Homogenization of non-uniformly bounded periodic diffusion energies in dimension two

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Abstract

This paper deals with the homogenization of two-dimensional oscillating convex functionals, the densities of which are equicoercive but not uniformly bounded from above. Using a uniform-convergence result for the minimizer, which holds for this type of scalar problems in dimension two, we prove in particular that the limit energy is local and recover the validity of the analog of the well-known periodic homogenization formula in this degenerate case. However, in the present context the classical argument leading to integral representation based on the use of cut-off functions is useless due to the unboundedness of the densities. In its place we build sequences with bounded energy, which converge uniformly to piecewise-affine functions, taking pointwise extrema of recovery sequences for affine functions.

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1 Introduction

General homogenization theorems ensure that the limit of oscillating functionals of the form

$$\int_\Omega f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u\big)\, dx$$

with domain some $W^{1,p}$ Sobolev space is a homogeneous integral of the same form

$$\int_\Omega f_{\rm hom}(\nabla u)\, dx$$

provided the function f is periodic in the first variable and satisfies the 'standard p-growth conditions' $c_1\,|\xi|^p - 1 \le f(y,\xi) \le c_2\,(1+|\xi|^p)$ (see, e.g., [4]). This result, up to the use of asymptotic homogenization formulas to describe $f_{\rm hom}$ in the vector case, is valid in any dimension and its proof is usually achieved using a technical argument due to De Giorgi, which consists in the use of 'cut-off' functions φ_n in the construction of recovery sequences of the form $v_n\varphi_n + (1-\varphi_n)u_n$ as a convex combination of two recovery sequences. The use of the p-growth condition allows to optimize the choice of these φ_n. This argument

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is used to 'glue' optimal sequences on overlapping sets, match boundary conditions, etc., and is stable under small variations of f under the above-mentioned growth conditions (see [4]).

For functionals not uniformly satisfying a p-growth condition, this result fails. In particular the limit of energies of the form

$$F_n(u) = \int_\Omega f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u\big)\, dx,$$

where f_n are periodic in the first variable and satisfy 'degenerate standard p-growth conditions' $c_1^n|\xi|^p - 1 \le f(y,\xi) \le c_2^n(1+|\xi|^p)$ with c_1^n possibly vanishing and c_2^n possibly diverging, a 'local' representation of the limit energy through the single variable u may fail. For quadratic energies it can be represented as a Dirichlet form (see [17]), or as a multi-phase energy (see [1], [6], [8], [9], [13], [15], [16]). Results by Camar-Eddine and Seppecher [10] determine that a wide class of local and non-local quadratic forms can be reached as Γ-limit of usual local Dirichlet-type integrals with degenerate coefficients.

The object of this paper is the homogenization of (nonlinear) integral functionals F_n as above, where Ω is a bounded open set of $\mathbb{R}^2$ and u is scalar, when f_n satisfies very mild growth conditions from above (see (2.1)–(2.3) below). In the simplest (linear and isotropic) case this can be translated into the Γ-convergence of oscillating functionals of the form

$$F_n(u) = \int_\Omega a_n\big(\tfrac{x}{\varepsilon_n}\big)\, |\nabla u|^2\, dx,$$

where $a_n \ge 1$ are 1-periodic but a_n are not bounded in L^∞. In this case many of the usual techniques of Γ-convergence hinted at above do not work as they are usually stated, but must be carefully modified. This can be seen by examining a sequence $w_n := \varphi_n u_n + (1-\varphi_n)v_n$ obtained by "joining" two sequences u_n and v_n with bounded energy. Its energy can be estimated by the energies along the sequences u_n and v_n, and a term depending on $\nabla\varphi_n$ and $u_n - v_n$. In the linear case above this remainder term takes the form

$$\int_\Omega a_n\big(\tfrac{x}{\varepsilon_n}\big)\, |\nabla\varphi_n|^2\, |u_n - v_n|^2\, dx,$$

and can be made arbitrarily small when $u_n - v_n$ tends to zero in L^2, upon suitably choosing φ_n, if a_n is bounded in L^∞. For unbounded coefficients, for such an argument to work some stronger convergence is required. In the two-dimensional case the compactness result of Briane and Casado-Diaz [7] ensures that we can restrict to sequences such that $u_n - v_n$ converges to zero uniformly, so that the error above is estimated by

$$\|\nabla\varphi_n\|_\infty^2\, \|u_n - v_n\|_\infty^2 \int_\Omega a_n\big(\tfrac{x}{\varepsilon_n}\big)\, dx \le |\Omega|\, \|\nabla\varphi_n\|_\infty^2\, \sup_n \|a_n\|_{L^1((0,1)^2)}\, \|u_n - v_n\|_\infty^2,$$

which shows that the L^1-boundedness of a_n can be used in the cut-off argument.

In place of an L^1-boundedness assumption we will suppose that

$$\lim_{n\to\infty} f_n^{\rm hom}(\xi) \le \bar b\,(1+|\xi|^p)$$

for all $\xi \in \mathbb{R}^2$, where the energy density $f_n^{\rm hom}$ is given by the *cell-problem formula* (2.4). This assumption clearly holds if f_n satisfies an L^1-boundedness hypothesis of the type

$$f_n(y,\xi) \le b_n(y)\,(1+|\xi|^p),$$

with $\sup_n \|b_n\|_{L^1((0,1)^2)} < \infty$, but is more general and covers the case of domains with strong inclusions.

Under such a general assumption we bypass the cut-off arguments above, using the specificity of the scalar setting coupled with the improved convergence of recovery sequences. To exemplify our approach, we can consider the simplest case of the construction of optimal sequences for a function of the form $u = u^1 \vee u^2$ ($\vee$ denotes the maximum) with u^i affine. If u^i_n are optimal sequences for u^i then we can simply set $u_n := u^1_n \vee u^2_n$. The uniform convergence of u^i_n allows then to estimate the error in terms of the size of a small neighbourhood of the set $\{u^1 = u^2\}$. A technical argument allows then to carry on this construction to optimal sequences for arbitrary piecewise-affine functions and then by density to the whole space $W^{1,p}$. This proves one of the two inequalities – namely, the Γ-limsup inequality – of Γ-convergence.

To prove the Γ-liminf inequality we have found it convenient to use the Fonseca-Müller blow-up technique, which allows to reduce to the study of converging sequences when the target function is linear $\xi \cdot x$. A similar argument as above allows then to modify such sequences so that it satisfies periodic boundary conditions, which allows an estimate with the energy densities $f_n^{\hom}(\xi)$. Again the scalar nature of the problem is heavily exploited both in the modification leading to periodic boundary conditions and in the reduction to a single cell-problem formula.

The paper is organized as follows. In Section 2 we state the main result which is proved in Section 3. Section 4 is devoted to a sufficient condition permitting to derive the boundedness of $f_n^{\hom}$ in $\mathbb{R}^2$.

Notation

- for any open set ω of $\mathbb{R}^2$, $\bar{\omega}$ denotes the closure of ω in $\mathbb{R}^2$;
- $Y := (0,1)^2$;
- $H_\sharp(Y)$ denotes the space of the Y-periodic functions which belong to $H_{\rm loc}(\mathbb{R}^2)$;

2 Statement of the results

Let $p > 1$, and let Ω be a bounded open set of $\mathbb{R}^2$ with a Lipschitz-continuous boundary. We consider a sequence of non-negative functions $f_n : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$, for $n \geq 1$, satisfying the following properties:

$$f_n(\cdot, \xi) \text{ is a } Y\text{-periodic measurable function for any } \xi \in \mathbb{R}^2, \tag{2.1}$$

$$f_n(y, \cdot) \text{ is convex with } f_n(y, \cdot) \geq f_n(y, 0) \text{ for a.e. } y \in \mathbb{R}^2, \tag{2.2}$$

there exists a non-negative sequence b_n such that

$$|\xi|^p - 1 \leq f_n(y, \xi) \leq b_n\ (1 + |\xi|^p), \quad \forall\, \xi \in \mathbb{R}^2, \text{ a.e. } y \in \mathbb{R}^2, \tag{2.3}$$

Remark 2.1. In (2.2) we can replace the convexity assumption by a continuity assumption. To this end, it is enough to replace the density $f_n(y, \cdot)$ by its convexification, which leads us to the same convergence result (see Theorem 2.3).

We define, for each fixed $n \geq 1$, the "homogenized" density $f_n^{\hom}$ by the classical minimization formula (see, e.g., Chapter 14 of [4]):

$$f_n^{\hom}(\xi) := \inf\left\{ \int_Y f_n(y, \xi + \nabla\varphi)\, dy \ :\ \varphi \in W^{1,p}_\sharp(Y) \right\}, \quad \text{for } \xi \in \mathbb{R}^2. \tag{2.4}$$

Thanks to the convexity and the bounds (2.3) satisfied by the function f_n, the infimum in problem (2.4) is attained, i.e.

$$\forall\,\xi \in \mathbb{R}^2,\ \exists\,\varphi_n^\xi \in W_\sharp^{1,p}(Y) \quad \text{such that} \quad f_n^{\hom}(\xi) = \int_Y f_n\big(y, \xi + \nabla\varphi_n^\xi\big)\,dy. \tag{2.5}$$

We will use the De Giorgi Γ-convergence theory. We refer to [11], [2] or [4] for a general presentation and the basic properties of Γ-convergence. Here, we simply recall the following definition:

Definition 2.2. A sequence of functionals $F_n : L^p(\Omega) \to [0,\infty]$ is said to Γ-converge to $F : L^p(\Omega) \to [0,\infty]$ for the strong topology of $L^p(\Omega)$ if, for any u in $L^p(\Omega)$,

(i) the Γ-*liminf inequality* holds

$$\forall\, u_n \longrightarrow u \ \text{ strongly in } L^p(\Omega), \quad F(u) \le \liminf_{n\to\infty} F_n(u_n), \tag{2.6}$$

(ii) the Γ-*limsup inequality* holds

$$\exists\, \bar{u}_n \longrightarrow u \ \text{ strongly in } L^p(\Omega), \quad F(u) = \lim_{n\to\infty} F_n(\bar{u}_n). \tag{2.7}$$

Any sequence satisfing (2.7) will be called a *recovery sequence* for F_n, of limit u.

Let ε_n be a sequence of positive numbers, which converges to 0 as $n \to \infty$. For any $n \ge 1$, we define the functional $F_n : L^p(\Omega) \to [0,\infty]$ by

$$F_n(u) := \begin{cases} \displaystyle\int_\Omega f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u\big)\,dx & \text{if } u \in W^{1,p}(\Omega) \\ \infty & \text{elsewhere.} \end{cases} \tag{2.8}$$

The main result of the paper is the following theorem:

Theorem 2.3. *Let Ω be a bounded open set of $\mathbb{R}^2$, with a Lipschitz continuous boundary. In addition to conditions* (2.1)–(2.3)*, assume that there exist a positive constant $\bar{b}$ and a function $f_\infty^{\hom} : \mathbb{R}^2 \to [0,\infty)$, such that*

$$\forall\,\xi \in \mathbb{R}^2, \quad \lim_{n\to\infty} f_n^{\hom}(\xi) = f_\infty^{\hom}(\xi) \le \bar{b}\,(1 + |\xi|^p)\,. \tag{2.9}$$

Then, the sequence of functionals F_n defined by (2.8) *Γ-converges for the strong topology of $L^p(\Omega)$, to the functional F_∞ defined by*

$$F_\infty(u) := \int_\Omega f_\infty^{\hom}(\nabla u)\,dx \tag{2.10}$$

for all $u \in W^{1,p}(\Omega)$.

Remark 2.4. Theorem 2.3 provides an extension of the periodic homogenization of energies even in the case of a single function; i.e., when the density $f_n(y,\xi) = f(y,\xi)$ does not depend on n and satisfies the growth condition

$$|\xi|^p - 1 \le f(y,\xi) \le b(y)\,(1 + |\xi|^p)\,, \quad \forall\,\xi \in \mathbb{R}^2, \text{ a.e. } y \in \mathbb{R}^2,$$

where $b \in L^1_\sharp(Y)$.

The classical framework of the periodic homogenization is based on the stronger assumption $b \in L^\infty_\sharp(Y)$, but holds true in any dimension and for non-convex vector-valued problems (see, e.g., Section 21.3 of [4]). The two-dimensional setting allows us to relax the right-hand side of the growth estimate (2.3), with a sequence b_n which is not necessarily bounded in $L^1_\sharp(Y)$. As a consequence we need to modify the definitions (2.8) of F_n and (2.4) of $f_n^{\hom}$ by assuming the continuity of the functions.

Remark 2.5. We can replace the assumption that 0 is an absolute minimizer of $f_n(y, \cdot)$ for a.e. $y \in \mathbb{R}^2$, by the following more general one:

There exist a function $\theta : [0, \infty) \to [0, \infty)$ and a sequence of functions φ_n in $C_\sharp(\varepsilon_n Y) \cap W^{1,p}_\sharp(\varepsilon_n Y)$, such that for any $n \geq 1$,

$$\lim_{t \to 0} \theta(t) = 0, \qquad \forall\, x_1, x_2 \in \mathbb{R}^2, \quad |\varphi_n(x_1) - \varphi_n(x_2)| \leq \theta(|x_1 - x_2|), \tag{2.11}$$

$$\nabla\varphi_n(\varepsilon_n y) \text{ is an absolute minimizer of } f_n(y, \cdot) \text{ for a.e. } y \in \mathbb{R}^2. \tag{2.12}$$

For example, the sequence defined by $\varphi_n(x) := \varepsilon_n\, \varphi(\frac{x}{\varepsilon_n})$, for $x \in \mathbb{R}^2$, where $\varphi \in W^{1,\infty}_\sharp(Y)$, satisfies condition (2.11) with $\theta(t) := \|\nabla\varphi\|_\infty t$.

3 Proof of the results

3.1 A uniform-convergence result

We have the following result which extends the uniform convergence result obtained in the linear framework of [7]:

Proposition 3.1. *Let Ω be a bounded open set of $\mathbb{R}^2$, with a Lipschitz continuous boundary. Let $f_n : \mathbb{R}^2 \times \mathbb{R} \to [0, \infty)$ be functions satisfying conditions* (2.1), (2.3) *and* (2.12). *Consider a function $u \in W^{1,p}(\Omega) \cap C(\bar\Omega)$, and a sequence $\hat u_n$ in $W^{1,p}(\Omega)$ which strongly converges to u in $L^p(\Omega)$, with*

$$\int_\Omega f_n\big(\tfrac{x}{\varepsilon_n}, \nabla \hat u_n\big)\, dx \leq c. \tag{3.1}$$

Let Ω' be an open subset of Ω. Then, there exist a subsequence of n, still denoted by n, and a sequence u_n in $W^{1,p}(\Omega)$ which satisfies the convergences

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega) \qquad \text{and} \qquad u_n \longrightarrow u \quad \text{strongly in } L^\infty_{\mathrm{loc}}(\Omega'), \tag{3.2}$$

and the energy estimate

$$\int_{\Omega'} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\, dx \leq \int_{\Omega'} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla \hat u_n\big)\, dx + o(1). \tag{3.3}$$

Moreover, for any open subsets $\omega, \tilde\omega$ of Ω, with $\bar\omega \subset \tilde\omega$, the sequence u_n satisfies

$$\int_{\omega} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\, dx \leq \int_{\tilde\omega} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla \hat u_n\big)\, dx + o(1). \tag{3.4}$$

Remark 3.2. In Proposition 3.1 the case $p \in (1, 2]$ is the most relevant, since in dimension two the embedding of $W^{1,p}(\Omega)$ in $C(\bar\Omega)$ is compact for $p > 2$.

The result of Proposition 3.1 also extends to the following periodic case with the sequence of functionals $F^{\sharp,\xi}_n$, for $\xi \in \mathbb{R}^2$, defined by

$$F^{\sharp,\xi}_n(\varphi) := \int_Y f_n\big(nx, \nabla\varphi(x)\big)\, dx, \quad \text{for } \varphi \in W^{1,p}_\sharp(Y). \tag{3.5}$$

Proposition 3.3. *For $n \geq 1$ and $\xi \in \mathbb{R}^2$, consider $\varphi_n^\xi \in W^{1,p}_\sharp(Y)$ satisfying* (2.5)*. Then, there exists a sequence ψ_n which converges to zero weakly in $W^{1,p}_\sharp(Y)$ and strongly in $L^\infty_\sharp(Y)$, such that*

$$\int_Y f_n\big(nx, \xi + \nabla\psi_n(x)\big)\, dx = \int_Y f_n\big(nx, \xi + \nabla\varphi_n^\xi(nx)\big)\, dx + o(1) = f_n^{\hom}(\xi) + o(1). \quad (3.6)$$

Moreover, for any regular bounded open sets $\omega, \tilde{\omega}$ of $\mathbb{R}^2$, with $\bar{\omega} \subset \tilde{\omega}$, we have

$$\int_\omega f_n\big(nx, \xi + \nabla\psi_n(x)\big)\, dx \leq |\tilde{\omega}|\, f_n^{\hom}(\xi) + o(1). \quad (3.7)$$

Proposition 3.1 is based on the following maximum principle result:

Lemma 3.4. *Let O be a bounded open subset of $\mathbb{R}^2$. Let φ be a function in $W^{1,p}(O)$ satisfying* (2.11)*. Let $g : O \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that*

(i) $g(\cdot, \xi)$ is measurable for any $\xi \in \mathbb{R}^2$,

(ii) $g(x, \cdot)$ is strictly convex for a.e. $x \in O$,

(iii) g satisfies the growth condition

$$|\xi|^p - 1 \leq g(x,\xi) \leq \beta(x)\,(1 + |\xi|^p)\,, \quad \forall \xi \in \mathbb{R}^2, \ \text{a.e. } x \in O,$$

where $\beta \in L^1(O)$,

(iv) $\nabla\varphi(x)$ is an absolute minimizer of $g(x, \cdot)$ for a.e. $x \in O$.

Let $G : W^{1,p}(O) \to [0, \infty]$ be the functional defined by

$$G(u) := \int_O g(x, \nabla u)\, dx, \quad \text{for } u \in W^{1,p}(O).$$

For $\hat{u} \in W^{1,p}(O) \cap C(\bar{O})$ with $G(\hat{u}) < \infty$, consider the function $u \in W^{1,p}(O)$ defined by the minimization problem

$$G(u) = \min\Big\{ G(v) \ : \ v - \hat{u} \in W^{1,p}_0(O) \Big\} < \infty.$$

Then, we have the following maximum principle

$$\min_{\partial O}\,(\hat{u} - \varphi) \leq u - \varphi \leq \max_{\partial O}\,(\hat{u} - \varphi) \quad \text{a.e. in } O.$$

Proof of Proposition 3.1. The proof is an adaptation of the proof of Theorem 2.1 in [7] to the present nonlinear framework. Therefore, we will give the main steps of the proof without specifying the details.

Define the function $g_n : \Omega \times \mathbb{R}^2 \to [0, \infty)$ by

$$g_n(x,\xi) := f_n\big(\tfrac{x}{\varepsilon_n}, \xi\big) + \frac{1}{n}\, |\xi - \nabla\varphi_n(x)|^p\,, \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}^2,$$

and the functional $G_n : W^{1,p}(\Omega) \to [0, \infty]$ by

$$G_n(u) := \int_{\Omega'} g_n(x, \nabla u)\, dx, \quad \text{for } u \in W^{1,p}(\Omega).$$

Note that, by the convexity of $f_n(y, \cdot)$ and (2.12), the function $g_n(x, \cdot)$ is a strictly convex function in $\mathbb{R}^2$ with $\nabla\varphi_n(x)$ as an absolute minimum.

Using a density argument and the continuity of the functional $v \mapsto \int_{\Omega'} f_n(x, \nabla v)\, dx$ in $W^{1,p}(\Omega)$, we can assume that $\hat{u}_n$ is regular without modifying the right-hand side of (3.3). By estimate (3.1) combined with the equicoercivity of $g_n(x, \cdot)$ (as a consequence of (2.3)) the sequence $\hat{u}_n$ is bounded in $W^{1,p}(\Omega)$ and thus weakly converges to u in $W^{1,p}(\Omega)$. Then, by virtue of the regularity of Ω, up to a subsequence, $\hat{u}_n$ converges uniformly to u in a relatively closed subset K of Ω, such that for a given $q \in (1, p)$, the q-capacity $C_q(\Omega \setminus K)$ of $\Omega \setminus K$ can be chosen arbitrarily small. By Lemma 2.8 of [7] (which is specific to dimension two) the diameter of any connected component O of $\Omega \setminus K$ is bounded by a constant times $C_q(\Omega \setminus K)^{\frac{1}{2-q}}$. Therefore, there exists an increasing sequence n_k, $k \geq 1$, of positive integers and a sequence K_k of relatively closed subsets of Ω such that

$$\forall\, n \geq n_k, \quad \|\hat{u}_n - u\|_{L^\infty(K_k)} \leq \frac{1}{k}, \tag{3.8}$$

and for any connected component O of $\Omega \setminus K_k$,

$$\operatorname{diam}(O) \leq \frac{1}{k}. \tag{3.9}$$

Now, for any $n \in [n_k, n_{k+1})$, define the function $u_n \in W^{1,p}(\Omega)$ by the following procedure:

- in any connected component O of $\Omega \setminus K_k$ such that $O \subset \Omega'$, u_n is defined by the minimization problem

$$\int_O g_n(x, \nabla u_n)\, dx = \min \left\{ \int_O g_n(x, \nabla v)\, dx \;:\; v - \hat{u}_n \in W_0^{1,p}(O) \right\}, \tag{3.10}$$

- $u_n := \hat{u}_n$ elsewhere.

Taking into account (3.1) it is easy to check that $u_n \in W^{1,p}(\Omega)$ and $u_n - \hat{u}_n \in W_0^{1,p}(\Omega)$. Thanks to Lemma 3.4 we have, for any connected component of $\Omega \setminus K_k$,

$$\forall\, n \in [n_k, n_{k+1}), \quad \min_{\partial O} (\hat{u}_n - \varphi_n) \leq u_n - \varphi_n \leq \max_{\partial O} (\hat{u}_n - \varphi_n) \quad \text{a.e. in } O. \tag{3.11}$$

Consider the increasing sequence of open subsets of Ω' defined by

$$\Omega'_k := \left\{ x \in \Omega' \;:\; \operatorname{dist}(x, \partial\Omega') > \frac{2}{k} \right\}, \quad \text{for } k \geq 1.$$

Note that by estimate (3.9) any connected component O such that $O \cap \Omega'_k \neq \emptyset$, satisfies $O \cap \partial\Omega = \emptyset$ and thus $\partial O \subset K_k$. Then, estimates (3.8), (3.11) and the triangle inequality imply that

$$\forall\, n \geq n_k, \quad \|u_n - u\|_{L^\infty(\Omega'_k)} \leq \frac{1}{k} + \sup_{\substack{x, y \in \Omega \\ |x-y| \leq \frac{1}{k}}} \Big(|u(x) - u(y)| + |\varphi_n(x) - \varphi_n(y)| \Big).$$

This, combined with the uniform continuity of u in $\bar{\Omega}$ and (2.11), yields

$$\lim_{k \to \infty} \left(\sup_{n \geq n_k} \|u_n - u\|_{L^\infty(\Omega'_k)} \right) = 0,$$

which implies the uniform convergence (3.2).

On the other hand, by the construction of $\hat{u}_n$ we have

$$\forall\, n \geq 1, \qquad u_n - \hat{u}_n \in W_0^{1,p}(\Omega') \quad \text{and} \quad G_n(u_n) = \int_{\Omega'} g_n(x, \nabla u_n)\, dx \leq G_n(\hat{u}_n). \tag{3.12}$$

Estimate (3.12) combined with the equicoercivity of $g_n(x,\cdot)$, estimate (3.1) and the boundedness of $\hat{u}_n$ in $W^{1,p}(\Omega)$, implies that u_n is also bounded in $W^{1,p}(\Omega)$. Therefore, u_n satisfies the weak convergence in (3.2). Again by (3.12) we get

$$\begin{aligned} G_n(u_n) &= \int_{\Omega'} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,dx + \frac{1}{n}\int_{\Omega'} |\nabla u_n - \nabla\varphi_n|^p\,dx \\ &= \int_{\Omega'} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,dx + o(1) \\ &\leq G_n(\hat{u}_n) + o(1) = \int_{\Omega'} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla \hat{u}_n\big)\,dx + o(1), \end{aligned}$$

which yields (3.3).

Finally, for k large enough, any connected component O of $\Omega\setminus K_k$ with $O\cap\bar{\omega}\neq\emptyset$, satisfies $O\subset\tilde{\omega}\setminus K_k$. Hence, from the definitions of g_n and u_n we deduce that for any $n\in[n_k,n_{k+1})$,

$$\int_{\omega\setminus K_k} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,dx \leq \sum_{O\subset\tilde{\omega}\setminus K_k}\int_O f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,dx \leq \int_{\tilde{\omega}\setminus K_k} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla \hat{u}_n\big)\,dx + o(1).$$

This combined with the equality $u_n=\hat{u}_n$ in K_k, implies (3.4) and concludes the proof.$\Box$

Proof of Proposition 3.3. Let us start by the following remark: In Proposition 3.1, when $\Omega := (-k,k)^2$, for an integer $k\geq 2$, and $\hat{u}_n$ is a sequence of Y-periodic functions which weakly converges to u in $W^{1,p}(\Omega)$, the closed sets K on which the convergence of $\hat{u}_n$ is uniform are Y-periodic. Indeed, the open sets $\Omega\setminus K$ of arbitrary small capacity are built from sets of the type $\{x\in\Omega : |\hat{u}_n(x)-u(x)|\geq\varepsilon\}$, $\varepsilon>0$, (see, e.g., Theorem 7 of [12]) which are clearly Y-periodic. Therefore, the sequence u_n defined by (3.10) is also Y-periodic. So, the procedure of Proposition 3.1 preserves the periodicity.

Let $\xi\in\mathbb{R}^2$. First of all, using a density argument and the continuity of the functional $\varphi\mapsto\int_Y f_n(y,\xi+\nabla\varphi)\,dy$ in $W^{1,p}_\sharp(Y)$, there exists a sequence $\hat{\psi}_n$ in $C^1_\sharp(Y)$ which is bounded in $W^{1,p}_\sharp(Y)$ and satisfies

$$\int_Y f_n\big(y,\xi+\nabla\hat{\psi}_n(y)\big)\,dy = \int_Y f_n\big(y,\xi+\nabla\varphi_n^\xi(y)\big)\,dy + o(1) = f_n^{\mathrm{hom}}(\xi)+o(1). \tag{3.13}$$

On the other hand, for any integer $k\geq 2$, the sequence $F_n^{\sharp,\xi}$ defined by (3.5) reads as

$$F_n^{\sharp,\xi}(\varphi) := \frac{1}{4k^2}\int_{(-k,k)^2} f_n\big(nx,\xi+\nabla\varphi(x)\big)\,dx, \quad \text{for } \varphi\in W^{1,p}_\sharp(Y),$$

and the continuous functions $\frac{1}{n}\,\hat{\psi}_n(nx)$ weakly converge to zero (continuous) in $W^{1,p}_\sharp(Y)$. Then, by the preliminary remark there exists a sequence ψ_n which weakly converges to zero in $W^{1,p}_\sharp(Y)$ and strongly in $L^\infty_\sharp(Y)$, such that

$$\begin{aligned} F_n^{\sharp,\xi}(\psi_n) &= \int_Y f_n\big(nx,\xi+\nabla\psi_n(x)\big)\,dx \\ &\leq F_n^{\sharp,\xi}\big(\tfrac{1}{n}\,\hat{\psi}_n(nx)\big) + o(1) = \int_Y f_n\big(nx,\xi+\nabla\hat{\psi}_n(nx)\big)\,dx + o(1). \end{aligned}$$

This, combined with (3.13) and the Y-periodicity of $\hat{\psi}_n$, yields the first estimate

$$\int_Y f_n\big(nx,\xi+\nabla\psi_n(x)\big)\,dx \leq f_n^{\mathrm{hom}}(\xi)+o(1). \tag{3.14}$$

On the other hand, let $\tilde\psi_n$ be the Y-periodic function defined by

$$\tilde\psi_n(y) := \frac{1}{n}\sum_{\kappa\in\{0,\dots,n-1\}^2}\psi_n\left(\frac{y+\kappa}{n}\right),\quad \text{for } y\in\mathbb{R}^2. \tag{3.15}$$

By the definition (2.4) of $f_n^{\hom}$, the Y-periodicity of $\tilde\psi_n, \psi_n, f_n(\cdot,\xi)$, and by the convexity of $f_n(x,\cdot)$, we have

$$\begin{aligned} f_n^{\hom}(\xi) &\le \int_Y f_n\big(y,\xi+\nabla\tilde\psi_n(y)\big)\,dy = \int_Y f_n\big(nx,\xi+\nabla\tilde\psi_n(nx)\big)\,dx \quad (y=nx)\\ &\le \frac{1}{n^2}\sum_{\kappa\in\{0,\dots,n-1\}^2}\int_Y f_n\big(nx,\xi+\nabla\psi_n(x+\tfrac{\kappa}{n})\big)\,dx\\ &= \frac{1}{n^2}\sum_{\kappa\in\{0,\dots,n-1\}^2}\int_{\frac{\kappa}{n}+Y} f_n\big(ny,\xi+\nabla\psi_n(y)\big)\,dy \quad (y=x+\tfrac{\kappa}{n})\\ &= \int_Y f_n\big(ny,\xi+\nabla\psi_n(y)\big)\,dy. \end{aligned} \tag{3.16}$$

Therefore, (3.14) and (3.16) imply the desired estimate (3.6).

On the other hand, similarly to (3.4) we obtain, owing to the construction of the function ψ_n from $\frac{1}{n}\hat\psi_n(nx)$, the inequality

$$\int_\omega f_n\big(nx,\xi+\nabla\psi_n(x)\big)\,dx \le \int_{\tilde\omega} f_n\big(nx,\xi+\nabla\hat\psi_n(nx)\big)\,dx + o(1).$$

Then, by the Y-periodicity of $\hat\psi_n$ combined with the regularity of $\tilde\omega$ we get

$$\int_\omega f_n\big(nx,\xi+\nabla\psi_n(x)\big)\,dx \le |\tilde\omega|\int_Y f_n\big(y,\xi+\nabla\hat\psi_n(y)\big)\,dy + o(1),$$

which implies inequality (3.7) by taking into account (3.13). □

Proof of Lemma 3.4. First note that the existence and the uniqueness of the function u is a consequence of the coerciveness and the strict convexity of $g(x,\cdot)$ combined with $G(\hat u)<\infty$. Set $m:=\min_{\partial O}(\hat u-\varphi)$. Since the negative part of $u-\varphi-m$, $(u-\varphi-m)^-$ belongs to $W_0^{1,p}(O)$ (see Lemma 2.7 of [7]) and $\nabla\varphi(x)$ is an absolute minimum of $g(x,\cdot)$, we have

$$\begin{aligned} G(u)\le G(u+(u-\varphi-m)^-) &= \int_{\{u-\varphi\ge m\}} g(x,\nabla u)\,dx + \int_{\{u-\varphi<m\}} g(x,\nabla\varphi)\,dx\\ &= \int_O g(x,\nabla u)\,dx + \int_{\{u-\varphi<m\}}\big(g(x,\nabla\varphi)-g(x,\nabla u)\big)\,dx\\ &\le G(u), \end{aligned}$$

Hence, by the convexity of G we deduce that

$$G(u)\le G\left(u+\tfrac12(u-\varphi-m)^-\right)\le\frac12\Big(G(u)+G\left(u+(u-\varphi-m)^-\right)\Big)\le G(u),$$

which yields

$$\int_O\left[\frac12\Big(g(x,\nabla u)+g\left(x,\nabla u+\nabla(u-\varphi-m)^-\right)\Big)-g\left(x,\nabla u+\tfrac12\nabla(u-\varphi-m)^-\right)\right]dx=0.$$

This combined with the strict convexity of $g(x,\cdot)$ implies that $\nabla(u-\varphi-m)^-=0$ a.e. in O. Therefore, we obtain $m\le u-\varphi$ a.e. in O. Similarly, we get $u-\varphi\le\max_{\partial O}(\hat u-\varphi)$ a.e. in O. □

3.2 Proof of Theorem 2.3

3.2.1 Proof of the Γ-limsup inequality

By condition (2.9) the functional F_∞ of (2.10) is continuous in $W^{1,p}(\Omega)$. Therefore, it is enough to prove the Γ-limsup inequality for piecewise-affine functions, which are a dense set in $W^{1,p}(\Omega)$ (see, e.g., [2] Remark 1.29).

Let D be a disk of $\mathbb{R}^2$ such that $\bar{\Omega} \subset D$, and consider a piecewise-affine function $u : D \to \mathbb{R}^2$ associated with a triangulation $(T_i)_{1\le i\le m}$ of D such that

$$u = \sum_{i=1}^{m} 1_{T_i}\, g^i, \qquad \text{where} \qquad g^i(x) = \xi^i \cdot x + c_i, \quad \text{for } \xi^i \in \mathbb{R}^2,\ c_i \in \mathbb{R},\ x \in D. \tag{3.17}$$

It is known (see, e.g., [18]) that there exist k subsets $J_1, \dots, J_k$ of $\{1, \dots m\}$, such that the following max-min representation holds:

$$u = \bigvee_{j=1}^{k} \bigwedge_{i\in J_j} g^i \quad \text{in } D. \tag{3.18}$$

Up to refining the triangulation (using the lines $\{g^i = g^j\}$ when $g^i \neq g^j$) we can assume that for any $\delta > 0$ small enough, the triangles T_i^δ defined by

$$T_i^\delta := \{x \in T_i \;:\; \operatorname{dist}(x, \partial T_i) \ge \delta\}, \quad \text{for } i \in \{1, \dots, m\}, \tag{3.19}$$

satisfy for any $i, j = 1, \dots, m$,

$$\forall\, x \in T_i^\delta, \quad \begin{cases} g^i(x) < g^l(x), & \forall\, l \in J_j \setminus \{i\} \text{ s.t. } g^l \neq g^i, \quad \text{if } i \in J_j \\ g^i(x) > \bigwedge_{l\in J_j} g^l(x), & \text{elsewhere.} \end{cases} \tag{3.20}$$

We denote by h the maximum of the diameters of T_i, and by Ω_h the union of the triangles T_i such that $T_i \cap \Omega \neq \emptyset$. For any $\xi \in \mathbb{R}^2$, consider a function $\varphi_n^\xi \in W_\sharp^{1,p}(Y)$ satisfyng (2.5).

By virtue of Proposition 3.1 applied to the functions $x \mapsto g^i(x) + \varepsilon_n\, \varphi_n^{\xi^i}(\frac{x}{\varepsilon_n})$, for $i = 1, \dots, m$, there exist sequences $v_n^i \in W^{1,p}(D)$ which weakly converge to g^i in $W^{1,p}(D)$ and strongly in $L^\infty_{\mathrm{loc}}(T_i)$, such that for any $i, j = 1, \dots, m$, with $T_i \subset \Omega_h$, we have

$$\begin{cases} \displaystyle\int_{T_i} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla v_n^i\big)\, dx & \le \displaystyle\int_{T_i} f_n\big(\tfrac{x}{\varepsilon_n}, \xi^i + \nabla\varphi_n^{\xi^i}(\tfrac{x}{\varepsilon_n})\big)\, dx + o(1) \\ \displaystyle\int_{T_i\setminus T_i^\delta} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla v_n^j\big)\, dx & \le \displaystyle\int_{\tilde{T}_i^\delta\setminus T_i^{2\delta}} f_n\big(\tfrac{x}{\varepsilon_n}, \xi^j + \nabla\varphi_n^{\xi^j}(\tfrac{x}{\varepsilon_n})\big)\, dx, \end{cases}$$

where $\tilde{T}_i^\delta$ are the enlarged triangles defined by

$$\tilde{T}_i^\delta := \big\{x \in \mathbb{R}^2 \;:\; \operatorname{dist}\big(x, T_i\big) < \delta\big\}, \quad \text{for } i \in \{1, \dots, m\}. \tag{3.21}$$

This combined with the periodicity of the functions φ_n^ξ implies that

$$\begin{cases} \displaystyle\int_{T_i} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla v_n^i\big)\, dx & \le |T_i|\, f_n^{\mathrm{hom}}(\xi^i) + o(1) \\ \displaystyle\int_{T_i\setminus T_i^\delta} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla v_n^i\big)\, dx & \le |\tilde{T}_j^\delta \setminus T_j^{2\delta}|\, f_n^{\mathrm{hom}}(\xi^i) + o(1). \end{cases} \tag{3.22}$$

In analogy to representation (3.18), we then define the function u_n, for $n \ge 1$, by

$$u_n = \bigvee_{j=1}^{k} \bigwedge_{i\in J_j} v_n^i \quad \text{a.e. in } \Omega_h. \tag{3.23}$$

Thanks to the uniform convergence of v_n^i in T_i^δ combined with property (3.20), we get that for n large enough,

$$\forall\, i \in \{1, \dots, m\}, \quad u_n(x) = v_n^i(x) \quad \text{a.e. } x \in T_i^\delta. \tag{3.24}$$

Using the following inequality, which is a consequence of definition (3.23) and of the bound from below of (2.3),

$$f_n(\tfrac{x}{\varepsilon_n}, \nabla u_n) \le \sum_{j=1}^{m} f_n(\tfrac{x}{\varepsilon_n}, \nabla v_n^j) + m - 1 \qquad \text{for a.e. } x \in \Omega_h, \tag{3.25}$$

we deduce from (3.24) and (3.22) that

$$\begin{aligned}
&\int_\Omega f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\, dx \le \sum_{T_i \subset \Omega_h} \int_{T_i^\delta} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\, dx + \int_{T_i \setminus T_i^\delta} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\, dx \\
&\le \sum_{T_i \subset \Omega_h} \int_{T_i^\delta} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla v_n^i\big)\, dx + \sum_{i,j=1}^{m} \int_{\tilde{T}_i^\delta \setminus T_i^{2\delta}} f_n\big(\tfrac{x}{\varepsilon_n}, \xi^j + \nabla \varphi_n^{\xi^j}\big(\tfrac{x}{\varepsilon_n}\big)\big)\, dx + O(\delta) \\
&\le \sum_{i=1}^{m} |T_i|\, f_n^{\text{hom}}(\xi^i) + \sum_{i,j=1}^{m} |\tilde{T}_i^\delta \setminus T_i^{2\delta}|\, f_n^{\text{hom}}(\xi^j) + o(1) + O(\delta)
\end{aligned}$$

Therefore, by the definitions (3.19), (3.21) of the triangles $T_i^\delta, \tilde{T}_i^\delta$ and the definition (3.17) of u together with convergence (2.9) we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \int_\Omega f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\, dx \quad &\le \quad \sum_{T_i \subset \Omega_h} |T_i|\, f_\infty^{\text{hom}}(\xi^i) + O(\delta) \\
&= \quad \int_\Omega f_\infty^{\text{hom}}(\nabla u)\, dx + O(h) + O(\delta),
\end{aligned}$$

which yields the Γ-limsup inequality.

3.2.2 Proof of the Γ-liminf inequality

The proof is based on the blow-up method due to Fonseca and Müller [14] and to Lemma 3.5 which leads us to periodic boundary conditions.

Since $L^p(\Omega)$ is separable, there exists a subsequence, still denoted by n, such that the sequence F_n in (2.8) Γ-converges to a functional F. Let $u \in L^p(\Omega)$ be such that $F(u) < \infty$. Then, consider a sequence u_n which strongly converges to u in $L^p(\Omega)$ and such that $F_n(u_n)$ is bounded. By the equicoercivity of F_n (as a consequence of (2.3)) the sequence u_n weakly converges to u in $W^{1,p}(\Omega)$.

Blow-up method of [14] (see also [5] for statement adapted to homogenization theory): Define the measure μ_n, ν_n by

$$\begin{cases} \mu_n(B) := \displaystyle\int_B f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\, dx \\ \nu_n(B) := \displaystyle\int_B |\nabla u_n|^p\, dx, \end{cases} \qquad \text{for any Borel set } B \subset \Omega. \tag{3.26}$$

Note that by the coercivity condition (2.3) of f_n, we have $\nu_n \le \mu_n + \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^2$. By the boundedness of $F_n(u_n) = \mu_n(\Omega)$, up to a subsequence

μ_n, ν_n weakly-$*$ converge respectively to the Radon measures μ, ν in $\mathcal{M}(\Omega)$. By lower semicontinuity and the Radon-Nikodym decomposition of μ, ν we have

$$\begin{cases} \liminf\limits_{n\to\infty} F_n(u_n) = \liminf\limits_{n\to\infty} \mu_n(\Omega) \geq \mu(\Omega) = \displaystyle\int_\Omega \frac{d\mu}{dx}\,dx + \mu_s(\Omega) \geq \int_\Omega \frac{d\mu}{dx}\,dx, \\ \liminf\limits_{n\to\infty} F_n(u_n) \geq \liminf\limits_{n\to\infty} \nu_n(\Omega) \geq \nu(\Omega) = \displaystyle\int_\Omega \frac{d\nu}{dx}\,dx + \nu_s(\Omega) \geq \int_\Omega \frac{d\nu}{dx}\,dx, \end{cases}$$

where μ_s, ν_s denote respectively the singular parts of μ, ν. Therefore, it remains to prove that the regular part of μ satisfies the pointwise inequality

$$\frac{d\mu}{dx}(x_0) \geq f_\infty^{\hom}\big(\nabla u(x_0)\big) \quad \text{a.e. } x_0 \in \Omega. \tag{3.27}$$

Now, fix a Lebesgue point x_0 common to $\frac{d\mu}{dx}, \frac{d\nu}{dx}$ and ∇u. The Besicovitch derivation theorem implies that

$$\begin{cases} \dfrac{d\mu}{dx}(x_0) = \lim\limits_{\rho\to 0} \dfrac{\mu(x_0+\rho Y)}{\rho^2} = \lim\limits_{\rho\to 0}\lim\limits_{n\to\infty} \dfrac{\mu_n(x_0+\rho Y)}{\rho^2} \\ \dfrac{d\mu}{dx}(x_0) = \lim\limits_{\rho\to 0} \dfrac{\nu(x_0+\rho Y)}{\rho^2} = \lim\limits_{\rho\to 0}\lim\limits_{n\to\infty} \dfrac{\nu_n(x_0+\rho Y)}{\rho^2}, \end{cases} \tag{3.28}$$

where the limits in n hold for any ρ but a countable set (since μ, ν are finite). Moreover, since x_0 is a Lebesgue point for ∇u, we have (see, e.g., Theorem 3.4.2. of [19])

$$\lim_{\rho\to 0} \frac{1}{\rho^2}\int_{x_0+\rho Y} \left|\frac{u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)}{\rho}\right|^p dx = 0.$$

Hence, by the strong convergence of u_n to u in $L^p(\Omega)$, we get that

$$\lim_{\rho\to 0}\lim_{n\to\infty} \frac{1}{\rho^2}\int_{x_0+\rho Y} \left|\frac{u_n(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)}{\rho}\right|^p dx = 0. \tag{3.29}$$

Then, using a diagonal extraction we deduce from (3.28) and (3.29) that there exist a subsequence of n, still denoted by n, and a positive sequence ρ_n such that ρ_n and $\eta_n := \varepsilon_n/\rho_n$ tend to zero, and such that the following limits hold

$$\begin{cases} \dfrac{d\mu}{dx}(x_0) = \lim\limits_{n\to\infty} \dfrac{1}{\rho_n^2}\displaystyle\int_{x_0+\rho_n Y} f_n\big(\tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,dx \\ \dfrac{d\nu}{dx}(x_0) = \lim\limits_{n\to\infty} \dfrac{1}{\rho_n^2}\displaystyle\int_{x_0+\rho_n Y} |\nabla u_n|^p\,dx, \end{cases} \tag{3.30}$$

$$\lim_{n\to\infty} \frac{1}{\rho_n^2}\int_{x_0+\rho_n Y} \left|\frac{u_n(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)}{\rho_n}\right|^p dx = 0. \tag{3.31}$$

Making the change of variables

$$\hat{z}_n(y) := \frac{u_n(x_0+\rho_n y) - u(x_0)}{\rho_n}, \quad \text{where} \quad y := \frac{x-x_0}{\rho_n}, \tag{3.32}$$

in (3.30) and (3.31), it follows that

$$\begin{cases} \dfrac{d\mu}{dx}(x_0) = \lim\limits_{n\to\infty} \displaystyle\int_Y f_n\big(\tfrac{y+\rho_n^{-1}x_0}{\eta_n}, \nabla\hat{z}_n\big)\,dy \geq \limsup_{n\to\infty} \int_{\eta_n[\eta_n^{-1}]Y} f_n\big(\tfrac{y+\rho_n^{-1}x_0}{\eta_n}, \nabla\hat{z}_n\big)\,dy \\ \dfrac{d\nu}{dx}(x_0) = \lim\limits_{n\to\infty} \displaystyle\int_Y |\nabla u_n(x_0+\rho_n y)|^p\,dy = \lim_{n\to\infty}\int_Y |\nabla\hat{z}_n|^p\,dy < \infty, \end{cases} \tag{3.33}$$

$$\lim_{n\to\infty}\int_Y |\hat{z}_n - \nabla u(x_0)\cdot y|^p\,dy = 0. \tag{3.34}$$

Therefore, the sequence $\hat{z}_n$ weakly converges to $\nabla u(x_0)\cdot y$ in $W^{1,p}(Y)$. In the same way this weak convergence holds in $W^{1,p}(RY)$ for any $R\geq 1$, since $\hat{z}_n$ is defined in the very large domain $\rho_n^{-1}(-x_0+\Omega)$.

Then, the following result allows us to recover periodic boundary conditions:

Lemma 3.5. *We have the inequality*

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\kappa_n Y} f_n\big(\tfrac{y+\rho_n^{-1}x_0}{\eta_n},\nabla\hat{z}_n\big)\,dy\\ &\geq \limsup_{n\to\infty}\left(\inf\left\{\int_{\kappa_n Y} f_n\big(\tfrac{y+\rho_n^{-1}x_0}{\eta_n},\nabla z\big)\,dy\ :\ z-\nabla u(x_0)\cdot y\in W^{1,p}_\sharp(\kappa_n Y)\right\}\right),\end{aligned}\tag{3.35}$$

where $\kappa_n := \eta_n[\eta_n^{-1}]$ *tends to* 1.

The proof of this result is postponed to the end of this section.

We can now conclude the proof. By a convexity argument and a translation (see, e.g., [3]) we obtain that

$$\begin{aligned}&\inf\left\{\int_{\eta_n[\eta_n^{-1}]Y} f_n\big(\tfrac{y+\rho_n^{-1}x_0}{\eta_n},\nabla z\big)\,dy\ :\ z-\nabla u(x_0)\cdot y\in W^{1,p}_\sharp\left(\eta_n[\eta_n^{-1}]Y\right)\right\}\\ &\geq \big(\eta_n[\eta_n^{-1}]\big)^2\,\inf\left\{\int_Y f_n\big(y,\nabla z\big)\,dy\ :\ z-\nabla u(x_0)\cdot y\in W^{1,p}_\sharp(Y)\right\}\\ &= \big(\eta_n[\eta_n^{-1}]\big)^2\, f_n^{\hom}\big(\nabla u(x_0)\big) = f_\infty^{\hom}\big(\nabla u(x_0)\big)+o(1)\end{aligned}$$

(by (2.9)). Combined with (3.35) and (3.33), this implies the desired inequality (3.27).

Proof of Lemma 3.5. Without loss of generality we can assume that $x_0=0$ and $\eta_n=\frac{1}{n}$. For $\delta\in(0,\frac{1}{2})$, set $Q_\delta:=(\delta,1-\delta)^2$ and consider the two Y-periodic functions $w^\pm$ defined by their restriction to Y:

$$w^\pm(y) := \pm\,\mathrm{dist}\,(y, Y\setminus Q_\delta)\,,\quad \text{for } y\in Y. \tag{3.36}$$

Each function $w^\pm$ is piecewise-affine and its graph restricted to Y is a tetrahedron the basis of which is Q_δ. Then, applying the proof of the Γ-limsup inequality with the functions $y\mapsto \xi\cdot y+\frac{1}{n}\varphi_n^\xi(ny)$, for $\xi\in\{\nabla u(x_0)+\nabla w^\pm\}$ (which is a set of 9 vectors), thanks to Proposition 3.3 we can construct two sequences $w_n^\pm$ which satisfy a max-min representation of type (3.23) and the following properties:

$$w_n^\pm \longrightarrow \nabla u(x_0)\cdot y + w^\pm \quad\text{weakly in } W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2) \text{ and strongly in } L^\infty_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.37}$$

$$w_n^\pm = \nabla u(x_0)\cdot y + \psi_n \quad\text{around } \partial Y,\quad \text{where } \psi_n\in W^{1,p}_\sharp(Y), \tag{3.38}$$

$$\int_{Y\setminus Q_{2\delta}} f_n\big(ny,\nabla w_n^\pm\big)\,dy \leq O(\delta)+o(1). \tag{3.39}$$

By construction, (3.38) is a consequence of the fact that $w^\pm=0$ in a neighborhood of ∂Y, while estimate (3.39) is deduced from (3.7).

On the other hand, by virtue of Proposition 3.1 there exists a sequence z_n in $W^{1,p}(Y)$ such that

$$z_n \longrightarrow \nabla u(x_0)\cdot y \quad\text{weakly in } W^{1,p}(Y) \text{ and strongly in } L^\infty_{\mathrm{loc}}(Y), \tag{3.40}$$

$$\int_Y f_n(ny, \nabla z_n)\, dy \le \int_Y f_n(ny, \nabla \hat{z}_n)\, dy + o(1). \tag{3.41}$$

Now, consider the function $\tilde{z}_n$ defined by

$$\tilde{z}_n := \big(w_n^+ \wedge z_n\big) \vee w_n^- \quad \text{in } Y, \tag{3.42}$$

namely z_n is "sandwiched" between w_n^+ and w_n^-. Since $w_n^+ = w_n^- = \nabla u(x_0)\cdot y + \psi_n$ around ∂Y, we have

$$\tilde{z}_n = \nabla u(x_0)\cdot y + \psi_n \quad \text{around } \partial Y. \tag{3.43}$$

Moreover, by the uniform convergence of $z_n - w_n^\pm$ to $-w^\pm$ in Q_δ combined with the fact that $\pm\, w^\pm$ is a positive continuous function in Q_δ, we get that for any n large enough,

$$\tilde{z}_n = z_n \quad \text{a.e. in } Q_{2\delta}. \tag{3.44}$$

Then, using that (similarly to (3.25))

$$f_n(ny, \nabla\tilde{z}_n) \le f_n(ny, \nabla z_n) + f_n(ny, \nabla w_n^+) + f_n(ny, \nabla w_n^-) + 2 \quad \text{a.e. in } Y,$$

we deduce from (3.44) and (3.39) that

$$\begin{aligned}
\int_Y f_n(ny, \nabla\tilde{z}_n)\, dy &= \int_{Q_{2\delta}} f_n(ny, \nabla z_n)\, dy + \int_{Y\setminus Q_{2\delta}} f_n(ny, \nabla\tilde{z}_n)\, dy \\
&\le \int_Y f_n(ny, \nabla z_n)\, dy + \int_{Y\setminus Q_{2\delta}} f_n(ny, \nabla w_n^+)\, dy \\
&\quad + \int_{Y\setminus Q_{2\delta}} f_n(ny, \nabla w_n^-)\, dy + 2\,|Y\setminus Q_{2\delta}| \\
&\le \int_Y f_n(ny, \nabla z_n)\, dy + o(1) + O(\delta).
\end{aligned}$$

Finally, combining the previous estimate with (3.43) and (3.41) we obtain that

$$\inf\left\{\int_Y f_n(ny, \nabla z)\, dy \;:\; z - \nabla u(x_0)\cdot y \in W^{1,p}_\sharp(Y)\right\} \le \int_Y f_n(ny, \nabla\hat{z}_n)\, dy + o(1) + O(\delta),$$

which yields the thesis. $\square$

4 A condition for the boundedness of f_n^{hom}

4.1 The main result

In this section we restrict ourselves to the sequence of functionals F_n (2.8) defined with the microscopic scale $\varepsilon_n = \frac{1}{n}$. Then, we have the following result:

Theorem 4.1. *Let Ω be a bounded open set of $\mathbb{R}^2$. In addition to conditions* (2.1), (2.2), *and* (2.3), *assume that there exists $C > 0$ such htat the density $f_n(y,\cdot)$ satisfies the estimate*

$$f_n(y, 2\,\xi) \le C\,\big(1 + f_n(y,\xi)\big), \quad \forall\, \xi\in\mathbb{R}^2, \text{ for a.e. } y\in\mathbb{R}^2. \tag{4.1}$$

Also assume that for any $\xi\in\mathbb{R}^2$, there exists a minimizer φ_n^ξ of (2.5) *such that*

$$\varphi_n^\xi \in C_\#(Y). \tag{4.2}$$

Let F be the Γ-limit of a subsequence of F_n defined by (2.8).

Then, a necessary and sufficient condition for the boundedness in $\mathbb{R}^2$ of the sequence f_n^{hom} in (2.4), *is that there exists a non-zero function $u\in W^{1,p}(\mathbb{R}^2)$, with compact support in Ω, such that $F(u) < \infty$.*

Theorem 2.3 clearly shows that the boundedness in $\mathbb{R}^2$ of $f_n^{\rm hom}$ implies that there exists a non-zero function $u \in W^{1,p}(\mathbb{R}^2)$, with compact support in Ω, such that $F(u) < \infty$ (F is actually finite on the whole space $W^{1,p}(\Omega)$). The present section is devoted to the proof of the converse. First of all, we will establish a general result in the convex case about the membership of regular functions in the domain of the Γ-limit.

4.2 A general result

Let Ω be a bounded open set of $\mathbb{R}^2$. Consider a sequence of functions $g_n : \Omega \times \mathbb{R}^2 \to [0, \infty)$ which satisfy the homogeneity condition (4.1) and the following ones:

$$g_n(\cdot, \xi) \text{ is measurable for any } \xi \in \mathbb{R}^2, \tag{4.3}$$

$$g_n(x, \cdot) \text{ is convex for a.e. } x \in \mathbb{R}^2, \tag{4.4}$$

there exists a function b_n in $L^\infty(\Omega)$ such that

$$|\xi|^p - 1 \le g_n(x, \xi) \le b_n(x)\,(1 + |\xi|^p)\,, \quad \forall\, \xi \in \mathbb{R}^2, \text{ for a.e. } x \in \Omega, \tag{4.5}$$

$$g_n(x, 2\,\xi) \le C\,\big(1 + g_n(x, \xi)\big), \quad \forall\, \xi \in \mathbb{R}^2, \text{ for a.e. } x \in \Omega. \tag{4.6}$$

Then, consider the sequence of convex functionals $G_n : L^p(\Omega) \to [0, \infty]$ defined by

$$G_n(v) := \begin{cases} \displaystyle\int_\Omega g_n\big(x, \nabla v\big)\,dx & \text{if } v \in W^{1,p}(\Omega) \\ \infty & \text{elsewhere.} \end{cases} \tag{4.7}$$

Thanks to the separability of $L^p(\Omega)$ we may assume that the sequence G_n Γ-converges to a functional $G : L^p(\Omega) \to [0, \infty]$ of domain $D(G)$. The following result gives a sufficient condition for regular functions to be in the domain of G:

Proposition 4.2. *Assume that there exist $\hat{x} \in \Omega$ and $w^0, w^1, w^2 \in C^1(\Omega)$ which satisfy*

$$0 \in \operatorname{int}\big(\operatorname{co}\big(\nabla w^0(\hat{x}), \nabla w^1(\hat{x}), \nabla w^2(\hat{x})\big)\big)\,, \tag{4.8}$$

and sequences w_n^i, for $i = 0, 1, 2$, which strongly converge to w^i in $L^\infty(\Omega)$, with

$$\limsup_{n\to\infty} \int_\Omega g_n(x, \nabla w_n^i)\,dx < \infty. \tag{4.9}$$

Then, there exists $\delta > 0$ such that $C_c^1\big(B(\hat{x}, \delta)\big) \subset D(G)$.

First note that all the L^∞-strong convergences in the sequel are a consequence of Proposition 3.1.

Proof. Consider $\varepsilon > 0$ small enough which will be chosen later, and define the function $z := (w^1 - w^0, w^2 - w^0)$. Since

$$\operatorname{int}\big(\operatorname{co}\big(\nabla w^0(\hat{x}), \nabla w^1(\hat{x}), \nabla w^2(\hat{x})\big)\big) \neq \emptyset,$$

the Jacobian matrix $Dz(\hat{x})$ is invertible. Then, there exists $\delta_0 > 0$ such that z is a C^1-diffeomorphism from $B(\hat{x}, \delta_0)$ into an open set $O \subset \mathbb{R}^2$. Taking δ_0 small enough, we can also assume that

$$\forall\, x \in B(\hat{x}, \delta_0), \quad \big|\nabla w^0(x) - \nabla w^0(\hat{x})\big| < \varepsilon \quad \text{and} \quad \big|Dz(x)^{-1} - Dz(\hat{x})^{-1}\big| < \varepsilon.$$

Now, consider $u \in C^1\big(\bar{B}(\hat{x},\delta_0)\big)$ with $\|\nabla u\|_{L^\infty(B(\hat{x},\delta_0))} < \varepsilon$, and define $R := (u - w^0) \circ z^{-1}$ which belongs to $C^1(O)$. Then, we have

$$\forall\, x \in B(\hat{x},\delta_0), \quad u(x) = w_0(x) + R(z(x)) \quad \text{and} \quad \nabla u(x) = \nabla w^0(x) + Dz(x)^T \nabla R(z(x)),$$

which gives

$$\nabla R(z(x)) = \big(Dz(x)^T\big)^{-1} \nabla(u - w^0)(x),$$

where ^{T} denoted the transposition. Defining $\eta := -\big(Dz(\hat{x})^T\big)^{-1} \nabla w^0(\hat{x})$, we get

$$\begin{aligned} |\nabla R(z(x)) - \eta| \;&\leq |\nabla u(x)|\,\big|Dz(x)^{-1}\big| + \big|Dz(x)^{-1}\big|\,\big|\nabla w^0(x) - \nabla w^0(\hat{x})\big| \\ &\quad + \big|\nabla w^0(\hat{x})\big|\,\big|Dz(x)^{-1} - Dz(\hat{x})^{-1}\big| \\ &< 2\,\varepsilon\left(\big|Dz(\hat{x})^{-1}\big| + \varepsilon\right) + \varepsilon\,\big|\nabla w^0(\hat{x})\big|. \end{aligned} \tag{4.10}$$

On the other hand, note that $\eta = (\eta_1, \eta_2)$ is also defined by the equality

$$0 = (1 - \eta_1 - \eta_2)\nabla w^0(\hat{x}) + \eta_1 \nabla w^1(\hat{x}) + \eta_2 \nabla w^2(\hat{x}),$$

which by (4.8) implies that $\eta_1 > 0$, $\eta_2 > 0$ and $\eta_1 + \eta_2 < 1$. Then, taking ε small enough in (4.10) we can assume that these strict inequalities also hold for the components of $\nabla R(z)$, i.e.

$$\partial_1 R(z) > 0, \quad \partial_2 R(z) > 0 \quad \text{and} \quad \partial_1 R(z) + \partial_2 R(z) < 1. \tag{4.11}$$

Now, define $z_n := (w_n^1 - w_n^0, w_n^2 - w_n^0)$ and $u_n := w_n^0 + R \circ z_n$ in $B(\hat{x},\delta)$, with $\delta = \delta_0/2$. The function u_n is well defined because $z\big(\bar{B}(\hat{x},\delta)\big)$ is a compact subset of O, hence its distance to ∂O is positive. Since z_n strongly converges to z in $L^\infty\big(B(\hat{x},\delta)\big)$, we have that for n large enough, $z_n\big(B(\hat{x},\delta)\big) \subset O$. Clearly, u_n strongly converges to u in $B(\hat{x},\delta)$ and satisfies

$$\nabla u_n = \big(1 - \partial_1 R(z_n) - \partial_2 R(z_n)\big)\nabla w_n^0 + \partial_2 R(z_n)\nabla w_n^1 + \partial_3 R(z_n)\nabla w_n^2.$$

Thanks to (4.11) and to the uniform convergence of $\partial_j R(z_n)$ to $\partial_j R(z)$, we get that ∇u_n is a convex combination of the ∇w_n^i, for $i = 1,2,3$, hence by (4.9) we obtain that

$$\limsup_{n\to\infty} \int_{B(\hat{x},\delta)} g_n(x, \nabla u_n)\, dx < +\infty. \tag{4.12}$$

Therefore, we have proved the existence of $\delta, \varepsilon > 0$ such that for any $u \in C^1\big(\bar{B}(\hat{x},2\delta)\big)$, with $\|\nabla u\|_{L^\infty(B(\hat{x},2\delta))} < \varepsilon$, there exists a sequence u_n in $W^{1,p}(B(\hat{x},\delta))$ which strongly converges to u in $L^\infty\big(B(\hat{x},\delta)\big)$ and satisfies (4.12). Moreover, if the support of u is contained in $B(\hat{x},\delta)$, then we can easily construct a function u_n with compact support in $B(\hat{x},\delta)$ so that u_n is defined in the whole set Ω. This establishes Proposition 4.2 for any $u \in C_c^1(\Omega)$ with $\|\nabla u\|_{L^\infty(\Omega)} < \varepsilon$.

If u does not satisfy this restriction, then we apply the result to $v := \varepsilon u / \big(2\,\|\nabla u\|_{L^\infty(\Omega)}\big)$, and we consider the sequence $u_n := 2\,\|\nabla u\|_{L^\infty(\Omega)}\, v_n/\varepsilon$, where v_n is the sequence relating to v. We use property (4.6) to conclude. □

As a consequence of Proposition 4.2 we have the following result in the periodic case:

Corollary 4.3. *In addition to conditions* (4.3)–(4.6) *assume that for all* $\xi \in \mathbb{R}^2$ *we have* $g_n(x,\xi) = f_n(nx,\xi)$ *for a.e.* $x \in \Omega$, *where* $f_n(\cdot,\xi)$ *is* Y*-periodic. Also assume that there exists a non-zero function in* $W^{1,p}(\Omega) \cap D(G)$ *with compact support in* Ω. *Then, we have* $C_c^1(\Omega) \subset D(G)$.

Proof. Let $u \in W^{1,p}(\Omega) \cap D(G)$ be with compact support in Ω, and consider a sequence u_n which weakly converges to u in $W^{1,p}(\Omega)$ and such that $G_n(u_n)$ is bounded. Then, by periodicity and by a translation argument, we have that for any $\tau \in \mathbb{R}^2$, with small enough norm, there exist a sequence u_n^τ in $W^{1,p}(\Omega)$ which weakly converges to $u(\cdot + \tau)$ in $W^{1,p}(\Omega)$, such that (see, e.g., Chapters 23-24 of [11] for more details)

$$\limsup_{n\to\infty} G_n(u_n^\tau) = \limsup_{n\to\infty} G_n(u_n).$$

Hence, we deduce that for any nonnegative $\rho \in C_c^\infty(\mathbb{R}^2)$ and any $\tau_1, \ldots, \tau_m \in \mathbb{R}^2$, with $\sum_{i=1}^m \rho(\tau_i) > 0$, the function

$$\frac{\displaystyle\sum_{i=1}^m \rho(\tau_i)\, u(\cdot + \tau_i)}{\displaystyle\sum_{i=1}^m \rho(\tau_i)}$$

also belongs to $D(G)$, as well as the function

$$x \longmapsto \frac{\displaystyle\int_{\mathbb{R}^2} u(x-y)\,\rho(y)\,dy}{\displaystyle\int_{\mathbb{R}^2} \rho(y)\,dy}.$$

Therefore, we are led to the case where u is a non-zero function in $C_c^\infty(\Omega) \cap D(G)$.

Now, from Lemma 4.4 below we deduce that for any $\xi \in \mathbb{R}^2$, with small enough norm, there exists $x \in \Omega$ such that $\nabla u(x) = \xi$. Using the translated functions $u(\cdot + \tau)$ as before, we thus get that any point of Ω satisfies the assumptions of Proposition 4.2, which implies that $C_c^1(\Omega) \subset D(G)$. $\square$

Lemma 4.4. *Let Ω a bounded open set of $\Omega \subset \mathbb{R}^2$. Consider a function $u \in C^1(\Omega) \cap C(\bar\Omega)$ with $u = 0$ on $\partial\Omega$, such that there exists $x_0 \in \Omega$ with $u(x_0) \neq 0$. Then, for any $\xi \in \mathbb{R}^2$ with*

$$|\xi| < \frac{|u(x_0)|}{\displaystyle\max_{x\in\partial\Omega} |x_0 - x|}, \tag{4.13}$$

there exists $x \in \Omega$ such that $\nabla u(x) = \xi$.

Proof. We can assume that $x_0 = 0$ and $u(0) > 0$. For $\xi \in \mathbb{R}^N$, we consider $y \in \bar\Omega$ such that

$$u(x) - \xi \cdot x = \max_{y\in\bar\Omega} \big(u(y) - \xi \cdot y\big).$$

If $x \in \partial\Omega$, then we have $u(x) = 0$ and

$$u(0) \le -\xi \cdot x \le |\xi| \max_{y\in\partial\Omega} |y|,$$

hence

$$|\xi| \ge \frac{u(0)}{\max_{y\in\partial\Omega} |y|}.$$

Conversely, if

$$|\xi| < \frac{u(0)}{\displaystyle\max_{y\in\partial\Omega} |y|},$$

then x is a maximizer of $\big(y \mapsto u(y) - \xi \cdot y\big)$ in Ω, which implies that $\nabla u(x) = \xi$. $\square$

4.3 Proof of Theorem 4.1

We need the following result which is essentially based on the continuity assumption (4.2):

Lemma 4.5. *Assume that the continuity condition* (4.2) *holds. Then, for any* $\xi \in \mathbb{R}^2$*, the sequence of functions* w_n^ξ *defined by* $w_n^\xi(x) := \xi \cdot x + \frac{1}{n}\,\varphi_n^\xi(nx)$, $x \in \mathbb{R}^2$*, strongly converges to* $\xi \cdot x$ *in* $L^\infty_{\rm loc}(\mathbb{R}^2)$.

Proof. Let Ω be a bounded open set of $\mathbb{R}^2$. The sequence w_n^ξ clearly converges to the continuous function $\xi \cdot x$ weakly in $W^{1,p}(\Omega)$. Moreover, since φ_n^ξ is a Y-periodic minimizer of (2.5), we have for any open set $O \subset \Omega$,

$$\int_O f_n\big(nx, \nabla w_n^\xi\big)\,dx = \min\left\{\int_O f_n(nx, \nabla w_n^\xi + \nabla\varphi)\,dx \;:\; \varphi \in W_0^{1,p}(O)\right\}. \tag{4.14}$$

Then, taking into account the continuity of w_n^ξ, the construction of the proof of Proposition 3.1 (compare (3.10) to (4.14)) shows that the sequence w_n^ξ strongly converges to $\xi \cdot x$ in $L^\infty_{\rm loc}(\Omega)$. □

As a consequence of Corollary 4.3 we have that $C_c^1(\Omega) \subset D(F)$ for any bounded open set of $\mathbb{R}^2$. Let Ω be the unit disk of $\mathbb{R}^2$, and fix $\delta > 0$. Let $\phi \in C_c^1\big((1+2\delta)\Omega\big)$ with $\phi = 1$ in $(1+\delta)\Omega$. Then, by Corollary 4.3 and Proposition 3.1 applied to the open set $(1+2\delta)\Omega$, there exists a sequence ζ_n which converges to $\phi(x)\,\xi \cdot x$ weakly in $W^{1,p}\big((1+2\delta)\Omega\big)$ and strongly in $L^\infty\big((1+\delta)\Omega\big)$, such that

$$\limsup_{n\to\infty} \int_{(1+\delta)\Omega} f_n(nx, \nabla\zeta_n)\,dx < \infty. \tag{4.15}$$

Similarly, for a function $\varphi \in C_c^1\big((1+\delta)\Omega\big)$ with $0 \le \varphi \le 1$ in $(1+\delta)\Omega$ and $\varphi = 1$ in Ω, there exists a sequence φ_n which converges to φ weakly in $W^{1,p}\big((1+2\delta)\Omega\big)$ and strongly in $L^\infty\big((1+\delta)\Omega\big)$, such that

$$\limsup_{n\to\infty} \int_{(1+\delta)\Omega} f_n(nx, \nabla\varphi_n)\,dx < \infty. \tag{4.16}$$

Using truncations we can also assume that $0 \le \varphi_n \le 1$ in $(1+\delta)\Omega$ and $\varphi_n = 1$ in Ω.

On the one hand, using successively the minimization property (4.14) of w_n^ξ and the convexity (2.2) of $f_n(nx,\cdot)$, we have

$$\begin{aligned}
&\int_{(1+\delta)\Omega} f_n\big(nx, \nabla w_n^\xi\big)\,dx \le \int_{(1+\delta)\Omega} f_n\big(nx, \nabla(w_n^\xi + \varphi_n(\zeta_n - w_n^\xi))\big)\,dx \\
&= \int_{(1+\delta)\Omega} f_n\big(nx, \varphi_n\nabla\zeta_n + (1-\varphi_n)\nabla w_n^\xi + (\zeta_n - w_n^\xi)\nabla\varphi_n\big)\,dx \\
&\le \frac{1}{2}\int_{(1+\delta)\Omega} \varphi_n\, f_n(nx, 2\nabla\zeta_n)\,dx + \frac{1}{2}\int_{(1+\delta)\Omega} f_n\big(nx, 2(\zeta_n - w_n^\xi)\nabla\varphi_n\big)\,dx \\
&+ \frac{1}{2}\int_{(1+\delta)\Omega} (1-\varphi_n)\, f_n\big(nx, 2\nabla w_n^\xi\big)\,dx,
\end{aligned}$$

hence by estimate (4.1) we get

$$\begin{aligned}
&\int_{(1+\delta)\Omega} f_n\big(nx, \nabla w_n^\xi\big)\,dx \\
&\le \frac{C}{2}\int_{(1+\delta)\Omega} f_n(nx, \nabla\zeta_n)\,dx + \frac{C}{2}\,\big\|\zeta_n - w_n^\xi\big\|^p_{L^\infty((1+\delta)\Omega)} \int_{(1+\delta)\Omega} f_n(nx, \nabla\varphi_n)\,dx \\
&+ \frac{C}{2}\int_{(1+\delta)\Omega\setminus\Omega} f_n\big(nx, \nabla w_n^\xi\big)\,dx \quad (\text{since } \varphi_n = 1 \text{ in } \Omega).
\end{aligned} \tag{4.17}$$

On the other hand, the Y-periodicity of ∇w_n^ξ implies that

$$\int_{(1+\delta)\Omega\setminus\Omega} f_n\big(nx, \nabla w_n^\xi\big)\, dx \underset{n\to\infty}{\approx} \frac{(1+\delta)^2-1}{(1+\delta)^2} \int_{(1+\delta)\Omega} f_n\big(nx, \nabla w_n^\xi\big)\, dx. \tag{4.18}$$

Moreover, the uniform convergence of ζ_n and Lemma 4.5 combined with estimates (4.15) and (4.16) give

$$\frac{C}{2}\int_{(1+\delta)\Omega} f_n(nx, \nabla\zeta_n)\, dx + \frac{C}{2}\left\|\zeta_n - w_n^\xi\right\|^p_{L^\infty((1+\delta)\Omega)} \int_{(1+\delta)\Omega} f_n(nx, \nabla\varphi_n)\, dx \le c. \tag{4.19}$$

Therefore, using estimates (4.18) and (4.19) in (4.17), and choosing

$$\frac{C}{2}\,\frac{(1+\delta)^2-1}{(1+\delta)^2} < 1$$

(which holds for δ small enough), it follows that

$$\int_{(1+\delta)\Omega} f_n\big(nx, \nabla w_n^\xi\big)\, dx \le c,$$

which by periodicity implies that the sequence $f_n^{\rm hom}(\xi)$ is bounded. $\square$

References

[1] M. Bellieud & G. Bouchitté: "Homogenization of elliptic problems in a fiber reinforced structure. Nonlocal effects", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **26** (4) (1998), 407-436.

[2] A. Braides: *Γ–convergence for Beginners*, Oxford University Press, Oxford, 2002.

[3] A. Braides, V. Chiadò Piat & A. Piatnitski: "A variational approach to double-porosity problems", *Asymptot. Anal.*, **39** (3-4) (2004), 281-308.

[4] A. Braides & A. Defranceschi: *Homogenization of Multiple Integrals*, Oxford University Press, Oxford 1998.

[5] A. Braides, M. Maslennikov & L. Sigalotti: "Homogenization by Blow-up", *Applicable Anal.*, to appear.

[6] M. Briane: "Homogenization of high-conductivity periodic problems: Application to a general distribution of one-directional fibers", *SIAM J. Math. Anal.*, **35** (1) (2003), 33-60.

[7] M. Briane & J. Casado-Díaz: "Asymptotic behaviour of equicoercive diffusion energies in two dimension", *Calc. Var. Part. Diff. Equa.*, **29** (4) (2007), 455-479.

[8] M. Briane & J. Casado-Díaz: "Compactness of sequences of two-dimensional energies with a zero-order term. Application to three-dimensional nonlocal effects", to appear in *Calc. Var. Part. Diff. Equa.*

[9] M. Briane & N. Tchou: "Fibered microstructures for some nonlocal Dirichlet forms", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **30** (4) (2001), 681-711.

[10] M. Camar-Eddine & P. Seppecher: "Closure of the set of diffusion functionals with respect to the Mosco-convergence", *Math. Models Methods Appl. Sci.*, **12** (8) (2002), 1153-1176.

[11] G. Dal Maso: *An introduction to Γ-convergence*, Birkhaüser, Boston 1993.

[12] L.C. Evans: *Weak Convergence Methods for Nonlinear Partial Differential Equations*, *AMS*, Providence (1990).

[13] V.N. Fenchenko & E.Ya. Khruslov: "Asymptotic of solution of differential equations with strongly oscillating matrix of coefficients which does not satisfy the condition of uniform boundedness", *Dokl. AN Ukr.SSR*, **4** (1981).

[14] I. Fonseca & S. Müller: "Quasiconvex integrands and lower semicontinuity in L^1", *SIAM J. Math. Anal.*, **23** (1992), 1081-1098.

[15] E.Ya. Khruslov: "Homogenized models of composite media", *Composite Media and Homogenization Theory*, ed. by G. Dal Maso and G.F. Dell'Antonio, in *Progress in Nonlinear Differential Equations and Their Applications*, Birkhaüser (1991), 159-182.

[16] E.Ya. Khruslov & V.A. Marchenko: *Homogenization of Partial Differential Equations*, Progress in Mathematical Physics, **46**, Birkhäuser, Boston 2006.

[17] U. Mosco: "Composite media and asymptotic Dirichlet forms", *J. Func. Anal.*, **123** (2) (1994), 368-421.

[18] S. Ovchinnikov: "Max-min representation of piecewise linear functions", *Beiträge Algebra Geom.*, **43** (1) (2002), 297-302.

[19] W. Ziemer: *Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation*, Springer-Verlag, New York 1989.