theta_0, \dots, \theta_{i-1}, \theta_i + \tilde{z}(x)).$$

This is an element of $AY(\{t_n^0, \dots, t_n^i\}, \boldsymbol{\varphi})$ and satisfies (7.2) and (7.3), hence we can use it as a competitor to obtain

$$\begin{aligned} &\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d(\boldsymbol{\nu}_n^i)_{t_n^i}(x, \theta, F) - \langle \boldsymbol{l}_n^i, \boldsymbol{v}_n^i(t_n^i) \rangle + \\ &\quad + \int_{D \times (\mathbb{R}^m)^{i+1}} H(\theta_i - \theta_{i-1}) d(\boldsymbol{\mu}_n^i)_{t_n^0 \dots t_n^i} \leq \\ &\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) d(\boldsymbol{\nu}_n^i)_{t_n^i}(x, \theta, F) - \langle \boldsymbol{l}_n^i, \boldsymbol{v}_n^i(t_n^i) + \tilde{u} \rangle + \\ &\quad + \int_{D \times (\mathbb{R}^m)^{i+1}} H(\theta_i + \tilde{z}(x) - \theta_{i-1}) d(\boldsymbol{\mu}_n^i)_{t_n^0 \dots t_n^i}. \end{aligned}$$

Thanks to the triangular inequality for H (which follows from (H.1)), this implies that

$$\begin{aligned} &\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F) - W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x))] d(\boldsymbol{\nu}_n^i)_{t_n^i}(x, \theta, F) \leq \\ &\quad \mathcal{H}(\tilde{z}) - \langle \boldsymbol{l}_n^i, \tilde{u} \rangle. \end{aligned}$$

By definition of ν_n , for every $t \in \Theta$ we obtain

$$\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F) - W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x))] d(\nu_n)_t(x, \theta, F) \leq \mathcal{H}(\tilde{z}) - \langle \mathbf{l}(\tau^n(t)), \tilde{u} \rangle$$

Since

$$|W(\theta, F) - W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x))| \leq \tilde{C}[(|\tilde{z}(x)| + |\nabla \tilde{u}(x)|)^2 + 1] + \tilde{C}(|\tilde{z}(x)| + |\nabla \tilde{u}(x)|)(|\theta| + |F|),$$

for a suitable positive constant \tilde{C} , and $(\nu_{n_k}^t)_t \rightarrow \nu_t$ 2-weakly*, for a suitable subsequence $((\nu_{n_k}^t)_t)_k$, we can deduce from Remark 4.3 that

$$\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F) - W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x))] d(\nu_{n_k}^t)_t(x, \theta, F) \rightarrow \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F) - W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x))] d\nu_t(x, \theta, F),$$

as $n \rightarrow \infty$. Since $\mathbf{l}(\tau^n(t)) \rightarrow \mathbf{l}(t)$, strongly in $H^1(D; \mathbb{R}^N)^*$, for every $t \in [0, T]$, we obtain (ev1) for $t \in \Theta$.

Let now $t \in [0, T] \setminus \Theta$. We can find a sequence $s^j \rightarrow t$, with $s^j \leq t$ and $s^j \in \Theta$, such that $\nu_{s^j} \rightarrow \nu_t$ 2-weakly*; as before we have that

$$\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F) - W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x))] d\nu_{s^j}(x, \theta, F) \rightarrow \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta, F) - W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x))] d\nu_t(x, \theta, F),$$

as $j \rightarrow \infty$. Again, since $\mathbf{l}(s^j) \rightarrow \mathbf{l}(t)$ strongly in $H^1(D; \mathbb{R}^N)^*$, using (ev1) for s^j , we obtain (ev1) for every $t \in [0, T]$.

Now we want to prove (ev2). Fix $t \in \Theta$ and let $(\nu_{n_k}^t)_t$ be a subsequence satisfying (7.19) and such that $(\nu_{n_k}^t)_t \rightarrow \nu_t$ 2-weakly* as $k \rightarrow \infty$; then, thanks to (7.18),

$$\mu_{n_k}^t \rightharpoonup \mu, \tag{7.26}$$

weakly* in $SY^2([0, T], D; \mathbb{R}^m)$; since the term containing \mathbf{l} is continuous with respect to 2-weak* convergence, the term containing W is weakly lower semicontinuous and the variation is weakly lower semicontinuous thanks to (7.26) and Lemma 4.11, we have

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d\nu_t(x, \theta, F) - \langle \mathbf{l}(t), \mathbf{v}(t) \rangle + \\ & \quad + \text{Var}_H(\mu; 0, t) \leq \\ & \leq \liminf_k \left[\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d(\nu_{n_k}^t)_t(x, \theta, F) - \langle \mathbf{l}(\tau^{n_k}(t)), \mathbf{v}_{n_k}^t(t) \rangle + \right. \\ & \quad \left. + \text{Var}_H(\mu_{n_k}^t; 0, t) \right]. \end{aligned}$$

Using (7.15) of Lemma 7.5, we can deduce that

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d\nu_t(x, \theta, F) - \langle \mathbf{l}(t), \mathbf{v}(t) \rangle + \\ & \quad + \text{Var}_H(\mu; 0, t) \leq \\ & \leq \mathcal{W}(z_0, v_0) - \langle \mathbf{l}(0), \mathbf{v}(0) \rangle - \int_0^t \langle \mathbf{l}(s), \dot{\varphi}(s) \rangle ds + \\ & \quad + \liminf_k \int_0^{\tau^{n_k}(t)} [\langle \sigma_{n_k}^t(s), \nabla \dot{\varphi}(s) \rangle - \langle \dot{\mathbf{l}}(s), \mathbf{v}_{n_k}^t(s) \rangle] ds. \end{aligned}$$

Since $\sup_{s \in [0, T]} \sup_n \|\boldsymbol{\sigma}_n(s)\|_2$ and $\sup_{s \in [0, T]} \sup_n \|\mathbf{v}_n(s)\|_2$ are finite, we can deduce, using Fatou Lemma, that

$$\begin{aligned} & \liminf_k \int_0^{\tau^{n_k}(t)} [\langle \boldsymbol{\sigma}_{n_k}(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle - \langle \dot{\mathbf{l}}(s), \mathbf{v}_{n_k}(s) \rangle] ds \leq \\ & \leq \limsup_n \int_0^{\tau^n(t)} [\langle \boldsymbol{\sigma}_n(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle - \langle \dot{\mathbf{l}}(s), \mathbf{v}_n(s) \rangle] ds \leq \\ & \leq \int_0^t \limsup_n [\langle \boldsymbol{\sigma}_n(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle - \langle \dot{\mathbf{l}}(s), \mathbf{v}_n(s) \rangle] ds. \end{aligned}$$

Thanks to (7.22) this implies that

$$\begin{aligned} & \liminf_k \int_0^{\tau^{n_k}(t)} [\langle \boldsymbol{\sigma}_{n_k}(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle - \langle \dot{\mathbf{l}}(s), \mathbf{v}_{n_k}(s) \rangle] ds \leq \\ & \leq \int_0^t [\langle \boldsymbol{\sigma}(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle - \langle \dot{\mathbf{l}}(s), \mathbf{v}(s) \rangle] ds. \end{aligned}$$

This implies (ev2) for $t \in \Theta$.

Let now $t \in [0, T] \setminus \Theta$ and let $s^j \rightarrow t$, satisfy $s_j \leq t$, and (7.21); it is easy to verify, using Lemma 4.11, that

$$\text{Var}_H(\boldsymbol{\mu}; 0, t) \leq \liminf_j \text{Var}_H(\boldsymbol{\mu}; 0, s^j),$$

hence (ev2) for t can be deduced from (ev2) for s^j .

8. AN ALTERNATIVE PROOF OF THE EXISTENCE RESULT IN THE QUASICONVEX CASE

In this section we focus on the particular case of $W(\theta, \cdot)$ quasiconvex, which can be treated using a spatial regularization depending on the gradient of the internal variable, as proposed in [7].

Definition 8.1. Let $\eta > 0$, $\boldsymbol{\varphi} \in AC([0, T]; H^1(D; \mathbb{R}^N))$, $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, $z_0 \in H^1(D; \mathbb{R}^m)$, $v_0 \in \mathcal{A}(0)$, and $T > 0$. A *solution of the η -regularized problem* with external load \mathbf{l} , boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0, v_0) , in the time interval $(0, T]$, is a pair $(\mathbf{z}_\eta, \mathbf{v}_\eta)$, with $\mathbf{z}_\eta: [0, T] \rightarrow H^1(D; \mathbb{R}^m)$ and $\mathbf{v}_\eta: [0, T] \rightarrow H^1(D; \mathbb{R}^N)$, satisfying the following properties:

(ev0)_{reg} *initial condition:* $(\mathbf{z}_\eta(0), \mathbf{v}_\eta(0)) = (z_0, v_0)$;

(ev1)_{reg} *kinematic admissibility:* $\mathbf{v}_\eta(t) \in \mathcal{A}(t)$ for every $t \in [0, T]$;

(ev2)_{reg} *global stability:* for every $t \in (0, T]$ and $(\hat{z}, \hat{v}) \in H^1(D; \mathbb{R}^m) \times \mathcal{A}(t)$ we have

$$\begin{aligned} & \mathcal{W}(\mathbf{z}_\eta(t), \mathbf{v}_\eta(t)) - \langle \mathbf{l}(t), \mathbf{v}_\eta(t) \rangle + \frac{\eta}{2} \|\nabla \mathbf{z}_\eta(t)\|_2^2 \leq \\ & \leq \mathcal{W}(\hat{z}, \hat{v}) - \langle \mathbf{l}(t), \hat{v} \rangle + \frac{\eta}{2} \|\nabla \hat{z}\|_2^2 + \mathcal{H}(\hat{z} - \mathbf{z}_\eta(t)); \end{aligned}$$

(ev3)_{reg} *energy inequality:* $\text{Var}_H(\mathbf{z}_\eta; 0, T) < \infty$, the map $t \mapsto [\langle \boldsymbol{\sigma}_\eta(t), \nabla \dot{\boldsymbol{\varphi}}(t) \rangle - \langle \dot{\mathbf{l}}(t), \mathbf{v}_\eta(t) \rangle]$ is measurable on $[0, T]$, and for every $t \in [0, T]$ we have

$$\begin{aligned} & \mathcal{W}(\mathbf{z}_\eta(t), \mathbf{v}_\eta(t)) - \langle \mathbf{l}(t), \mathbf{v}_\eta(t) \rangle + \frac{\eta}{2} \|\nabla \mathbf{z}_\eta(t)\|_2^2 + \text{Var}_H(\mathbf{z}_\eta; 0, t) \leq \\ & \leq \mathcal{W}(z_0, v_0) - \langle \mathbf{l}(0), v_0 \rangle + \frac{\eta}{2} \|\nabla z_0\|_2^2 - \int_0^t \langle \mathbf{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle ds + \\ & \quad + \int_0^t [\langle \boldsymbol{\sigma}_\eta(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle - \langle \dot{\mathbf{l}}(s), \mathbf{v}_\eta(s) \rangle] ds, \end{aligned}$$

where $\boldsymbol{\sigma}_\eta(s) := \frac{\partial W}{\partial F}(\mathbf{z}_\eta(s), \nabla \mathbf{v}_\eta(s))$ for every $s \in [0, T]$.

The proof of Theorem 4.1 in [7] can be repeated in our case to obtain the following result.

Theorem 8.2. *Let η , φ , \mathbf{l} , (z_0, v_0) , and T be as in Definition 8.1. Then there exists a solution of the η -regularized problem with external load \mathbf{l} , boundary datum φ , and initial condition (z_0, v_0) , in the time interval $(0, T]$. Moreover there exists a positive constant K_η such that*

$$\sup_{t \in [0, T]} \|\mathbf{z}_\eta(t)\|_2 \leq K_\eta, \quad (8.1)$$

$$\sup_{t \in [0, T]} \|\mathbf{v}_\eta(t)\|_{H^1} \leq K_\eta. \quad (8.2)$$

Lemma 8.3. *Let φ , \mathbf{l} , v_0 , and T be as in Definition 8.1, and let $(z_0^\eta)_{\eta > 0}$ be a family of functions in $H^1(D; \mathbb{R}^m)$, such that*

$$\sup_{\eta} \|z_0^\eta\|_2 \leq M, \quad (8.3)$$

$$\sup_{\eta} \eta \|\nabla z_0^\eta\|_2^2 \leq M, \quad (8.4)$$

for a suitable positive constant M . Then there exists a positive constant K such that the solutions $(\mathbf{z}_\eta, \mathbf{v}_\eta)$ of the η -regularized problems with initial condition (z_0^η, v_0) satisfy the following conditions

$$\sup_{t \in [0, T]} \sup_{\eta} \|\mathbf{z}_\eta(t)\|_2 \leq K, \quad \sup_{t \in [0, T]} \sup_{\eta} \|\nabla \mathbf{v}_\eta(t)\|_2 \leq K, \quad (8.5)$$

$$\sup_{\eta} \text{Var}_H(\mathbf{z}_\eta; 0, T) \leq K, \quad \sup_{t \in [0, T]} \frac{\eta}{2} \|\nabla \mathbf{z}_\eta(t)\|_2^2 \leq K. \quad (8.6)$$

Proof. Using the fact that $\sup_{t \in [0, T]} \|\mathbf{l}(t)\|_{(H^1)^*}$, $\int_0^T \|\dot{\mathbf{l}}(t)\|_{(H^1)^*} dt$, and $\int_0^T \|\dot{\varphi}(t)\|_{H^1} dt$ are finite, the hypotheses on W , the hypotheses (8.3) and (8.4), and the inequalities (8.1) and (8.2), we can deduce from (ev3)_{reg} that, for every $\eta > 0$,

$$c(\|\mathbf{z}_\eta(t)\|_2^2 + \|\nabla \mathbf{v}_\eta(t)\|_2^2) - C \leq \tilde{C} + \tilde{c} \sup_{s \in [0, T]} (\|\mathbf{z}_\eta(s)\|_2 + \|\mathbf{v}_\eta(s)\|_{H^1}),$$

for suitable positive constants \tilde{c}, \tilde{C} independent of t and η . Since this can be repeated for every $t \in [0, T]$, Poincaré inequality implies (8.5). Inequalities (8.6) come now from (ev3)_{reg} and (8.5). \square

Remark 8.4. From (8.6) we can deduce that

$$\eta_n \nabla \mathbf{z}_{\eta_n}(t) \rightarrow 0$$

strongly in $L^2(D; \mathbb{R}^{m \times d})$, for every positive sequence $\eta_n \rightarrow 0$ and every $t \in [0, T]$.

Definition 8.5. Given an external load $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, a boundary datum $\varphi \in AC([0, T]; H^1(D; \mathbb{R}^N))$, an initial condition $(z_0, v_0) \in L^2(D; \mathbb{R}^m) \times \mathcal{A}(0)$, and $T > 0$, a *quasistatic evolution obtained by spatial regularizations* in the time interval $[0, T]$ is a pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{[0, T]} \times SY^2([0, T], D; \mathbb{R}^m)$ with

$$\tilde{\pi}_{\mathbb{R}^m}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t, \quad \text{for every } t \in [0, T], \quad (8.7)$$

for which there exist:

- a sequence $(z_0^n)_n \subset H^1(D; \mathbb{R}^m)$, with

$$z_0^n \rightarrow z_0 \quad \text{strongly in } L^2(D; \mathbb{R}^m), \quad (8.8)$$

- a positive sequence η_n converging to 0, with

$$\eta_n \|\nabla z_0^n\|_2^2 \rightarrow 0, \quad (8.9)$$

- a subset Θ of $[0, T]$, with $0 \in \Theta$ and $\mathcal{L}^1([0, T] \setminus \Theta) = 0$,

such that $\boldsymbol{\nu}$ and Θ satisfy property (appr) of Remark 6.16, and the solutions $(\mathbf{z}_{\eta_n}, \mathbf{v}_{\eta_n})$ of the η_n -regularized problems with initial conditions (z_0^n, v_0) in the time interval $(0, T]$ satisfy the following conditions:

(conv1)_{reg} for every finite sequence $t_1 < \dots < t_l$ in Θ

$$\delta_{(\mathbf{z}_{\eta_n(t_1)}, \dots, \mathbf{z}_{\eta_n(t_l)})} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_l} \quad \text{2-weakly}^*, \quad (8.10)$$

as $n \rightarrow \infty$;

(conv2)_{reg} for every $t \in \Theta$ there exists a subsequence $(\mathbf{z}_{\eta_{n_k}^t}, \mathbf{v}_{\eta_{n_k}^t})_k$ of $(\mathbf{z}_{\eta_n}, \mathbf{v}_{\eta_n})_n$, possibly depending on t , with

$$\begin{aligned} \limsup_n [\langle \boldsymbol{\sigma}_{\eta_n}(t), \nabla \boldsymbol{\varphi}(t) \rangle - \langle \dot{\mathbf{l}}(t), \mathbf{v}_{\eta_n}(t) \rangle] &= \\ &= \lim_k [\langle \boldsymbol{\sigma}_{\eta_{n_k}^t}(t), \nabla \boldsymbol{\varphi}(t) \rangle - \langle \dot{\mathbf{l}}(t), \mathbf{v}_{\eta_{n_k}^t}(t) \rangle], \end{aligned} \quad (8.11)$$

and

$$\delta_{(\mathbf{z}_{\eta_{n_k}^t}(t), \nabla \mathbf{v}_{\eta_{n_k}^t}(t))} \rightharpoonup \boldsymbol{\nu}_t \quad \text{2-weakly}^*, \quad (8.12)$$

as $k \rightarrow \infty$.

In the next theorem we will show that the quasistatic evolution obtained by spatial regularizations fulfills the requirements of Definition 6.14.

Theorem 8.6. *Let $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, $\boldsymbol{\varphi} \in AC([0, T]; H^1(D; \mathbb{R}^N))$, (z_0, v_0) , and $T > 0$ be as in Definition 8.5, and assume that*

$$\mathcal{W}(z_0, v_0) \leq \mathcal{W}(\hat{z}, \hat{v}) - \langle \mathbf{l}(0), \hat{v} - v_0 \rangle + \mathcal{H}(\hat{z} - z_0), \quad (8.13)$$

for every $\hat{z} \in L^2(D; \mathbb{R}^m)$ and every $\hat{v} \in \mathcal{A}(0)$. Then a quasistatic evolution obtained by spatial regularizations in the time interval $[0, T]$ is a globally stable quasistatic evolution of Young measures with the same data in the time interval $[0, T]$.

Proof. Let $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{[0, T]} \times SY_-^2([0, T], D; \mathbb{R}^m)$ be a quasistatic evolution obtained by spatial regularizations.

First of all we show that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY([0, T], \boldsymbol{\varphi})$: thanks to (8.10) and (8.12), it is immediate to see that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(\Theta, \boldsymbol{\varphi})$; as in the proof of the main theorem, (appr) allows now to prove that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY([0, T], \boldsymbol{\varphi})$. In particular, for every $t \in [0, T]$ there exists a function $\mathbf{v}(t) \in \mathcal{A}(t)$ such that $\nabla \mathbf{v}(t) = \text{bar}(\tilde{\pi}_{\mathbb{R}^N \times d}(\boldsymbol{\nu}_t))$, thanks to Remark 6.8.

Let η_n and z_0^n be the sequences appearing in Definition 8.5, and let $(\mathbf{z}_{\eta_n}, \mathbf{v}_{\eta_n})$ be the solutions of the η_n -regularized problems in the time interval $(0, T]$; (ev0) comes immediately from (ev0)_{reg}, (8.8), and (8.12).

Now we prove that $(\boldsymbol{\nu}, \boldsymbol{\mu})$ satisfies (ev1).

For $t = 0$ it comes immediately from (8.13).

For $t \neq 0$, we first show that (ev1) holds for test functions $\tilde{z} \in H^1(D; \mathbb{R}^m)$. Let $t \in \Theta \setminus 0$, $(\eta_{n_k}^t)_k$ be the sequence appearing in (conv2)_{reg}, and $(\tilde{z}, \tilde{u}) \in H^1(D; \mathbb{R}^m) \times H_{\Gamma_0}^1(0)$. By (ev1)_{reg}, we have

$$\begin{aligned} &\mathcal{W}(\mathbf{z}_{\eta_{n_k}^t}(t), \mathbf{v}_{\eta_{n_k}^t}(t)) + \frac{\eta_{n_k}^t}{2} \|\nabla \mathbf{z}_{\eta_{n_k}^t}(t)\|_2^2 \leq \\ &\leq \mathcal{W}(\mathbf{z}_{\eta_{n_k}^t}(t) + \tilde{z}, \mathbf{v}_{\eta_{n_k}^t}(t) + \tilde{u}) - \langle \mathbf{l}(t), \tilde{u} \rangle + \frac{\eta_{n_k}^t}{2} \|\nabla \mathbf{z}_{\eta_{n_k}^t}(t) + \nabla \tilde{z}\|_2^2 + \mathcal{H}(\tilde{z}). \end{aligned}$$

Thanks to Remark 8.4,

$$\frac{\eta_{n_k}^t}{2} \|\nabla \mathbf{z}_{\eta_{n_k}^t}(t) + \nabla \tilde{z}\|_2^2 - \frac{\eta_{n_k}^t}{2} \|\nabla \mathbf{z}_{\eta_{n_k}^t}(t)\|_2^2 = \frac{\eta_{n_k}^t}{2} \|\nabla \tilde{z}\|_2^2 + \langle \eta_{n_k}^t \nabla \mathbf{z}_{\eta_{n_k}^t}(t), \nabla \tilde{z} \rangle, \rightarrow 0,$$

as $k \rightarrow \infty$. On the other hand we have

$$\begin{aligned} &\mathcal{W}(\mathbf{z}_{\eta_{n_k}^t}(t) + \tilde{z}, \mathbf{v}_{\eta_{n_k}^t}(t) + \tilde{u}) - \mathcal{W}(\mathbf{z}_{\eta_{n_k}^t}(t), \mathbf{v}_{\eta_{n_k}^t}(t)) = \\ &= \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) - W(\theta, F)] d\delta_{(\mathbf{z}_{\eta_{n_k}^t}(t), \nabla \mathbf{v}_{\eta_{n_k}^t}(t))}(x, \theta, F); \end{aligned}$$

since

$$\begin{aligned} & |W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) - W(\theta, F)| \leq \\ & \tilde{C}[(|\tilde{z}(x)| + |\nabla \tilde{u}(x)|)^2 + 1] + \tilde{C}(|\tilde{z}(x)| + |\nabla \tilde{u}(x)|)(|\theta| + |F|), \end{aligned} \quad (8.14)$$

thanks to Remark 4.3 and (8.12) we can deduce that

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) - W(\theta, F)] d\delta_{(\mathbf{z}_{\eta_{n_k}^t}(t), \nabla \mathbf{v}_{\eta_{n_k}^t}(t))}(x, \theta, F) \rightarrow \\ & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) - W(\theta, F)] d\boldsymbol{\nu}_t(x, \theta, F), \end{aligned}$$

and hence we can conclude that

$$\begin{aligned} & \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) d\boldsymbol{\nu}_t(x, \theta, F) \leq \\ & \leq \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) d\boldsymbol{\nu}_t(x, \theta, F) - \langle \mathbf{l}(t), \tilde{u} \rangle + \mathcal{H}(\tilde{z}). \end{aligned} \quad (8.15)$$

Using (appr) it is easy to extend the previous inequality to $t \in [0, T] \setminus \Theta$ reasoning as in the proof of the main theorem.

Now we prove that (8.15) holds also for test functions $\tilde{z} \in L^2(D; \mathbb{R}^m)$. Let $t \in (0, T]$, $\tilde{z} \in L^2(D; \mathbb{R}^m)$, $\tilde{u} \in H_{\Gamma_0}^1(0)$, and assume that $(\tilde{z}_h)_h$ is a sequence in $H^1(D; \mathbb{R}^m)$ with $\tilde{z}_h \rightarrow \tilde{z}$ strongly in $L^2(D; \mathbb{R}^m)$; then we have

$$\mathcal{H}(\tilde{z}_h) \rightarrow \mathcal{H}(\tilde{z}),$$

and

$$\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} [W(\theta + \tilde{z}(x), F + \nabla \tilde{u}(x)) - W(\theta + \tilde{z}_h(x), F + \nabla \tilde{u}(x))] d\boldsymbol{\nu}_t(x, \theta, F) \rightarrow 0,$$

thanks to the hypotheses on H , and to (W.2) and (2.2), respectively; hence (ev1) for test functions \tilde{z} and \tilde{u} can be deduce from (8.15) applied to \tilde{z}_h and \tilde{u} .

Finally we prove (ev2).

Thanks to (8.8) and (8.9), we have

$$\mathcal{W}(z_0^n, v_0) + \frac{\eta_n}{2} \|\nabla z_0^n\|_2^2 \rightarrow \mathcal{W}(z_0, v_0),$$

as $n \rightarrow \infty$; therefore we can argue as in the proof of the main theorem using (ev2)_{reg} and the approximation properties of $(\boldsymbol{\nu}, \boldsymbol{\mu})$. \square

Theorem 8.7. *Let $\mathbf{l} \in AC([0, T]; H^1(D; \mathbb{R}^N)^*)$, $\boldsymbol{\varphi} \in AC([0, T]; H^1(D; \mathbb{R}^N))$, $T > 0$, and $(z_0, v_0) \in L^2(D; \mathbb{R}^m) \times \mathcal{A}(0)$. Then there exists a quasistatic evolution obtained by spatial regularizations with external load \mathbf{l} , boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0, v_0) , in the time interval $[0, T]$.*

Proof. Fix a sequence $(z_0^n)_n$ in $H^1(D; \mathbb{R}^m)$ satisfying (8.8) and a positive sequence $\eta_n \rightarrow 0$ satisfying (8.9); let $(\mathbf{z}_{\eta_n}, \mathbf{v}_{\eta_n})$ be the solutions of the η_n -regularized problems with external load \mathbf{l} , boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0^n, v_0) . Thanks to (8.5), (8.6), and (H.2), we can apply Theorem 4.10 to obtain a subset Θ of $[0, T]$ with $0 \in \Theta$ and $\mathcal{L}^1([0, T] \setminus \Theta) = 0$, a compatible system $\boldsymbol{\mu} \in SY_-^2([0, T], D; \mathbb{R}^m)$, and a subsequence still indicated by $(\eta_n)_n$ satisfying (conv1)_{reg}. For every $t \in \Theta$ we select a subsequence $(\eta_{n_k}^t)_k$ of $(\eta_n)_n$, possibly depending on t , satisfying (8.11). Thanks to (8.5), we can apply Lemma 4.13 to obtain another subsequence still denoted by $(\eta_{n_k}^t)_k$ and $\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{[0, T]}$ which verify conditions (conv2)_{reg}, (appr), and (8.7). \square

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REFERENCES

- [1] J.M. Ball, A version of the fundamental theorem for Young measures. *PDE's and continuum models of phase transitions (Nice, 1988)*, 207-215, *Lecture Notes in Physics, Springer-Verlag, Berlin*, 1989.
- [2] E. Acerbi and N. Fusco, Semicontinuity problems in the calculus of variations. *Arch. Ration. Mech. Anal.* **86** (1984) 125-145.
- [3] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland, Amsterdam-London; American Elsevier, New York (1973).
- [4] G. Dal Maso, A. De Simone, M.G. Mora and M. Morini, Time-dependent systems of generalized Young measures. *Netw. Heterog. Media* **2** (2007) 1-36.
- [5] G. Dal Maso, A. De Simone, M.G. Mora and M. Morini, Globally stable quasistatic evolution in plasticity with softening. Preprint SISSA 27/2007/M.
- [6] I. Fonseca, S. Müller and P. Pedregal, Analysis of concentration and oscillation effects generated by gradients. *Siam J. Math. Anal.* **29** (1998) 736-756.
- [7] G. Francfort and A. Mielke, Existence results for a class of rate-independent material models with nonconvex elastic energy. *J. Reine und Angew. Math.* **595** (2006) 55-91.
- [8] A.N. Kolmogorov, *Foundations of the Theory of Probability*. Chelsea Publishing Company, II edition, New York 1956.
- [9] M. Kočvara, A. Mielke and T. Roubíček, A rate-independent approach to the delamination problem. *Math. Mech. Solids* **11** (2006) 423-447.
- [10] C. Miehe and M. Lambrecht, Analysis of microstructure development in shearbands by energy relaxation of incremental stress potentials: large-strain theory for standard dissipative solids. *Internat. J. Numer. Methods Engrg.* **58** (2003) 1-41.
- [11] C. Miehe, J. Schotte and M. Lambrecht, Computational homogenization of materials with microstructures based on incremental variational formulations. *IUTAM Symposium on Computational Mechanics of Solid Materials at Large Strains (Stuttgart, 2001)*, 87-100, *Solid Mech. Appl., Kluwer Acad. Publ., Dordrecht*, 2003.
- [12] A. Mielke, Evolution of rate-independent systems. In: *Evolutionary equations. Vol. II*. Edited by C. M. Dafermos and E. Feireisl, 461-559, *Handbook of Differential Equations*. Elsevier/North-Holland, Amsterdam, 2005.
- [13] A. Mielke and T. Roubíček, Rate-independent damage processes in nonlinear elasticity. *M³ As Math. Models Methods Appl. Sci.* **16** (2006) 177-209.
- [14] A. Mielke, F. Theil and V.I. Levitas, A variational formulation of rate-independent phase transformations using an extremum principle. *Arch. Ration. Mech. Anal.* **162** (2002) 137-177.
- [15] M. Ortiz and E. Repetto, Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Physics Solids* **47(2)** (1999), 397-462.
- [16] P. Pedregal, *Parametrized measures and variational principles. Progress in Nonlinear Differential Equations and their Applications* **30**. Birkhäuser Verlag, Basel, 1997.
- [17] M. Valadier, Young measures. *Methods of nonconvex analysis (Varenna, 1989)*, 152-188, *Lecture Notes in Math., Springer-Verlag, Berlin*, 1990.

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