THE SET OF REGULAR VALUES (IN THE SENSE OF CLARKE) OF A LIPSCHITZ MAP. A SUFFICIENT CONDITION FOR THE RECTIFIABILITY OF CLASS C^2.

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Abstract. We prove a result about the rectifiability of class C^2 of the set of regular values (in the sense of Clarke) of a Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^N$ (with $n < N$).

1. Introduction and statement of main result

A Borel subset S of $\mathbb{R}^N$ is said to be a $(\mathcal{H}^n, n)$ rectifiable set of class C^H if there exist countably many n-dimensional submanifolds M_j of $\mathbb{R}^N$ of class C^H such that

$$\mathcal{H}^n\Big(S \backslash \bigcup_j M_j\Big) = 0.$$

Observe that for $H = 1$ this is equivalent to say that S is a countably n-rectifiable set, e.g. by [14, Lemma 11.1].

Such a notion has been introduced in [3] and provides a natural setting for the description of singularities of convex functions and convex surfaces, [1, 2]. More generally, it can be used to study the singularities of surfaces with generalized curvatures, [2]. Rectifiability of class C^2 is strictly related to the context of Legendrian rectifiable subsets of $\mathbb{R}^N \times \mathbb{S}^{N-1}$, [11, 12, 6, 7]. The level sets of a $W^{k,p}_{\mathrm{loc}}$ mapping between manifolds are rectifiable sets of class C^k, [4]. Applications of rectifiable sets of class C^H to geometric variational problems can be found in [8].

This paper is devoted to prove a result about the rectifiability of class C^2 of the set of regular values (in the sense of Clarke) of a Lipschitz map

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \qquad (n < N).$$

Before we state it, let us introduce some notation. For $\gamma \in I(n, N)$ and $s \in \mathbb{R}^n$, let $\partial\varphi^\gamma(s)$ denote the Clarke subdifferential of the map

$$\varphi^\gamma := (\varphi^{\gamma_1}, \dots, \varphi^{\gamma_n}) : \mathbb{R}^n \to \mathbb{R}^n$$

namely

$$\partial\varphi^\gamma(s) := \mathrm{co}\Big\{ \lim_{i\to\infty} D\varphi^\gamma(s_i) \,\Big|\, D\varphi^\gamma(s_i) \text{ exists}, s_i \to s \Big\}$$

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compare [5, p.133]. Then let

$$\mathcal{R} := \{s \in \mathbb{R}^n \,|\, \partial\varphi^\gamma(s) \text{ is nonsingular for some } \gamma\}.$$

Our main goal is to prove the following theorem.

Theorem 1.1. *Let be given a family of bounded functions*

$$c_i : \mathbb{R}^n \to \mathbb{R}\backslash\{0\} \qquad (i = 1, \dots , n),$$

a family of Lipschitz maps

$$\varphi_i : \mathbb{R}^n \to \mathbb{R}^N \qquad (i = 1, \dots , n)$$

and denote by A the set of points $t \in \mathbb{R}^n$ satisfying the following conditions:

(i) *The map φ and all the maps φ_i are differentiable at t;*
(ii) *The equality*

$$D_i\varphi(t) = c_i(t)\varphi_i(t) \tag{1.1}$$

holds for all $i = 1, \dots , n$.

Also assume that

(iii) *For almost every $a \in A$ there exists a non-trivial ball B centered at a and such that*

$$\mathcal{L}^n(B\backslash A) = 0.$$

Then $\varphi(A \cap \mathcal{R})$ is a $(\mathcal{H}^n, n)$ rectifiable set of class C^2.

Remark 1.1. As an immediate corollary of Theorem 1.1, we get this result. Let be given a set of Lipschitz maps

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N, \qquad \varphi_i : \mathbb{R}^n \to \mathbb{R}^N \ (i = 1, \dots , n)$$

and a set of bounded functions

$$c_i : \mathbb{R}^n \to \mathbb{R}\backslash\{0\} \qquad (i = 1, \dots , n)$$

such that

$$D_i\varphi = c_i\varphi_i \qquad (i = 1, \dots , n)$$

almost everywhere in $\mathbb{R}^n$. Then the image $\varphi(\mathcal{R})$ is a $(\mathcal{H}^n, n)$ rectifiable set of class C^2.

Remark 1.2. Let E be any subset of $\mathcal{R}$ and define

$$E^\gamma := \{s \in E \,|\, \partial\varphi^\gamma(s) \text{ is nonsingular}\}, \qquad \gamma \in I(n, N).$$

Then one obviously has

$$\bigcup_{\gamma \in I(n,N)} E^\gamma = E.$$

Remark 1.3. If $s \in \mathcal{R}^\gamma$, by the Lipschitz inverse function Theorem (e.g. [5, Theorem 3.12]), there exist a neighborhood U (in $\mathbb{R}^n$) of s and a neighborhood V (in $\mathbb{R}^n$) of $\varphi^\gamma(s)$ such that

- $V = \varphi^\gamma(U)$ and $\varphi^\gamma|U : U \to V$ is invertible;
- $(\varphi^\gamma|U)^{-1}$ is Lipschitz.

Let $\overline{\gamma}$ denote the multi-index in $I(N-n,N)$ which complements γ in $\{1,2,\dots,N\}$ in the natural increasing order and set (for $x \in \mathbb{R}^N$)

$$x^\gamma := (x^{\gamma_1},\dots,x^{\gamma_n}), \qquad x^{\overline{\gamma}} := (x^{\overline{\gamma}_1},\dots,x^{\overline{\gamma}_{N-n}}).$$

Then the map

$$f := \varphi^{\overline{\gamma}} \circ (\varphi^\gamma|U)^{-1} : V \to \mathbb{R}^{N-n}$$

is Lipschitz and its graph

$$G_f^\gamma := \{x \in \mathbb{R}^N \mid x^\gamma \in V \text{ and } x^{\overline{\gamma}} = f(x^\gamma)\}$$

coincides with $\varphi(U)$.

By virtue of Remark 1.2 (with $E = A \cap \mathcal{R}$) and Remark 1.3, and recalling that the graph of a Lipschitz map is a rectifiable set (e.g. [14, Theorem 5.3]), we are reduced to prove the following claim.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let $\gamma \in I(n,N)$ and consider a map*

$$g : \mathbb{R}^n \to \mathbb{R}^{N-n}$$

of class C^1. Then $\varphi((A\cap\mathcal{R})^\gamma) \cap G_g^\gamma$ is a $(\mathcal{H}^n, n)$ rectifiable set of class C^2.

Remark 1.4. The remainder of our paper is devoted to proving Theorem 1.2. With no loss of generality, we can restrict our attention to the particular case when $\gamma = \{1,\dots,n\}$.

2. Preliminaries (under the assumptions of Theorem 1.2, with $\gamma = \{1,\dots,n\}$)

2.1. **Further reduction of the claim.** From now on, for simplicity, $G_g^{\{1,\dots,n\}}$, $(A\cap\mathcal{R})^{\{1,\dots,n\}}$ and $\varphi^{\{1,\dots,n\}}$ will be donoted by G_g, F and λ, respectively.

Define

$$L := \varphi^{-1}(G_g) \cap F.$$

Without loss of generality, we can assume that $\mathcal{L}^n(L) < \infty$. Then, by a well-known regularity property of $\mathcal{L}^n$, for any given real number $\varepsilon > 0$ there exists a closed subset L_ε of $\mathbb{R}^n$ with

$$L_\varepsilon \subset L, \qquad \mathcal{L}^n(L \backslash L_\varepsilon) \le \varepsilon, \tag{2.1}$$

compare e.g. [13, Theorem 1.10]. Moreover, since L_ε is closed, one has

$$L_\varepsilon^* \subset L_\varepsilon \tag{2.2}$$

where L_ε^* is the set of density points of L_ε. Recall that

$$\mathcal{L}^n(L_\varepsilon \backslash L_\varepsilon^*) = 0 \tag{2.3}$$

by a well-known result of Lebesgue. In the special case that L has measure zero, we define $L_\varepsilon := \emptyset$, hence $L_\varepsilon^* := \emptyset$.

Observe that

$$G_g \cap \varphi(F) \backslash \varphi(L_\varepsilon^*) \subset \varphi\left(\varphi^{-1}(G_g) \cap F \backslash L_\varepsilon^*\right) = \varphi(L \backslash L_\varepsilon^*)$$

hence

$$\begin{aligned}\mathcal{H}^n\left(G_g\cap\varphi(F)\backslash\varphi(L_\varepsilon^*)\right) &\le \mathcal{H}^n\left(\varphi\left(L\backslash L_\varepsilon^*\right)\right)\\ &\le \int_{L\backslash L_\varepsilon^*} J_n\varphi\, d\mathcal{L}^n\\ &\le \mathrm{Lip}(\varphi)^n\mathcal{L}(L\backslash L_\varepsilon^*)\\ &\le \varepsilon\,\mathrm{Lip}(\varphi)^n\end{aligned}$$

by the area formula (compare [10, §3.2.], [14, §8]), (2.1), (2.2) and (2.3). It follows that

$$\mathcal{H}^n\Big(G_g\cap\varphi(F)\backslash\bigcup_{j=1}^{\infty}\varphi(L_{1/j}^*)\Big)=0.$$

Thus, to prove Theorem 1.2, it suffices to show that

$$\varphi(L_\varepsilon^*)\text{ is a }(\mathcal{H}^n,n)\text{ rectifiable set of class }C^2$$

for all $\varepsilon>0$.

2.2. **Further notation.** Let us consider the projection

$$\Pi:\mathbb{R}^N\to\mathbb{R}^{N-n},\qquad (x_1,\dots,x_N)\mapsto(x_{n+1},\dots,x_N).$$

For $i\in\{1,\dots,n\}$ and $s,\sigma\in\mathbb{R}^n$, define

$$\Phi_{i;s}(\sigma):=\Pi\varphi_i(\sigma)-\sum_{j=1}^{n}\frac{\partial g}{\partial x^j}(\lambda(s))\varphi_i^j(\sigma),\tag{2.4}$$

$$R_s^{(0)}(\sigma):=g(\lambda(\sigma))-g(\lambda(s))-\sum_{j=1}^{n}\frac{\partial g}{\partial x^j}(\lambda(s))\left[\varphi^j(\sigma)-\varphi^j(s)\right]$$

and

$$R_{i;s}^{(1)}(\sigma):=\frac{\partial g}{\partial x^i}(\lambda(\sigma))-\frac{\partial g}{\partial x^i}(\lambda(s)).$$

Remark 2.1. All the maps $\sigma\mapsto\Phi_{i;s}(\sigma)$ are Lipschitz.

3. Lemmas (under the assumptions of Theorem 1.2, with $\gamma=\{1,\dots,n\}$)

Lemma 3.1. *Consider the square-matrix field*

$$\rho\mapsto M(\rho):=\begin{pmatrix}\varphi_1^1(\rho)&\cdots&\varphi_1^n(\rho)\\ \vdots&&\vdots\\ \varphi_n^1(\rho)&\cdots&\varphi_n^n(\rho)\end{pmatrix},\qquad \rho\in\mathbb{R}^n.$$

and let $t\in F$. Then there exists a nontrivial ball B, centered at t, such that

- *The matrix $M(\rho)$ is invertible for all $\rho\in B$;*
- *The map*

  $$\rho\mapsto M(\rho)^{-1},\qquad \rho\in B$$

  is Lipschitz.

Proof. One has

$$M(t) = \left(\prod_{i=1}^{n} c_i(t)\right)^{-1} \begin{pmatrix} D_1\varphi^1(t) & \cdots & D_1\varphi^n(t) \\ \vdots & & \vdots \\ D_n\varphi^1(t) & \cdots & D_n\varphi^n(t) \end{pmatrix}$$

by (1.1). Since $D\lambda(t) \in \partial\lambda(t)$ and $t \in \mathcal{R}^{\{1,\cdots,n\}}$, one has

$$\det M(t) \neq 0.$$

But the function $\rho \mapsto \det M(\rho)$ is continuous, hence there exists a nontrivial ball B centered at t and such that

$$|\det M(\rho)| \geq \frac{|\det M(t)|}{2}$$

for all $\rho \in B$, hence the two claims easily follow. □

Lemma 3.2. *If $s \in L_\varepsilon^*$ then*

(1) *One has*

$$\Phi_{i;s}(s) = 0$$

for all $i \in \{1, \dots, n\}$;

(2) *Moreover, for $l \in \{1, \dots, N-n\}$*

$$\frac{\partial g^l}{\partial x^i}(\lambda(s)) = \left[M(s)^{-1}\right]_i \bullet \varphi_*^{n+l}(s)$$

where $[\cdot]_i$ denotes the i^{th} row in the argument matrix and

$$\varphi_*^{n+l} := (\varphi_1^{n+l}, \dots, \varphi_n^{n+l}).$$

Proof. (1) First of all, observe that

$$g(\lambda(t)) = \Pi\varphi(t)$$

for all $t \in \varphi^{-1}(G_g)$. Since $L_\varepsilon^* \subset A$ the two members of this equality are both differentiable at s. Moreover s is a limit point of $L_\varepsilon \subset \varphi^{-1}(G_g)$. It follows that (for $i = 1, \dots, n$)

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s)) D_i\varphi^j(s) = \Pi D_i\varphi(s)$$

namely

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s)) c_i(s)\varphi_i^j(s) = c_i(s)\Pi\varphi_i(s)$$

by (1.1). Recalling that $c_i(s) \neq 0$, we get

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(\lambda(s))\varphi_i^j(s) = \Pi\varphi_i(s) \tag{3.1}$$

i.e. $\Phi_{i;s}(s) = 0$.

(2) The system (3.1) is equivalent to

$$M(s)\nabla g^l(\lambda(s)) = \varphi_*^{n+l}(s)^T, \qquad l \in \{1, \dots, N-n\}$$

hence the conclusion follows. □

Lemma 3.3 (**Main lemma**). *Let $s \in L_\varepsilon^*$ and $t \in A$ be such that*

$$\mathcal{H}^1([s;t]\backslash A) = 0 \tag{3.2}$$

where $[s;t]$ denotes the segment joining s and t. Define the map parametrizing $[s;t]$ as

$$\sigma : [0,1] \to \mathbb{R}^n, \qquad \rho \mapsto s + \rho(t-s).$$

If $t \in \varphi^{-1}(G_g)$ then

$$R_s^{(0)}(t) = \sum_{i=1}^n (t^i - s^i) \int_0^1 c_i(\sigma(\rho))\, \Phi_{i;s}(\sigma(\rho))\, d\rho;$$

Proof. First of all, observe that:

- Since $s, t \in \varphi^{-1}(G_g)$ one has $g(\lambda(s)) = \Pi\varphi(s)$ and $g(\lambda(t)) = \Pi\varphi(t)$;
- The function $\rho \mapsto \varphi(\sigma(\rho))$ is Lipschitz, hence it is differentiable almost everywhere in $[0,1]$. Moreover the assumption (3.2) implies that
$$(\varphi \circ \sigma)'(\rho) = \sum_{i=1}^n (t^i - s^i) D_i\varphi(\sigma(\rho))$$
at a.e. $\rho \in [0,1]$.

Recalling also (1.1), we obtain

$$\begin{aligned} R_s^{(0)}(t) &= \Pi\varphi(t) - \Pi\varphi(s) - \sum_{j=1}^n \frac{\partial g}{\partial x^j}(\lambda(s)) \left[\varphi^j(t) - \varphi^j(s)\right] \\ &= \sum_{i=1}^n (t^i - s^i) \int_0^1 \left[\Pi D_i\varphi(\sigma(\rho)) - \sum_{j=1}^n \frac{\partial g}{\partial x^j}(\lambda(s)) D_i\varphi^j(\sigma(\rho))\right] d\rho \\ &= \sum_{i=1}^n (t^i - s^i) \int_0^1 c_i(\sigma(\rho)) \left[\Pi\varphi_i(\sigma(\rho)) - \sum_{j=1}^n \frac{\partial g}{\partial x^j}(\lambda(s)) \varphi_i^j(\sigma(\rho))\right] d\rho. \end{aligned}$$

The conclusion follows at once from (2.4). □

Lemma 3.4. *Let Z be a null-measure subset of $\mathbb{R}^n$ and $s \in \mathbb{R}^n$. Then there exists a null-measure subset W of $\mathbb{R}^n$ such that*

$$\mathcal{H}^1(Z \cap [s;t]) = 0 \tag{3.3}$$

for all $t \in \mathbb{R}^n \backslash W$.

Proof. Let φ_Z denote the characteristic function of Z. By a standard application of the coarea formula (e.g. [9, §3.4.4], [10, §3.2.13]), we obtain

$$0 = \int_{\mathbb{R}^n} \varphi_Z = \int_{\mathbb{S}^{n-1}} \left(\int_0^{+\infty} \varphi_Z(s + \rho u)\rho^{n-1} d\rho \right) d\mathcal{H}^{n-1}(u)$$

hence

$$\int_0^{+\infty} \varphi_Z(s + \rho u)\rho^{n-1} d\rho = 0 \tag{3.4}$$

for all $u \in \mathbb{S}^{n-1} \backslash Q$, where Q is a measurable subset of $\mathbb{S}^{n-1}$ such that $\mathcal{H}^{n-1}(Q) = 0$. Define

$$W := s + \mathbb{R}^+ Q = \{s + \rho u \mid \rho \in \mathbb{R}^+, \, u \in Q\}.$$

By invoking again the coarea formula, we find (denoting with $B(0, R)$ the ball of radius R centered at the origin)

$$\begin{aligned} \mathcal{L}^n(W \cap B(0,R)) &= \int_{B(0,R)} \varphi_W = \int_{\mathbb{S}^{n-1}} \left(\int_0^R \varphi_W(s + \rho u) \rho^{n-1} d\rho \right) d\mathcal{H}^{n-1}(u) \\ &= \int_{\mathbb{S}^{n-1}} \left(\int_0^R \varphi_Q(u) \rho^{n-1} d\rho \right) d\mathcal{H}^{n-1}(u) = \frac{R^n}{n} \int_{\mathbb{S}^{n-1}} \varphi_Q d\mathcal{H}^{n-1} \\ &= 0 \end{aligned}$$

for all $R > 0$. It follows that $\mathcal{L}^n(W) = 0$. Finally the formula (3.3) follows at once from (3.4). □

4. Proof of Theorem 1.2

As we observed in Remark 1.4 above, we can assume $\gamma = \{1, \dots, n\}$ and the notation introduced in sections 2, 3. Moreover let A' be the set of $a \in A$ such that there exists a non-trivial ball B centered at a satisfying

$$\mathcal{L}^n(B \backslash A) = 0.$$

One has

$$\mathcal{L}^n(A \backslash A') = 0 \tag{4.1}$$

by assumption (iii) in Theorem 1.1.

For each positive integer j define $\Gamma_{\varepsilon,j}$ as the set of $s \in L_\varepsilon^* \cap A'$ such that

$$\|R_s^{(0)}(t)\| \le j \|\lambda(t) - \lambda(s)\|^2 \tag{4.2}$$

and

$$\|R_{i;s}^{(1)}(t)\| \le j \|\lambda(t) - \lambda(s)\| \qquad (i = 1, \dots, n) \tag{4.3}$$

for all $t \in L_\varepsilon^*$ satisfying

$$\|t - s\| \le \frac{1}{j}.$$

Proposition 4.1. *One has*

$$\bigcup_j \Gamma_{\varepsilon,j} = L_\varepsilon^* \cap A'.$$

Proof. Since (obviously!)

$$\Gamma_{\varepsilon,j} \subset \Gamma_{\varepsilon,j+1} \subset L_\varepsilon^* \cap A'$$

for all positive integers j, we get at once

$$\bigcup_j \Gamma_{\varepsilon,j} \subset L_\varepsilon^* \cap A'.$$

In order to prove the opposite inclusion, consider $s \in L^*_\varepsilon \cap A'$ and let U and V be as in Remark 1.3. Observe that

$$\|t-s\| = \left\|(\lambda|U)^{-1}(\lambda(t)) - (\lambda|U)^{-1}(\lambda(s))\right\| \leq \mathrm{Lip}(\lambda|U)^{-1} \, \|\lambda(t)-\lambda(s)\| \tag{4.4}$$

for all $t \in U$.

Since $s \in A'$, there exists a non-trivial ball B centered at s such that

$$B \subset U, \qquad \mathcal{L}^n(B\backslash A) = 0.$$

By applying Lemma 3.4 with $Z := B\backslash A$, we find

$$\mathcal{H}^1([s;t]\backslash A) = \mathcal{H}^1(Z \cap [s;t]) = 0$$

for a.e. $t \in B$. Then Lemma 3.3 and Lemma 3.2(1) imply

$$\begin{aligned}
\|R^{(0)}_s(t)\| &\leq \sum_{i=1}^n |t^i - s^i| \left\| \int_0^1 c_i(\sigma(\rho)) \left[\Phi_{i;s}(\sigma(\rho)) - \Phi_{i;s}(s)\right] d\rho \right\| \\
&\leq \sum_{i=1}^n \mathrm{Lip}\left(\Phi_{i;s}\right) \, |t^i - s^i| \, \|c_i\|_\infty \int_0^1 \|\sigma(\rho) - s\| \, d\rho \\
&= \frac{\|t-s\|}{2} \sum_{i=1}^n \mathrm{Lip}\left(\Phi_{i;s}\right) \, |t^i - s^i| \, \|c_i\|_\infty \\
&\leq C \, \|t-s\|^2
\end{aligned}$$

for a.e. $t \in B \cap \varphi^{-1}(G_g)$, where C is a suitable number which does not depend on t. By continuity we get

$$\|R^{(0)}_s(t)\| \leq C \, \|t-s\|^2$$

for all $t \in B \cap \varphi^{-1}(G_g)$. Recalling (4.4) we conclude that

$$\|R^{(0)}_s(t)\| \leq C_0 \, \|\lambda(t) - \lambda(s)\|^2, \qquad C_0 := C \left[\mathrm{Lip}(\lambda|U)^{-1}\right]^2$$

for all $t \in B \cap \varphi^{-1}(G_g)$. By shrinking B (if need be!) we can also deduce the existence of a number C_1 which does not depend on t and is such that

$$\|R^{(1)}_{i;s}(t)\| \leq C_1 \, \|\lambda(t) - \lambda(s)\| \qquad (i = 1, \dots, n)$$

for all $t \in L^*_\varepsilon \cap B$, by Lemma 3.1, Lemma 3.2(2) and (4.4). Hence

$$s \in \Gamma_{\varepsilon,j}$$

provided j is big enough. □

Since $L^*_\varepsilon \subset A$, from Proposition 4.1 it follows that

$$\varphi(L^*_\varepsilon) = \varphi(L^*_\varepsilon \cap A) = \varphi(L^*_\varepsilon \cap (A\backslash A')) \cup \varphi(L^*_\varepsilon \cap A') = \varphi(L^*_\varepsilon \cap (A\backslash A')) \cup \bigcup_j \varphi(\Gamma_{\varepsilon,j})$$

where $\varphi(L^*_\varepsilon \cap (A\backslash A'))$ has measure zero, by (4.1). Hence it will be enough to prove that (for all ε and j)

$$\varphi(\Gamma_{\varepsilon,j}) \text{ is a } (\mathcal{H}^n, n) \text{ rectifiable set of class } C^2. \tag{4.5}$$

To prove this claim, first consider a countable measurable covering $\{Q_l\}_{l=1}^{\infty}$ of $\Gamma_{\varepsilon,j}$ such that

$$\mathrm{diam}\ Q_l \ \leq \ \frac{1}{j}$$

for all l, and define

$$F_l := \overline{\lambda(\Gamma_{\varepsilon,j} \cap Q_l)}.$$

If $\xi, \eta \in F_l$, then there exist two sequences

$$\{s_k\}, \{t_k\} \subset \Gamma_{\varepsilon,j} \cap Q_l$$

such that

$$\lim_k \lambda(s_k) = \xi, \qquad \lim_k \lambda(t_k) = \eta.$$

By (4.2) and (4.3) we get

$$\|R^{(0)}_{s_k}(t_k)\| \leq j\|\lambda(t_k) - \lambda(s_k)\|^2$$

and

$$\|R^{(1)}_{i,s_k}(t_k)\| \leq j\|\lambda(t_k) - \lambda(s_k)\| \qquad (i = 1, \dots, n)$$

for all k. Letting $k \to \infty$, we conclude that

$$\left\| g(\eta) - g(\xi) - \sum_{h=1}^{n} \frac{\partial g}{\partial x^h}(\xi)(\eta^h - \xi^h) \right\| \leq j\|\eta - \xi\|^2$$

and

$$\left\| \frac{\partial g}{\partial x^i}(\eta) - \frac{\partial g}{\partial x^i}(\xi) \right\| \leq j\|\eta - \xi\| \qquad (i = 1, \dots, n)$$

for all $\xi, \eta \in F_l$. By the Whitney extension Theorem [15, Ch. VI, §2.3] it follows that each $g|F_l$ can be extended to a map in $C^{1,1}(\mathbb{R}^n, \mathbb{R}^{N-n})$. Then the Lusin type result [10, §3.1.15] implies that $\varphi(\Gamma_{\varepsilon,j} \cap Q_l)$ is a $(\mathcal{H}^n, n)$ rectifiable set of class C^2. Finally, claim (4.5) follows observing that

$$\varphi(\Gamma_{\varepsilon,j}) = \bigcup_l \varphi(\Gamma_{\varepsilon,j} \cap Q_l).$$

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