A NOTE ON SETS OF FINITE PERIMETER IN THE PLANE

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Abstract. The aim of this paper is to study the minimal perimeter problem for sets containing a fixed set E in $\mathbb{R}^2$ in a very general setting, and to give the explicit solution.

This Article is Dedicated to Paolo Marcellini
on the Occasion of His Sixtieth Birthday

Introduction

It is well known that if E is a sufficiently smooth bounded open set in the plane, the convex hull $\mathrm{co}(E)$ of E is the bounded connected open set of minimal perimeter containing E. In this paper we study the same kind of minimization problem in a much more general framework by minimizing the perimeter in the class of indecomposable sets containing a fixed measurable set E.

To this aim, we recall that a set of finite perimeter $E \subset \mathbb{R}^N$ is said to be *decomposable* if there exists a partition of E in two measurable sets A, B with strictly positive measure such that

$$P(E) = P(A) + P(B),$$

where $P(\cdot)$ denotes the perimeter of a measurable set. Therefore, the notion of *indecomposable* set, i.e. not decomposable set, clearly extends the topological notion of connectedness. Besides, it can be easily shown that if $\Omega \subset \mathbb{R}^N$ is a connected open set of finite perimeter then Ω is indecomposable.

In the sequel, given a measurable set $E \subset \mathbb{R}^N$, we shall denote by E^λ, λ in $[0,1]$, the set of all points where the density of E is equal to λ. Moreover, we say that a property holds modulo $\mathcal{H}^k$ if it holds apart from a set of $\mathcal{H}^k$-measure zero, where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

With this notation our main result reads as follows.

Theorem 1. *Let $E \subset \mathbb{R}^2$ be a bounded measurable set. The problem*

$$\inf\{P(F) : F \supset E \ (\mathrm{mod}\ \mathcal{H}^2),\ F \text{ indecomposable and bounded}\} \tag{1}$$

has the unique solution $F_0 = \mathrm{co}(E^1)$.

Notice that the statement of Theorem 1 reproduces exactly what is known in the smooth case provided we take as a representative of the measurable set E the set E^1 of its points of density 1, which coincides $\mathcal{H}^2$-a.e. with E.

In the particular case when E is an indecomposable set the result above can be improved by showing that $\mathrm{co}(E^1)$ is the unique minimizer of the perimeter also in the larger class of all bounded sets of finite perimeter containing E almost everywhere. Moreover, it can be also proved that $\mathrm{co}(E^1)$ coincides with the convex envelope of a connected open set of finite perimeter essentially obtained by removing all "holes" from the set E (for a precise statement see Theorem 6).

A quick comment about the proofs. The precise analysis of the structure of sets of finite perimeter in the plane carried on by Ambrosio-Caselles-Masnou-Morel in [1] permits to give much simpler proofs of our results than one could have expected in such a general framework.

1. Sets of finite perimeter

Given a measurable set $E \subset \mathbb{R}^N$ and a point $x \in \mathbb{R}^N$, we recall that the *density* of E at x is defined as

$$D(E, x) := \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|}.$$

The set $E^1 = \{x \in \mathbb{R}^N : D(E,x) = 1\}$ is called the *essential interior* of E. Similarly, E^0 is the *essential exterior* of E and $\partial^M E = \mathbb{R}^N \setminus (E^1 \cup E^0)$ is the *essential boundary* of E.

We say that a set E is of finite perimeter if $\mathcal{H}^{N-1}(\partial^M E)$ is finite. Its perimeter $P(E)$ is then given by

$$P(E) := \mathcal{H}^{N-1}(\partial^M E).$$

Notice that this definition of perimeter is equivalent to the distributional one (see Definition 3.35 and Theorem 3.61 in [2] together with Theorem 4.5.11 in [4]).

By definition, the essential boundary of E contains the points where the density of E is equal to 1/2. However, (see Theorem 3.61 in [2]) if E is a set of finite perimeter, then

$$\mathcal{H}^{N-1}(\partial^M E \setminus E^{1/2}) = 0. \tag{2}$$

The following result is proved in [1] (Theorem 1).

Theorem 2. *Let E be a set of finite perimeter in $\mathbb{R}^N$. Then there exists a unique finite or countable family of pairwise disjoint indecomposable sets $\{E_i\}_{i \in I}$ such that $|E_i| > 0$, for every $i \in I$, and*

$$P(E) = \sum_i P(E_i).$$

Moreover, if $F \subset E$ is an indecomposable set then F is contained (mod $\mathcal{H}^N$) *in some set E_i.*

2. Sets of finite perimeter in the plane

Let us turn to sets of finite perimeter in the plane.

We recall that Γ is a Jordan curve if $\Gamma = \gamma([a,b])$, for some $a < b$ in $\mathbb{R}$, and some continuous, one-to-one on $[a,b)$, map γ such that $\gamma(a) = \gamma(b)$. According to the Jordan curve theorem, Γ splits $\mathbb{R}^2 \setminus \Gamma$ in two open components, a bounded one $\mathrm{int}(\Gamma)$ and an unbounded one $\mathrm{ext}(\Gamma)$, both having common boundary Γ.

Next result is proved in [1] (see Corollary 1). It states that a bounded indecomposable set in the plane essentially coincides with the interior of a Jordan curve minus a finite or countable number of holes.

Theorem 3. *Let E be an indecomposable bounded set of $\mathbb{R}^2$. Then, there exists a unique decomposition* (mod $\mathcal{H}^1$) *of $\partial^M E$ into a finite or countable number of* *rectifiable Jordan curves C^+, C_i^-, $i \in I$, such that int$(C_i^-) \subset$ int(C^+), the int(C_i^-) are pairwise disjoint,*

$$E{=}int(C^+) \setminus \bigcup_i int(C_i^-) \pmod{\mathcal{H}^2} \quad \text{and} \quad P(E) = \mathcal{H}^1(C^+) + \sum_i \mathcal{H}^1(C_i^-).$$

Lemma 4. *Let $E \subset \mathbb{R}^2$ be an indecomposable bounded set. If E is not equivalent $\pmod{\mathcal{H}^2}$ to a convex open set, then there exists a bounded set $F \supset E \pmod{\mathcal{H}^2}$ such that*

$$P(F) < P(E).$$

Proof. By Theorem 3,

$$E{=}\mathrm{int}(C^+) \setminus \bigcup_i \mathrm{int}(C_i^-) \quad \text{and} \quad P(E) = \mathcal{H}^1(C^+) + \sum_i \mathcal{H}^1(C_k^-).$$

Define $G := \mathrm{int}(C^+)$. If there exists i such that $\mathcal{H}^1(C_i^-) > 0$, then $P(G) < P(E)$. We can therefore suppose $E = \mathrm{int}(C^+)$.

Suppose that G is not convex. We claim that there exist $a, b \in C^+$ such that the segment (a, b) is contained in $\mathrm{ext}(C^+)$.

In fact, since G is not convex, there exist two points x, y in G such that the segment $[x, y]$ is not contained in G. Then, either there is a point $z \in [x, y] \cap \mathrm{ext}(C^+)$ and in this case the claim immediately follows, or the segment $[x, y]$ is all contained in $\overline{G}$ and then, by slightly tilting the segment $[x, y]$, one reduces to the previous case.

Let us now denote by $\gamma : [0, 1] \to \mathbb{R}^2$ a parametrization of C^+. Without loss of generality, we may assume that $\gamma(0) = \gamma(1) = a$, $\gamma(1/2) = b$. Set $\Gamma_1 := \gamma((0, 1/2))$, $\Gamma_2 := \gamma((1/2, 1))$. Clearly $\Gamma_1 \cup [a, b], \Gamma_2 \cup [a, b]$ are Jordan curves, thus we may define

$$A_1 := \mathrm{int}(\Gamma_1 \cup [a, b]), \qquad A_2 := \mathrm{int}(\Gamma_2 \cup [a, b]).$$

We claim that

$$\text{either} \quad A_1 \cap \Gamma_2 \neq \emptyset \quad \text{or} \quad A_2 \cap \Gamma_1 \neq \emptyset. \tag{3}$$

Let us argue by contradiction. If (3) does not hold, let us set

$$A := A_1 \cup A_2 \cup (a, b).$$

We show that A is a bounded connected open set. Fix $x \in A$. If x belongs to A_1 or A_2, then it is obvious that x belongs to the interior of A. Therefore, let us consider the case when x belongs to the open segment (a, b). In this case, there exists a ball B centered in x and not intersecting $\Gamma_1 \cup \Gamma_2 \cup \{a, b\}$. Denote by B^+ and B^- the two opens half balls in which B is divided by (a, b). Since $x \in \partial A_1$, there exists a point $y_1 \in A_1 \cap B \setminus (a, b)$. To fix the ideas, let us assume $y_1 \in B^+$. Similarly there exists $y_2 \in A_2 \cap B \setminus (a, b)$. If also y_2 belongs to B^+ then, by a simple connectedness argument, $B^+ \subset A_1 \cap A_2$. Thus $A_1 \cap A_2 \neq \emptyset$ and $A_1 \neq A_2$ (since they have different boundaries). Therefore, either $A_1 \setminus A_2 \neq \emptyset$ or $A_2 \setminus A_1 \neq \emptyset$. Assume the former is true. Then there exist $x_1 \in A_1 \setminus A_2$ and $x_2 \in A_1 \cap A_2$ Moreover, since we are assuming that $A_1 \cap \partial A_2 = \emptyset$, $x_1 \in A_1 \cap \mathrm{ext}(\Gamma_2 \cup [a, b])$. Since A_1 is connected, we can find an arc $\widehat{x_1 x_2} \subset A_1$ with extreme points x_1, x_2. Then, $\widehat{x_1 x_2} \cap \Gamma_2 \neq \emptyset$ which is impossible since we are assuming $A_1 \cap \Gamma_2 = \emptyset$. This shows that if $B^+ \subset A_1$ necessarily $B^- \subset A_2$, hence $B \subset A$, thus completely proving that A is an open set.

To show that A is connected, let us fix two points $x_1, x_2 \in A$. If $x_1, x_2 \in A_1 \cup (a, b)$, clearly there exists an arc connecting the two points contained in $A_1 \cup (a, b)$. Similarly for the case $x_1, x_2 \in A_2 \cup (a, b)$. Finally, if $x_1 \in A_1$ and $x_2 \in A_2$, we can fix a point $x \in (a,b)$ and connect first x_1 to x by an arc contained in A_1 and then x_2 to x by an arc contained in A_2. Therefore, in all possible case, there is an arc connecting x_1 to x_2 contained in A.

Let us finally prove that $\partial A = C^+$. Then, from this equality, it will immediately follow that $A = G$ and thus that $(a,b) \subset G$ which is impossible. This contradiction will prove (3). Let us first take $x \in \partial A$. Then we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x and contained either in A_1 or in A_2 or in the segment (a,b). Therefore, x necessarily belongs to $\Gamma_1 \cup \Gamma_2 \cup [a,b]$. On the other hand $x \notin (a,b)$ hence $x \in C^+$, thus showing that $\partial A \subset C^+$. To prove the opposite inclusion, take a point x in $\Gamma_1 \cup \{a,b\}$ (the case $\Gamma_2 \cup \{a,b\}$ is similar). Then, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ contained in A_1 and converging to x. Thus $x \in \overline{A}_1 \setminus A \subset \overline{A} \setminus A = \partial A$.

Having proved (3), let us assume, without loss of generality, that there exists $x \in A_1 \cap \Gamma_2$, hence there exists a point $y \in A_1 \cap G$. Again, by connectedness, this implies that

$$G \subset \operatorname{int}(\Gamma_1 \cup [a,b]).$$

Thus, defining $F := \operatorname{int}(\Gamma_1 \cup [a,b])$, we obtain that

$$\begin{aligned} P(E) = \mathcal{H}^1(C^+) \quad &= \mathcal{H}^1(\Gamma_1) + \mathcal{H}^1(C^+ \setminus \Gamma_1) \\ &> \mathcal{H}^1(\Gamma_1) + \mathcal{H}^1([a,b]) = \mathcal{H}^1(\Gamma_1 \cup [a,b]) = P(F), \end{aligned}$$

as we stated. □

Proposition 5. *Let $E \subset \mathbb{R}^2$ be an indecomposable bounded set. The problem*

$$\inf\{P(F) : F \supset E \ (\mathrm{mod}\ \mathcal{H}^2), F \ \textit{bounded}\} \tag{4}$$

has the unique solution $F_0 = \mathrm{co}(E^1)$.

Proof. Let B a ball such that $E \subset\subset B$ and consider the problem

$$\inf\{P(F) : F \supset E \ (\mathrm{mod}\ \mathcal{H}^2), F \subset B\}. \tag{5}$$

The assertion will easily follow by proving that $\mathrm{co}(E^1)$ is the unique minimizer for (5). Let $\{F_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for problem (5). Since the measures of F_n are equibounded, by Theorem 3.39 in [2], passing possibly to a subsequence, we may assume that

$$\chi_{F_n} \text{ converges a.e. to } \chi_F,$$

where F is a set of finite perimeter such that $F \supset E$ (mod $\mathcal{H}^2$). By the lower semicontinuity of perimeter, F is minimal.

Denoting by $\{E_i\}_{i\in I}$ the decomposition of F in indecomposable sets provided by Theorem 2, there exists i such that

$$E_i \supset E \ (\mathrm{mod}\ \mathcal{H}^2)$$

and thus by the minimality of F, $E_i = F$ (mod $\mathcal{H}^2$), hence, F is indecomposable. By Lemma 4, F is also convex.

If F were not equivalent to $\mathrm{co}(E^1)$, the intersection $F \cap \mathrm{co}(E^1)$ would be a convex set containing E (mod $\mathcal{H}^2$) with measure strictly smaller than the measure of F and therefore, by Lemma 2.4 in [3], we would have $P(F \cap \mathrm{co}(E^1)) < P(F)$. This contradiction proves that $F = \mathrm{co}(E^1)$. □

Let us now prove Theorem 1.

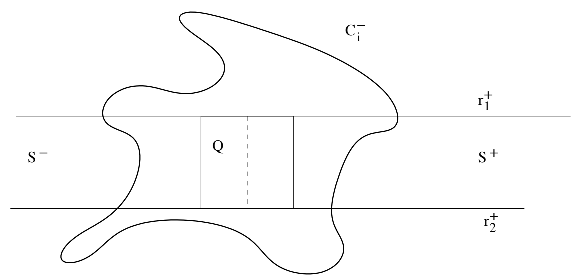

FIGURE 1

Proof of Theorem 1. As before, given any ball B such that $E \subset\subset B$, it is enough to show that $\mathrm{co}(E^1)$ is the unique minimizer of problem

$$\inf\{P(F) : F \supset E \ (\mathrm{mod}\ \mathcal{H}^2), F \text{ indecomposable and } F \subset B\}. \tag{6}$$

Let us consider a minimizing sequence $\{F_n\}_{n\in\mathbb{N}}$ for problem (6). By Proposition 5, we may assume that each F_n is convex and that the sequence χ_{F_n} converges almost everywhere. Setting

$$F := \{x \in \mathbb{R}^2 : \chi_{F_n}(x) \to 1\},$$

clearly F is a convex set and the same argument used in the proof of Proposition 5 shows that $\mathrm{co}(E^1) = F$. □

Let us now show that, if E is indecomposable, $\mathrm{co}(E^1)$ coincides (in the usual pointwise sense) with the convex envelope of $\mathrm{int}(C^+)$.

Theorem 6. *Let E be an indecomposable bounded set of $\mathbb{R}^2$ such that*

$$E = int(C^+) \setminus \bigcup_i int(C_i^-) \ (\mathrm{mod}\ \mathcal{H}^2),$$

where C^+, C_i^- are as in Theorem 3. Then,

$$\mathrm{co}(E^1) = \mathrm{co}(\mathrm{int}(C^+)).$$

Proof. Notice that $E^1 \subset \mathrm{co}(\mathrm{int}(C^+))$, hence, $\mathrm{co}(E^1) \subset \mathrm{co}(\mathrm{int}(C^+))$. In fact, if $x \in E^1 \setminus \mathrm{int}(C^+)$, there exists a ball B centered in x such that

$$|B \cap \mathrm{int}(C^+)| > \frac{1}{2}|B|.$$

Assuming without loss of generality that $x = 0$ and denoting by r the radius of B,

$$|B \cap \mathrm{int}(C^+)| = \int_0^\pi d\theta \int_0^r \rho[\chi_{\mathrm{int}(C^+)}(\theta,\rho) + \chi_{\mathrm{int}(C^+)}(\theta+\pi,\rho)]d\rho > \frac{\pi r^2}{2}.$$

Therefore there exists $\theta \in (0,\pi)$ such that

$$\int_0^r \rho[\chi_{\mathrm{int}(C^+)}(\theta,\rho) + \chi_{\mathrm{int}(C^+)}(\theta+\pi,\rho)]d\rho > r^2/2.$$

Similarly, there exists a value of $\rho > 0$ such that

$$\chi_{\mathrm{int}(C^+)}(\theta,\rho) + \chi_{\mathrm{int}(C^+)}(\theta+\pi,\rho) > 1.$$

This proves that there exists at least one diameter of the ball containing a point $y_1 \in \mathrm{int}(C^+)$ on one side with respect to x and a point $y_2 \in \mathrm{int}(C^+)$ on the other side. Thus $x \in [y_1, y_2] \subset \mathrm{co}(\mathrm{int}(C^+))$.

Denote by F the closure of $\mathrm{co}(E^1)$. We claim that

$$|\mathrm{int}(C_i^-) \setminus F| = 0, \text{ for every } i \in I. \tag{7}$$

Indeed, if this is false for some i, since $\mathrm{int}(C_i^-) \setminus F$ is an open set, there exists a closed square $Q \subset \mathrm{int}(C_i^-) \setminus F$ with sides parallel to the coordinate axes and length l. Let us denote by S^+ and S^- the two open strips shown in Figure 1.

Since the square Q is contained in $\mathrm{int}(C^+)$, all the half lines r^+ parallel to r_1^+, r_2^+ and contained in S^+ intersect C^+ in at least one point. Therefore, we can easily conclude that $\mathcal{H}^1(C^+ \cap S^+) \geq l$. Since $C^+ \subset \partial^M E$ (mod $\mathcal{H}^1$) and E has density $1/2$ at $\mathcal{H}^1$-a.e. point of $\partial^M E$, we may conclude that $|E^1 \cap S^+| > 0$. Similarly, one proves that $|E^1 \cap S^-| > 0$ and thus, there exists a point in $Q \cap F$. This contradiction proves (7).

Therefore, we may conclude that $F \supset \mathrm{int}(C^+)$ (mod $\mathcal{H}^2$), hence, since $\mathrm{int}(C^+)$ is open and F is closed, the inclusion $F \supset \mathrm{int}(C^+)$ is also true in the usual pointwise sense. Thus F contains $\mathrm{co}(\mathrm{int}(C^+))$ and we have

$$\mathrm{co}(E^1) \subset \mathrm{co}(\mathrm{int}(C^+)) \subset \overline{\mathrm{co}(E^1)}.$$

Since $\mathrm{co}(\mathrm{int}(C^+))$ is open, we conlude that $\mathrm{co}(E^1) = \mathrm{co}(\mathrm{int}(C^+))$. □

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