Attractors for gradient flows of non convex functionals and applications

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Abstract

Abstract. This paper addresses the long-time behaviour of gradient flows of non convex functionals in Hilbert spaces. Exploiting the notion of *generalized semiflows* by J. M. Ball, we provide some sufficient conditions for the existence of a global attractor. The abstract results are applied to various classes of non convex evolution problems. In particular, we discuss the long-time behaviour of solutions of quasi-stationary phase field models and prove the existence of a global attractor.

Key words: gradient flow, generalized semiflow, global attractor, Young measures, quasi-stationary phase field models.

AMS Subject Classification: 37L30, 35K55, 80A22.

1 Introduction

The aim of this paper is to address the long-time behaviour of strong solutions of the gradient flow equation

$$u' + \partial_s\phi(u) \ni 0 \quad \text{a.e. in} \quad (0,+\infty), \quad u(0) = u_0, \tag{GF}$$

associated with the *(strong) limiting subdifferential* $\partial_s\phi : \mathscr{H} \to 2^{\mathscr{H}}$ of a functional

$$\phi : \mathscr{H} \to (-\infty,+\infty] \quad \text{proper and lower semicontinuous}, \tag{1.1}$$

possibly non convex, defined on a separable Hilbert space $\mathscr{H}$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. The strong limiting subdifferential $\partial_s\phi$ of ϕ is a suitably generalized gradient notion (see below), related to the sequential strong closure in $\mathscr{H} \times \mathscr{H}$ of the graph of the *Fréchet subdifferential* $\partial\phi$ of ϕ. The latter is defined, letting $D(\phi) := \{u \in \mathscr{H} : \phi(u) < +\infty\}$, as

$$\xi \in \partial\phi(v) \quad \text{iff} \quad v \in D(\phi), \ \ \liminf_{w\to v} \frac{\phi(w)-\phi(v)-\langle\xi, w-v\rangle}{|w-v|} \geq 0. \tag{1.2}$$

Existence and approximation results for (GF) have been obtained in [29, 30] for the non-autonomous situation $u' + \partial_s\phi(u) \ni f$, where $f \in L^2_{loc}(0,+\infty;\mathscr{H})$ and an initial datum $u_0 \in D(\phi)$ is given. The arguments of [30] are based on the theory of *Minimizing Movements* [1, 17] and of *Curves of Maximal Slope* [2, 14, 18, 24], as well as on Young measures in Hilbert spaces. A remarkable result of [30] is that solutions of (GF) fulfil the *energy identity*

$$\phi(u(t)) + \int_s^t |u'(r)|^2\,dr = \phi(u(s)) \quad \forall 0 \leq s \leq t < +\infty. \tag{1.3}$$

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The main issue of this paper is to show that, under suitable assumptions, the set of solutions of (GF) admits a *global attractor*. Equation (1.3) entails that the functional ϕ decreases along trajectories. Hence, we shall focus our attention on the metric phase space (X, d_X) given by

$$X := D(\phi), \quad d_X(u,v) := |u-v| + |\phi(u) - \phi(v)| \ \ \forall u, v \in X.$$

Indeed, we define this phase space in terms of the functional ϕ, which turns out to be a *Lyapunov function* for the system (see [28, 35] for some analogous choices).

Due to the possible non convexity of the functional ϕ, uniqueness for (GF) may genuinely fail. Hence, (GF) does not generate a semigroup, and we cannot rely on the well-established theory of [39] for the study of the long-term dynamics of the solutions. In recent years, several approaches have been developed in order to address the asymptotic behaviour of solutions of differential problems without uniqueness. Without any claim of completeness, we may refer the reader to, e.g., the results by Sell [33, 34], Chepyzhov & Vishik [15], Melnik & Valero [26], to the survey by Caraballo, Marín-Rubio & Robinson [13], and to the work of J. M. Ball [5, 6].

In particular, we will focus here on the theory of *generalized semiflows* proposed in [5]. A generalized semiflow is a family of functions on $[0, +\infty)$ taking values in the phase space and complying with suitable existence, stability for time translation, concatenation, and upper semicontinuity axioms (see Section 2.2). Within this setting, it is possible to introduce a suitable notion of global attractor and to characterize the existence of such an attractor in terms of boundedness and compactness properties.

The main results of this paper state that, under suitable assumptions, the solution set to (GF) is a generalized semiflow in the space (X, d_X) (Theorem 1), and that it possesses a global attractor (Theorem 2). The key point in our proofs involves passing to the limit in the energy identity (1.3) by means of compactness results for Young measures in Hilbert spaces.

A large part of the paper is devoted to a discussion on the applications of the aforementioned abstract results to evolution problems with a gradient flow structure. First of all, we show the existence of a global attractor in the case of ϕ being a suitable perturbation of a convex functional. In fact, our results apply to C^1 perturbations as well as to dominated concave perturbations of convex functionals (see Section 4 below).

Secondly, we investigate the long-time behaviour of a class of solutions of the so-called *quasi-stationary phase field* system

$$\begin{cases} \partial_t(\vartheta + \chi) - \Delta\vartheta = 0, \\ F'(\chi) = \vartheta, \end{cases} \tag{1.4}$$

in $\Omega \times (0, +\infty)$, where Ω is a bounded domain and F' is the Gâteaux derivative of a functional F, (possibly neither smooth nor convex). The model (1.4) arises as a suitable generalization of the (formal) quasi-stationary asymptotics of the standard parabolic phase field model [12], which describes the phase transition in an ice-water system. In this connection, ϑ is the relative temperature of the system, while the order parameter χ yields the local proportion of the liquid versus the solid phase. The usual choice for F is

$$F(\chi) := \frac{1}{2}\int_\Omega |\nabla\chi|^2 dx + \frac{1}{4}\int_\Omega (\chi^2 - 1)^2 dx. \tag{1.5}$$

The existence of solutions of some initial and boundary value problem for (1.4) with F as in (1.5) was firstly proved by Plotnikov & Starovoitov in [27]. The latter paper addresses the case of homogeneous Dirichlet conditions on ϑ and homogeneous Neumann conditions on χ, and exploits a compactness method and a non standard unique continuation result. Let us mention that the latter technique heavily relies on the precise form of (1.5) and cannot be easily extended to a more general situation. A second result in the direction of the existence of a solution of (1.4)-(1.5) is due to Schätzle [32]. The argument devised in [32] for proving existence for (1.4)-(1.5), supplemented with homogeneous Neumann-Neumann boundary conditions on both ϑ and χ, exploits some spectral analysis results and the analyticity of $\chi \mapsto (\chi^2-1)^2/4$. Once again, this technique is especially tailored to the form of (1.5) and cannot be reproduced for general functionals F. We may observe (see, e.g., [41]) that, indeed, (1.4) stems as the formal gradient entropy flow for the phase field system.

The latter gradient flow approach to the problem of existence of solutions of (1.4) has been fully considered in detail by Rossi & Savaré in [29, 30]. In particular, the existence results in [29, 30] provide a unified frame and extend the previous aforementioned contributions on existence results for quasi-stationary phase fields.

The gradient flow structure of (1.4) is enlightened by introducing the variable $u := \vartheta + \chi$. Following [30], one can rigorously prove that (1.4), along with the boundary conditions $u - \chi = \partial_n \chi = 0$ on $\partial\Omega$ for instance, may be interpreted as the gradient flow equation in the Hilbert space $H^{-1}(\Omega)$ of the functional $\phi : H^{-1}(\Omega) \to (-\infty, +\infty]$ defined by

$$\phi(u) := \inf_{\chi \in H^1(\Omega)} \left(\frac{1}{2} \int_\Omega |u - \chi|^2 \, dx + F(\chi) \right), \quad D(\phi) := L^2(\Omega). \tag{1.6}$$

Namely, in [30] it has been shown that the solutions of (GF), with the choice (1.6) for ϕ, provide a family of solutions of (1.4), supplemented with homogeneous Dirichlet-Neumann conditions.

In Section 5.2 we show that the solutions of (1.4) arising from the gradient flow of the functional ϕ (1.6) indeed form a generalized semiflow, which admits a global attractor. Let us stress that this gradient flow approach does not provide the description of the long-term behaviour of the *whole* set of solutions of (1.4), but is rather concerned with a proper subclass of solutions. Moreover, we present some result on the long-time behaviour of solutions in the weakly coercive case of Neumann-Neumann boundary conditions. The latter situation is more delicate, since (1.4) fails to have a gradient flow structure. However, the existence of solutions may be deduced by suitably approximating the system by means of more regular problems of gradient flow type. The latter approximation procedure has been in fact detailed in [30] and is here reconsidered from the point of view of the long-time dynamics. In particular, in the weakly coercive case, the set of solutions of (1.4) obtained as mentioned above fails to be a generalized semiflow. Nevertheless, by slightly extending Ball's theory (see Section 2.2), in Section 5.2.2 we prove the existence of a suitable notion of weak global attractor for the weakly coercive problem as well. Indeed, denoting by $\mathcal{A}_\lambda$ for $\lambda \in (0, 1)$ the family of global attractors for the approximate problems and by $\mathcal{A}$ the weak global attractor for the (weakly coercive) limit problem, we also prove in Section 5.2.3 the convergence of $\mathcal{A}_\lambda$ to $\mathcal{A}$, as the approximation parameter $\lambda \downarrow 0$, with respect to a suitable Hausdorff semidistance.

Plan of the paper. We present some introductory material in Section 2. In particular, Section 2.1 concerns the existence of solutions of (GF) and reports a result from [30], while in Section 2.2 we recall some results on Ball's theory on generalized semiflows and develop additional material, in the direction of studying a weak semiflow structure and a weak notion of attractor. Section 3 contains the statement and the proof of our main abstract results (Theorems 1 and 2). The ensuing Sections 4-5 are devoted to applications. In particular, Section 4 deals with the long-time behaviour of solutions of gradient flows of suitably perturbed convex functionals. We consider both the case of C^1 perturbations and that of (suitably dominated) concave perturbations. Moreover, some PDE examples are provided within these classes of problems. Section 5 is focused on the long-time behaviour of solutions of the quasi-stationary phase field model (1.4). Since our approach to the long-time behaviour of (1.4) is substantially based on the gradient flow strategy developed in [30], we shall briefly recall the techniques and results of the latter paper in Section 5. Then, Theorems 1 and 2 are applied to the quasi-stationary problem (1.4) in Section 5.2.

Acknowledgment. The authors would like to thank Prof. Giuseppe Savaré for some valuable and inspiring conversations.

2 Preliminary results

2.1 Existence for gradient flows of non convex functionals

In this section we gain some insight into an existence result for (GF) that has been obtained in [30]. To this aim, let us start by reviewing the results on gradient flows in the convex case. Given $T > 0$

and $f : (0,T) \to \mathscr{H}$, we consider the problem

$$u'(t) + \partial\phi(u(t)) \ni f(t) \quad \text{for a.e. } t \in (0,T), \quad u(0) = u_0. \tag{2.1}$$

When ϕ is a *convex* functional, the Fréchet subdifferential of ϕ coincides with the subdifferential $\partial\phi$ of ϕ in the sense of Convex Analysis (so we shall use the same notation for both subdifferential notions). The latter is defined by

$$\xi \in \partial\phi(v) \quad \text{iff} \quad v \in D(\phi), \quad \phi(w) - \phi(v) - \langle \xi, w - v \rangle \geq 0 \quad \forall\, w \in \mathscr{H}. \tag{2.2}$$

The literature on existence, uniqueness, regularity, and approximation of solutions of (2.1) is well-established and dates back to the early 70s (see the seminal references [8, 9, 16, 23]). In particular, it is well-known that, if $u_0 \in D(\phi)$ and $f \in L^2(0,T;\mathscr{H})$, then the Cauchy problem (2.1) admits a unique solution $u \in H^1(0,T;\mathscr{H})$, which complies with the energy identity

$$\phi(u(t)) + \int_s^t |u'(r)|^2\, dr = \phi(u(s)) + \int_s^t \langle f(r), u'(r) \rangle dr \quad \forall 0 \leq s \leq t \leq T. \tag{2.3}$$

Indeed, relation (2.3) follows from the *chain rule* property of convex subdifferentials, i.e.,

$$\begin{aligned} &\text{if } \ u \in H^1(0,T;\mathscr{H}),\ \xi \in L^2(0,T;\mathscr{H}),\ \xi(t) \in \partial\phi(u(t)) \text{ for a.e. } t \in (0,T), \\ &\text{then } \ \phi \circ u \in AC(0,T), \quad \tfrac{d}{dt}\phi(u(t)) = \langle \xi(t), u'(t) \rangle \text{ for a.e. } t \in (0,T). \end{aligned} \tag{2.4}$$

In fact, the *strong-weak closure* of $\partial\phi$ in the sense of graphs, i.e.,

$$u_n \to u,\ \xi_n \rightharpoonup \xi \text{ in } \mathscr{H},\ \xi_n \in \partial\phi(u_n)\ \forall n \quad \Rightarrow \quad \phi(u_n) \to \phi(u),\ \xi \in \partial\phi(u), \tag{2.5}$$

the elementary continuity property

$$u_n \to u, \quad \sup_n |\partial\phi^\circ(u_n)| < +\infty \quad \Rightarrow \phi(u_n) \to \phi(u), \tag{2.6}$$

(where we use the notation $|A^\circ| := \inf_{a\in A} |a|$ for all non-empty sets $A \subset \mathscr{H}$), and the chain rule (2.4) play a crucial role in the proof of the existence of solutions of (2.1). Furthermore, the long-time behaviour of (2.1) from the point of view of the theory of universal attractors, see e.g. Temam [39], is quite well-understood, even in the non autonomous case (see also [37]).

Let us now turn to the case of a proper, lower semicontinuous, and non convex functional ϕ, cf. (1.1). One shall observe that, even in the non convex case, for any $u \in D(\partial\phi)$ the Fréchet subdifferential $\partial\phi(u)$ is a convex subset of $\mathscr{H}$. On the other hand, the elementary example $\phi(x) := \min\{(x-1)^2, (x+1)^2\}$ (with $\partial\phi(x) := 2(x+1)$ for $x < 0$, $\partial\phi(x) := 2(x+1)$ for $x > 0$, but $\partial\phi(0) = \emptyset$) shows that, unlike the convex case (see (2.5)), the graph of the Fréchet subdifferential of a non convex functional may not be strongly-weakly closed.

Therefore, following [30], we define the *strong limiting subdifferential* $\partial_s\phi$ of ϕ at a point $v \in D(\phi)$ as the set of the vectors ξ such that there exist sequences

$$v_n, \xi_n \in \mathscr{H} \quad \text{with} \quad \xi_n \in \partial\phi(v_n),\ v_n \to v,\ \xi_n \to \xi,\ \phi(v_n) \to \phi(v), \tag{2.7}$$

as $n \to +\infty$. Furthermore, we define the *weak limiting subdifferential* $\partial_\ell\phi$ of ϕ at $v \in D(\phi)$ as the set of all vectors ξ such that there exist sequences

$$v_n, \xi_n \in \mathscr{H} \quad \text{with} \quad \xi_n \in \partial\phi(v_n),\ v_n \to v,\ \xi_n \rightharpoonup \xi,\ \sup_n \phi(v_n) < +\infty. \tag{2.8}$$

Of course, $\partial_\ell\phi$ and $\partial_s\phi$ reduce to the subdifferential $\partial\phi$ of ϕ in the sense of Convex Analysis whenever ϕ is convex, due to (2.5) and (2.6).

Note that the strong limiting subdifferential $\partial_s\phi$ of ϕ fulfils this closure property:

$$\begin{gathered} \forall\, \{u_k\}, \{\xi_k\} \text{ such that } \ u_k \to u,\ \xi_k \to \xi,\ \phi(u_k) \to \phi(u), \text{ as } k \uparrow +\infty, \quad \xi_k \in \partial_s\phi(u_k)\ \forall k \in \mathbb{N}, \\ \text{then } \xi \in \partial_s\phi(u). \end{gathered} \tag{2.9}$$

Instead, $\partial_\ell\phi$ is not strongly-weakly closed in the sense of graphs. Actually, $\partial_\ell\phi$ can be characterized as a version of the *strong-weak* closure of $\partial_s\phi$, as the following result shows.

Lemma 2.1. *Let $\phi : \mathscr{H} \to (-\infty, +\infty]$ comply with* (1.1)*. Then, for any $u \in \mathscr{H}$*

$$\xi \in \partial_\ell \phi(u) \iff \exists \{u_k\}, \{\xi_k\} \subset \mathscr{H} : \ u_k \to u, \ \xi_k \rightharpoonup \xi, \ \sup_k \phi(u_k) < +\infty, \ \xi_k \in \partial_s \phi(u_k) \, \forall k \in \mathbb{N} \quad (2.10)$$

i.e., $\partial_\ell \phi$ coincides with the (sequential) strong-weak closure of $\partial_s \phi$ along sequences with bounded energy.

Proof. The left-to-right implication in (2.10) follows immediately from the definition of $\partial_\ell \phi$, noting that $\partial \phi(u) \subset \partial_s \phi(u)$ for any $u \in \mathscr{H}$. In order to prove the converse implication, we recall that in separable Hilbert spaces (more in general, in reflexive spaces and dual of separable spaces, cf. [10, Chap. 3]), it is possible to introduce a norm $|||\cdot|||$, and thus a metric, inducing weak convergence on every bounded set. Thus, let us fix a sequence $\{(u_k, \xi_k)\}$ as in (2.10): necessarily, there exists $M \geq 0$ such that $|\xi_k| \leq M$, and $\xi_k \rightharpoonup \xi$ may be rephrased as $|||\xi_k - \xi||| \to 0$. In order to prove that the limit pair (u, ξ) fulfils $\xi \in \partial_\ell \phi(u)$, we are going to construct by a diagonalization procedure a sequence $\{(v_k, \omega_k)\} \subset \mathscr{H} \times \mathscr{H}$ such that

$$v_k \to u, \ \omega_k \rightharpoonup \xi \text{ as } k \uparrow +\infty, \quad \sup_k \phi(v_k) < +\infty, \quad \omega_k \in \partial \phi(v_k) \ \forall k \in \mathbb{N}. \quad (2.11)$$

Note that the relation $\xi_k \in \partial_s \phi(u_k)$ for all $k \in \mathbb{N}$ can be rephrased in the following way: for any $k \in \mathbb{N}$ there exist sequences $\{u_n^k\}$, $\{\xi_n^k\}, \subset \mathscr{H}$ with

$$u_n^k \to u_k, \ \xi_n^k \to \xi_k, \ \phi(u_n^k) \to \phi(u_k) \quad \text{as } n \uparrow +\infty, \ \text{ and } \quad \xi_n^k \in \partial \phi(u_n^k) \ \forall n \in N.$$

In particular, for any $k \in \mathbb{N}$ we may find $n(k) \in \mathbb{N}$ such that

$$|u_{n(k)}^k - u_k| + |\xi_{n(k)}^k - \xi_k| + |\phi(u_{n(k)}^k) - \phi(u_k)| \leq \frac{1}{k}.$$

Then, let us set $v_k := u_{n(k)}^k$ and $\omega_k := \xi_{n(k)}^k$. Obviously, $\omega_k \in \partial \phi(v_k)$, $v_k \to u$ as $k \uparrow +\infty$, and $\sup_k \phi(v_k) < +\infty$. On the other hand, we remark that the sequence $\{\omega_k\}$ lies in a bounded set of $\mathscr{H}$, since for all $k \in \mathbb{N}$ $|\omega_k| \leq |\omega_k - \xi_k| + |\xi_k| \leq 1 + M$. Therefore, (2.11) follows by noting that

$$|||\omega_k - \xi||| \leq |||\omega_k - \xi_k||| + |||\xi_k - \xi||| \leq \sqrt{2}|\omega_k - \xi_k| + |||\xi_k - \xi||| \to 0 \quad \text{as } k \uparrow +\infty. \qquad \square$$

Under the assumption that $\partial_\ell \phi$ satisfies a chain rule property analogous to the chain rule (2.4) of the subdifferential of Convex Analysis, in [30] existence and approximation results have been obtained for (GF), supplemented with some initial datum $u_0 \in D(\phi)$ and source term f. Let us now recall one of the existence results proved in [30].

Theorem 2.2. *Suppose that $\phi : \mathscr{H} \to (-\infty, +\infty]$ complies with* (1.1)*, with the* coercivity *assumption*

$$\exists \, \kappa \geq 0 : \quad v \mapsto \phi(v) + \kappa |v|^2 \quad \textit{has compact sublevels,} \quad (\text{COMP})$$

and with the chain rule *condition*

$$\begin{gathered} \text{if } v \in H^1(a, b; \mathscr{H}), \ \xi \in L^2(a, b; \mathscr{H}), \ \xi \in \partial_\ell \phi(v) \ \textit{a.e. in } (a, b), \\ \textit{and } \phi \circ v \textit{ is bounded, then } \phi \circ v \in AC(a, b) \textit{ and} \\ \frac{d}{dt} \phi(v(t)) = \langle \xi(t), v'(t) \rangle \quad \textit{for a.e.} \quad t \in (a, b). \end{gathered} \quad (\text{CHAIN})$$

Then, for any $u_0 \in D(\phi)$, $T > 0$ and $f \in L^2(0, T; \mathscr{H})$ the Cauchy problem

$$u'(t) + \partial_s \phi(u(t)) \ni f(t) \quad \textit{a.e. in } (0, T), \quad u(0) = u_0,$$

admits a solution $u \in H^1(0, T; \mathscr{H})$. Moreover, one has the energy identity

$$\int_s^t |u'(\sigma)|^2 \, d\sigma + \phi(u(t)) = \phi(u(s)) + \int_s^t \langle f(\sigma), u'(\sigma) \rangle \, d\sigma \quad \forall 0 \leq s \leq t \leq T. \quad (2.12)$$

The chain rule (CHAIN), which is indeed classical in the convex case (2.4), holds true in a variety of non convex situations as well. First of all, (CHAIN) is fulfilled by C^1 perturbations of convex functionals. In particular, letting $\phi = \phi_1 + \phi_2$, where ϕ_1 is convex and ϕ_2 is C^1, and exploiting Lemma 4.2, one readily checks that $\partial_\ell \phi = \partial\phi_1 + D\phi_2$ and (CHAIN) follows. A second class of functionals complying with the chain rule (CHAIN) is provided by dominated concave perturbations of (convex) functionals. In particular, in [30, Thm. 4] it is proved that (CHAIN) holds for all proper, lower semicontinuous functionals ϕ admitting the decomposition

$$\begin{gathered}\phi = \psi_1 - \psi_2 \text{ in } D(\phi), \quad \text{with} \quad \psi_1 : D(\phi) \to \mathbb{R} \text{ l.s.c. and satisfying (CHAIN)}, \\ \psi_2 : \mathrm{co}\left(D\phi\right) \to \mathbb{R} \text{ convex and l.s.c. in } D(\phi), \quad D(\partial_\ell\psi_1) \subset D(\partial\psi_2),\end{gathered} \tag{2.13}$$

(where $\mathrm{co}\left(D\phi\right)$ denotes the convex hull of $D(\phi)$), and fulfilling

$$\begin{gathered}\forall M \geq 0 \quad \exists\, \rho < 1,\, \gamma \geq 0 \quad \text{such that} \quad \sup_{\xi_2 \in \partial\psi_2(u)} |\xi_2| \leq \rho |(\partial_\ell \psi_1(u))^\circ| + \gamma \\ \text{for every } u \in D(\partial_\ell\psi_1) \text{ with } \max(\phi(u), |u|) \leq M.\end{gathered} \tag{2.14}$$

Namely, if ψ_1 is itself convex, we are requiring the domain of $\partial\psi_1$ to be included in $D(\partial\psi_2)$. This in fact implies that $\partial\psi_1$ somehow dominates $\partial\psi_2$.

2.2 Generalized semiflows

For the reader's convenience, we recall here the main definitions and results on the theory of *attractors* for *generalized semiflows*, closely following [5]. Our final aim is to apply Ball's theory to the Cauchy problem (GF), and slightly extend it in view of applications.

Notation. Let (X, d_X) be a (not necessarily complete) metric space. We recall that the *Hausdorff semidistance* or *excess* $e(A, B)$ of two non-empty subsets $A, B \subset X$ is given by $e(A, B) := \sup_{a\in A} \inf_{b\in B} d_X(a, b)$. For all $\varepsilon > 0$, we also denote by $B(0, \varepsilon)$ the ball $B(0, \varepsilon) := \{x \in X \,:\, d_X(x, 0) < \varepsilon\}$, and by $N_\varepsilon(A) := A + B(0, \varepsilon)$ the ε-neighborhood of a subset A.

Definition 2.3 (Generalized semiflow)**.** *A* generalized semiflow *G on X is a family of maps $g : [0, +\infty) \to X$ (referred to as "solutions"), satisfying:*

(H1) (Existence) *for any $g_0 \in X$ there exists at least one $g \in G$ with $g(0) = g_0$,*

(H2) (Translates of solutions are solutions) *for any $g \in G$ and $\tau \geq 0$, the map $g^\tau(t) := g(t+\tau)$, $t \in [0, +\infty)$, belongs to G,*

(H3) (Concatenation) *for any g, $h \in G$ and $t \geq 0$ with $h(0) = g(t)$, then $z \in G$, z being the map defined by $z(\tau) := g(\tau)$ if $0 \leq \tau \leq t$, and $h(\tau - t)$ if $t < \tau$.*

(H4) (Upper-semicontinuity w.r.t. initial data) *If $\{g_n\} \subset G$ and $g_n(0) \to g_0$, then there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in G$ such that $g(0) = g_0$ and $g_{n_k}(t) \to g(t)$ for all $t \geq 0$.*

The application of the theory of generalized semiflows to suitable classes of differential problems is often delicate. Indeed, one usually needs to choose carefully the correct notion of solution of the problem in order to check the validity of the properties (H1)-(H4). This process may not be straightforward whenever one considers some suitably weak notion of solvability. On the one hand, solutions have indeed to be weak enough to comply with (H1) (assumption (H2) is generally easy to meet in actual situations). On the other hand, the notion of solution has to be *robust* enough in order to fulfil (H4). This robustness may turn out to be in conflict with (H3). For instance, this may occur when the existence of weak solutions of a differential problem is proved by approximation (like e.g. for the solutions of the quasi-stationary phase field Problem 5.1 in the weakly coercive case, cf. Theorem 5.5). Then, one is naturally led to define the candidate semiflow as the set of all solutions which are limits in a suitable topology of sequences of approximate solutions. Axioms (H1) and (H2) will be trivially checked, and, if the aforementioned topology is strong enough, one can hopefully verify (H4) as well. However, due to this approximation procedure, the concatenation in (H3) may not hold (the approximating sequences may not have the same indices). This is particularly the case of the set of *limiting energy solutions* of Problem 5.1 in the weakly coercive case (cf. Definition 5.10).

Therefore, in the setting of the phase space (X, d_X) we aim at partially extending the standard theory of generalized semiflows to the case of a non-empty set $\mathcal{G}$ of functions $g : [0, +\infty) \to X$, complying with (H1), (H2), (H4), but not necessarily with (H3). In this framework, we shall introduce a weakened notion of attractor, for objects which are slightly more general than semiflows. Before moving on, let us explicitly stress that we do not claim originality for the notion of *weak generalized semiflow* we present below. Indeed, the possibility of studying the long-time dynamics of differential systems by considering (multivalued) solution operators fulfilling (2.16) has been recently considered in [25, 26]. In particular, this multivalued approach has also been applied to the standard phase field system by Kapustyan, Melnik & Valero [22].

Definition 2.4 (Weak generalized semiflow). *We say that a non-empty family $\mathcal{G}$ of maps $g : [0, +\infty) \to X$ is a* weak generalized semiflow *on X if $\mathcal{G}$ complies with the properties* (H1), (H2), *and* (H4).

Continuity property (C4). We say that a (weak) generalized semiflow fulfills $(C4)$ if for any $\{g_n\} \subset \mathcal{G}$ with $g_n(0) \to g_0$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in \mathcal{G}$ such that $g(0) = g_0$ and $g_{n_k} \to g$ uniformly on the compact subsets of $[0, +\infty)$,
Orbits, ω-limits, and attractors. Given a weak generalized semiflow $\mathcal{G}$ on X, we may introduce for every $t \geq 0$ the operator $T(t) : 2^X \to 2^X$ defined by

$$T(t)E := \{g(t) \ : \ g \in \mathcal{G} \ \text{ with } \ g(0) \in E\}, \quad E \subset X. \tag{2.15}$$

The family of operators $\{T(t)\}_{t \geq 0}$ fulfils the following property

$$T(t+s)B \subset T(t)T(s)B \quad \forall s, t \geq 0 \quad \forall\, B \subset X, \tag{2.16}$$

and in general does not define a *semigroup* on the power set 2^X. Note that (2.16) improves to a semigroup relation when $\mathcal{G}$ is a *generalized semiflow.* Given a solution $g \in \mathcal{G}$, we introduce the *positive orbit of g* as the set $\gamma^+(g) := \{g(t) \ : \ t \geq 0\}$, while its *$\omega$-limit* $\omega(g)$ is defined by

$$\omega(g) := \{x \in X \ : \ \exists\{t_n\}, \ t_n \to +\infty, \ \text{such that} \ \ g(t_n) \to x\}.$$

We say that $w : \mathbb{R} \to X$ is a *complete orbit* if, for any $s \in \mathbb{R}$, the translate map $w^s \in \mathcal{G}$ (cf. (H2)). Moreover, we may consider the *positive orbit of a subset* $E \subset X$, i.e. the set $\gamma^+(E) := \cup_{t \geq 0} T(t)E = \cup\{\gamma^+(g) \ : \ g \in \mathcal{G}, \ g(0) \in E\}$, and, for every $\tau \geq 0$, we define $\gamma^\tau(E) := \cup_{t \geq \tau} T(t)E = \gamma^+(T(\tau)E)$. Finally, the *$\omega$-limit* of E is defined as

$$\begin{aligned} \omega(E) := \big\{x \in X \ : \ \exists\{g_n\} \subset \mathcal{G} \text{ such that } \ \{g_n(0)\} \subset E, \\ \{g_n(0)\} \ \text{is bounded, and} \ \ \exists t_n \to +\infty \text{ with } \ g_n(t_n) \to x\big\}. \end{aligned}$$

Given subsets $U, E \subset X$, we say that U *attracts* E if $e(T(t)E, U) \to 0$ as $t \to +\infty$. Further, we say that U is *positively invariant* if $T(t)U \subset U$ for every $t \geq 0$, that U is *quasi-invariant* if for any $v \in U$ there exists a complete orbit w with $w(0) = v$ and $w(t) \in U$ for all $t \in \mathbb{R}$, and finally that U is *invariant* if $T(t)U = U$ for every $t \geq 0$ (equivalently, if it is both positively and quasi-invariant).

Definition 2.5 (Weak Global Attractor and Global Attractor.). *Let $\mathcal{G}$ be a* weak generalized semiflow. *We say that a non-empty set $\mathcal{A}$ is a* weak global attractor *for $\mathcal{G}$ if it is compact, quasi-invariant, and attracts all the bounded sets of X. We say that a set $A \subset X$ is a* global attractor *for a generalized semiflow G if A is compact, invariant, and attracts all the bounded sets of X.*

The price of dropping the semigroup property for T consists in the fact that the notion of weak attractor introduced above will be quasi-invariant but will fail to be invariant. Moreover, we may observe that a weak global attractor (if existing), is minimal in the set of the closed subsets of X attracting all bounded sets, hence it is unique, cf. [25].
Compactness and dissipativity properties. Let $\mathcal{G}$ be a *weak generalized semiflow.* We say that $\mathcal{G}$ is

eventually bounded if for every bounded $B \subset X$ there exists $\tau \geq 0$ such that $\gamma^\tau(B)$ is bounded,

point dissipative if there exists a bounded set $B_0 \subset X$ such that for any $g \in \mathcal{G}$ there exists $\tau \geq 0$ such that $g(t) \in B_0$ for all $t \geq \tau$,

compact if for any sequence $\{g_n\} \subset \mathcal{G}$ with $\{g_n(0)\}$ bounded, there exists a subsequence $\{g_{n_k}\}$ such that $\{g_{n_k}(t)\}$ is convergent for any $t > 0$.

The notions we have just introduced are not independent one from each other cf. [5, Prop. 3.1 & 3.2].
Lyapunov function. The notion of *Lyapunov function* can be introduced starting from the following definitions: we say that a complete orbit $g \in \mathcal{G}$ is *stationary* if there exists $x \in X$ such that $g(t) = x$ for all $t \in \mathbb{R}$ - such x is then called a *rest point*. Note that the set of rest points of $\mathcal{G}$, denoted by $Z(\mathcal{G})$, is closed in view of (H4). A function $V : X \to \mathbb{R}$ is said to be a *Lyapunov function* for $\mathcal{G}$ if: V is continuous, $V(g(t)) \leq V(g(s))$ for all $g \in \mathcal{G}$ and $0 \leq s \leq t$ (i.e., V decreases along solutions), and, whenever the map $t \mapsto V(g(t))$ is constant for some complete orbit g, then g is a stationary orbit.

Finally, we say that a global attractor A for G is *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \subset X$ with $e(E, A) \leq \delta$, then $e(T(t)E, A) \leq \varepsilon$ for all $t \geq 0$.
Existence of the global attractor. We recall the main results from Ball [5] (cf. Thms. 3.3, 5.1, and 6.1 therein), which provide criteria for the existence of a global attractor A for a generalized semiflow G. More precisely, Theorem 2.6 gives a characterization of A, whereas Theorem 2.7 states a sufficient condition for the existence of A in the case in which G also admits a Lyapunov function.

Theorem 2.6. *An eventually bounded, point dissipative, and compact generalized semiflow G has a global attractor. Moreover, the attractor A is unique, it is the maximal compact invariant subset of X, and it can be characterized as*

$$A = \cup\{\omega(B) \ : \ B \subset X, \ \textit{bounded}\} = \omega(X). \tag{2.17}$$

Besides, if all elements of G are continuous functions in $(0, +\infty)$ and (C4) *is fulfilled, then A is Lyapunov stable.*

Theorem 2.7. *Assume that G is eventually bounded and compact, admits a Lyapunov function V, and that the sets of its rest points $Z(G)$ is bounded. Then, G is also point dissipative, and thus admits a global attractor A. Moreover, $\omega(u) \subset Z(G)$ for all trajectories $u \in G$.*

Existence of the weak global attractor.

Theorem 2.8. *Let $\mathcal{G}$ be a weak generalized semiflow. Moreover, assume that $\mathcal{G}$ is eventually bounded, point dissipative and compact. Then, $\mathcal{G}$ possesses a unique weak global attractor $\mathcal{A}$. Moreover, $\mathcal{A}$ can be characterized as*

$$\mathcal{A} = \{\xi \in X : \ \textit{there exists a bounded complete orbit } w : \ w(0) = \xi\}\,. \tag{2.18}$$

Clearly, one can replace 0 in formula (2.18) with any $s \in \mathbb{R}$.

Concerning the first part of the statement, it is sufficient to check that the argument developed in [5] for the proof of Theorem 2.6 goes through without the concatenation condition (H3). As for (2.18), the fact that the global attractor is generated by all complete bounded trajectories is well-known for semigroups and semiflows (cf. [39]), and, up to our knowledge, it has been observed in some generalized framework in [15, 19]. Note that this characterization also holds for the global attractors of the standard generalized semiflow constructed in Theorems 2.6 and 2.7. As already mentioned, we shall apply the weak global attractor machinery to a class of differential problems for which the existence of solutions is proved by means of an approximation argument. In this framework, in Section 5.2.3 the structure formula (2.18) will play a basic role in the proof that the sequence of global attractors of the approximate problems converges in a suitable sense to the weak global attractor of the limit problem (see also [36] for an analogous approximation result).

Proof. Arguing as in [5, Thm. 3.3], one has to preliminarily show the following two facts: their proof simply consists in repeating the arguments of [5, Lemmas 3.4, 3.5], which are valid independently of (H3).

Claim 1. *If $\mathcal{G}$ fulfills* (H1), (H2), (H4) *and is asymptotically compact, then for any non-empty and bounded set $B \subset X$, $\omega(B)$ is non-empty, compact, quasi-invariant, and attracts B.*

Claim 2. *If $\mathcal{G}$ fulfills* (H1), (H2), (H4), *it is asymptotically compact and point dissipative, then there exists a bounded set $\mathcal{B}$ such that for any compact set $K \subset X$ there exist $\tau = \tau(K) > 0$ and $\varepsilon = \varepsilon(K) > 0$ with $T(t)(N_\varepsilon(K)) \subset \mathcal{B}$ for all $t \geq \tau(K)$.*

Hence, let us define $\mathcal{A} := \omega(\mathcal{B})$ where $\mathcal{B}$ is exactly the bounded set of Claim 2. Owing to Claim 1, $\mathcal{A}$ is non-empty, compact, quasi-invariant, and attracts $\mathcal{B}$. Let us now fix any bounded set B and consider its compact ω-limit $K := \omega(B)$, which attracts B by Claim 1. Using Claim 2, one readily exploits (2.16) and adapts the proof of [5, Thm. 3.3] in order to infer that $\mathcal{B}$ attracts B as well. Thus, also $\mathcal{A}$ attracts B and, being B arbitrary among bounded sets, we have checked that $\mathcal{A}$ is the weak global attractor.

Let us now prove (2.18). To this aim, we fix $\xi \in \mathcal{A}$. Then, the *quasi-invariance* of $\mathcal{A}$ entails that there exists a complete orbit w such that $w(0) = \xi$ and $w(t) \in \mathcal{A}$ for any t. In particular, w is also bounded since $\mathcal{A}$ is bounded and we shave shown one inclusion in (2.18). To prove the converse inclusion, consider any bounded and complete orbit w in $\mathcal{G}$ and set $\mathcal{O} := \{w(t),\ \ t \in \mathbb{R}\}$. The set $\mathcal{O}$ is clearly bounded in the phase space and quasi-invariant, and the following chain of inclusions holds

$$\mathcal{O} \subset T(t)\mathcal{O} \subset \omega(\mathcal{O}) \subset \mathcal{A}. \tag{2.19}$$

In fact, the first inclusion is due to the quasi-invariance of $\mathcal{O}$, while the second one holds since the ω-limit set of any bounded set attracts the set itself, Finally, the last inclusion follows from (2.17). Thus, we conclude that, for any bounded and complete orbit w of $\mathcal{G}$, $w(0) \in \mathcal{A}$. which clearly implies (2.18). □

Finally, by adapting the proof of [5, Thm. 5.1], one may obtain the analogue of Theorem 2.7 for weak global attractors, namely

Theorem 2.9. *Let $\mathcal{G}$ be an eventually bounded and compact weak generalized semiflow. Moreover, suppose that $\mathcal{G}$ admits a Lyapunov function, and that there exists a non-empty subset $\mathcal{D}$ of X such that*

$$T(t)\mathcal{D} \subset \mathcal{D} \quad \forall t \geq 0, \tag{2.20}$$

$$\textit{the set } Z(\mathcal{G}) \cap \mathcal{D} \textit{ is bounded in } X. \tag{2.21}$$

Then, $\mathcal{G}$ possesses a unique weak global attractor $\mathcal{A}$ in $\mathcal{D}$. Furthermore, for any trajectory $u \in \gamma^+(\mathcal{D})$ we have $\omega(u) \subset Z(\mathcal{G})$ and the weak global attractor $\mathcal{A}$ complies with (2.18).

Indeed, Theorem 2.9 directly corresponds to Theorem 2.7 with the choice $\mathcal{D} = X$. On the other hand, the need for restricting the natural phase space X to a proper subset $\mathcal{D}$ is well motivated by applications and the reader is referred to Section 5.2 for some example in this direction.

3 Main results

In view of the assumption $u_0 \in D(\phi)$ in the existence Theorem 2.2, we are naturally led to work in the phase space

$$X := D(\phi), \quad \text{with} \quad d_X(u,v) := |u - v| + |\phi(u) - \phi(v)| \quad \forall u, v \in X. \tag{3.1}$$

Note that (X, d_X) is not, in general, a *complete* metric space.

For the sake of simplicity, we will assume that $0 \in D(\phi)$ and $\phi(0) = 0$, but it is clear that this assumption is not at all restrictive, since with a proper translation we can deal with the general case in which $0 \notin D(\phi)$. Hence, a subset $B \subset X$ is d_X-bounded iff it is contained in a d_X-ball $B(0, R)$ for some $R > 0$, i.e.

$$|u| + |\phi(u)| \leq R \quad \forall u \in B. \tag{3.2}$$

Definition 3.1. *We denote by $\mathcal{S}$ the set of all functions $u : [0,+\infty) \to \mathscr{H}$ such that $u \in H^1(0,T;\mathscr{H})$ for all $T > 0$ and*

$$u'(t) + \partial_s\phi(u(t)) \ni 0 \quad \textit{for a.e. } t \in (0,+\infty). \tag{3.3}$$

Remark 3.2. We could include a *constant* source term $f \in \mathscr{H}$ in (3.3) by replacing ϕ with the functional ϕ_f defined by $\phi_f(v) := \phi(v) - \langle f, v\rangle$ for all $v \in \mathscr{H}$.

Theorem 1 (Generalized semiflow)**.** *Let ϕ comply with the assumptions* (1.1)*,* (COMP)*, and* (CHAIN) *of Theorem 2.2. In addition, assume that*

$$\exists K_1, K_2 \geq 0 : \quad \phi(u) \geq -K_1|u| - K_2 \quad \forall u \in \mathscr{H}. \tag{3.4}$$

Then, $\mathcal{S}$ is a generalized semiflow on X, whose elements are continuous functions on $[0,+\infty)$ *and comply with* (C4)*.*

In order to study the long-time behaviour of our gradient flow equation, we assume an additional continuity property of the potential ϕ, that is

$$v_n \to v, \quad \sup_n \big(|(\partial_\ell\phi(v_n))^\circ|, \phi(v_n)\big) < +\infty \quad \Rightarrow \phi(v_n) \to \phi(v). \tag{CONT}$$

Note that (CONT) is readily fulfilled by lower semicontinuous convex functionals (cf. (2.6)). Let $\{T(t)\}_{t\geq 0}$ be the family of operators (2.15) associated with the generalized semiflow $\mathcal{S}$. We have

Theorem 2 (Global attractor)**.** *Let ϕ fulfil* (1.1)*,* (COMP)*,* (CHAIN)*,* (CONT)*, and*

$$\liminf_{|u|\to+\infty} \phi(u) = +\infty. \tag{3.5}$$

Further, let $\mathcal{D}$ be a non-empty subset of X satisfying

$$T(t)\mathcal{D} \subset \mathcal{D} \quad \forall t \geq 0, \tag{3.6}$$

$$\textit{the set } Z(\mathcal{S}) \cap \mathcal{D} := \{u \in D(\partial_s\phi) \,:\, 0 \in \partial_s\phi(u)\} \cap \mathcal{D} \textit{ is bounded in } X. \tag{3.7}$$

Then, there exists a unique attractor A for $\mathcal{S}$ in *$\mathcal{D}$, given by*

$$A := \cup \{\omega(D) \,:\, D \subset \mathcal{D} \textit{ bounded}\}.$$

Moreover, A is Lyapunov stable.

With respect to applications, let us stress that assumptions (3.6)-(3.7) are of course to be checked for all current choices of the functional ϕ. In order to fix ideas, let us remark that, in the convex case, (3.7) follows for instance from (3.5).

The fundamental theorem of Young measures for weak topologies. Before developing the proof of Theorems 1, 2, we report a compactness result for Young measures in the framework of the weak topology, which shall play a crucial role in the sequel. Hence, for the reader's convenience let us recall the definition of *(time-dependent) parametrized (or Young) measures.* Denoting by $\mathcal{L}$ the σ-algebra of the Lebesgue measurable subsets of $(0,T)$ and by $\mathscr{B}(\mathscr{H})$ the Borel σ-algebra of $\mathscr{H}$, we define a *parametrized (Young) measure* in $\mathscr{H}$ to be a family $\boldsymbol{\nu} := \{\nu_t\}_{t\in(0,T)}$ of Borel probability measures on $\mathscr{H}$ such that for all $B \in \mathscr{B}(\mathscr{H})$ the map $t \in (0,T) \mapsto \nu_t(B)$ is $\mathcal{L}$-measurable. We denote by $\mathcal{Y}(0,T;\mathscr{H})$ the set of all parametrized measures. The following result has been proved in [30] (cf. Thm. 3.2 therein), as a consequence of the so-called fundamental compactness theorem for Young measures, [3, Thm. 1] (see also [4]).

Theorem 3.3. *Let $\{v_n\}_{n\in\mathbb{N}}$ be a* bounded *sequence in $L^p(I;\mathscr{H})$, for some $p > 1$. Then, there exists a subsequence $k \mapsto v_{n_k}$ and a parametrized measure $\boldsymbol{\nu} = \{\nu_t\}_{t\in I} \in \mathcal{Y}(I;\mathscr{H})$ such that for a.e. $t \in I$*

$$\nu_t \textit{ is concentrated on the set } L(t) \textit{ of the } \text{weak limit points } \textit{of } \{v_{n_k}(t)\}, \tag{3.8}$$

$$\int_I \left(\int_{\mathscr{H}} |\xi|^p \, d\nu_t(\xi) \right) dt \leq \liminf_{k\to\infty} \int_I |v_{n_k}(t)|^p \, dt < +\infty. \tag{3.9}$$

Moreover, setting $v(t) := \int_{\mathscr{H}} \xi \, d\nu_t(\xi)$, *we have*

$$v_{n_k} \rightharpoonup v \text{ in } L^p(I;\mathscr{H}) \text{ if } p < \infty \quad \text{and} \quad v_{n_k} \rightharpoonup^* v \text{ in } L^\infty(I;\mathscr{H}) \text{ if } p = \infty. \tag{3.10}$$

Henceforth, we will denote by C any positive constant coming into play throughout the following proofs, pointing out the occurring exceptions.

3.1 Proof of Theorem 1

It follows from Theorem 2.2 that for any $u_0 \in X$ there exists $u : (0,+\infty) \to \mathscr{H}$ fulfilling $u(0) = u_0$ and (3.3). Moreover, the energy identity

$$\int_0^t |u'(\sigma)|^2 \, d\sigma + \phi(u(t)) = \phi(u_0) \quad \forall t \in [0,+\infty)$$

yields that $u' \in L^2(0,T;\mathscr{H})$ for all $T > 0$, whence $u \in H^1(0,T;\mathscr{H})$ for all $T > 0$: therefore, $\mathcal{S}$ complies with (H1). It is easy to check that $\mathcal{S}$ satisfies (H2) and (H3) as well. Besides, the elements of $\mathcal{S}$ are continuous functions on $[0,\infty)$: in fact, $u \in C^0([0,T];\mathscr{H})$ for all $T > 0$ and, in view of (CHAIN), $\phi \circ u \in AC(0,T)$ for any $T > 0$.

Proof of (H4). Let us fix a sequence $\{u_0^n\} \subset D(\phi)$ converging to $u_0 \in D(\phi)$ w.r.t. the metric of X, i.e.

$$|u_0^n - u_0| + |\phi(u_0^n) - \phi(u_0)| \to 0 \quad \text{as } n \uparrow +\infty, \tag{3.11}$$

and let $u_n \in H^1(0,T;\mathscr{H})$ for all $T > 0$ be the corresponding sequence of solutions in $\mathcal{S}$. We split the proof of (H4) into steps.

A priori estimates on $\{u_n\}$. *For any $T > 0$ there exists a positive constant C_T, only depending on u_0 and T, such that*

$$\|u_n\|_{L^\infty(0,T;\mathscr{H})} + \|u_n'\|_{L^2(0,T;\mathscr{H})} \leq C_T, \tag{3.12}$$

$$\sup_{[0,T]} |\phi(u_n(t))| \leq C_T. \tag{3.13}$$

Indeed, it follows from the energy identity and from (3.11) that

$$\int_0^t |u_n'(\sigma)|^2 \, d\sigma + \phi(u_n(t)) = \phi(u_0^n) \leq C \tag{3.14}$$

for any $n \in N$ and $t \in (0,+\infty)$. On the other hand, for any fixed $T > 0$ and $t \in (0,T]$,

$$\frac{1}{2}\int_0^t |u_n'(\sigma)|^2 \geq \frac{1}{2t}|u_n(t) - u_0^n|^2 \geq \frac{1}{4T}|u_n(t)|^2 - \frac{1}{2T}|u_0^n|^2 \geq K_1|u_n(t)| - TK_1^2 - \frac{1}{2T}|u_0^n|^2.$$

Therefore, (3.14) yields

$$\frac{1}{2}\int_0^t |u_n'(\sigma)|^2 + K_1|u_n(t)| + \phi(u_n(t)) \leq C + \frac{1}{2T}|u_0^n|^2 + TK_1^2,$$

for any $t \in (0,T]$, whence (3.12) (in view of (3.4)), as well as (3.13).

Convergence results for $\{u_n\}$. *There exist a subsequence $\{u_{n_k}\}$, a function $u \in H^1(0,T;\mathscr{H})$ for all $T > 0$, and a limit Young measure $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,+\infty)} \in \mathcal{Y}(0,+\infty;\mathscr{H})$ associated with $\{u_{n_k}'\}$, such that*

$$u_{n_k} \rightharpoonup u \quad \text{in } H^1(0,T;\mathscr{H}) \quad \forall T > 0, \tag{3.15}$$

$$u_{n_k} \to u \quad \text{in } C^0([0,T];\mathscr{H}) \quad \forall T > 0, \tag{3.16}$$

$$\nu_t \text{ is concentrated on } -\partial_\ell\phi(u(t)), \text{ for a.e. } t \in (0,+\infty), \text{ and} \tag{3.17}$$

$$\int_0^T \left(\int_{\mathscr{H}} |\xi|^2 \, d\nu_t(\xi) \right) dt \leq \liminf_{k\to\infty} \int_0^T |u_{n_k}'(t)|^2 \, dt < +\infty \quad \forall T > 0. \tag{3.18}$$

The estimates (3.12), (3.13), and the assumption (COMP) yield that for any fixed $T > 0$ there exists a compact set $\mathscr{K}(T) \subset \mathscr{H}$ such that $\cup_{n\in\mathbb{N}} \{u_n(t) \ : \ t \in [0,T]\} \subset \mathscr{K}(T)$. Hence, taking into account the estimate (3.12) and applying the generalized Ascoli theorem [38, Lemma 1], we conclude that there exist a subsequence u_n (which we do not relabel) and a limit $u \in H^1(0,T;\mathscr{H})$ fulfilling (3.15) and (3.16) on $(0,T)$. On the other hand, using the compactness result for Young measures Theorem 3.3, up to a further extraction we also find a limit Young measure $\boldsymbol{\nu} \in \mathcal{Y}(0,T;\mathscr{H})$ such that the lower-semicontinuity relation (3.9) with $p = 2$, and the concentration property (3.8) hold for $\boldsymbol{\nu}$ and $\{u_n'\}$. Note that (3.8) yields relation (3.17) on $(0,T)$ for all $T > 0$. Indeed, the set $L(t)$ of the weak limit points of $u_n'(t)$ fulfils

$$L(t) \subset -\partial_\ell\phi(u(t)) \quad \text{for a.e. } t \in (0,T)$$

in view of the convergence (3.16) for $\{u_n(t)\}$, of the a priori estimate (3.13) for $\{\phi(u_n(t))\}$, and of Lemma 2.1. Then, by a diagonal argument, we extend the maps $t \mapsto u(t)$ and $t \mapsto \nu_t$ to $(0,+\infty)$, finding that $u \in H^1(0,T;\mathscr{H})$ for all $T > 0$, $\boldsymbol{\nu} \in \mathcal{Y}(0,+\infty;\mathscr{H})$ and fulfils (3.17), and we extract a subsequence u_{n_k} for which (3.15), (3.16), and (3.18) hold.

Passage to the limit. *The limit function u belongs to $\mathcal{S}$, fulfils $u(0) = u_0$, and*

$$d_X(u_{n_k}(t), u(t)) \to 0 \quad \text{as } k \uparrow +\infty \quad \forall t \geq 0. \tag{3.19}$$

Let us now fix an arbitrary $t > 0$: taking the $\liminf$ of both sides of (3.14) in view of (3.11), (3.15)-(3.18), and of the lower semicontinuity of ϕ, we find

$$\frac{1}{2}\int_0^t |u'(s)|^2 ds + \frac{1}{2}\int_0^t \left(\int_{\mathscr{H}} |\xi|^2 \, d\nu_s(\xi)\right) ds + \phi(u(t)) \leq \phi(u_0). \tag{3.20}$$

On the other hand, (3.17) and (3.18) ensure that

$$\int_0^t |(\partial_\ell\phi(u(s)))^\circ|^2 \, ds < +\infty.$$

Hence, by [30, Thm. 3.3, Prop. 3.4], there exists a selection $\xi(\cdot) \in \partial_\ell\phi(u(\cdot))$ in $L^2(0,t;\mathscr{H})$. Also, (3.20) yields that $\phi \circ u$ is bounded on $(0,T)$. Thus, we may apply the chain rule (CHAIN) and conclude that $\phi \circ u \in AC(0,T)$. Owing to [30, Thm. 3.3], the limit Young measure $\boldsymbol{\nu}$, fulfilling (3.17), also complies with a chain rule formula, yielding that (here we set $\eta(s) := \int_{\mathscr{H}} \xi d\nu_s(\xi)$):

$$\phi(u_0) - \phi(u(t)) = \int_0^t \langle u'(s), \eta(s)\rangle ds. \tag{3.21}$$

Note that, owing to Theorem 3.3 and, in particular, (3.10), we have that $u' = \eta$ almost everywhere. Combining (3.20) and (3.21), we deduce

$$\frac{1}{2}\int_0^t \int_{\mathscr{H}} |\xi - u'(s)|^2 d\nu_s(\xi) = \frac{1}{2}\int_0^t |u'(s)|^2 ds + \frac{1}{2}\int_0^t \int_{\mathscr{H}} |\xi|^2 d\nu_s(\xi) - \int_0^t \left(u'(s), \int_{\mathscr{H}} \xi d\nu_s(\xi)\right) ds \leq 0,$$

whence $\nu_s = \delta_{u'(s)}$ for a.e. $s \in (0,t)$. Therefore, (3.17) yields

$$u'(s) \in -\partial_\ell\phi(u(s)) \quad \text{for a.e. } s \in (0,t).$$

Being t arbitrary, we infer that u solves

$$u'(t) + \partial_\ell\phi(u(t)) \ni 0 \quad \text{for a.e. } t \in (0,+\infty). \tag{3.22}$$

The initial condition $u(0) = u_0$ of course ensues from (3.11) and (3.16). Finally, let us take the $\limsup_{k\uparrow+\infty}$ of (3.14). By (3.11),

$$\limsup_{k\uparrow+\infty} \left(\int_0^t |u_{n_k}'(\sigma)|^2 \, d\sigma + \phi(u_{n_k}(t))\right) \leq \limsup_{k\uparrow+\infty} \phi(u_0^{n_k}) = \phi(u(0)) = \phi(u(t)) + \int_0^t |u'(\sigma)|^2 \, d\sigma,$$

where we have used that u fulfils (3.22) and applied the chain rule (CHAIN) to the selection $-u'(t) \in \partial_\ell \phi(u(t))$. Therefore, we deduce that for every $t > 0$

$$\int_0^t |u'(\sigma)|^2 \, d\sigma \leq \liminf_{k\uparrow+\infty} \int_0^t |u'_{n_k}(\sigma)|^2 \, d\sigma \leq \limsup_{k\uparrow+\infty} \int_0^t |u'_{n_k}(\sigma)|^2 \, d\sigma \leq \int_0^t |u'(\sigma)|^2 \, d\sigma,$$
$$\phi(u(t)) \leq \liminf_{k\uparrow+\infty} \phi(u_{n_k}(t)) \leq \limsup_{k\uparrow+\infty} \phi(u_{n_k}(t)) \leq \phi(u(t)).$$

Finally,

$$\begin{aligned} u_{n_k} \to u \quad &\text{strongly in } H^1(0,T;\mathscr{H}) \text{ for any } T > 0, \\ u_{n_k}(t) \to u(t) \quad &\text{in } X \text{ for any } t > 0, \end{aligned} \tag{3.23}$$

whence it is easy to infer that u in fact solves (3.3). We can conclude that $\mathcal{S}$ fulfils (H4).
Conclusion of the proof. Note that we can in fact improve (3.23). By subtracting the energy identity for u from the energy identity for u_{n_k}, we get that

$$\begin{aligned} |\phi(u_{n_k}(t)) - \phi(u(t))| &\leq |\phi(u_0^{n_k}) - \phi(u_0)| + \int_0^t \left| |u'_{n_k}(s)|^2 - |u'(s)|^2 \right| ds \\ &\leq |\phi(u_0^{n_k}) - \phi(u_0)| + \int_0^t \left(|u'_{n_k}(s))| + |u'(s)| \right) |u'_{n_k}(s)) - u'(s)| ds \\ &\leq |\phi(u_0^{n_k}) - \phi(u_0)| + C\|u'_{n_k} - u'\|_{L^2(0,t;\mathscr{H})}, \end{aligned} \tag{3.24}$$

and the above right-hand side goes to zero, as $k \uparrow +\infty$, uniformly in t on the compact subsets of $[0, +\infty)$, so that we may conclude that $\phi \circ u_{n_k} \to \phi \circ u$ uniformly on the compact subsets of $[0, +\infty)$. We have thus also proved the continuity property (C4). □

3.2 Proof of Theorem 2

Note that any of the trajectories $u \in \mathcal{S}$ complies with the energy identity (2.12) (with $f \equiv 0$). In particular, $\phi(u(t)) \leq \phi(u(s))$ for all $0 \leq s \leq t < +\infty$. Let us check that ϕ is a Lyapunov function for $\mathcal{S}$. In fact, ϕ is trivially continuous w.r.t. the topology of X. Let now $w : \mathbb{R} \to X$ be a complete orbit for $\mathcal{S}$. Note that, by Definition 3.1, $w \in H^1_{\mathrm{loc}}(\mathbb{R};\mathscr{H})$, and that it fulfils the energy identity (2.12) on $\mathbb{R}$. Suppose that the function $t \in \mathbb{R} \mapsto \phi(w(t))$ is constant: hence,

$$\int_s^t |w'(\sigma)|^2 d\sigma = \phi(w(s)) - \phi(w(t)) = 0 \quad \forall\, s,t \in \mathbb{R},\ s \leq t.$$

Thus, $w'(t) = 0$ for a.e. $t \in \mathbb{R}$; as w is absolutely continuous, we deduce that w is a stationary orbit.

Therefore, in view of Theorem 2.7 and of the assumptions (3.6)-(3.7), it is sufficient to show that $\mathcal{S}$ is *eventually bounded* and *compact*.
Eventually boundedness. In order to check that $\mathcal{S}$ is eventually bounded, we fix a ball $B(0,R)$ centered at 0 of radius R in X: we will show that there exists $R' > 0$ such that the evolution of the ball $B(0,R)$ is contained in the ball $B(0,R')$. Indeed, let $u \in \mathcal{S}$ be a trajectory starting from some $u_0 \in B(0,R)$, cf. (3.2). By the energy identity,

$$\phi(u(t)) \leq \int_0^t |u'(s)|^2 ds + \phi(u(t)) \leq \phi(u_0) \leq R \quad \forall t \geq 0. \tag{3.25}$$

Therefore, taking into account our coercivity assumption (3.5), we deduce that

$$|u(t)| \leq R'' \quad \forall t \geq 0, \tag{3.26}$$

for some $R'' > 0$ and the eventual boundedness follows with $R' := R + R''$.

Compactness. In order to verify that $\mathcal{S}$ is compact, we consider a sequence $u_n \in \mathcal{S}$ such that $u_n(0)$ is bounded in X: we will show that

$$\text{there exists a subsequence } u_{n_k} \text{ such that } u_{n_k} \text{ is convergent in } X \textit{ for all } t > 0. \tag{3.27}$$

In fact, since $u_n(0)$ is bounded in X, we may argue as in the proof of (H4) in Theorem 1, and obtain that there exists a subsequence u_{n_k} and a limit function $u \in H^1(0,T;\mathscr{H})$ for all $T > 0$ such that the a priori bounds (3.12)-(3.13) and the convergences (3.15)-(3.18) hold. However, unlike in the proof of Theorem 1, we cannot directly conclude (3.20) anymore, since now we only have $u_n(0) \to u_0$ in $\mathscr{H}$. Actually, we will prove (3.27) by combining the assumed (cf. (CONT)) continuity of ϕ along the sequences with equibounded slope with Helly's compactness principle for monotone functions with respect to the pointwise convergence (for the proof of this result, the reader is referred to, e.g., [2, Chap. 4]. Indeed, thanks to the energy identity

$$\int_s^t |u_n'(\sigma)|^2 \, d\sigma + \phi(u_n(t)) = \phi(u_n(s)) \quad \forall s, t \in [0,T], \tag{3.28}$$

the function $t \mapsto \phi(u_n(t))$ is non-increasing. Thus, Helly's Theorem applies, and we obtain that there exists a function $\varphi : [0,+\infty) \to (-\infty,+\infty]$, which is non-increasing, such that

$$\varphi(t) := \lim_{k\uparrow+\infty} \phi_{n_k}(u(t)) \quad \forall t \geq 0, \tag{3.29}$$

for a proper subsequence n_k of n. Now, by (3.3) and (3.12), we have

$$\sup_{k\in\mathbb{N}} \int_0^T |(\partial_\ell \phi(u_{n_k}(t)))^\circ|^2 \, dt < +\infty.$$

Hence, by Fatou's Lemma,

$$\liminf_{k\uparrow+\infty} |(\partial_\ell \phi(u_{n_k}(t)))^\circ|^2 < +\infty \quad \text{for a.e. } t \in (0,T). \tag{3.30}$$

Therefore, for almost any t we can select a proper subsequence n_{k_λ} of n_k (note that, at this stage, the latter extraction depends on t) such that $(\partial_\ell \phi(u_{n_{k_\lambda}}(t)))^\circ$ is bounded as $\lambda \uparrow +\infty$. Also in view of (3.13) and (3.16), we can now exploit (CONT) and conclude that

$$\lim_{\lambda\uparrow+\infty} \phi(u_{n_{k_\lambda}}(t)) = \phi(u(t)). \tag{3.31}$$

Actually, the extraction of the subsequence in (3.31) does not in fact depend on t, since, by the lower semicontinuity of ϕ, we have for a.e. $t \in (0,T)$

$$\liminf_{k\uparrow+\infty} \phi(u_{n_k}(t)) \leq \liminf_{\lambda\uparrow+\infty} \phi(u_{n_{k_\lambda}}(t)) = \phi(u(t)) \leq \liminf_{k\uparrow+\infty} \phi(u_{n_k}(t)), \tag{3.32}$$

yielding

$$\phi(u(t)) = \lim_{k\uparrow+\infty} \phi(u_{n_k}(t)) = \varphi(t) \quad \text{for a.e. } t \in (0,T). \tag{3.33}$$

In the next lines, we will actually show that (3.33) holds *for all* $t > 0$, thus concluding (3.27) thanks to (3.29). To this aim, we will use the same technique devised for proving the upper semicontinuity property (H4). First, we take the $\liminf$ as $k \uparrow +\infty$ of both sides of (3.28). In view of the convergences (3.15)-(3.18), (3.29), and of the fact that $\varphi(t) = \phi(u(t))$ for almost every $t > 0$, we obtain

$$\frac{1}{2}\int_s^t |u'(\sigma)|^2 ds + \frac{1}{2}\int_s^t \left(\int_{\mathscr{H}} |\xi|^2 \, d\nu_\sigma(\xi) \right) ds + \phi(u(t)) \leq \phi(u(s)) \tag{3.34}$$

for all $t \in (0,T]$ and for a.e. $0 < s \leq t$. Now, by arguing exactly as in the proof of (H4) (with the sole difference that all the time integrals are now considered between s and t, with $s > 0$, since we do not have the convergence in X for the sequence of the initial values $u_n(0)$), we deduce that the limit function u solves

$$u'(t) + \partial_\ell \phi(u(t)) \ni 0 \quad \text{for a.e. } t \in (0, +\infty). \tag{3.35}$$

Thus, in view of (CHAIN), the function u also verifies the energy identity on the interval (s, t), with $0 < s \le t \le T$ (see [30, Theorem 3]). In particular, this means that the map $t \mapsto \phi(u(t))$ is continuous and non-increasing, and thus $\varphi(t) = \phi(u(t))$ *for any* $t > 0$, as desired. □

4 Applications: perturbations of convex functionals

In this section, we apply our abstract theory to some concrete examples of parabolic partial differential equations. More precisely, we will deal with the long-time dynamics of gradient flows of various non convex perturbations of convex functionals. First, we shall consider the case of C^1 perturbations. Secondly, we apply our abstract results to gradient flows of functionals ϕ given by the difference of two convex and lower semicontinuous functionals.

4.1 C^1 perturbations of convex functions

We consider functionals $\phi : \mathscr{H} \to (-\infty, +\infty]$ of the type

$$\phi = \phi_1 + \phi_2, \text{ with } \phi_1 \text{ proper, l.s.c., and convex on } D(\phi_1) \subset \mathscr{H}, \ \phi_2 \in C^1(\mathscr{H}). \tag{4.1}$$

The problem of the existence of solutions of gradient flow equations for functionals ϕ of this form has been addressed in [24] (see also the lectures notes [1] and [2]). The uniqueness of solutions is an open problem owing to the possible non convexity of the perturbation ϕ_2. Here, we prove that the set of all solutions of

$$u'(t) + \partial\phi_1(u(t)) + D\phi_2(u(t)) = 0 \ \text{ for a.e. } t \in (0, +\infty) \tag{4.2}$$

is a *generalized semiflow* in the phase space $X = D(\phi_1)$, endowed with the metric

$$d_X(u, v) := |u - v| + |\phi_1(u) - \phi_1(v)| \ \ \forall u, v \in \mathscr{H}. \tag{4.3}$$

Moreover, we show that this generalized semiflow has a unique global attractor in the phase space $D(\phi_1)$. The following proposition is a consequence of Theorems 1 and 2, with the choice $\mathcal{D} = \mathscr{H}$.

Proposition 4.1 (Global attractor for C^1-perturbations of convex functions)**.** *Let $\phi : \mathscr{H} \to (-\infty, +\infty]$ be as in* (4.1)*. Suppose that ϕ complies with the assumptions* (COMP) *and* (3.5)*. Moreover, we assume that*

$$\forall T > 0 \quad v \in H^1(0, T; \mathscr{H}) \Rightarrow D\phi_2(v) \in L^2(0, T; \mathscr{H}), \tag{4.4}$$

$$\text{the set } \ \{v \in \mathscr{H} : \partial\phi_1(v) + D\phi_2(v) \ni 0\} \ \text{ is bounded in } D(\phi_1). \tag{4.5}$$

Then, the set of all solutions in $H^1(0, T; \mathscr{H})$, $\forall T > 0$, of (4.2) *is a* generalized semiflow *on $(D(\phi_1), d_X)$ (see* (4.3)*) and possesses a unique global attractor. Moreover, the attractor is Lyapunov stable.*

Preliminarily, we need the following

Lemma 4.2. *Let $\phi_1 : \mathscr{H} \to (-\infty, +\infty]$ be a proper, lower semicontinuous functional, and let $\phi_2 : \mathscr{H} \to (-\infty, +\infty]$ be continuous and Gâteau differentiable, with $D\phi_2 : \mathscr{H} \to \mathscr{H}$ demicontinuous. Set $\phi : \phi_1 + \phi_2$. Then,*

$$\partial_\ell \phi(u) = \partial_\ell \phi_1(u) + D\phi_2(u) \quad \forall\, u \in \mathscr{H}. \tag{4.6}$$

The same conclusion holds for $\partial_s \phi$.

For the proof of this lemma, we refer the interested reader to [31].

Proof of Proposition 4.1. First of all we note that, since $\partial_s\phi(v) = \partial\phi_1(v) + D\phi_2(v)$ for all $v \in \mathscr{H}$ thanks to Lemma 4.2, the set of all solutions of (4.2) coincides with set of all solutions of (3.3), with $\phi = \phi_1 + \phi_2$. In order to apply our Theorems 1 and 2, we only need to verify the validity of the chain rule (CHAIN), of the continuity condition (CONT), and of (3.7). For any given functions v, ξ like in the hypothesis of (CHAIN), condition (4.4) and the fact that $\phi_2 \in C^1(\mathscr{H})$ entail the validity of the chain rule for ϕ_2. Moreover, since $D\phi_2(v) \in L^2(0,T;\mathscr{H})$, by (4.4) we also have by comparison that $\partial\phi_1(v) \ni \xi - D\phi_2(v) \in L^2(0,T;\mathscr{H})$. Since the chain rule (CHAIN) holds for ϕ_1 by convexity, we conclude that it holds as well for ϕ. The continuity property (CONT) easily follows from the continuity of ϕ_2 and from the convexity of ϕ_1 (see (2.6)). Finally, the condition (4.5) on the solutions of the stationary equation ensures the validity of (3.7). Indeed, again thanks to Lemma 4.2, we have

$$Z(\mathcal{S}) = \{v \in \mathscr{H} : \partial_s\phi(v) \ni 0\} = \{v \in \mathscr{H} : \partial\phi_1(v) + D\phi_2(v) \ni 0\}\,, \tag{4.7}$$

and the latter set is bounded by assumption. Thus, the assertion follows. □

4.2 Dominated concave perturbations of convex functions

In this section, we apply our results to gradient flows of functionals ϕ given by

$$\phi = \psi_1 - \psi_2, \text{ with } \psi_1,\, \psi_2 \text{ proper, l.s.c., and convex on } D(\psi_i) \subset \mathscr{H}, i = 1,2. \tag{4.8}$$

Of course, $D(\phi) = D(\psi_1) \cap D(\psi_2)$. The starting point of our analysis is the following Lemma, which sheds light on the structure of the limiting subdifferential for a functional ϕ as in (4.8), and states a sufficient condition for the validity of the chain rule (CHAIN) (its proof is to be found in [30, Lemma 4.8 and Lemma 4.9]).

Lemma 4.3 (Subdifferential decomposition and chain rule)**.** *Let $\phi : \mathscr{H} \to (-\infty, +\infty]$ fulfil* (4.8)*,* (COMP)*, and*

$$\forall M \geq 0,\ \exists \rho < 1, \gamma \geq 0 \quad \textit{such that} \sup_{\xi \in \partial\psi_2(u)} |\xi| \leq \rho |(\partial\psi_1(u))^\circ| + \gamma$$
$$\textit{for every } u \in D(\partial\psi_1) \textit{ with } \max(\phi(u), |u|) \leq M. \tag{4.9}$$

Then, every $g \in \partial\phi(u)$ with $\max(\phi(u), |u|) \leq M$ can be decomposed as

$$g = \lambda^1 - \lambda^2, \quad \lambda^i \in \partial\psi_i(u), \tag{4.10}$$

where ρ, γ are given in terms of M by (4.9)*; moreover, ϕ satisfies the chain rule* (CHAIN)*.*

As a consequence of this lemma, we have that the solutions of the gradient flow equation (3.3) with $\phi = \psi_1 - \psi_2$ (whenever they exist) indeed solve the equation

$$u'(t) + \partial\psi_1(u(t)) - \partial\psi_2(u(t)) \ni 0 \text{ for a.e. } t \in (0, +\infty). \tag{4.11}$$

We note that the existence of a global attractor for the solutions of equations of the form (4.11) has been addressed by VALERO [40]. The approach in [40] is different from the present one since it is based on the abstract theory, developed by MELNIK & VALERO in [25], of attractors for multivalued semiflows. Our result is however sharper although less general. On the one hand, we do not focus on the whole class of solutions of equation (4.11), but rather on the set of solutions which can be obtained from the gradient flow approach. Secondly, VALERO [40] tackles the problem in the phase space $\overline{D(\psi_1)}$, endowed with the metric of $\mathscr{H}$. On the other hand, our analysis is performed in the phase space given by the domain of the potential ϕ, that is $D(\phi)$, endowed with metric d_X (see (3.1)), which is stronger than that of $\mathscr{H}$. In the following concrete examples of PDEs, taken from [40], the difference between Valero's results and the present ones will be clarified.

Henceforth, we fix the Hilbert space $\mathscr{H}$ to be $\mathscr{H} := L^2(\Omega)$, Ω being a bounded domain of $\mathbb{R}^d$ with smooth boundary $\partial\Omega$; we shall denote by $|\cdot|$ the norm in $L^2(\Omega)$. In particular, all subdifferentials are computed with respect to the metric in $L^2(\Omega)$. Let $x = (x_1, \ldots, x_d)$ represent the variable in Ω.

Example 1. We shall consider gradient flow solutions of the equation

$$\begin{aligned} u'(t) - \Delta_p u(t) - |u(t)|^\alpha u(t) = 0 \quad \text{for a.e. } t \in (0,+\infty), \\ \text{where} \quad \Delta_p u := \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right), \end{aligned} \tag{4.12}$$

and $p > 2$ and $\alpha > 0$ fulfil

$$\begin{cases} 2+\alpha < p, \\ 2(1+\alpha) \leq \frac{dp}{d-p}, \text{ if } d > p. \end{cases} \tag{4.13}$$

Then, let us consider the following functionals defined in $L^2(\Omega)$:

$$\psi_1(v) := \frac{1}{p}\sum_{i=1}^d \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^p dx \quad \text{if } v \in W_0^{1,p}(\Omega), \quad \psi_1(v) := +\infty \quad \text{otherwise}, \tag{4.14}$$

$$\psi_2(v) := \frac{1}{\alpha+2}\int_\Omega |v|^{\alpha+2} dx \quad \text{if } v \in L^{\alpha+2}(\Omega), \quad \psi_2(v) := +\infty \quad \text{otherwise}. \tag{4.15}$$

It is clear that, with these choices of ψ_1 and ψ_2, equation (4.12) can be rewritten in the form of (4.11). Consequently, we let $\phi = \psi_1 - \psi_2$, and we study the long-time behaviour of the solutions of the gradient flow for ϕ in the framework of the phase space

$$\begin{cases} X := D(\phi) = W_0^{1,p}(\Omega) = \mathcal{D} \text{ (see (4.13)), with} \\ d_X(u,v) := \|u-v\|_{L^2(\Omega)} \\ +\left|\frac{1}{p}\left(\|\nabla u\|_p^p - \|\nabla v\|_p^p\right) - \frac{1}{2+\alpha}\left(\|u\|_{2+\alpha}^{2+\alpha} - \|v\|_{2+\alpha}^{2+\alpha}\right)\right|, \quad u,v \in W_0^{1,p}(\Omega). \end{cases} \tag{4.16}$$

Thanks to (4.13), it is not difficult to prove that ϕ is lower semicontinuous, has compact sublevels in $L^2(\Omega)$, and satisfies the coercivity condition (3.5) (which clearly entails (3.4)). Concerning the lower semicontinuity, we have to prove that given a sequence $\{u_n\}_{n=1}^{+\infty} \subset W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$, one has $\liminf_{n\to+\infty} \phi(u_n) \geq \phi(u)$. Without loss of generality, we may suppose that $\sup_n \phi(u_n) < +\infty$. Hence, the first relation in (4.13) gives the following chain of inequalities for a positive constant C independent of n

$$C \geq \phi(u_n) \geq \|u_n\|^p_{W_0^{1,p}(\Omega)} - C\|u_n\|^{2+\alpha}_{W_0^{1,p}(\Omega)} \geq C\|u_n\|^p_{W_0^{1,p}(\Omega)} - C, \tag{4.17}$$

where we have also used the Young inequality. Thus, by standard weak compactness results and possibly extracting some not relabeled subsequence, $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and, by the compact embedding $W_0^{1,p}(\Omega) \subset L^{2+\alpha}(\Omega)$ (see (4.13)), we have $u_n \to u$ strongly in $L^{2+\alpha}(\Omega)$. The lower semicontinuity of ϕ is now an easy consequence of this strong convergence and of the lower semicontinuity of the norms w.r.t. the weak convergence. The estimate (4.17) also shows that the sublevels of ϕ, being bounded in $W_0^{1,p}(\Omega)$, are compact in $L^2(\Omega)$. The coercivity condition (3.5) easily follows from (4.17) by noting that, since $p > 2$ by assumption, we have

$$\phi(v) \geq C\|u_n\|^p_{W_0^{1,p}(\Omega)} - C \geq C|u| - C \ \ \forall\, v \in W_0^{1,p}(\Omega).$$

In order to apply Theorems 1 and 2 and find that the set of all the solutions of

$$u'(t) + \partial_s(\psi_1 - \psi_2)(u(t)) \ni 0 \text{ in } \mathscr{H} \quad \text{for a.e. } t \in (0,+\infty) \tag{4.18}$$

generates a *generalized semiflow* on (X, d_X) (see (4.16)), possessing a unique global attractor, we still need to check that $\phi = \psi_1 - \psi_2$ complies with the chain rule (CHAIN) and with (3.7) (in fact, (3.6) is valid since we have chosen $\mathcal{D} = X$). As for proving the validity of the chain rule for ϕ, we have to check that the subdifferentials of ψ_1 and ψ_2 comply with condition (4.9). To this aim, we note that for any $u \in D(\partial\psi_1)$ with $\max(\phi(u), |u|) \leq M$, we have that $\|u\|_{W_0^{1,p}(\Omega)} \leq \gamma$, where γ is a positive constant depending on M and on Ω. Thus, (4.9) follows with any choice of $\rho \in (0,1)$ by simply noting that $|\partial\psi_2(u)| = \|u\|_{2(\alpha+1)}^{\alpha+1}$, and recalling that $W_0^{1,p}(\Omega) \subset L^{2(\alpha+1)}(\Omega)$ with continuous injection (see (4.13)). In order to prove (3.7) for ϕ, we have to check (see (4.16)) that the set

$$\{v \in \mathscr{H} : \partial_s(\psi_1 - \psi_2)(v) \ni 0\} \text{ is bounded in } (W_0^{1,p}(\Omega), d_X). \tag{4.19}$$

Note that Lemma 4.3 and the definition of ψ_1 and ψ_2 entail that

$$\{v \in \mathscr{H} : \partial_s(\psi_1 - \psi_2)(v) \ni 0\} \subseteq \left\{v \in W_0^{1,p}(\Omega) : \Delta_p v - |v|^\alpha v = 0 \ \text{ a.e. in } \Omega\right\}. \tag{4.20}$$

Then, we only need to prove that the latter set is bounded in $(W_0^{1,p}(\Omega), d_X)$. This follows by simply testing in $L^2(\Omega)$ the equation $\Delta_p v - |v|^\alpha v = 0$ with v and performing the same computations as for proving (4.17). This produces a bound in $W_0^{1,p}(\Omega)$ for the solutions of the aforementioned stationary equation, which entails the bound in the phase space (4.16) by using the embedding $W_0^{1,p}(\Omega) \subset L^{\alpha+2}(\Omega)$ again. We have thus proved the following.

Proposition 4.4. *Let α and p satisfy* (4.13) *and $\phi = \psi_1 - \psi_2$ with ψ_1 and ψ_2 as in* (4.14)-(4.15). *Then, the solutions of the gradient flow equation*

$$u'(t) + \partial_s(\psi_1 - \psi_2)(u(t)) \ni 0 \quad \textit{for a.e. } t \textit{ in } (0, +\infty) \tag{4.21}$$

generate a generalized semiflow *in $(W_0^{1,p}(\Omega), d_X)$ which possesses a unique global attractor. This attractor is also Lyapunov stable.*

Example 2. In this example, still taken from [40], we consider gradient flow solutions of

$$u'(t) - \Delta u(t) - f(u(t)) \in \lambda H(u(t) - 1) \quad \text{for a.e. } t \in (0, +\infty), \tag{4.22}$$

where H is the Heaviside graph, i.e. the maximal multivalued monotone graph in $\mathbb{R} \times \mathbb{R}$ given by

$$H(v) := 1 \text{ if } v > 0, \quad H(v) := [0,1] \text{ if } v = 0, \quad \text{and } \ H(v) = 0 \text{ if } v < 0, \tag{4.23}$$

and λ is a non-negative constant. Finally, $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous function such that

$$|f(s)| \leq k_1 + k_2|s|, \text{ with } k_1 \geq 0 \text{ and } 0 \leq k_2 < \lambda_1, \tag{4.24}$$

with λ_1 the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. We introduce the following functionals, defined in $L^2(\Omega)$,

$$\psi_1(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 dx \quad \text{if } v \in H_0^1(\Omega), \quad \psi_1(v) := +\infty \text{ otherwise}, \tag{4.25}$$

$$\psi_2(v) := \int_\Omega F(u)dx + \lambda\int_\Omega (u-1)^+ dx, \tag{4.26}$$

where $F' = f$. As in Example 1, we aim to consider the dynamics of the gradient flow for the functional $\phi = \psi_1 - \psi_2$ in the phase space

$$\begin{cases} X := D(\phi) = H_0^1(\Omega) = \mathcal{D}, \text{ with} \\ d_X(u,v) := \|u - v\|_{L^2(\Omega)} \\ +\left|\frac{1}{2}\left(\|\nabla u\|_2^2 - \|\nabla v\|_2^2\right) - \left(\int_\Omega (F(u) - F(v))dx + \lambda\int_\Omega (u-1)^+ dx - \lambda\int_\Omega (v-1)^+ dx\right)\right|, \\ u, v \in H_0^1(\Omega). \end{cases}$$

The functional ϕ is lower semicontinuous in $L^2(\Omega)$. In fact, if we are given a sequence $\{u_n\}_{n=1}^{+\infty} \subset H_0^1(\Omega)$ with $u_n \to u$ in $L^2(\Omega)$ and $\sup\limits_n \phi(u_n) < +\infty$, then the growth condition on f entails that

$$F(u_n) \leq C + C|u_n|^2 \quad \text{a.e. in } \Omega. \tag{4.27}$$

Thus, by a variant of the Dominated Convergence Theorem (see, e.g., [20, Theorem 4]), there holds $F(u_n) \to F(u)$ in $L^1(\Omega)$. The lower semicontinuity of ϕ now descends from the lower semicontinuity of norms w.r.t. the weak convergence. Moreover, ϕ has compact sublevels in $L^2(\Omega)$. In fact, by the Poincaré inequality, combined with the growth condition on f (recall that $k_2 < \lambda_1$), we have that

$$\begin{aligned}\phi(u) &\geq \frac{1}{2}\|\nabla v\|_2^2 - \int_\Omega F(v)dx - \lambda\int_\Omega (v-1)^+ dx \\ &\geq C\|\nabla v\|_2^2 - C \quad \text{for a given } C > 0 \text{ and } \forall v \in H_0^1(\Omega).\end{aligned} \tag{4.28}$$

Thus, the sublevels of ϕ are bounded in $H_0^1(\Omega)$, which is clearly compact in $L^2(\Omega)$. Note that (4.28) entails the coercivity assumption (3.5) (and thus (3.4)). Again, in order to apply our Theorems 1 and 2, we need to check (CHAIN) and (3.7) ((3.6) is again trivial since we take $\mathcal{D} = X$). The chain rule (CHAIN) easily follows from Lemma 4.3. In fact, the subdifferential $\partial\psi_2$ of ψ_2 is simply

$$\partial\psi_2(v) = f(v) + \lambda H(v-1) \quad \forall\, v \in D(\partial\psi_2) = L^2(\Omega),$$

thanks to the growth condition (4.24), while the subdifferential of ψ_1 is clearly $-\Delta$, with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Thus, condition (4.9) easily follows with $\gamma = M$ and with any choice of $\rho \in (0,1)$. Finally, the condition on the rest points follows from the same argument used in (4.19)-(4.20). In this case, the analogue of the stationary equation in (4.20) is the following

$$-\Delta v - f(v) \in \lambda H(v-1) \quad \text{a.e. in } \Omega, \quad v = 0 \quad \text{a.e. on } \partial\Omega. \tag{4.29}$$

We thus have

Proposition 4.5. *Let us consider the functional $\phi = \psi_1 - \psi_2$, with ψ_1 and ψ_2 as in* (4.25)-(4.26). *Then, the solutions of the gradient flow equation*

$$u'(t) + \partial_s(\psi_1 - \psi_2)(u(t)) \ni 0 \quad \textit{for a.e. } t \textit{ in } (0, +\infty) \tag{4.30}$$

generate a generalized semiflow *in $(H_0^1(\Omega), d_X)$ which possesses a unique global attractor. The attractor is also Lyapunov stable.*

Example 3. We are interested in the study of the long-time behaviour for the gradient flow solutions of the following equation

$$u'(t) - \Delta u(t) + \partial I_K(u(t)) + f_1(u(t)) - f_2(u(t)) \ni 0 \quad \text{for a.e. t in } (0, +\infty). \tag{4.31}$$

In the latter equation, the symbol ∂I_K represents the subdifferential of the indicator function of the closed and convex set K (see definition (4.35) below), while $f_i : \mathbb{R} \to \mathbb{R}$ (i=1, 2) are two non-decreasing continuous functions which satisfy the following growth and compatibility conditions

$$\begin{gathered}\text{there exist} \quad 0 \leq k_1, k_2, k_3 < \lambda_1,\ k_4 \geq 0,\ \varepsilon > 0 \quad \text{such that} \\ |f_1(s)| \leq k_1(|s|^{d/(d-2)} + 1) \quad \forall s \in \mathbb{R} \quad \text{if } d \geq 3 \\ |f_2(s)| \leq k_2 + k_3|s|, \\ (f_1(s) - f_2(s))s \geq (-\lambda_1 - \varepsilon)s^2 - k_4,\end{gathered} \tag{4.32}$$

where λ_1 is the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. We then denote by F_1 and F_2 the primitives of f_1 and f_2 respectively. Consequently, F_1 and F_2 are differentiable convex functions in $\mathbb{R}$ such that $F_i' = f_i$, $i = 1, 2$. Without loss of generality, we assume that $F_1(0) = 0$. We will only consider the case in which Ω is a bounded domain of $\mathbb{R}^d$ with $d \geq 3$, the one-dimensional and the two-dimensional cases being easier. The only difference is in the growth condition imposed on f_1, which may be weakened. More precisely, in two dimensions we can deal with a function f_1 growing at most like a polynomial with order ν, ν being any real number $1 \leq \nu < +\infty$. In one dimension, we do not need any additional growth condition.

Now, let us consider the following functionals on $L^2(\Omega)$

$$
\begin{aligned}
\psi_1(v) &:= \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega F_1(v)dx \quad \text{if } v \in K, \quad \psi_1(v) := +\infty \text{ otherwise,} && (4.33)\\
\psi_2(v) &:= \int_\Omega F_2(v)dx, && (4.34)
\end{aligned}
$$

where K is the following closed and convex subset

$$K := \left\{v \in H^1_0(\Omega) : v(x) \geq 0, \text{ a.e. in } \Omega\right\}. \tag{4.35}$$

It is not difficult to show (see [9, Prop. 2.17]) that the subdifferential of ψ_1 in $L^2(\Omega)$ has the following expression

$$
\begin{gathered}
w \in \partial\psi_1(u) \Leftrightarrow w \in -\Delta u + \partial I_K(u) + f_1(u),\\
\text{with} \quad D(\partial\psi_1) = H^2(\Omega) \cap K.
\end{gathered}
\tag{4.36}
$$

Thus, it is clear that (4.31) could be rewritten in the form of (4.11), with ψ_1 and ψ_2 as in (4.33)-(4.34).

Again, we are interested in the long-time dynamics of the gradient flow for the functional $\phi = \psi_1 - \psi_2$, in the framework of the phase space

$$
\begin{cases}
X := D(\phi) = K = \mathcal{D} \text{ with}\\
d_X(u,v) := \|u - v\|_{L^2(\Omega)}\\
+\left|\frac{1}{2}\left(\|\nabla u\|_2^2 - \|\nabla v\|_2^2\right) + \left(\int_\Omega (F_1(u) - F_1(v))dx - \int_\Omega (F_2(u) - F_2(v))dx\right)\right|.
\end{cases}
$$

Thus, we have to check the validity of the hypotheses of Theorems 1 and 2. The proof of the lower semicontinuity follows exactly the same lines of Example 2, with some minor modifications due to presence of the two extra terms $I_K(v)$ and $\int_\Omega F_1(v)dx$ in the definition of (4.33). However, these two terms, being positive, could be easily handled by lower semicontinuity. Moreover, arguing as in (4.28) we find that ϕ complies with (COMP) and (3.5) with respect to the norm of $L^2(\Omega)$. The chain rule property (CHAIN) is an easy consequence of Lemma 4.3, of the structure of the subdifferential of ψ_1 (see (4.36)), and of the growth condition on the perturbation f_2, which entails that the subdifferential $\partial\psi_2$ of ψ_2 is simply given by $\partial\psi_2(v) = f_2(v)$ for all $v \in L^2(\Omega)$. Thus, in order to check the validity of (4.9) and of (3.7), we proceed as in the former examples. In this case, the analogue of the stationary equation in (4.20) reads

$$-\Delta v + \partial I_K(v) + f_1(v) - f_2(v) \ni 0 \quad \text{a.e. in } \Omega. \tag{4.37}$$

Hence, by simply testing (4.37) with v (which belongs to K, being a solution of (4.37)) and using the last condition in (4.32), we get a bound for v in $H^1_0(\Omega)$. Thus, by the growth condition (4.32) on f_1 we have a similar bound for the growth of F_1. More precisely, there holds

$$|F_1(v)| \leq C(|v| + |v|^{(2d-2)/(d-2)}) \quad \text{for some } C > 0. \tag{4.38}$$

Thus, since $H^1_0(\Omega)$ is continuously embedded in $L^{(2d-2)/(d-2)}(\Omega)$, by (4.38) F_1 maps bounded sets in $H^1_0(\Omega)$ to bounded sets of $L^1(\Omega)$. To conclude that the set of the rest points is bounded, it remains to show the boundedness of the solutions of the stationary inclusion (4.37), also with respect to the F_2-part of the metric d_X (see (4.37)). But this is simpler, thanks to the linear growth of f_2, which entails a quadratic growth for its primitive F_2.

We thus have the following.

Proposition 4.6. *Let K be as in* (4.35) *and the functional ϕ be given by $\phi = \psi_1 - \psi_2$ with ψ_1 and ψ_2 as in* (4.33)-(4.34). *Then, the solutions of the gradient flow equation*

$$u'(t) + \partial_s(\psi_1 - \psi_2)(u(t)) \ni 0 \quad \textit{for a.e. } t \textit{ in } (0, +\infty) \tag{4.39}$$

generate a generalized semiflow *in K which possesses a unique global attractor. The attractor is also Lyapunov stable.*

Note that, as we have already mentioned, our choice of the phase space brings to a more regular attractor, attracting with respect to the $W^{1,p}$-norm (Example 1) and to the H^1_0-norm (Examples 2-3), whereas in [40] the attraction holds with respect to the L^2-metric. Furthermore, our phase space keeps track of the constraint imposed on the unknowns.

5 Applications: long-time behaviour of quasi-stationary evolution systems

General Setup. The functional setting we deal with features a standard Hilbert triplet

$$V \subset H \equiv H' \subset V', \quad \text{with dense and compact inclusions.} \tag{5.1}$$

We denote by ${}_{V'}\langle \cdot\,, \cdot \rangle_V$ the duality pairing between V' and V and by $(\cdot\,, \cdot)_H$ the scalar product in H, recalling that ${}_{V'}\langle u, v\rangle_V = (u, v)_H \quad \forall\, u \in H, v \in V$. Furthermore, let $a : V \times V \to \mathbb{R}$ be a *non-negative, symmetric, and continuous* bilinear form, and let $A : V \to V'$ be the continuous linear operator associated with a, i.e.

$${}_{V'}\langle Au, v\rangle_V := a(u, v) \quad \forall\, u, v, \in V. \tag{5.2}$$

We also consider a proper functional $F : H \to [0, +\infty]$ whose sublevels

$$\big\{\chi \in H : F(\chi) \leq s\big\} \quad \text{are } \textit{strongly} \text{ compact in } H, \tag{5.3}$$

and we denote by by $\partial F : H \to 2^H$ the Fréchet subdifferential of F in H, namely

$$\theta \in \partial F(\chi) \quad \Leftrightarrow \quad \chi \in D(F) \subset H, \quad \liminf_{\|\eta-\chi\|_H \to 0} \frac{F(\eta) - F(\chi) - (\theta, \eta - \chi)_H}{\|\eta - \chi\|_H} \geq 0.$$

We aim to investigate the long-time behaviour of (a class of solutions of) the following evolution system, coupling a diffusion equation with a quasi-stationary condition, for which an existence result was obtained in [30, Sect. 5].

Problem 5.1. *Given $T > 0$ and $u_0 \in H$, find a pair $u, \chi : (0, T) \to H$, with $u(t) - \chi(t) \in V$ for a.e. $t \in (0, T)$, which satisfies at a.e. $t \in (0, T)$ the system*

$$\begin{cases} u'(t) + A(u(t) - \chi(t)) = 0 \quad \text{in } V', \\ \chi(t) + \partial F(\chi(t)) \ni u(t) \quad \text{in } H, \\ u(0) = u_0. \end{cases} \tag{5.4}$$

In Section 5.1, we briefly summarize for the reader's convenience the techniques developed in [30, Sec. 5] for Problem 5.1. Hence, we distinguish the two following cases:

1. the form a is *coercive,* i.e., there exists a constant $\alpha > 0$ such that

$$a(u, u) \geq \alpha \|u\|_V^2 \quad \forall\, u \in V, \tag{5.5}$$

2. a is *weakly coercive,* namely, there exist λ, $\alpha_\lambda > 0$ s.t.

$$a(u, u) + \lambda \|u\|_H^2 \geq \alpha_\lambda \|u\|_V^2 \quad \forall\, u \in V. \tag{5.6}$$

In fact, whenever a is weakly coercive, *for all* $\lambda > 0$ it is possible to find a constant $\alpha_\lambda > 0$ fulfilling (5.6).

5.1 Existence results for Problem 5.1

5.1.1 The coercive case: existence by a gradient flow approach

Assume that (5.5) holds. Then, we endow V with the norm $\|v\|_V^2 := a(v,v)$ for all $v \in V$ and A turns out to be an isometry between the spaces V and V'. Let us introduce the functional $\phi : V' \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\phi(u) := \inf_{\chi \in H} \mathscr{F}(u,\chi), \quad \mathscr{F}(u,\chi) := \begin{cases} \frac{1}{2}\|u-\chi\|_H^2 + F(\chi) & \text{if } u,\chi \in H, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.7}$$

Clearly, $D(\phi) = H$; further, for $u \in D(\phi)$ we denote by $M(u)$ the set of the elements $\chi \in H$ attaining the minimum in (5.7), i.e.

$$M(u) := \big\{\chi \in H : \mathscr{F}(u,\chi) = \phi(u)\big\}. \tag{5.8}$$

Note that $M(u) \neq \emptyset$ for all $u \in H$, since F is l.s.c. and has compact sublevels. Further, the following formula

$$\phi(u) = \frac{1}{2}\|u\|_H^2 - \sup_{\chi \in H}\left((u,\chi)_H - \Big(\frac{1}{2}\|\chi\|_H^2 + F(\chi)\Big)\right) \tag{5.9}$$

shows that ϕ is in fact a concave perturbation of a quadratic functional (cf. with (2.13)).

Proposition 5.2 and Corollary 5.3 below (which we recall from [30, Sec.5]) ensure that Problem 5.1 may be interpreted as the Cauchy problem

$$u'(t) + \partial_s\phi(u(t)) \ni 0 \quad \text{a.e. in } (0,T), \quad u(0) = u_0, \tag{5.10}$$

for the functional ϕ in the Hilbert space

$$\begin{gathered} \mathscr{H} := V', \quad \text{endowed with the scalar product} \\ \langle u,v\rangle_{\mathscr{H}} := a(A^{-1}u, A^{-1}v) = {}_{V'}\langle u, A^{-1}v\rangle_V = {}_{V'}\langle v, A^{-1}u\rangle_V \quad \forall u,v \in V'. \end{gathered} \tag{5.11}$$

Note that, in this framework, the Fréchet and the (strong and weak) limiting subdifferentials of ϕ have to be considered with respect to the scalar product (5.11).

Proposition 5.2. *The functional $\phi : \mathscr{H} \to [0,+\infty]$ defined by* (5.7) *has $D(\phi) = H$, is lower semicontinuous on the Hilbert space $\mathscr{H}$* (5.11)*, and complies with* (COMP) *and* (CONT)*. Moreover, for every $u \in H$,*

$$\chi \in M(u) \quad \Rightarrow \quad \chi + \partial F(\chi) \ni u, \tag{5.12}$$

while for every $u \in D(\partial_\ell\phi)$

$$\xi \in \partial_\ell\phi(u) \quad \Rightarrow \quad \exists\, \chi \in M(u) : \quad u - \chi \in V, \quad \xi = A(u-\chi), \tag{5.13}$$

and the same result holds for $\partial_s\phi$.

Corollary 5.3 (Gradient flows solve the system)**.** *Suppose that $u_0 \in H$. Then, any solution $u \in H^1(0,T;\mathscr{H})$ of the Cauchy problem* (5.10) *in the Hilbert space* (5.11) *fulfils*

$$\begin{gathered} u \in L^\infty(0,T;H) \textit{ and there exists } \chi \in L^\infty(0,T;H) \textit{ with} \\ u - \chi \in L^2(0,T;V), \quad \chi(t) \in M(u(t)) \quad \textit{for a.e. } t \in (0,T), \\ \textit{and the pair } (u,\chi) \textit{ solves the system } (5.4). \end{gathered} \tag{5.14}$$

In view of the above results, in [30] the existence of solutions of Problem 5.1 is deduced from the general Theorem 2.2, applied to the Cauchy problem (5.10) (with the choice (5.7) for ϕ). As a consequence, the following result has been obtained (see [30, Thm. 5.8])

Theorem 5.4. *In the setting of* (5.1), (5.2), (5.5)*, suppose that F complies with* (5.3) *and* either *with*

$$\forall\, M \geq 0\ \exists\, \rho < 1,\ \gamma \geq 0 \text{ such that this a priori estimate holds:}$$
$$\left.\begin{matrix} u \in V, \quad \chi \in M(u), \\ \max(\|u\|_H, F(\chi)) \leq M \end{matrix}\right\} \quad \Rightarrow \quad \chi \in V, \quad \|A\chi\|_{V'} \leq \rho \|Au\|_{V'} + \gamma. \tag{5.15}$$

or *with*

$$\begin{gathered} \text{there exists a Banach space } W \text{ such that } V \subset W \subset H \text{ with continuous inclusions,} \\ H \text{ satisfies the interpolation property } \quad (W, V')_{1/2,2} \subset H, \end{gathered} \tag{5.16}$$

and for every $M \geq 0$ there exists $C > 0$ such that this a priori estimate *holds:*

$$\left.\begin{matrix} u - \chi \in V, \quad \chi \in M(u) \\ \max(\|u\|_H, F(\chi)) \leq M \end{matrix}\right\} \quad \Rightarrow \quad \|\chi\|_W \leq C\left(1 + \|A(u-\chi)\|_{V'}\right). \tag{5.17}$$

Then, for every $u_0 \in H$ and $T > 0$, Problem 5.1 admits a solution (u, χ), with $u \in H^1(0,T;V') \cap L^\infty(0,T;H)$, $\chi \in L^\infty(0,T;H)$, $u - \chi \in L^2(0,T;V)$, fulfilling $u(0) = u_0$, the system

$$\begin{cases} u'(t) + A(u(t) - \chi(t)) = 0 \quad \text{in } V' \quad \text{for a.e. } t \in (0,T), \\ \chi(t) \in M(u(t)) \quad \text{in } H \quad \forall t \in (0,T), \end{cases} \tag{5.18}$$

and the energy identity

$$\int_s^t a(u(r) - \chi(r))\, dr + \mathscr{F}(u(t), \chi(t)) = \mathscr{F}(u(s), \chi(s)) \quad \forall\, 0 \leq s \leq t \leq T. \tag{5.19}$$

Let us stress that Theorem 5.4 yields the existence of a special class of solutions of Problem 5.1, satisfying in particular the energy identity (5.19).

5.1.2 The weakly coercive case: existence by an approximation argument

In [30, Sec. 5], it has been shown that, in the setting of (5.1)-(5.3) and (5.6), the same conclusions of Theorem 5.4 hold. The proof of this result is performed by an approximation technique which we briefly recall. In fact, this procedure has inspired our approach to the study of the long-time behaviour of the solutions of Problem 5.1 in the weakly coercive case (cf. Section 5.2.2 later on).

For any $\lambda > 0$ we consider the *coercive* bilinear forms $a_\lambda(u,v) := a(u,v) + \lambda(u,v)_H \quad \forall\, u, v \in V$ and the related operators $A_\lambda : V \to V'$. Theorem 5.4 yields the existence of a solution pair $(u_\lambda, \chi_\lambda)$ to the Cauchy problem

$$\begin{cases} u_\lambda'(t) + A_\lambda(u_\lambda(t) - \chi_\lambda(t)) = 0 \quad \text{in } V' \quad \text{for a.e. } t \in (0,T), \\ \chi_\lambda(t) \in M(u_\lambda(t)) \quad \text{in } H \quad \forall t \in (0,T), \\ u_\lambda(0) = u_0, \end{cases} \tag{5.20}$$

fulfilling for any $T > 0$ the energy identity

$$\int_s^t a_\lambda(u_\lambda(r) - \chi_\lambda(r))\, dr + \mathscr{F}(u_\lambda(t), \chi_\lambda(t)) = \mathscr{F}(u_\lambda(s), \chi_\lambda(s)) \quad \forall\, 0 \leq s \leq t \leq T. \tag{5.21}$$

Then, it is possible to show that the sequences $\{u_\lambda\} \subset H^1(0,T;V') \cap L^\infty(0,T;H)$ and $\{\chi_\lambda\} \subset L^\infty(0,T;H)$ in fact approximate a solution of Problem 5.1. We have the following existence and approximation result (cf. [30, Thm. 5.9]).

Theorem 5.5. *Assume* (5.1)-(5.3) *and* (5.6)*, and let F fulfil either* (5.15) *or* (5.16)-(5.17)*. Let $\{(u_\lambda, \chi_\lambda)\}_\lambda$ be the sequence of solution pairs to* (5.20)*. Then, there exists a subsequence $\lambda_k \downarrow 0$ as $k \uparrow$*

$+\infty$ *and a pair* (u, χ) *such that* $u \in H^1(0,T;V') \cap L^\infty(0,T;H)$, $\chi \in L^\infty(0,T;H)$, $u - \chi \in L^2(0,T;V)$, *and the following convergences hold:*

$$
\begin{aligned}
u_{\lambda_k} \to u \quad & \textit{strongly in } C^0([0,T];V'), \\
\phi \circ u_{\lambda_k} \to \phi \circ u \quad & \textit{uniformly on } [0,T].
\end{aligned}
\tag{5.22}
$$

Moreover, the pair (u, χ) *fulfils* $u(0) = u_0$, *the system* (5.18), *and the energy identity* (5.19).

5.2 Long-time behaviour for general quasi-stationary evolution systems

This section is devoted to the investigation of the long-time behaviour of the solutions of the evolution problem

$$
\begin{cases}
u'(t) + A(u(t) - \chi(t)) = 0 & \text{in } V' \quad \text{for a.e.}\, t \in (0,+\infty), \\
\chi(t) + \partial F(\chi(t)) \ni u(t) & \text{in } H \quad \text{for a.e.}\, t \in (0,+\infty).
\end{cases}
\tag{5.23}
$$

In doing so, we maintain the distinction between the two cases: 1. the form a is *coercive* and 2. the form a is *weakly coercive*.

In the coercive case, we shall keep to the abstract gradient flow approach of [30] (cf. Section 5.1.1), and analyze the long-term behaviour of the solutions of (5.23) derived from the related gradient flow equation (5.10). We shall refer to such solutions as *energy solutions* (cf. Definition 5.6 below). More precisely, by using the abstract results presented in the former Section 3, we will show that the set of the *energy solutions* of (5.23) is a generalized semiflow, which possesses a Lyapunov stable global attractor. On the other hand, in the weakly coercive case we shall follow the approximation approach outlined in Section 5.1.2. Specifically, we will only consider the solutions of (5.23) which are limits of *energy solutions* of the approximate coercive problem (5.35) below. These *limiting energy solutions* form a weak generalized semiflow (in the sense of Section 2.2), which possesses a weak global attractor.

5.2.1 The coercive case

Definition 5.6 (Energy solutions). *We say that a function* $u \in H^1(0,T;V') \cap L^\infty(0,T;H)$ $\forall T > 0$ *is an* energy solution *of* (5.23) *in the coercive case if* u *solves the gradient flow equation*

$$
\begin{gathered}
u'(t) + \partial_s \phi(u(t)) \ni 0 \quad \textit{for a.e. } t \in (0,+\infty), \\
\textit{in the Hilbert space } \mathscr{H} := V', \textit{ for the functional} \\
\phi(u) := \begin{cases} \inf_{\chi \in H} \left(\frac{1}{2}\|u - \chi\|_H^2 + F(\chi) \right) & u \in H, \\ +\infty & u \in V' \setminus H. \end{cases}
\end{gathered}
\tag{5.24}
$$

We denote by $\mathcal{E}$ *the set of all energy solutions.*

Note that this definition focuses on the role of the solution component u, rather than on χ. In order to study the long-time behaviour of the energy solutions of (5.23), we shall apply our abstract results Theorem 1 and Theorem 2 in the framework of the phase space (cf. with (3.1))

$$
X := D(\phi) = H, \quad \text{with} \quad d_X(u,v) := \sqrt{a(A^{-1}(u-v))} + |\phi(u) - \phi(v)| \quad \forall u, v \in H,
\tag{5.25}
$$

where as usual we have used the notation $a(w) := a(w,w)$ for $w \in V$.

As we have recalled in Section 5.1.1 (cf. Proposition 5.2), under the assumption (5.3) the potential ϕ in (5.24) is lower semicontinuous on $\mathscr{H}$ and complies with (COMP) and with the coercivity condition (3.4) (since it takes positive values). On the other hand, the chain rule (CHAIN) holds true for ϕ once we assume (5.15) or (5.16)-(5.17). Hence, Theorem 1 guarantees that $\mathcal{E}$ is a generalized semiflow.

In order to apply Theorem 2, we shall check that ϕ complies with (3.5) and with (3.7), with the choice $\mathcal{D} = X = H$, cf. (5.25). Preliminarily, we need the following lemma (in fact, a direct corollary of Proposition 5.2), which sheds light on the set $Z(\mathcal{E})$ of the rest points of the semiflow $\mathcal{E}$.

Lemma 5.7. *Assume* (5.1)-(5.3) *and* (5.5)*. Then,*

$$\forall\, \bar{u} \in Z(\mathcal{E}) = \{u \in H \,:\, \partial_s\phi(u) \ni 0\} \quad \exists\, \bar{\chi} \in M(\bar{u}) : \quad \bar{u} - \bar{\chi} \in V, \quad A(\bar{u} - \bar{\chi}) = 0. \tag{5.26}$$

Proposition 5.8. *Under the assumptions of Lemma* 5.7*, suppose further that the functional* $F : H \to [0, +\infty]$ *fulfils:*

1. *there exist constants* $\kappa_1,\ \kappa_2 > 0$ *such that for all* $\chi \in D(F)$
$$F(\chi) \geq \kappa_1 \|\chi\|_H^2 - \kappa_2, \tag{5.27}$$
2. *and either one of the following*
   (a) *the proper domain of* F
   $$D(F) \text{ is bounded in the metric space } (X, d_X), \tag{5.28}$$
   (b) *there exist two constants* $L_1,\ L_2 > 0$ *such that for all* $\chi \in D(\partial F)$
   $$(\xi, \chi)_H \geq L_1 \|\chi\|_H - L_2 \quad \forall \xi \in \partial F(\chi). \tag{5.29}$$

Then, the potential ϕ *in* (5.24) *satisfies the coercivity condition* (3.5)*. Furthermore, the set* $Z(\mathcal{E})$ *of the rest points for* $\mathcal{E}$ *fulfils*

$$Z(\mathcal{E}) \text{ is bounded in } (X, d_X). \tag{5.30}$$

Proof. Preliminarily, let us recall the representation formula (5.9) for ϕ, and let us fix an element $\overline{\chi} \in D(F)$. Noting that

$$\sup_{\chi \in H} \left((u, \chi)_H - \left(\frac{1}{2} \|\chi\|_H^2 + F(\chi) \right) \right) \geq -\frac{1}{4} \|u\|_H^2 - \frac{3}{2} \|\overline{\chi}\|_H^2 - F(\overline{\chi}), \tag{5.31}$$

we deduce from (5.9) that there exists a constant $J_3 \geq 0$, only depending on the chosen $\overline{\chi}$, such that

$$\phi(u) \leq \frac{3}{4} \|u\|_H^2 + J_3 \quad \forall u \in H, \tag{5.32}$$

i.e., ϕ has at most a quadratic growth. In order to show (3.5), let us note that, by elementary computations and (5.27), there holds

$$\begin{aligned} \frac{1}{2} \|u - \chi\|_H^2 + F(\chi) &\geq \frac{1}{2} \|u\|_H^2 + \frac{1}{2} \|\chi\|_H^2 - (u, \chi)_H + F(\chi) \\ \geq \frac{\kappa_1}{1 + 2\kappa_1} \|u\|_H^2 - \kappa_1 \|\chi\|_H^2 + F(\chi) &\geq \frac{\kappa_1}{1 + 2\kappa_1} \|u\|_H^2 - \kappa_2 \quad \forall \chi \in L^2(\Omega). \end{aligned} \tag{5.33}$$

Hence, by taking the infimum with respect to χ and recalling the definition (5.7) of ϕ, we deduce that ϕ controls the H-norm and (3.5) ensues.

Now, we have to prove the boundedness of the set $Z(\mathcal{E})$ under either the assumption (5.28) or (5.29). We start by showing that $Z(\mathcal{E}) \subset D(F)$. Indeed, let $\bar{u}$ be an arbitrary element of $Z(\mathcal{E})$. It follows from Lemma 5.7 and from the coercivity of A that there exists $\chi \in M(\bar{u})$ such that $\chi = \bar{u}$. In particular, $\bar{u} \in M(\bar{u}) \subset D(F)$. Thus, if (5.28) holds, (5.30) is trivially proved. Let us alternatively assume (5.29). From $\bar{u} \in M(\bar{u})$ we infer $0 \in \partial F(\bar{u})$. Then, (5.29) yields $\|\bar{u}\|_H \leq L_2/L_1$, whence we deduce (5.30) owing to (5.32). □

In view of Proposition 5.8, Lemma 5.7, and Theorem 2.7, we have the following

Theorem 5.9. *Let* (5.1)-(5.3), (5.5), (5.27)*, and either* (5.28) *or* (5.29) *hold. Further, assume that* F *complies either with* (5.15)*, or with* (5.16)-(5.17)*. Then, the set* $\mathcal{E}$ *of the energy solutions of the evolution problem* (5.23) *is a generalized semiflow in the phase space* $X = D(\phi) = H$*, endowed with the metric* (5.25)*, and* $\mathcal{E}$ *satisfy the continuity property* (C4)*. Moreover,* $\mathcal{E}$ *possesses a unique global attractor* $A_{\mathcal{E}}$*, which is Lyapunov stable. Finally, for any trajectory* $u \in \mathcal{E}$ *and all* $u_\infty \in \omega(u)$*, there holds* $0 \in \partial F(u_\infty)$*.*

5.2.2 The weakly coercive case

In the setting of (5.1)-(5.2) and (5.6), we shall work in the phase space

$$X = D(\phi) = H, \quad d_X^w(u,v) := \|u - v\|_{V'} + |\phi(u) - \phi(v)| \quad \forall u, v \in H, \tag{5.34}$$

where ϕ is defined by (5.24). Along the lines of the approximation procedure outlined in Section 5.2.2, for any $\lambda > 0$ we consider the set $\mathcal{E}_\lambda$ of the *energy solutions* (cf. Definition 5.6) of the approximate problems (cf. with (5.20))

$$\begin{cases} u_\lambda'(t) + A_\lambda(u_\lambda(t) - \chi_\lambda(t)) = 0 \quad \text{in } V' \quad \text{for a.e. } t \in (0, +\infty), \\ \chi_\lambda(t) \in M(u_\lambda(t)) \quad \text{in } H \quad \forall t \in (0, +\infty). \end{cases} \tag{5.35}$$

Now, we may introduce the class of solutions of (5.23) to which we shall restrict our investigation.

Definition 5.10 (Limiting energy solutions.)**.** *We say that a function $u \in H^1(0,T;V') \cap L^\infty(0,T;H)$ for all $T > 0$ is a* limiting energy solution *to the evolution problem* (5.23) *in the weakly coercive case, if u fulfils the system* (5.18) *a.e. on* $(0, +\infty)$*, the energy identity* (5.19) *for all $0 \le s \le t < +\infty$, and there exists a sequence $\{\lambda_k\}$, $\lambda_k \downarrow 0$ as $k \uparrow +\infty$, and a sequence $u_{\lambda_k} \in \mathcal{E}_{\lambda_k}$ for all k, such that*

$$u_{\lambda_k} \to u \quad \textit{in } X \textit{ locally uniformly on } \ [0, +\infty). \tag{5.36}$$

We denote by $\overline{\mathcal{E}}$ the set of all limiting energy solutions.

Once again, in this definition we only focus on the role of the variable u. In fact, as it will be clear from the sequel, for any $u \in \overline{\mathcal{E}}$ there exists a function $\chi \in L^\infty(0,T;H)$ for all $T > 0$ such that $u - \chi \in L^2(0,T;V)$ for all $T > 0$ and (5.18), (5.19) hold on $[0, +\infty)$, cf. the proof of Proposition 5.11. Of course, Definition 5.10 has been inspired by the existence Theorem 5.5, ensuring that the set $\overline{\mathcal{E}}$ is non-empty and indeed complies with the axiom (H1) of the definition of a generalized semiflow. In the forthcoming Propositions 5.11, 5.12 we shall get further insight into the semiflow properties of $\overline{\mathcal{E}}$.

Proposition 5.11. *Assume* (5.1)-(5.3) *and* (5.6)*, and let F fulfil either* (5.15) *or* (5.16)- (5.17)*. Then, $\overline{\mathcal{E}}$ is a weak generalized semiflow complying with* (C4)*, and its elements are continuous functions on* $[0, +\infty)$.

Proof. Axiom (H2) can be trivially checked. The elements of $\overline{\mathcal{E}}$ are continuous on $[0, +\infty)$ since $u \in C^0([0,T];V')$ for all $T > 0$ and the energy identity (5.19) ensures that $\phi \circ u$ is locally absolutely continuous on $[0, +\infty)$.

In order to verify (C4) (which obviously yields (H4)), let us fix a sequence $\{u_n\} \subset \overline{\mathcal{E}}$ such that $u_n(0) \to u_0$ in X, i.e. $u_n(0) \to u_0$ in V' and $\phi(u_n(0)) \to \phi(u_0)$. We aim to show that there exists $u \in \overline{\mathcal{E}}$ such that, up to a subsequence,

$$u_n \text{ converges to } u \text{ in } X \text{ locally uniformly on } [0, +\infty). \tag{5.37}$$

To this purpose, we note that, by definition of $\overline{\mathcal{E}}$, for all n there exists a sequence $\{u_n^{\lambda_k}\}_k \subset \mathcal{E}_{\lambda_k}$ such that $u_n^{\lambda_k} \to u_n$ as $k \uparrow +\infty$ locally uniformly on $[0, +\infty)$. In particular, we can choose some increasing sequence $\{\lambda_{k_n}\}$ (in short: $\{\lambda_n\}$) in such a way that

$$\sup_{t \in [0,n]} d_X^w\big(u_n(t), u_n^{\lambda_n}(t)\big) \le \frac{1}{n}. \tag{5.38}$$

Whence, in particular, $u_n^{\lambda_n}(0) \to u_0$ in X. Thus we have that $\phi(u_n^{\lambda_n}(0)) \le C$ for a constant independent of $n \in \mathbb{N}$. The energy identity (5.21) for the pair $(u_n^{\lambda_n}, \chi_n^{\lambda_n})$ reads on the interval $[0, n]$:

$$\int_s^t a_{\lambda_n}(u_n^{\lambda_n}(r) - \chi_n^{\lambda_n}(r))\,dr + \mathscr{F}(u_n^{\lambda_n}(t), \chi_n^{\lambda_n}(t)) = \mathscr{F}(u_n^{\lambda_n}(s), \chi_n^{\lambda_n}(s)) \tag{5.39}$$

for all $0 \le s \le t \le n$. Using that $\mathscr{F}(u_n^{\lambda_n}, \chi_n^{\lambda_n}) \ge \frac{1}{2}\|u_n^{\lambda_n} - \chi_n^{\lambda_n}\|_H^2$, that the sublevels of F are bounded in H and the first of (5.35), we deduce the a priori estimates

$$\|u_n^{\lambda_n}\|_{H^1(0,n;V')} + \|u_n^{\lambda_n} - \chi_n^{\lambda_n}\|_{L^2(0,n;V)\cap L^\infty(0,n;H)} + \|\chi_n^{\lambda_n}\|_{L^\infty(0,n;H)} \le C$$

for a constant independent of $n \in \mathbb{N}$. Thus, suitable compactness results and a diagonal argument yield that there exist subsequences $\{u_{n_j}^{\lambda_{n_j}}\}$ and $\{\chi_{n_j}^{\lambda_{n_j}}\}$ (we will use the short-hand notation $\{\lambda_j\}$, $\{u_j\}$, and $\{\chi_j\}$), and a pair of functions (u, χ_*), with $u \in H^1(0,T;V') \cap L^\infty(0,T;H)$, $\chi_* \in L^\infty(0,T;H)$ and $u - \chi_* \in L^2(0,T;V)$ for all $T > 0$, for which the following convergences hold as $j \uparrow \infty$:

$$\begin{gathered} u_j \rightharpoonup^* u \quad \text{in } H^1(0,T;V') \cap L^\infty(0,T;H), \qquad u_j \to u \quad \text{in } C^0([0,T];V'),\ \ \forall T > 0, \\ u_j(t) \rightharpoonup u(t) \ \text{ in } H \text{ for any } t \in (0,+\infty), \\ \chi_j \rightharpoonup^* \chi_* \quad \text{in } L^\infty(0,T;H) \quad \text{and} \quad u_j - \chi_j \rightharpoonup u - \chi_* \text{ in } L^2(0,T;V) \quad \forall T > 0. \end{gathered} \tag{5.40}$$

Note that the pointwise weak convergence of u_j follows from the generalized Ascoli theorem [38, Cor. 4]. In particular, $u(0) = u_0$. Hence, (u, χ_*) fulfils

$$u'(t) + A(u(t) - \chi_*(t)) = 0 \quad \text{and} \quad \chi_*(t) \in \mathrm{co}(M(u(t))) \quad \text{for a.e. } t \in (0,+\infty).$$

Moreover, taking the lim sup as $j \uparrow +\infty$ of the energy identity (5.39) with $s = 0$, we get for all $T > 0$

$$\begin{aligned} \int_0^t a(u(r) - \chi_*(r))\,dr + \phi(u(t)) &\le \limsup_{j\uparrow+\infty} \int_0^t a_{\lambda_j}(u_j(r) - \chi_j(r))\,dr + \phi(u_j(t)) \le \phi(u_0) \\ = \phi(u(t)) + \limsup_{j\uparrow+\infty} &\int_0^t a_{\lambda_j}(u(r) - \chi_*(r))\,dr = \phi(u(t)) + \int_0^t a(u(r) - \chi_*(r))\,dr \end{aligned} \tag{5.41}$$

$\forall t \in [0,T]$. Indeed, in (5.41) we have used that, thanks to either (5.15) or to (5.16)-(5.17), for any $T > 0$ the map $\phi \circ u \in AC(0,T)$, and that, for any fixed $j \in \mathbb{N}$, the following chain rule holds:

$$\frac{d}{dt}(\phi \circ u) = \langle u', u - \chi_* \rangle = \langle A_{\lambda_j}(u - \chi_*), u - \chi_* \rangle = a_{\lambda_j}(u - \chi_*) \quad \text{a.e. in } (0,T),$$

see also the proof of [30, Thm. 5.9]. Finally, the last passage in (5.41) follows from the trivial convergence $\lambda_j(u_j - \chi_j) \to 0$ in $L^2(0,T;H)$ as $j \uparrow \infty$. Thanks to the lower semicontinuity argument also exploited in the final part of the proof of Theorem 1, we easily infer from (5.41) that for all $T > 0$

$$A(u_j - \chi_j) \to A(u - \chi_*) \ \text{ strongly in } L^2(0,T;V'), \quad \phi(u_j(t)) \to \phi(u(t)) \ \forall t \in [0,T]. \tag{5.42}$$

By a careful measurable selection argument, detailed in the proof of [30, Thm.5.9], it is possible to show that there exists a function $\chi \in L^\infty(0,T;H)$ fulfilling

$$\chi(t) \in M(u(t)) \quad \forall t \in (0,T), \qquad u - \chi \in L^2(0,T;V), \tag{5.43}$$

$$A(u(t) - \chi_*(t)) = A(u(t) - \chi(t)) \quad \text{for a.e. } t \in (0,T). \tag{5.44}$$

Being T arbitrary, we conclude that the pair (u, χ) fulfils (5.18) a.e. on $(0,+\infty)$. Furthermore, from the energy identities (5.39) and (5.41) we infer for all $t > 0$

$$\begin{aligned} |\phi(u_j(t)) - \phi(u(t))| &\le |\phi(u_j(0)) - \phi(u_0)| + \int_0^t \big| \|A(u_j(s) - \chi_j(s))\|_{V'}^2 - \|A(u(s) - \chi(s))\|_{V'}^2 \big|\,ds \\ &\le |\phi(u_j(0)) - \phi(u_0)| + \Big(\|A(u_j - \chi_j)\|_{L^2(0,t;V')} \\ &\qquad + \|A(u - \chi)\|_{L^2(0,t;V')} \Big) \|A(u_j - \chi_j) - A(u - \chi)\|_{L^2(0,t;V')}. \end{aligned}$$

Hence, in view of (5.38) and of (5.42), we easily conclude (cf. (3.24)), that $\phi(u_j) \to \phi(u)$ locally uniformly on $[0,+\infty)$. Combining the latter convergence with the first of (5.40), we find that

$$u_j \to u \quad \text{in } X \text{ locally uniformly on } [0,+\infty). \tag{5.45}$$

Finally, owing to (5.42)-(5.45), we pass to the limit in the energy identity (5.39), and we deduce that the pair (u, χ) fulfils the energy identity (5.19) for all $0 \leq s \leq t < +\infty$. By the previous construction, u is approximated in the sense of (5.36), whence $u \in \overline{\mathcal{E}}$.

In the end, one directly checks that, for all $T < +\infty$ and $n_j > T$,

$$\sup_{t\in[0,T]} d_X^w\big(u(t), u_{n_j}(t)\big) \leq \sup_{t\in[0,T]} d_X^w\big(u(t), u_j(t)\big) + \sup_{t\in[0,T]} d_X^w\big(u_j(t), u_{n_j}(t)\big)$$
$$\leq \sup_{t\in[0,T]} d_X^w\big(u(t), u_j(t)\big) + \frac{1}{n_j},$$

also in view of (5.38). Owing to (5.45), we conclude the convergence (5.37), and (C4) ensues. □

Proposition 5.12. *Under the same hypotheses of Proposition* 5.11*, assume further that F complies with* (5.27)*. Then, $\overline{\mathcal{E}}$ is compact and eventually bounded.*

Proof. Let us point out that, by Definition 5.10, the limiting energy solutions of Problem 5.1 comply with the energy identity (5.19) just like the energy solutions deriving from the gradient flow equation (5.24). Thus, the eventually boundedness of $\overline{\mathcal{E}}$ follows exactly by the same argument developed in the proof of our abstract Theorem 2 (cf. (3.25)-(3.26)), since assumption (5.27) provides the sufficient coercivity (cf. the proof of Proposition 5.8).

In order to prove that $\overline{\mathcal{E}}$ is compact, we fix a sequence $u_n \in \overline{\mathcal{E}}$ such that $u_n(0)$ is bounded in X. The same computations as in the proof of Proposition 5.11 yield that there exists an increasing sequence $\{\lambda_n\}$ and $u_{\lambda_n} \in \mathcal{E}_{\lambda_n}$ for which (5.38) holds. In particular, note that $\{u_n^{\lambda_n}(0)\}$ is bounded in X. Hence, again exploiting the energy identity (5.39) for the pair $(u_n^{\lambda_n}, \chi_n^{\lambda_n})$, we infer that there exists a subsequence (which we do not relabel) and a limit pair $(u, \bar{\chi})$ for which the convergences (5.40) hold true on $(0, +\infty)$. However, since in this case we cannot conclude anymore that $\{u_n^{\lambda_n}(0)\}$ converges, we cannot exploit the proof of Proposition 5.11 in order to conclude that $u_n^{\lambda_n}$ converges to u locally uniformly on $[0, +\infty)$. Instead, we will argue in the same way as in the proof of the compactness property in Theorem 2. Let us sketch this procedure. First, the energy identity (5.21) yields that the map $t \mapsto \phi(u_n^{\lambda_n}(t))$ is non-increasing. By Helly's Theorem, for all $t > 0$ the function $\varphi(t) := \lim_{n\uparrow+\infty} \phi(u_n^{\lambda_n}(t))$ is well-defined. Moreover, (5.21) and Fatou's Lemma entail that

$$\liminf_{n\uparrow+\infty} \|A(u_n^{\lambda_n}(t) - \chi_n^{\lambda_n}(t))\|_{V'}^2 + \sup_n \left(\frac{1}{2}\|u_n^{\lambda_n}(t) - \chi_n^{\lambda_n}(t)\|_H^2 + F(\chi_n^{\lambda_n}(t))\right) < +\infty$$

(where $\chi_n^{\lambda_n} \in M(u_n^{\lambda_n})$) for almost every $t > 0$. Also using the compactness of the sublevels of F (5.3), one easily infers that for almost any $t > 0$ there exist a subsequence $j \mapsto n_j$, possibly depending on t, and a pair $(\hat{u}(t), \hat{\chi}(t))$ for which (using short-hand notation) $\chi_j(t) \to \hat{\chi}(t)$ and $u_j(t) - \chi_j(t) \to \hat{u}(t) - \hat{\chi}(t)$ strongly in H. Thus, $u_j(t) \to \hat{u}(t)$ in H, whence necessarily $\hat{u}(t) = u(t)$ for a.e. $t \in (0, T)$ thanks to (5.40). Finally, it is not difficult to check that $\hat{\chi}(t) \in M(u(t))$, and that

$$\lim_{j\uparrow+\infty} \phi(u_j(t)) = \phi(u(t)) \quad \text{for a.e.}\, t \in (0, T),$$

cf. with (3.31). Arguing as in (3.32), we finally deduce that $\varphi(t) = \phi(u(t))$ for a.e. $t \in (0, +\infty)$. Thus, exactly as in the proof of Theorem 2 we may pass to the limit in (5.21) for all $t > 0$ and for a.e. $s \in (0, t)$ for which $\varphi(s) = \phi(u(s))$. We can now develop the same energy identity argument of (5.41)-(5.42) (of course replacing u_0 with $u(s)$), and we deduce $\phi(u_j(t)) \to \phi(u(t))$ $\forall t > 0$. Then, exploiting (5.38), we complete the proof of the compactness property. □

Long-time behaviour of the limiting energy solutions. We shall prove that the weak generalized semiflow $\overline{\mathcal{E}}$ of the limiting energy solutions of (5.23) possesses a weak global attractor in the particular case (which is however meaningful in view of the applications):

$$V = H^1(\Omega), \quad H = L^2(\Omega), \quad {}_{H^1(\Omega)'}\langle Au, v\rangle_{H^1(\Omega)} = \int_\Omega \mathsf{A}_1 \nabla u \nabla v \quad \forall\, u, v \in H^1(\Omega). \tag{5.46}$$

Here, $\mathsf{A}_1 : \Omega \to \mathbb{M}^{m\times m}$ is a field of symmetric matrices, with bounded and measurable coefficients, satisfying the usual uniform ellipticity condition

$$\mathsf{A}_1(x)\eta\cdot\eta \geq \rho > 0 \quad \forall\, x\in\Omega,\ \eta\in\mathbb{R}^m,\ |\eta| = 1. \tag{5.47}$$

Let us point out that, according to Definition 5.10 and to (5.46), any limiting energy solution u of (5.23) fulfils the system

$$\begin{cases} u'(t) - \operatorname{div}\mathsf{A}_1\nabla(u(t)-\chi(t)) = 0 \ \text{ in } \Omega\times(0,+\infty), \\ \chi(t)\in M(u(t)) \ \text{ in } \Omega\times(0,+\infty), \\ \mathsf{A}_1\nabla(u-\chi)\cdot\boldsymbol{n} = 0 \ \text{ in } \partial\Omega\times(0,+\infty). \end{cases} \tag{5.48}$$

Note that u is a conserved parameter. Indeed, taking the integral in space of the first equation in (5.48), one finds that the map $t\mapsto \int_\Omega u(t)$ is constant along the evolution. This in particular implies that the semiflow corresponding to the limiting energy solutions of (5.48) is not point dissipative. In other words, the set of stationary solutions of (5.48) is unbounded in $H^1(\Omega)'$. Eventually, no global attractor in the phase space $X = H^1(\Omega)'$ is to be expected (this kind of difficulty is well-known and is, for instance, discussed in [39, Chapter 3] in connection with the long-time analysis of the Cahn-Hilliard equation). Hence, we shall consider some modification of the phase space by fixing explicit bounds on the conserved quantity $\int_\Omega u$. To this aim, we use the notation

$$m(u) := \frac{1}{|\Omega|}\, {}_{H^1(\Omega)'}\langle u, 1\rangle_{H^1(\Omega)}, \quad \mathcal{D}(\bar m) := \left\{ u\in H^1(\Omega)' \,:\, m(u)\leq \bar m\right\}, \tag{5.49}$$

for given $u\in H^1(\Omega)'$ and $\bar m > 0$ (here $|\Omega|$ stands for the volume of Ω).

Note that the energy identity (5.19) suggests that another choice for the invariant region $\mathcal{D}$ could be, for a given positive $C_\phi > 0$,

$$\mathcal{D}_\phi = \{ v\in X : \phi(v)\leq C_\phi\}. \tag{5.50}$$

In the next Theorem, we apply the abstract results of Theorem 2.9 to the set $\overline{\mathcal{E}}$ of the limiting energy solutions of (5.48). Although we give the proof in the case in which $\mathcal{D}$ is as in (5.49), the same results hold also when we choose $\mathcal{D}$ in (5.50).

Theorem 5.13. *In the setting of* (5.46)*, let F comply with* (5.3) *and either with* (5.15) *or with* (5.16)-(5.17)*. Further, suppose that*

$$D(\partial F) \ \text{ is bounded in } L^2(\Omega). \tag{5.51}$$

Then, for any $\bar m > 0$ the set $\overline{\mathcal{E}}$ of the limiting energy solutions of (5.48) *admits the weak global attractor $\mathcal{A}_{\overline{\mathcal{E}}}$ in the set $\mathcal{D}(\bar m)$. Moreover, for any trajectory $u\in\gamma^+(\mathcal{D}(\bar m))$ and for any $u_\infty\in\omega(u)$ there exists $\chi_\infty\in M(u_\infty)$ such that*

$$\begin{cases} -\operatorname{div}\mathsf{A}_1\nabla(u_\infty-\chi_\infty) = 0 \quad \text{in } \Omega, \\ \chi_\infty\in M(u_\infty) \quad \text{in } \Omega \\ \mathsf{A}_1\nabla(u_\infty-\chi_\infty)\cdot\boldsymbol{n} = 0 \quad \text{in } \partial\Omega. \end{cases} \tag{5.52}$$

Note that the assumptions (5.27) and (5.28)-(5.29) of Theorem 5.9 have been replaced by the stronger coercivity condition (5.51).

Proof. Preliminarily, it is easy to see that assumption (5.27) in Proposition 5.12 may be replaced by (5.51). Then, relying on Propositions 5.11, 5.12 we conclude that the weak generalized semiflow $\overline{\mathcal{E}}$ is eventually bounded and compact. Furthermore, since any $u\in\overline{\mathcal{E}}$ complies with the energy identity (5.19) and with (5.18), we have that ϕ is a Lyapunov function for $\overline{\mathcal{E}}$, in fact arguing as in the proof of Theorem 2. Then, in view of Theorem 2.9 it is sufficient to see that for any $\bar m > 0$ the set $\mathcal{D}(\bar m)$ complies with conditions (2.20)-(2.21).

As already observed, for any trajectory u starting from the set $\mathcal{D}(\bar m)$ we have $m(u'(t)) = 0$ for a.e. $t\in(0,+\infty)$. Thus, the invariance condition (2.20) ensues. In order to check (2.21), let us fix

$\bar{u} \in Z(\overline{\mathcal{E}}) \cap \mathcal{D}(\bar{m})$. Recalling (5.48), we easily see that there exists $\bar{\chi} \in M(\bar{u})$ such that the pair $(\bar{u}, \bar{\chi})$ fulfils the system (5.52). In particular, $\bar{\chi} \in D(\partial F)$, so by (5.51) there exists a constant $\bar{r} > 0$ such that $|m(\bar{\chi})| \leq \bar{r}$. Thus, $|m(\bar{u} - \bar{\chi})| \leq \bar{m} + \bar{r}$. Combining this with the first of (5.52) and with Poincaré's inequality, we infer that there exists a positive constant C independent of $\bar{u}$ and $\bar{\chi}$ such that $\|\bar{u} - \bar{\chi}\|_{H^1(\Omega)} \leq C$. Since $\bar{\chi}$ is bounded in $L^2(\Omega)$ by (5.51), we conclude that $\bar{u}$ is bounded in $L^2(\Omega)$. Thus, the set $Z(\overline{\mathcal{E}}) \cap \mathcal{D}(\bar{m})$ is bounded in the phase space (5.34), as ϕ is controlled by the norm on $L^2(\Omega)$, cf. the growth estimate (5.32).

Therefore, the existence of a weak attractor $\mathcal{A}_{\overline{\mathcal{E}}}$ in the set $Z(\overline{\mathcal{E}}) \cap \mathcal{D}(\bar{m})$ is established, and (5.52) follows from the last part of the statement of Theorem 2.9. □

5.2.3 Approximation of the weak global attractor

In this section we discuss the approximation of the weak global attractor of the limiting energy solutions with the global attractor of the weal generalized semiflow $\mathcal{E}_\lambda$, generated by the solutions of the approximating scheme (5.35). We shall denote by X_ϕ the subset $X \cap \mathcal{D}_\phi$ of the phase space $X = D(\phi)$, endowed with the distance d_X (5.25). For any $\lambda > 0$, let $\mathcal{A}_\lambda$ be the global attractor of the generalized semiflow $\mathcal{E}_\lambda$ in the phase space (X_ϕ, d_X), whose existence is ensured by Theorem 5.9. Further, let $\mathcal{A}_{\overline{\mathcal{E}}}$ be the weak global attractor of the set $\overline{\mathcal{E}}$ of the limiting energy solutions of (5.48) in the phase space (X_ϕ, d_X^w) (5.34). Finally, we denote by e_ϕ the Hausdorff semidistance (or *excess*) associated with the distance d_X^w. We have the following

Theorem 5.14. *In the setting of* (5.46)*, let F comply with* (5.3) *and either with* (5.15)*, or with* (5.16)-(5.17)*. Further, assume* (5.51)*. Then,*

$$\lim_{\lambda \downarrow 0} \, e_\phi(\mathcal{A}_\lambda, \mathcal{A}_{\overline{\mathcal{E}}}) = 0. \tag{5.53}$$

Proof. In order to prove (5.53) we argue by contradiction along the lines of Hale & Raugel, cf. [21]. Assume that (5.53) does not hold: then, we can find $r_0 > 0$ and sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \downarrow 0$ and for all $n \in \mathbb{N}$

$$\xi_n \in \mathcal{A}_{\lambda_n}, \quad \inf_{\xi \in \mathcal{A}_{\overline{\mathcal{E}}}} d_X^w(\xi_n, \xi) \geq r_0. \tag{5.54}$$

Now, the invariance of $\mathcal{A}_{\lambda_n}$ (but actually the sole quasi-invariance would be sufficient, see the proof of Theorem 2.8) entails that there exists a complete orbit u_n with $u_n(0) = \xi_n$ and $u_n(t) \in \mathcal{A}_{\lambda_n}$ for all $t \in \mathbb{R}$. It is not difficult to see that this orbit is bounded independently of λ_n with respect to d_X^w. In fact, the energy identity (recall that u_n is in particular an energy solution), (5.50) and the translation invariance of the complete orbit u_n entail that

$$\int_{-T}^{T} |u_n'(s)|^2 \, ds + \phi(u_n(T)) \leq C_\phi \qquad \forall T > 0. \tag{5.55}$$

The proof of Theorem 5.5 in [30] (see also Propositions 5.11 and 5.12) shows that this estimate is sufficient to pass to the limit as $\lambda_n \downarrow 0$, obtaining in the limit a complete and bounded orbit u of the set $\overline{\mathcal{E}}$ of the limiting energy solutions to (5.23). In particular, there holds $\xi_n = u_n(0) \to u(0)$ in X. Now, since by (2.18) the weak global attractor is generated by the complete and bounded orbits of $\overline{\mathcal{E}}$, we conclude that $u(0) \in \mathcal{A}_{\overline{\mathcal{E}}}$. This leads to contradiction with (5.54). □

5.2.4 Applications to quasi-stationary phase field models

Let us consider the following quasi-stationary system, which generalizes the quasi-stationary phase field model (cf. system (1.4)-(1.5)):

$$\begin{cases} \partial_t u - \operatorname{div} \mathsf{A}_1 \nabla(u - \chi) = 0, \\ -\operatorname{div} \mathsf{A}_2 \nabla \chi + \partial \mathcal{W}(\chi) \ni u, \end{cases} \qquad \text{in } \Omega \times (0, +\infty). \tag{5.56}$$

Here, $\mathsf{A}_2 : \Omega \to \mathbb{M}^{m\times m}$ is a field of symmetric matrices, with bounded and measurable coefficients, satisfying the uniform ellipticity condition (5.47). On the other hand, $\mathcal{W}$ is either an arbitrary C^1 real function with superlinear growth (in this case ∂F reduces to $\mathcal{W}'$), or a semi-convex function valued in $\mathbb{R}\cup\{+\infty\}$. Meaningful examples of $\mathcal{W}$ are:

$$\mathcal{W}(\chi) := \frac{(\chi^2-1)^2}{4}, \tag{5.57}$$

$$\mathcal{W}(\chi) := I_{[-1,1]}(\chi) + (1-\chi)^2; \tag{5.58}$$

$$\mathcal{W}(\chi) := c_1\left((1+\chi)\ln(1+\chi) + (1-\chi)\ln(1-\chi)\right) - c_2\chi^2 + c_3\chi + c_4, \tag{5.59}$$

with $c_1, c_2 > 0$ and $c_3, c_4 \in \mathbb{R}$ (see e.g. [11, 4.4, p.170] for (5.59), [7], [41] for (5.58)). The symbol $I_{[-1,1]}$ denotes the indicator function of $[-1,1]$, which forces the constraint $-1 \le \chi \le 1$. In the sequel, we shall employ the notation $D(\mathcal{W}) := \left\{\chi \in L^2(\Omega) : \mathcal{W}(\chi) \in L^2(\Omega)\right\}$.

In [30, Sec. 5], existence results have been obtained for some initial boundary-value problems for (5.56) on a finite time interval. Specifically, (5.56) has been supplemented with the natural homogeneous Neumann boundary condition on χ, and with homogeneous, either Dirichlet or Neumann, boundary conditions on $u-\chi$, and the existence results of [27] and of [32] have been respectively recovered. Here, we shall focus on the long-time behaviour of (5.56), supplemented with both kinds of boundary conditions. In fact, we shall apply the abstract results of Sections 5.2.1, 5.2.2 to suitable families of solutions of the related boundary value problems.

Attractor for the quasi-stationary phase field model with Dirichlet-Neumann boundary conditions. We supplement (5.56) with the boundary conditions

$$u-\chi = 0, \quad A_2\nabla\chi\cdot\boldsymbol{n} = 0 \quad \text{in } \partial\Omega\times(0,+\infty). \tag{5.60}$$

Note that the system (5.56), (5.60) may be reformulated as the abstract evolution system (5.23) with the choices $V := H^1_0(\Omega)$, $H := L^2(\Omega)$, $V' := H^{-1}(\Omega)$, $A := -\operatorname{div}(\mathsf{A}_1\nabla\cdot)$, and with $F : L^2(\Omega) \to [0,+\infty]$ given by

$$F(\chi) : \begin{cases} \int_\Omega \left(\frac{1}{2}\mathsf{A}_2(x)\nabla\chi(x)\cdot\nabla\chi(x) + \mathcal{W}(\chi(x))\right)dx & \chi \in H^1(\Omega)\cap D(\mathcal{W}), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.61}$$

As A is coercive on V, we will focus on the *energy solutions* of (5.56)-(5.60). They stem from the gradient flow equation (5.24), in the space $\mathscr{H} = H^{-1}(\Omega)$, for the functional $\phi : H^{-1}(\Omega) \to (-\infty,+\infty]$

$$\phi(u) := \begin{cases} \inf_{\chi\in H^1(\Omega)}\Big\{ \int_\Omega \frac{1}{2}|u(x)-\chi(x)|^2 + \dfrac{1}{2}\mathsf{A}_2(x)\nabla\chi(x)\cdot\nabla\chi(x) + \mathcal{W}(\chi(x))dx\Big\}, & u \in L^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.62}$$

with $\mathcal{W}$ as in (5.57)-(5.59), for instance. Hence, let us check that the assumptions of Theorem 5.9 are fulfilled within this framework. Since the matrix field A_2 is uniformly elliptic, F has strongly compact sublevels in $L^2(\Omega)$ for all the examples (5.57)-(5.59). Concerning condition (5.27), it is sufficient to show that there exist constants $\kappa_1,\ \kappa_2 > 0$ such that $\int_\Omega \big(\mathcal{W}(\chi(x)) - \kappa_1|\chi(x)|^2\big)dx \ge -\kappa_2$, which is satisfied in all cases (5.57)-(5.59). Also note that F complies with (5.16)-(5.17) (with the choice $W = H^1(\Omega)$). Instead, the validity of (5.28) (or (5.29)) depends on the particular choice of the potential $\mathcal{W}$. More precisely, if we choose the singular potentials (5.58) or (5.59), then (5.28) is easily satisfied, since the domain of F fulfils

$$D(F) \subseteq H^1(\Omega)\cap\left\{v \in L^2(\Omega) :\ -1 \le v(x) \le 1, \ \text{ for a.e. } x\in\Omega\right\} \tag{5.63}$$

(the two sets coincide if we choose the potential $\mathcal{W}$ in (5.58)). Thus, $D(F)$ is clearly bounded in $L^2(\Omega)$. On the other hand, it is not difficult to control that the usual double well potential (5.57) complies with (5.29). Eventually, we conclude that the set of the energy solutions of (5.56), (5.60) is a generalized semiflow. Such a semiflow possesses a Lyapunov stable global attractor in the phase space $D(\phi) = L^2(\Omega)$, endowed with the distance defined by the functional ϕ (5.62).

Attractor for the quasi-stationary phase field model with Robin-Neumann boundary conditions. We supplement (5.56) with the conditions

$$\mathsf{A}_1\nabla(u-\chi)\cdot\boldsymbol{n}+\omega(u-\chi)=0, \quad \mathsf{A}_2\nabla\chi\cdot\boldsymbol{n}=0 \quad \text{in } \partial\Omega\times(0,+\infty), \tag{5.64}$$

where $\omega>0$. This problem may be recast in the form (5.23) by setting $V:=H^1(\Omega)$, $H:=L^2(\Omega)$,

$$_{V'}\langle Au,v\rangle_V := \int_\Omega \mathsf{A}_1(x)\nabla(u(x))\cdot\nabla v(x)dx+\omega\int_{\partial\Omega}u(s)v(s)ds,$$

and choosing F as in (5.61). Since A is coercive on $H^1(\Omega)$, we may again consider the energy solutions of (5.56), (5.64) in the sense of Definition 5.6. In this setting, the ambient space $\mathscr{H}$ is $(H^1(\Omega))'$, with ϕ defined by (5.62). Hence, we may argue exactly in the same way as for the Dirichlet-Neumann problem, with the sole difference that now F complies with (5.15). Therefore, Theorem 5.9 applies and we conclude the existence of a global attractor for the semiflow of the energy solutions of (5.56), (5.64). This gradient flow approach could also be extended to tackle more general boundary conditions on $u-\chi$, such as homogeneous Dirichlet (or Robin) on a portion of $\partial\Omega$, and non-homogeneous Neumann on the remaining part.

Attractor for the quasi-stationary phase field model with Neumann-Neumann boundary conditions. We supplement the system (5.56) with the boundary conditions

$$\mathsf{A}_1\nabla(u-\chi)\cdot\boldsymbol{n}=0, \quad \mathsf{A}_2\nabla\chi\cdot\boldsymbol{n}=0 \quad \text{in } \partial\Omega\times(0,+\infty). \tag{5.65}$$

Problem (5.56), (5.65) can be rephrased in the form of Problem 5.23 by setting

$$V:=H^1(\Omega), \quad H:=L^2(\Omega), \quad {}_{V'}\langle Au,v\rangle_V := \int_\Omega \mathsf{A}_1(x)\nabla(u(x))\cdot\nabla v(x)dx, \tag{5.66}$$

and F as in (5.61). Note that A is only weakly coercive on $H^1(\Omega)$. Following the outline of Section 5.2.2, we shall focus on the long-time behaviour of the set $\overline{\mathcal{E}}_{\mathrm{neu}}$ of the limiting energy solutions of (5.56), (5.65). Let us check the conditions of Theorem 5.13. First, note that F satisfies (5.16)-(5.17), with $W=H^1(\Omega)$, for the potential $\mathcal{W}$ as in (5.57)-(5.59). On the other hand, in view of (5.63), condition (5.51) holds true only in the cases of (5.58)-(5.59). Arguing as for the Dirichlet-Neumann and Robin-Neumann cases, it is not difficult to see that F complies with the remaining assumptions of Theorem 5.13. Thus, we conclude that for all $\bar{m}>0$ $\overline{\mathcal{E}}_{\mathrm{neu}}$ admits a unique weak global attractor $\mathcal{A}_{\overline{\mathcal{E}}_{\mathrm{neu}}}$ in the set $\mathcal{D}(\bar{m})$, and that (5.52) holds for $\omega-$limit points of the trajectories. Finally, referring to the notation of Section 5.2.3 (with ϕ defined by (5.62)), we have that the sequence $\{\mathcal{A}_\lambda\}$ of the global attractors of the solutions of the approximate problems (5.35) converges to the weak global attractor $\mathcal{A}_{\overline{\mathcal{E}}_{\mathrm{neu}}}$ in the sense that $\lim_{\lambda\downarrow 0}\ e_\phi(\mathcal{A}_\lambda,\mathcal{A}_{\overline{\mathcal{E}}_{\mathrm{neu}}})=0$.

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