The Isoperimetric Profile of a Noncompact Riemannian Manifold for Small Volumes

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Contents

1 Introduction

Let M be an n-dimensional Riemannian manifold. We deal, mainly, with the problem of finding a relatively compact domain $D \subset\subset M$ that minimizes $Area(\partial D)$ among domains of the same volume, for sufficiently small values of volume. We reformulate the problem in the context of currents, of geometric measure theory. Given $0 < v < Vol(M)$, consider all integral currents T in M with volume v, and denote the mass of the boundary as $Area(\partial T)$. From now on we think to the problem of finding minimizing currents with a fixed volume constraint. This problem is referred as the **isoperimetric problem**, throughout the paper.

When we speak about area and volume, respectively $Area(\cdot)$ and $Vol(\cdot)$, we do not mention the metric when this is clear from the context, but some time it will be necessary to specify the metric for the sake of clarity and according to this convention we can write $Area_g$ and Vol_g where g will be the involved metric.

The principal achievements of this paper concern the link between the theory of pseudo-bubbles and the isoperimetric problem for small volumes, in a complete Riemannian manifold with some kind of boundedness at infinity, on the metric and its fourth derivatives. This task was carried out by the same author in the context of manifolds for which there is existence of minimizers in all volumes, in particular for manifolds with cocompact isometry group or manifolds with finite volume, compare with [RR04] . In this paper, we deal with the same questions, but the technics employed to encompass the difficulties arisen from the lack of existence of minimizers, are completely new. Namely, we embed isometrically the manifold M into a metric space composed of the disjoint union of pieces $(M_\infty, p_\infty, g_\infty)$ that are limit manifolds of sequences $(M, p_j, g)_j$, with $p_j \in M$, in some suitable pointed $C^{k,\alpha}$ topology. The arguments presented here are useful because they permit to show nontrivial propositions for M complete, noncompact, possibly without existence of minimizers, only provided that sufficiently many sequences (M, p_j, g) have a limit in a $C^{k,\alpha}$ toplogy. For the convenience of the reader we repeat the relevant material from [Nar09a],[BM82], [Pet98], [PX09] and [Nar09b] without proofs, thus making our exposition self-contained.

In first we recall the definition of a pseudo-bubble. Let $Q = id - P$, where P is orthogonal projection of $L^2(T^1_pM)$ on the first eigenspace of the Laplacian T^1_pM is the fiber over p of the unit tangent bundle of the Riemannian manifold M.

Definition 1.1. *[Nar09a] A* ***pseudo-bubble*** *is an hypersurface $\mathcal{N}$ embed-*

ded in M such that there exists a point $p \in M$ and a function u belonging to $C^{2,\alpha}(T_p^1 M \backsimeq \mathbb{S}^{n-1}, \mathbb{R})$, such that $\mathcal{N}$ is the graph of u in normal polar coordinates centered at p, i.e. $\mathcal{N} = \left\{exp_p(u(\theta)\theta),\ \theta \in T_p^1 M\right\}$ and $Q(H(u))$ is a real constant, where H is the mean curvature operator.

To state a uniqueness theorem for pseudo-bubbles we need the notion of *center of mass.*

Definition 1.2. *Let (Ω, μ) be a probability space and $f : \Omega \to M$ a measurable function. We consider the following function $\mathcal{E} : M \to [0, +\infty[$:*

$$\mathcal{E}(x) := \frac{1}{2} \int_{\Omega} d^2(x, f(y)) d\mu(y).$$

The ***center of mass*** *of f with respect to the measure μ is the minimum of $\mathcal{E}$ on M, provided that it exists and is unique.*

In particular, we can speak about the center of mass of a hypersurface of small diameter (we apply this definition to the $(n-1)$-dimensional measure of the boundary). The main result on pseudo-bubbles is the following theorem.

Theorem 1.1 ([Nar09a], Theorem 1)**.** *Let M be a complete Riemannian manifold. Denote $\mathcal{F}^{k,\alpha}$ be the fiber bundle on M where the fiber over p is the space of $C^{k,\alpha}$ functions on the unit tangent sphere $T_p^1 M$. There exists a C^∞ map, $\beta : M \times]0, Vol(M)[\to \mathcal{F}^{2,\alpha}$ such that for all $p \in M$, and all sufficiently small $v > 0$, the hypersurface $exp_p(\beta(p, v)(\theta)\theta)$ is the unique pseudo-bubble with center of mass p enclosing a volume v.*

Remark: If g is an isometry of M, g sends pseudo-bubbles to pseudo-bubbles and $g \circ \beta = \beta \circ g$ (g acts only on the first factor M).

1.1 Main Results

According to [MJ00], small solutions of the isoperimetric problem in compact Riemannian manifolds, or noncompact manifolds with cocompact isometry group, are close to geodesic balls. In fact they are graphs, in normal coordinates, of $C^{2,\alpha}$ small functions. This holds as well also for noncompact manifolds under a C^4 bounded geometry assumption, as will be proven in section 3. In any case, it follows that these small isoperimetric domains are pseudo-bubbles.

Remark: C^4 boundedness is due only to the technical limits of the methods employed for proving theorem 3.2. A slight change in the proof actually shows that, this assumption can be relaxed.

The main result of this paper is theorem 1, which provides a criterion for existence of minimizers having sufficiently small volume. Now, let us recall the basic definitions from the theory of convergence of manifolds, as exposed in [Pet98], to state correctly theorem 1.

Definition 1.3 (Petersen [Pet98])**.** *A sequence of pointed complete Riemannian manifolds is said to converge in the pointed $C^{m,\alpha}$ topology $(M_i, p_i, g_i) \to (M, p, g)$ if for every $R > 0$ we can find a domain Ω_R with $B(p, R) \subseteq \Omega \subseteq M$, a natural number $\nu_R \in \mathbb{N}$, and embeddings $F_{i,R} : \Omega_R \to M_i$ for large $i \geq \nu_R$ such that $B(p_i, R) \subseteq F_{i,R}(\Omega_R)$ and $F^*_{i,R}(g) \to g$ on Ω_R in the $C^{m,\alpha}$ topology.*

It is easy to see that this type of convergence implies pointed Gromov-Hausdorff convergence. When all manifolds in question are closed, then the maps F_i are diffeomorphisms. So for closed manifolds we can speak about unpointed convergence. In this case, convergence can therefore only happen if all the manifolds in the tail end of the sequence are diffeomorphic. In particular, classes of closed Riemannian manifolds that are precompact in some $C^{m,\alpha}$ topology contain at most finitely many diffeomorphism types. For the precise definition of $C^{m,\alpha}$ bounded geometry, see the definition below.

Definition 1.4 (Petersen [Pet98])**.** *Suppose A is a subset of a Riemannian n-manifold (M, g). We say that the $C^{m,\alpha}$-norm on the scale of r of $A \subseteq (M, g)$: $||A||_{C^{m,\alpha},r} \leq Q$, if we can find charts $\psi_s : \mathbb{R}^n \supseteq B(0, r) \to U_s \subseteq M$ such that*

- *(i): For all $p \in A$ there exists U_s such that $B(p, \frac{1}{10}e^{-Q}r) \subseteq U_s$.*
- *(ii): $|D\psi_s| \leq e^Q$ on $B(0, r)$ and $|D\psi_s^{-1}| \leq e^Q$ on U_s.*
- *(iii): $r^{|j|+\alpha}||D^j_{g_s}||_\alpha \leq Q$ for all multi indices j with $0 \leq |j| \leq m$.*
- *(iv): Here g_s is the matrix of functions of metric coefficients in the ψ_s coordinates regarded as a matrix on $B(0, r)$.*

Definition 1.5. *For given $Q > 0$, $n \geq 2$, $m \geq 0$, $\alpha \in]0, 1]$, and $r > 0$ define $\mathcal{M}^{m,\alpha}(n, Q, r)$ as the class of complete, pointed Riemannian n-manifolds (M, p, g) with $||(M, g)||_{C^{m,\alpha},r} \leq Q$.*

In the sequel, $n \geq 2$, $r, Q > 0$, $m \geq 4$, $\alpha \in [0, 1]$.

Theorem 1. *There exists $0 < v^* = v^*(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, for every v such that $0 < v < v^*$ then*

(I): The two following statements are equivalent,

(a): the function $p \mapsto f_M(p,v)$ *attains its minimum,*

(b): there exists solutions of the isoperimetric problem at volume v*,*

(II): $I_M(v) = Min\{f_{M_\infty}(p_\infty, v) |\, (M, p_j, g) \to (M_\infty, p_\infty, g)\ for\ some\ (p_j)\}$.

Here $p_j \in M$ *and the function* $p \mapsto f_M(p,v)$ *gives the area of pseudo-bubbles contained in a given manifold* M*, with center of mass* $p \in M$ *and enclosed volume* v*. Moreover, every solution* D *of the isoperimetric problem is of the form* $\beta(p_0, v)$ *where* p_0 *is a minimum of* $p \mapsto f_M(p,v)$ *and conversely. With* β *obtained in theorem 1.1.* f_M *is invariant and* β *equivariant under the group of isometries of* M*.*

The proof of theorem 1 will be achieved at the end of section 3.

Remark: The interest in theorem 1 is the reduction of minimizer's existence problem, with fixed volume, for the area functional, from the original infinite dimensional minimum problem to a finite dimensional one, say to find the minima of a smooth function defined on the manifold M.

Let us mention one important consequence (theorem 2) for the isoperimetric profile defined below.

Definition 1.6. *Let* M *be a Riemannian manifold of dimension* n *(possibly with infinite volume). Denote by* τ_M *the set of relatively compact open subsets of* M *with smooth boundary. The function* $I : [0, Vol(M)[\to [0, +\infty[$ *such that* $I(0) = 0$

$$I : \begin{cases}]0, Vol(M)[& \to \quad [0, +\infty[\\ v & \mapsto \quad Inf_{\left\{ \begin{array}{l} \Omega \in \tau_M \\ Vol(\Omega) = v \end{array} \right\}} \{Area(\partial\Omega)\} \end{cases}$$

is called the ***isoperimetric profile function*** *(or shortly the* ***isoperimetric profile****) of the manifold* M*.*

In this respect, we need to compute an asymptotic expansion of the function $v \mapsto f(p,v)$. We use results of [PX09]. For completeness'sake, the statement of the following theorem is included. Furthermore, we agree that any term denoted $\mathcal{O}(r^k)$ is a smooth function on $\mathbb{S}^{n-1}$ that might depend on p but which is bounded by a constant independent of p times r^k in the C^2 topology.

Definition 1.7. *We denote by* $c_n := \frac{Area(\mathbb{S}^{n-1})}{[Vol(\mathbb{B}^n)]^{\frac{n-1}{n}}}$ *the constant in the Euclidean isoperimetric profile.*

Lemma 1.1 ([Nar09a])**.** *Asymptotic expansion of the area of pseudo-bubbles as a function of the enclosed volume.*

$$f(p,v)=c_n v^{\frac{n-1}{n}}\left\{1+a_p\left(\frac{v}{\omega_n}\right)^{\frac{2}{n}}+\mathcal{O}(v^{\frac{4}{n}})\right\},\tag{1}$$

with $a_p:=-\frac{1}{2n(n+2)}Sc(p)$.

Theorem 2. *For all* $M\in\mathcal{M}^{m,\alpha}(n,Q,r)$*, let*

$$S=Sup_{p\in M}\{Sc(p)\}.$$

Then the isoperimetric profile $I_M(v)$ *has the following asymptotic expansion in a neighborhood of the origin*

$$I_M(v)=c_n v^{\frac{n-1}{n}}\left(1-\frac{S}{2n(n+2)}\left(\frac{v}{\omega_n}\right)^{\frac{2}{n}}+o(v^{\frac{2}{n}})\right).\tag{2}$$

In theorem 2 and lemma 1.1, $\mathcal{O}(t^\alpha)$ and $o(t^\alpha)$ are functions that depend only on t. The asymptotic expansion of the volume of pseudo-bubbles and the volume of their boundary can be computed with theorem 1.1, this yields an expansion for the profile.

1.2 Plan of the article

1. Section 2 describes why and in what sense approximate solutions of the isoperimetric problem, in the case of small volumes, are close to Euclidean balls, providing a decomposition theorem for domains belonging to an almost minimizing sequences in small volumes.

2. In section 3 we prove theorem 1, generalizing to the case of C^4-bounded geometry manifolds some results of [Nar09a], in particular corollary 3.1 that constitutes the only known proof at my knowledge of the fact that for small volumes minimizers are invariant under the action of the groups of isometries of M that fix their barycenters.

3. In section 4 the results of preceding sections and those of [Nar09b], [MJ00], [PX09] are applied to obtain the first two nonzero coefficients of the asymptotic expansion of the isoperimetric profile in the noncompact case under C^4-bounded geometry assumption on M.

1.3 Acknowledgements

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2 Partitions of domains

2.1 Introduction

In this section it is assumed that

1. M has bounded geometry ($|\mathcal{K}| \leq \Lambda$ and $inj_M \geq \varepsilon > 0$) where inj_M is the injectivity radius of M ,
2. the domains $D_j \in \tau_M$ are **approximate solutions** i.e. $\frac{Area(\partial D_j)}{I(Vol(D_j))} \to 1$ for $j \to +\infty$.

We prove in this section the following theorem.

Theorem 3. *Let (M, g) be a Riemannian manifold with bounded geometry, D_j a sequence of approximate solutions of the isoperimetric problem such that $Vol_g(D_j) \to 0$. Then there exist $p_j \in M$, and radii $R_j \to 0$ such that*

$$\lim_{j \to +\infty} \frac{Vol(D_j \Delta B(p_j, R_j))}{Vol(D_j)} \to 0. \tag{3}$$

The proof of theorem 3 occupies the rest of this section.

2.2 Euclidean version of theorem 3

Roughly speaking, we have that in $\mathbb{R}^n$ approximate solutions of the isoperimetric problem are close to balls in the mass norm, as stated in the following theorem. A good reference for the following theorem is [LR03].

Theorem 2.1. *Let $\{T_j\} \subset \mathbb{I}_n(\mathbb{R}^n)$ be a sequence of integral currents, satisfying*

$$\lim_{j \to +\infty} \frac{\mathbf{M}(\partial T_j)}{\mathbf{M}(T_j)^{\frac{n-1}{n}}} = c_n.$$

Then there exist balls W_j such that

$$\frac{\mathbf{M}(T_j - W_j)}{\mathbf{M}(T_j)} \to 0.$$

Sketch of proof: We use here the BV function theory and that of finite perimeter sets as stated in [Giu84] because for all polyhedral chain P, $||\chi_{Spt||P||}||_{BV(\mathbb{R}^n)} < +\infty$. In what follows we translate our problem in the language of BV functions.

Let $|\cdot|$ be the Lebesgue measure on $\mathbb{R}^n$. Now we give an argument for minimizing sequences that will be useful in the sequel. Let $(E_k)_{k\geq 1}$ be a minimizing sequence of domains for the functional $\mathcal{H}^{n-1}(\partial(\cdot))$ such that $|E_k| = 1$.

1. A compactness theorem stated in [Giu84][page 17] ensures that there exists a set E such that a subsequence

$$\chi_{E_k} \to \chi_E$$

in $L^1_{loc}(\mathbb{R}^n)$.
2. By lower semicontinuity of Lebesgue measure and of perimeter it follows $|E| \leq \liminf_{k\to+\infty} |E_k| \leq 1$, $P(E, \mathbb{R}^n) \leq \liminf_{k\to+\infty} P(E_k, \mathbb{R}^n) \leq c_n$.

Now if we show that $|E| = 1$ then we finish the proof because Euclidean isoperimetric domains are round balls, so E is the Euclidean ball of volume 1. This and $L^1(B(0,2))$ convergence together ensure that the mass outside this Euclidean ball goes to zero and that the volume of the set-theoretic symmetric difference $|E\Delta E_k|$ goes to zero.
To prove that $|E| = 1$ is done clearly for Carnot-Caratheodory groups in [LR03] and for this reason I will not repeat it here. It divides into two steps:

- first to show that there exist translates of E_k having an intersection with the ball of radius 1 of mass not less than a constant $m_0 > 0$ (Lemma 4.1 of [LR03]),

- we cannot find a nonnegligible subset of E_k far away from this radius 1 ball because E_k is almost perimeter minimizing among all sets of measure 1 (Lemma 4.2, [LR03]).

To prove the theorem it is sufficient to apply the preceding argument to sets E_j obtained by $supp||T_j||$ by a dilatation of a factor $\frac{1}{\mathbf{M}(T_j)^{\frac{1}{n}}}$ and setting W_j equal to $\mathbf{M}(T_j)^{\frac{1}{n}}E$. ■

Remark: We observe incidentally that the arguments used here don't make use of the monotonicity formula (see next section) but only of the Euclidean isoperimetric inequality.
I want to thank Frank Morgan for suggesting to me a more general and in some respect simpler proof of this result for bubbles clusters in the fourth 2008 edition of his book [Mor08], pages 129-131. This can help in the understanding of earlier work of Almgren [AJ76].

2.3 Lebesgue numbers

Let (M, g) be a Riemannian manifold with bounded geometry. We can construct a good covering of M by balls having the same radius.

Lemma 2.1. *Let (M, g) be a Riemannian manifold with bounded geometry. There exist an integer N, some constants C, $\epsilon > 0$ and a covering $\mathcal{U}$ of M by balls having the same radius 3ϵ and having also the following properties.*

1. *ϵ is a Lebesgue number for $\mathcal{U}$, i.e. every ball of radius ϵ is entirely contained in at least one element of $\mathcal{U}$ and meets at most N elements of $\mathcal{U}$.*
2. *For every ball B of this covering, there exists a C bi-Lipschitz diffeomorphism on an Euclidean ball of the same radius.*

Proof: Let $\epsilon = \frac{inj_M}{2}$. Let $\mathcal{B} = \{B(p, \epsilon)\}$ be a maximal family of balls of M of radius ϵ that have the property that any pair of distinct members of $\mathcal{B}$ have empty intersection. Then the family $2\mathcal{B} := \{B(p, 2\epsilon)\}$ is a covering of M. Furthermore, for all $y \in M$, there exist $B(p, \varepsilon) \subset \mathcal{B}$ such that $y \in B(y, 2\epsilon)$ and thus $B(y, \varepsilon) \subseteq B(p, 3\epsilon)$. Hence ϵ is a Lebesgue number for the covering $3\mathcal{B}$. Let $B(p, 3\epsilon)$ and $B(p', 3\epsilon)$ be two balls of $3\mathcal{B}$ having nonempty intersection. Then $d(p, p') < 6\epsilon$, hence $B(p', \epsilon) \subseteq B(p, 7\epsilon)$. The ratios $Vol(B(p, 7\epsilon))/Vol(B(p, \epsilon))$ are uniformly bounded because the Ricci curvature of M is bounded from below, and hence the Bishop-Gromov inequality applies. The number of disjoints balls of radius ϵ, contained in $B(p, 7\epsilon)$, is bounded and does not depend on p. Thus the number of balls of $3\mathcal{B}$ that intersect one of these balls is uniformly bounded by an integer N. We conclude the proof by taking $\mathcal{U} := 3\mathcal{B}$. In fact by Rauch's comparison theorem, for every ball $B(p, \epsilon)$, the exponential map is C bi-Lipschitz with a constant C that depends only on ϵ and on upper bounds for the sectional curvature $\mathcal{K}$. ■

2.4 Partition domains in small diameter subdomains

This section is inspired by the article of Bérard and Meyer [BM82] lemma II.15 and the theorem of appendix C, page 531.

Proposition 2.1. *Let I be the isoperimetric profile of M. Then*

$$\limsup_{a \to 0} \frac{I(a)}{a^{\frac{n-1}{n}}} \leq c_n.$$

Proof: Fix a point $p \in M$.

$$\limsup_{a\to 0} \frac{I(a)}{a^{\frac{n-1}{n}}} \le \limsup_{a\to 0} \frac{Area(\partial B(p,r(a)))}{Vol(B(p,r(a)))^{\frac{n-1}{n}}}$$

with $r(a)$ such that $Vol(B(p,r(a))) = a$. Changing variables in the limits, we find

$$\begin{aligned} \limsup_{a\to 0} \frac{Area(\partial B(p,r(a)))}{Vol(B(p,r(a)))^{\frac{n-1}{n}}} &= \limsup_{r\to 0} \frac{Area(\partial B(p,r))}{Vol(B(p,r))^{\frac{n-1}{n}}} \\ \limsup_{r\to 0} \frac{r^{n-1}Area(\mathbb{S}^{n-1}) + \cdots}{\left[r^n Vol(\mathbb{B}^n) + \cdots\right]^{\frac{n-1}{n}}} &= c_n. \end{aligned}$$

■

Definition 2.1. *Let $r > 0$. We define the* unit grid *of $\mathbb{R}^n$, G_1, as the set of points which have at least one integer coordinate. We call G a* grid of mesh r *if G is of the form $v + rG_1$ where $v \in \mathbb{R}^n$. We denote by $\mathcal{G}_r := ([0,r]^n, \mathcal{L}^n)$ the set of all grids of mesh r, endowed with its natural Lebesgue measure.*

Proposition 2.2. *Let D be an open subset of $\mathbb{R}^n$.*

$$\frac{1}{r^n}\int_{\mathcal{G}_r} Area(D\cap G)\mathcal{L}^n(dG) = \frac{n}{r}Vol(D).$$

Proof: We observe that every grid G decomposes as a union of n sets $G^{(i)}$ of the type $v + tG_1^{(i)}$ where $G_1^{(i)}$ is the set of points with integer $i-$th coordinate.
Moreover $G^{(i)} \cap G^{(j)}$ has $(n-1)$-dimensional Hausdorff measure equal to zero.

$$\begin{aligned} \frac{1}{r^n}\int_{\mathcal{G}_r} Area(D\cap G)\mathcal{L}^n(dG) &= \frac{1}{r^n}\sum_{i=1}^{n}\int_{[0,r]^n} Area(D\cap G^{(i)})\mathcal{L}^n(dG) \\ &= \frac{1}{r^n}\sum_{i=1}^{n}\int_0^r r^{n-1}Area(D\cap G^{(i)})\mathcal{L}^n(dG) \\ &= \frac{n}{r}Vol(D). \end{aligned}$$

■

Corollary 2.1. *Let $r > 0$. Let D be an open set of $\mathbb{R}^n$. There exists a grid G of mesh r such that*

$$Area(D \cap G) \leq \frac{n}{r} Vol(D). \tag{4}$$

Proposition 2.3. *We denote $D_{G,k}$ the connected components of $D \setminus G$. Then*

$$\frac{\sum_k Area(\partial D_{G,k}) - Area(\partial D)}{Vol(D)^{\frac{n-1}{n}}} \to 0$$

as $\dfrac{Vol(D)^{\frac{1}{n}}}{r} \to 0$.

Proof: For every grid G,

$$\sum_k Area(\partial D_{G,k}) - Area(\partial D) = 2Area(D \cap G).$$

By corollary 2.2, there exists a grid G such that $Area(D \cap G) \leq \frac{n}{r} Vol(D)$. We deduce that

$$0 \leq \frac{\sum_k Area(\partial D_{G,k}) - Area(\partial D)}{Vol(D)^{\frac{n-1}{n}}} \leq \frac{\frac{2n}{r} Vol(D)}{Vol(D)^{\frac{n-1}{n}}} = \frac{2nVol(D)^{\frac{1}{n}}}{r}.$$

Thus if r is very large with respect to $Vol(D)^{\frac{1}{n}}$ then

$$\frac{\sum_k Area(\partial D_{G,k}) - Area(\partial D)}{Vol(D)^{\frac{n-1}{n}}}$$

is close to 0. ■

Proposition 2.4. *Let M be a Riemannian manifold with bounded geometry. Let D_j be a sequence of domains of M so that*

1. $Vol(D_j) \to 0$.
2. $\limsup_{j \to +\infty} \frac{Area(\partial D_j)}{Vol(D_j)^{\frac{n-1}{n}}} \leq c_n$.

For any sequence (r_j) of positive real numbers that tends to zero ($r_j \to 0$) and $\frac{Vol(D_j)^{\frac{1}{n}}}{r_j} \to 0$, there exists a partition $D_j = \bigcup_k D_{j,k}$ of D_j in domains $D_{j,k}$ with $Diam(D_{j,k}) \leq const_M \cdot r_j$ such that

$$\limsup_{j \to +\infty} \frac{\sum_k Area(\partial D_{j,k})}{(\sum_k Vol(D_{j,k}))^{\frac{n-1}{n}}} \leq c_n.$$

Proof: We apply lemma 2.1 and we take a covering $\{\mathcal{U}\}$ of M by balls of radius 3ϵ, of multiplicity N and Lebesgue number $\epsilon > 0$. For every ball $B(p, 3\epsilon)$ of this family, we fix a diffeomorphism $\phi_p : B(p, 3\epsilon) \to B_{\mathbb{R}^n}(0, 3\epsilon)$ of Lipschitz constant C. For every j we fix also a radius $r_j >> Vol(D_j)^{\frac{1}{n}}$ and we map the grids of mesh r_j of $\mathbb{R}^n$ in $B(p, 3\epsilon)$ via ϕ_p, i.e. for $G \in \mathcal{G}_{r_j}$, we have

$$G_p = \phi_p^{-1}(G).$$

Let us denote by $D_{j,k}$ the connected components of $D_j \setminus (\cup_p G_p)$. We are looking for an estimate of the supplementary boundary volume introduced by the partition in this $D_{j,k}$,

$$\sum_k Area(\partial D_{j,k}) - Area(\partial D_j) = 2Area(D_j \cap (\cup_l G_l)).$$

First estimate the average $m = \frac{1}{r_j^n} \int_{\mathcal{G}_{r_j}} Area(D_j \cap (\cup_l G_l))\mathcal{L}^n(dG)$ of this volume over all possible choices of the grids $G \in \mathcal{G}_{r_j}$.

$$\begin{aligned}
m &\leq \frac{1}{r_j^n} \sum_p \int_{\mathcal{G}_{r_j}} Area(D_j \cap G_p)\mathcal{L}^n(dG) \\
&\leq \frac{1}{r_j^n} \sum_p \int_{\mathcal{G}_{r_j}} Area_{(\mathbb{R}^n, \phi_p^{-1*}(g))}(\phi_p(D_j) \cap G)\mathcal{L}^n(dG) \\
&\leq \frac{C}{r_j^n} \sum_p \int_{\mathcal{G}_{r_j}} Area_{(\mathbb{R}^n, can)}(\phi_p(D_j \cap \mathcal{U}_p) \cap G)\mathcal{L}^n(dG) \\
&\leq C\frac{n}{r_j} \sum_p Vol(\phi_p(D_j \cap B(p, 3\epsilon))) \\
&\leq C^2\frac{n}{r_j} \sum_p Vol(D_j \cap B(p, 3\epsilon)) \\
&\leq C^2\frac{n}{r_j} NVol(D_j).
\end{aligned}$$

This is true because every point of M is contained in at most N balls $B(p, 3\epsilon)$. Then there exists G in $\mathcal{G}_{r_j}$ such that

$$Area(D_j \cap (\cup_p G_p)) \leq C^2\frac{n}{r_j} NVol(D_j),$$

and so

$$0 \leq \frac{\sum_k Area(\partial D_{j,k}) - Area(\partial D_j)}{Vol(D_j)^{\frac{n-1}{n}}} \leq 2C^2\frac{n}{r_j} NVol(D_j)^{\frac{1}{n}}.$$

From the last inequality we obtain

$$\limsup_{j\to+\infty} \frac{\sum_k Area^M(\partial D_{j,k})}{(\sum_k Vol^M(D_{j,k}))^{\frac{n-1}{n}}} = \limsup_{j\to 0} \frac{Area^M(\partial D_j)}{Vol^M(D_j)^{\frac{n-1}{n}}} \le c_n.$$

Now, fix $x \in D_j$. By construction, ϵ is a Lebesgue number of the covering $\{\mathcal{U}\}$, and there exists a ball $B(p, 3\epsilon)$ that contains $B_M(x,\epsilon)$. Let $D_{j,k}$ denote each connected component of $D \setminus (\cup_p G_p)$ that contains x, and $D'_{j,k}$ each connected component of $\phi_p(B(p,\epsilon)) \setminus G$ that contains $\phi_p(x)$. We observe that $D'_{j,k}$ is a cube of edge r_j; if j is large enough so that $r_j \le \epsilon/C\sqrt{n}$, then $D'_{j,k}$ is contained in $\phi_p(B(p,\epsilon))$, hence $D_{j,k}$ is contained in $\phi_p^{-1}D'_{j,k}$, which has diameter at most $C\,r_j$. ■

2.5 Selecting a large subdomain

We first show that an almost Euclidean isoperimetric inequality can be applied to small domains.

Lemma 2.2. *Let M be a Riemannian manifold with bounded geometry. Then*

$$\frac{Area(\partial D)}{Vol(D)^{\frac{n-1}{n}}} \ge c_n(1 - \eta(diam(D))) \tag{5}$$

with $\eta \to 0$ as $diam(D) \to 0$.

Proof: In a ball of radius $r < inj(M)$, we reduce to the Euclidian isoperimetric inequality via the exponential map, that is a C bi-Lipschitz diffeomorphism with $C = 1+\mathcal{O}(r^2)$. This implies for all domains of diameter $< r$,

$$\frac{Area(\partial D)}{Vol(D)^{\frac{n-1}{n}}} \ge c_n C^{-2n+2} = c_n(1 - \mathcal{O}(r^2)).$$

■

Second, we have a combinatorial lemma that tells that in a partition the largest domain contains almost all the volume.

Lemma 2.3. *Let $f_{j,k} \in [0,1]$ be numbers such that for all j, $\sum_k f_{j,k} = 1$. Then*

$$\limsup_{j\to+\infty} \sum_k f_{j,k}^{\frac{n-1}{n}} \le 1$$

implies that

$$\lim_{j\to+\infty}\max_k f_{j,k}=1.$$

Proof: We argue by contradiction. Suppose there exists $\varepsilon>0$ for which there exists $j_\varepsilon\in\mathbb{N}$ so that for all $j\geq j_\varepsilon$, we have $\max_k\{f_{j,k}\}\leq 1-\varepsilon$. Then for all $j\geq j_\varepsilon$, we have $f_{j,k}\leq 1-\varepsilon$. From this inequality,

$$\sum_k f_{j,k}^{\frac{n-1}{n}}=\sum_k f_{j,k}f_{j,k}^{\frac{-1}{n}}\geq\frac{\sum_k f_{j,k}}{(1-\varepsilon)^{\frac{1}{n}}}\geq\frac{1}{(1-\varepsilon)^{\frac{1}{n}}},$$

hence

$$\limsup_{j\to+\infty}\sum_k f_{j,k}^{\frac{n-1}{n}}\geq\frac{1}{(1-\varepsilon)^{\frac{1}{n}}}>1,$$

which is a contradiction. ■

Proposition 2.5. *Let M be a Riemannian manifold with bounded geometry. Let D_j be a sequence of approximate solutions in M with volumes that tend to zero. Let r_j be a sequence of positive real numbers such that $r_j\to 0$ and $\frac{Vol(D_j)^{\frac{1}{n}}}{r_j}\to 0$. There exist $p_j\in M$ and $\varepsilon_j\leq const_M r_j$ and subdomains $D'_j\subset D_j$ such that*

1. $D'_j\subseteq B(p_j,\varepsilon_j)$
2. $\frac{Area(\partial D'_j)}{Vol(D'_j)^{\frac{n-1}{n}}}\to 0$
3. $\lim_{j\to+\infty}\frac{Vol^M(D'_j)}{Vol^M(D_j)}=1.$

Proof: Apply proposition 2.4. By the definition of isoperimetric profile and lemma 2.2 we have

$$Area(\partial D_{j,k})\geq I(Vol(D_{j,k}))\geq c_n Vol(D_{j,k})^{\frac{n-1}{n}}(1-\eta_j)$$

where $\eta_j\to 0$. Since

$$\limsup_{j\to+\infty}\frac{\sum_k c_n Vol(D_{j,k})^{\frac{n-1}{n}}(1-\eta_j)}{Vol(D_j)^{\frac{n-1}{n}}}\leq\limsup_{j\to+\infty}\frac{\sum_k Area(\partial D_{j,k})}{Vol(D_j)^{\frac{n-1}{n}}}\leq c_n,$$

$$\limsup_{j\to+\infty}\frac{\sum_k Vol(D_{j,k})^{\frac{n-1}{n}}}{Vol(D_j)^{\frac{n-1}{n}}}\leq\limsup_{j\to+\infty}\frac{1}{1-\eta_j}=1.$$

Now, set $f_{j,k} = \frac{Vol(D_{j,k})}{Vol(D_j)}$. We can suppose that $f_{j,1} = max_k\{f_{j,k}\}$. We apply lemma 2.3 and we deduce that

$$\frac{Vol(D_{j,1})}{Vol(D_j)} \to 1.$$

But by construction $D_{j,1} \subset B_M(p_j, const_M r_j)$ for some sequence of points p_j in M. Finally, proposition 2.4 gives

$$\limsup \frac{Area(\partial D_j, 1)}{Vol(D_j)^{\frac{n-1}{n}}} \leq \limsup \ \leq c_n.$$

Thus one can take $D'_j = D_{j,1}$. ■

2.6 End of the proof of theorem 3

In this subsection we terminate the proof of theorem 3.

Proof: Let D_j be a sequence of approximate solutions with $Vol(D_j) \to 0$. According to proposition 2.5 there exist subdomains $D'_j \subseteq D_j$, points $p_j \in M$ and radii $\varepsilon_j \to 0$ such that

(i): $D'_j \subseteq B(p_j, \varepsilon_j)$.

(ii): $\frac{Vol(D'_j)}{Vol(D_j)} \to 1$.

(iii): $\frac{Area(\partial D'_j)}{Vol(D_j)^{\frac{n-1}{n}}} \to 0$.

We identify all tangent spaces $T_{p_j}M$ with a fixed Euclidean space $\mathbb{R}^n$ and consider the domains $D''_j = exp^{-1}(D'_j)$ in $\mathbb{R}^n$. Since the pulled back metrics $\tilde{g}_j = exp^*_{p_j}(g_M)$ converge to the Euclidean metric,

$$\frac{Area(\partial D''_j)}{Vol(D''_j)^{\frac{n-1}{n}}} \to c_n.$$

According to theorem 2.1, there exist Euclidean balls $W_j = B_{eucl.}(\tilde{q}_j, R_j)$ in $\mathbb{R}^n$ such that

$$\frac{Vol_{eucl.}(D''_j \Delta W_j)}{Vol_{eucl.}(D''_j)} \to 0.$$

Note that $\tilde{g}_j$-balls are close to Euclidean balls,

$$\frac{Vol_{eucl.}(D''_j \Delta W_j)}{Vol_{eucl.}(W_j)} \rightarrow 0.$$

Thus

$$\frac{Vol_{eucl.}(D''_j \Delta B^{\tilde{g}_j}(\tilde{q}_j, R_j))}{Vol_{eucl.}(D''_j)} \rightarrow 0,$$

and then, for $q_j = exp_{p_j}(\tilde{q}_j)$,

$$\frac{Vol_{eucl.}(D'_j \Delta B^{g}(\tilde{q}_j, R_j))}{Vol_{eucl.}(D'_j)} = \frac{Vol_{eucl.}(D''_j \Delta B^{\tilde{g}_j}(\tilde{q}_j, R_j))}{Vol_{\tilde{g}}(W_j)} \rightarrow 0.$$

Finally, since $\frac{Vol(D_j \Delta D'_j)}{Vol(D_j)} \rightarrow 0$, $\frac{Vol_g(D_j \Delta B(q_j, R_j))}{Vol_g(D_j)} \rightarrow 0$.
This completes the proof of theorem 3. ■

2.7 Case of exact solutions

Remark: When we consider the *solutions* of the isoperimetric problem (this is the case treated in [MJ00]), and not *approximate solutions*, the conclusion is stronger. In fact we can prove directly by the monotonicity formula that D_j is of small diameter and this simplifies a lot the arguments showing that D_j are close in flat norm to a round ball.

Lemma 2.4. *Assume D_j is a sequence of solution of the isoperimetric problem. The dilated domains $D'''_j := \frac{exp_{p_j}^{-1}(D_j)}{Vol_g(D_j)^{\frac{1}{n}}}$ are of bounded diameter and hence we can find a positive constant $R > 0$ in the proof of the preceding theorem so that for all $j \in \mathbb{N}$ we have*

$$D'''_j \subseteq B(0, R).$$

Proof: For the domains D'''_j, the mean curvature of the boundary in $(\mathbb{R}^n, eucl)$ $h_j^{eucl} \leq M = const.$ for all j (apply the Lévy-Gromov isoperimetric inequality [Gro86a], [Gro86b]) and hence the monotonicity formula of [All72][5.1 (3)] page 446 gives for a fixed r_0 and all j

$$||\partial D'''_j||(B(a_j, r_0)) \geq e^{-Mr_0} \Theta^{n-1}(||\partial D'''_j||, a_j)\omega_{n-1} r_0^{n-1} \tag{6}$$

$a_j \in spt||\partial D_j'''||$, r_0 for a fixed r_0 and all j. We argue

$$const \geq Area_{g_{can}}(\partial D_j''') \geq \left[\frac{Diam_{g_{can}}(D_j''')}{2r_0}\right]\omega_{n-1}r_0^{n-1}$$

and we can conclude that $Diam_{g_{can}}(D_j''')$ is uniformly bounded. ■

3 Existence for small volumes.

For compact manifolds, the regularity theorem of [MJ00] applies, and there is no need to use the more general theorem 3.2. For noncompact manifolds the situation is quite involved.

3.1 Minimizers are pseudo-bubbles.

When M is noncompact, the regularity theorem of [MJ00] has to be replaced by a more general statement, for the following reasons.

1. Solutions of the isoperimetric problem need not exist in M.
2. Minimizing sequences may escape to infinity, therefore varying ambient metrics cannot be avoided.

Now, let us recall the basic result from the theory of convergence of manifolds, as exposed in [Pet98].

Theorem 3.1 (Fundamental Theorem of Convergence Theory. [Pet98] Theorem 72). *$\mathcal{M}^{m,\alpha}(n,Q,r)$ is compact in the pointed $C^{m,\beta}$ topology for all $\beta<\alpha$.*

In subsequent arguments will be needed a regularity theorem, in a variable metrics context.

Theorem 3.2. *[Nar09b] Let M^n be a compact Riemannian manifold, g_j a sequence of Riemannian metrics of class C^∞ that converges to a fixed metric g_∞ in the C^4 topology. Assume that B is a domain of M with smooth boundary ∂B, and T_j is a sequence of currents minimizing area under volume contraints in (M^n,g_j) satisfying*

$$(*): Vol_{g_\infty}(B\Delta T_j)\to 0.$$

Then ∂T_j is the graph in normal exponential coordinates of a function u_j on ∂B. Furthermore, for all $\alpha\in]0,1[$, $u_j\in C^{2,\alpha}(\partial B)$ and $||u_j||_{C^{2,\alpha}(\partial B)}\to 0$ as $j\to+\infty$.

Remark: Roughly speaking, theorem 3.2 says that if an integral rectifiable current T is minimizing and sufficiently close in flat norm to a smooth current then T is smooth too. In [Nar09b] there is a precise computation of the constants coming from an effective proof of the theorem.
Remark: Theorems 3.1 and 3.2 are the main reason for assuming to work under C^4 bounded geometry assumptions in this paper.

In the sequel we use often the following classical isoperimetric inequality due to Pierre Berard and Daniel Meyer.

Theorem 3.3. *([BM82] Appendix C]). Let M^{n+1} be a smooth, complete Riemannian manifold, possibly with boundary, of bounded geometry (bounded sectional curvature and positive injectivity radius). Then, given $0 < \delta < 1$, (the interesting case is when δ is close to 1) there exists $v_0 > 0$ such that any open set U of volume $0 < v < v_0$ satisfies*

$$Area(\partial U) \geq \delta c_n v^{\frac{n-1}{n}}. \tag{7}$$

Remark: The preceding theorem implies in particular that for a complete Riemannian manifold with bounded sectional curvature and strictly positive injectivity radius holds $I_M(v) \sim c_n v^{\frac{n-1}{n}}$ as $v \to 0$.

Lemma 3.1. *Let $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, and (D_j) a sequence of solutions of the isoperimetric problem with $Vol_g(D_j) \to 0$. Then possibly extracting a subsequence, there exist points $p_j \in M$ such that the domains D_j are graphs in polar normal coordinates of functions u_j of class $C^{2,\alpha}$ on the unit sphere of $T_{p_j}M$ of the form $u_j = r_j(1 + v_j)$ with $||v_j||_{C^{2,\alpha}(\partial B_{T_pM}(0,1))} \to 0$ and radii $r_j \to 0$.*

Proof: We consider tangent spaces $T_{p_j}M$ in this situation we identify them with a fixed copy of $\mathbb{R}^n$ and in this fixed space we carry almost the same analysis as already done in [Nar09a]. In fact we take domains T_j to be $exp_{p_j}^{-1}(D_j)$ rescaled by $\frac{1}{r_j}$ in the same fixed copy of $\mathbb{R}^n$ then T_j is a solution of the isoperimetric problem for the rescaled pulled-back metric $g_j = \frac{1}{r_j^2} exp_p^*(g)$ which converges volumewise to a unit ball. Since the sequence g_j converges at least C^4 to a Euclidean metric, because of the C^4 bounded geometry assumption on g the same arguments as in the preceding lemma applies. ■

Lemma 3.2. *For all $n, r, Q, m \geq 4, \alpha$, there exists $0 < v_1 = v_1(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, for every domain D solution of the isoperimetric problem with $0 < Vol(D) \leq v_1$, there exists a point $p_D \in M$ (depending on D) such that D is the normal graph of a function $u_D \in C^{2,\alpha}(\mathbb{S}^{n-1})$ with $u_D = r_D(1 + v_D)$ and $||v_D||_{C^{2,\alpha}(\mathbb{S}^{n-1})} \to 0$ as $Vol(D) \to 0$.*

Proof: Otherwise there exists a sequence D_j of solutions of the isoperimetric problem with volumes $Vol(D_j) \to 0$ for which ∂D_j is not the graph on the sphere $\mathbb{S}^{n-1}$ of T_pM of a function $u_j = r_j(1+v_j)$ where $||v_j||_{C^{2,\alpha}}$ goes to 0. This contradicts lemma 3.1. ■

Theorem 3.4. *For all n, r, Q, m, α there exists $0 < v_2 = v_2(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, $0 < v < v_2$, if $D \subseteq M$ has volume v and $I_M(v) = Area(\partial D)$ then ∂D is a pseudo-bubble.*

Proof: An analysis of the proof of theorem 1 of [Nar09a] shows how this application of the implicit function theorem gives a constant, say C_0 depending on n, r, Q, m, α such that the normal graph of a function u on the unit tangent sphere centered at $p \in M$ with $||u||_{C^{2,\alpha}} \leq C_0$, solution of the pseudo-bubbles equation is of the form $\beta(p, r)$, $r < r_0$ then the argument given in theorem 3.1 of [Nar09a] applies. ■

Corollary 3.1. *Let $0 < v < v_2$, then for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, suppose that there exist a minimizing current T for the isoperimetric problem with small enclosed volume v, $p \in M$ being its center of mass, and $St_p \leq Isom(M)$ being the stabilizer of p for the canonical action of the group of isometries $Isom(M)$ of M. Then for all $k \in St_p$, we have $k(T) = T$.*

Proof: Following theorem 1.1, ∂T is the pseudo-bubble $\beta(p, r)$ where $\omega_n \rho^n = Vol(T)$. If $k \in St_p$, then, $k(\beta(p, r)) = \beta(k(p), r*)$ for some small $r*$. For small volumes parameter r is in one to one correspondence with parameter v, but v is the enclosed volume and this does not change under the action of an isometry so by uniqueness of pseudo-bubbles we have that $r* = r$ hence $\beta(k(p), r) = \beta(p, r)$ and $k(T) = T$. ■

3.2 Proof of theorem 1.

For what follows it will be useful to give the definitions below.

Definition 3.1. *Let $(D_j)_j \subseteq \tau_M$ we say that $(D_j)_j$ is an* ***almost minimizing sequence in volume*** *$v > 0$ if*

(i): $Vol(D_j) \to v$,

(ii): $Area(\partial D_j) \to I_M(v)$.

Definition 3.2. *Given $\phi : M \to N$ be a diffeomorphism between two Riemannian manifolds and $\varepsilon > 0$. We say that ϕ is a $(1+\varepsilon)$-**isometry** if for every $x, y \in M$ holds $\frac{1}{1+\varepsilon} d_M(x,y) \leq d_N(\phi(x), \phi(y)) \leq (1+\varepsilon) d_M(x,y)$.*

For the convenience of the reader we have divided the proof into a sequence of lemmas. To this aim we start with a very general question about the continuity of the isoperimetric profile function. The following lemma will be useful in many places in the sequel.

Lemma 3.3. *Let M be a complete Riemannian manifold with $|K_M| \leq K$, $inj_M > 0$. Then $I_M : [0, Vol(M)[\to [0, +\infty[$ is continuous.*

Proof: Fix $\varepsilon > 0$ and take a domain D with smooth boundary ε-almost minimizer in volume $0 < v' < Vol(M)$ i.e.:

$$Vol(D) = v',$$

and

$$I_M(v') \leq Area(\partial D) \leq I_M(v') + \varepsilon. \tag{8}$$

Consider a small volume $w' > 0$ and take the domain $D \cup B(p,r)$ with $Vol(B(p,r)) = w'$ and $B(p,r) \cap D = \emptyset$. This yields to

$$\begin{aligned} I_M(v' + w') &\leq Area(\partial(D \cup B(p,r))) \\ &= Area(\partial D) + c_n w'^{\frac{n-1}{n}} \\ &\leq I_M(v') + \varepsilon + c_n w'^{\frac{n-1}{n}}. \end{aligned} \tag{9}$$

The reason for involve in the preceding formula the constant c_n is a consequence of the asymptotic expansion of area of a geodesic balls as a function of volume enclosed. Let $f(r) = \sup_{p \in M} \{Vol(D \cap B(p,r))\}$, hence we get the existence a positive function ϕ, with

$$f(r) \geq \phi(v', r) > 0.$$

It is easy to see that for every $0 < w' < \phi(v', r)$ there exists a point p with $Vol(B(p,r) \cap D) = w'$. Now, we want to consider domains $D - B(p,r)$ and evaluate their boundary area to obtain

$$\begin{aligned} I_M(v' - w') &\leq Area(\partial(D - B(p,r))) \\ &\leq Area(\partial D) + Cr(w')^{n-1} \\ &\leq I_M(v') + \varepsilon + Cr(w')^{n-1}, \end{aligned} \tag{10}$$

where $r(w') = inf\{\rho | \phi(v', \rho) > w'\} \to 0$, as $w' \to 0$, since $r \mapsto \phi(v', r)$ is a strictly increasing positive function and $r(w')$ is its inverse function. Letting

ε tend to zero the following two inequalities hold

$$\begin{aligned} I_M(v'+w') &\leq Area(\partial(D\cup B(p,r))) \\ &= Area(\partial D)+c_n w'^{\frac{n-1}{n}} \\ &\leq I_M(v')+c_n w'^{\frac{n-1}{n}}, \end{aligned} \tag{11}$$

$$\begin{aligned} I_M(v'-w') &\leq Area(\partial(D-B(p,r))) \\ &\leq Area(\partial D)+Cr(w')^{n-1} \\ &\leq I_M(v')+Cr(w')^{n-1}. \end{aligned} \tag{12}$$

From (11) applied to $v'=v$, $w'=w$, and once more applied to $v'=v-w$, $w'=w$, we obtain

$$I_M(v)-c_n w^{\frac{n-1}{n}} \leq I_M(v+w) \leq I_M(v)+c_n w^{\frac{n-1}{n}}, \tag{13}$$

which gives

$$I_M(v)=\lim_{w\to 0^+} I_M(v+w). \tag{14}$$

Applying (12) in first to $v'=v$, $w'=w$, and in second to $v'=v-w$, $w'=w$ we get

$$I_M(v)-Cr(w)^{n-1} \leq I_M(v-w) \leq I_M(v)+Cr(w)^{n-1}, \tag{15}$$

which implies

$$I_M(v)=\lim_{w\to 0^-} I_M(v+w). \tag{16}$$

Combining (14) with (16) we conclude that

$$I_M(v)=\lim_{w\to 0} I_M(v+w).$$

Which is our claim. ■

3.2.1 Existence of a minimizer in a $C^{m,\alpha}$ limit manifold

Lemma 3.4. *Let M be with bounded sectional curvature and positive injectivity radius. $(M,p_j)\to(M_\infty,p_\infty)$ in $C^{m,\alpha}$ topology, $m\geq 1$. Then*

$$I_{M_\infty}\geq I_M. \tag{17}$$

Proof: Fix $0<v<Vol(M)$. Let $D_\infty\subseteq M_\infty$ an arbitrary domain of volume v. Put $r:=d_H(D_\infty,p_\infty)$, where d_H denotes the Hausdorff distance. Consider the sequence $\varphi_j: B(p_\infty,r)\to M$, of $(1+\varepsilon_j)$-isometry given by the convergence of pointed manifolds, for some sequence $\varepsilon_j\searrow 0$. Set $D_j:=\varphi_j(D_\infty)$ and $v_j:=Vol(D_j)$ it is easy to see that

(i): $v_j \to v$,

(ii): $Area_g(\partial D_j) \to Area_{g_\infty}(\partial D_\infty)$.

(i)-(ii) are true because φ_j are $1+\varepsilon_j$ isometries. After this very general preliminary construction that doesn't requires any bounded geometry assumptions on M, we proceed to the proof of (17) by contradiction. In this respect suppose that there exist a volume $0 < v < Vol(M)$ satisfying

$$I_{M_\infty}(v) < I_M(v). \tag{18}$$

Then there is a domain $D_\infty \subseteq M_\infty$ such that

$$I_{M_\infty}(v) \leq Area_{g_\infty}(\partial D_\infty) < I_M(v).$$

As above we can find domains (D_j) satisfying (i)-(ii). But by definition $I_M(v_j) \leq Area_g(\partial D_j)$ hence passing to the limit we get

$$I_M(v) = \lim_{j\to+\infty} I_M(v_j) \leq \lim_{j\to+\infty} Area_g(\partial D_j) = Area_{g_\infty}(\partial D_\infty) < I_M(v). \tag{19}$$

(19) shows that (18) is incompatible with the assumption of the theorem. ■

The next lemma is simply a restatement of theorem 3.

Lemma 3.5. *For all n, r, Q, m, α, and $\varepsilon > 0$ there exists $0 < v_3 = v_3(n, r, Q, m, \alpha, \varepsilon)$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, there is a positive number $\eta = \eta(\varepsilon, M) > 0$ with the following properties if $0 < v = Vol(D) < v_3$, $\frac{Area(\partial D)}{I_M(Vol(D))} < 1 + \eta$ it follows that there exists $p = p_D \in M$, $R = C(n, r, Q, m, \alpha)v^{\frac{1}{n}}$ satisfying*

$$\frac{Vol(D\Delta B(p,R))}{Vol(D)} \leq \varepsilon. \tag{20}$$

Proof: As it is easy to check this lemma is a restatement of theorem 3 in an ε-δ language with a little extra effort about uniformity in the class $\mathcal{M}^{m,\alpha}(n, Q, r)$, after having observed that the constant C used in the proof of lemma 2.4 depends only on n, r, Q, m, α. ■

Definition 3.3. *Let M be a Riemannian manifold. $0 < v < Vol(M)$ we say that $I_M(v)$ is **achieved** if there exists an integral current $D \subseteq M$ such that $Vol(D) = v$ and $Area(\partial D) = I_M(v)$.*

Lemma 3.6. *For all n, r, Q, m, α there exist $0 < v_4 = v_4(n, r, Q, m, \alpha)$, $C_1 = C_1(n, r, Q, m, \alpha) > 0$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, $0 < v < v_4$, with $I_M(v)$ achieved then*

$$I_M(v+h) \leq I_M(v) + C_1 h v^{-\frac{1}{n}}, \tag{21}$$

provided that $v + h < v_4$.

Proof: Let us define, $v_4 = Min\{1, v_0, v_1, v_2\}$. Put $\psi_{M,p}(\tilde{v}) = Area(\beta)^{\frac{n}{n-1}}$ where β is the pseudo-bubble of M, centered at p and enclosing volume $\tilde{v}$. Then $\tilde{v} \mapsto \psi_{M,p}(\tilde{v})$ is C^1 and $||\psi_{M,p}||_{C^1([0,v_4])} \leq C$ uniformly with respect to M and p, i.e., $C = C(n, r, Q, m, \alpha)$, this is a nontrivial consequence of the proof of the existence of pseudo-bubbles that could be found in [Nar09a]. When $v + h < v_4$,

$$\psi_{M,p}(v+h) \leq \psi_{M,p}(v) + Ch.$$

$$\begin{array}{rcl} I_M(v+h) & \leq & \psi_{M,p}(v+h)^{\frac{n-1}{n}} \\ & \leq & \psi_{M,p}(v)^{\frac{n-1}{n}}\left(1 + \frac{Ch}{\psi_{M,p}(v)}\right)^{\frac{n-1}{n}} \\ & \leq & \psi_{M,p}(v)^{\frac{n-1}{n}}\left(1 + \frac{n-1}{n}C'h\right) \\ & \leq & \psi_{M,p}(v)^{\frac{n-1}{n}} + C_1 h v^{-\frac{1}{n}} \\ & \leq & I_M(v) + C_1 h v^{-\frac{1}{n}}. \end{array} \tag{22}$$

■

Now we want to apply the theory of convergence of manifolds suitably mixed with geometric measure theory to the isoperimetric problem for small volumes. Some parts of the proof are inspired from [RR04]

Lemma 3.7. *For all n, r, Q, m, α, there exists $0 < v_6 = v_6(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$, and for all v, with $0 < v < v_6$ there is a sequence of points p_j, a limit manifold $(M_\infty, p_\infty, g_\infty) \in \mathcal{M}^{m,\alpha}(n, Q, r)$ such that*

(I): $(M, p_j, g) \to (M_\infty, p_\infty, g_\infty)$ in $C^{m,\beta}$ topology for $\beta < \alpha$,

(II): $I_{M_\infty}(v)$ is achieved,

(III): D_∞ is a pseudo-bubble,

(IV): $I_M(v) = I_{M_\infty}(v)$.

Proof: Fix $1 > \delta > 0$, and $\varepsilon > 0$ such that

$$\frac{1}{2}\delta\frac{c_n}{C_1} > \gamma(\varepsilon)^{\frac{1}{n}} > 0, \tag{23}$$

with $\gamma = \gamma(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}$. Observe that this is possible because $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Set $v_6 = Min\{v_0, v_2, v_3, v_4\}$ as obtained respectively in lemma 3.3, 3.5, 3.6 and theorem 3.4. Let $0 < v < v_6$. Let D_j be a minimizing sequence in volume v i.e. $Vol(D_j) = v$ and $Area(\partial D_j) \to I_M(v)$. Take now j large enough to have $\frac{Vol(\partial D_j)}{I_M(v)} < 1 + \eta_\varepsilon$ with $\eta_\varepsilon > 0$ as in theorem 3.5. There exist p_j, R s.t.

$$\frac{Vol(D_j \Delta B(p_j, R))}{Vol(D_j)} \leq \varepsilon.$$

By theorem 3.1 applied to the sequence of pointed manifolds $(M, p_j, g)_j \subset M^{m,\alpha}(n, Q, r)$ we obtain the existence of a pointed manifold $(M_\infty, p_\infty, g_\infty)$ s.t. $(M, p_j) \to (M_\infty, p_\infty, g_\infty)$ in $C^{4,\beta}$ topology.

What we want to do in the sequel is to define domains $\tilde{D}_j^c \subseteq M_\infty$ (passing to a subsequence if necessary), that are images via the diffeomorphisms F_j of $C^{4,\beta}$ convergence of a suitable truncation D'_j of D_j with balls whose radii t_j are given by the coarea formula (because it is needed to control the amount of area added in the truncation procedure), to obtain an integral current $D_\infty \subseteq M_\infty$ s.t. $\tilde{D}_j^c \to D_\infty$ in $\mathcal{F}_{loc}(M_\infty)$ topology. This goal will be achieved by taking an exhaustion of M_∞ by geodesic balls, applying a standard compactness argument of geometric measure theory in each of these balls and using a diagonal process.

Take a sequence of scales (r_i), $i \geq 0$ satisfying $r_0 \geq R$ and $r_{i+1} \geq r_i + 2i$, consider an exhaustion of M_∞ by balls of center p_∞ and radius r_i, i.e. $M_\infty = \bigcup_i B(p_\infty, r_i)$. Then for every i the convergence in $C^{4,\beta}$ topology gives existence of $\nu_{r_i} > 0$ and diffeomorphisms $F_{j,r_i} : B(p_\infty, r_i) \to B(p_j, r_i)$ for all $j \geq \nu_{r_i}$, that are $(1 + \varepsilon_j)$-isometries for some sequence $0 \leq \varepsilon_j \to 0$.

At this stage we start the diagonal process, determining a suitable double sequence of cutting radii $t_{i,j} > 0$ with $i \geq 1$ and $j \in S_i \subseteq \mathbb{N}$ for some sequence of infinite sets $S_1 \supseteq ... \supseteq S_{i-1} \supseteq S_i \supseteq S_{i+1} \supseteq ...$, defined inductively. Before to proceed we recall the argument of coarea used in this proof repeatedly. For every domain $D \subseteq M$, every point $p \in M$, and interval $J \subseteq \mathbb{R}$ there exists $t \in J$ such that

$$Area(D \cap (\partial B(p, t))) = \frac{1}{|J|}\int_J Area((\partial B(p, s)) \cap D)ds \leq \frac{Vol(D)}{|J|}. \tag{24}$$

We proceed as follow, cut by coarea with radii $t_{1,j} \in]r_1, r_1 + j[$ for $j \geq \nu_{r_2}$ we get domains $D'_{1,j} = D_j \cap B(p_j, t_{1,j})$, $D''_{1,j} = D_j - D'_{1,j}$ for j large enough (i.e., $j \geq \nu_{r_1}$), satisfying

$$\left|Area(\partial D'_{1,j}) + Area(\partial D''_{1,j}) - Area(\partial D_j)\right| \leq \frac{v}{1}. \tag{25}$$

Consider the sequence of domains $\left(\tilde{D}_{1,j} = F^{-1}_{j,r_2}(D'_{1,j})\right)_j$ for $j \geq \nu_{r_2}$, it is true that

1. $Area(\partial D'_{1,j}) \leq Area(\partial D_j) + 2\frac{v}{1} \leq I_M(v) + 2\frac{v}{1}$,
2. $Vol(D'_{1,j}) \leq v$,

so we have volume and boundary area, of the sequence of domains, bounded by a constant. A standard argument of geometric measure theory allows us to extract a subsequence $D'_{1,j}$ with $j \in S_1 \subseteq \mathbb{N}$, converging on $B(p_\infty, r_2)$ to a domain $D_{\infty,1}$ in $\mathcal{F}_{B(p_\infty, r_2)}$. Now we look at the subsequence D_j with $j \in S_1$ and repeat the preceding argument to obtain radii $t_{2,j} \in]r_2, r_3[$ and a subsequence $D'_{2,j} = D_j \cap B(p_j, t_{2,j})$ for $j \in S_1$ and $j \geq \nu_{r_3}$ such that

$$\left|Area(\partial D'_{2,j}) + Area(\partial D''_{2,j}) - Area(\partial D_j)\right| \leq \frac{v}{2}. \tag{26}$$

Analogously, the sequence $\left(\tilde{D}_{2,j} = F^{-1}_{j,r_3}(D'_{2,j})\right)_j$ for j running in S_1 has bounded volume and bounded boundary area, so there is a convergent subsequence $\left(\tilde{D}_{2,j}\right)$ defined on some subset $S_2 \subseteq S_1$ that is convergent on $B(p_\infty, r_3)$ to a domain $D_{\infty,2}$ in $\mathcal{F}_{B(p_\infty, r_3)}$. Continuing in this way, we obtain the existence of $S_1 \supseteq ... \supseteq S_{i-1} \supseteq S_i$, radii $t_{k,j} \in]r_k, r_k + k[$, domains $D'_{i,j} = D_j \cap B(p_j, t_{i,j})$, $D''_{i,j} = D_j - D'_{i,j}$ satisfying

$$\left|Area(\partial D'_{kj}) + Area(\partial D''_{kj}) - Area(\partial D_j)\right| \leq \frac{v}{k}, \tag{27}$$

for all $1 \leq k \leq i$ and $j \in S_k$ and for all $i \geq 1$. Moreover, putting $\tilde{D}_{k,j} = F^{-1}_{j,r_{k+1}}(D'_{k,j})$ for all $1 \leq k \leq i$ and $j \in S_k$ we have convergence of $(\tilde{D}_{k,j})_{j \in S_k}$ on $B(p_\infty, r_{k+1})$ to a domain $D_{\infty,k}$ in $\mathcal{F}_{B(p_\infty, r_{k+1})}$ for all $i \geq 1$ and $k \leq i$. Let j_i be chosen inductively so that

$$j_i < j_{i+1} \tag{28}$$

$$Vol(\tilde{D}_{i,\sigma_i(j_i)} \Delta D_{\infty,i}) \leq \frac{1}{i}, \tag{29}$$

define $\sigma(i) = \sigma_i(j_i)$, then the sequence $\tilde{D}^c_i := F^{-1}_{\sigma(i), r_{i+1}}(D'_{i,\sigma(i)})$ converges to $D_\infty = \bigcup_i D_{\infty,i}$ in $\mathcal{F}_{loc}(M_\infty)$ topology. Observe, here that $|t_{i+1} - t_i| > i$.

From now on, we restrict our attention to the sequences $\bar{D}_i = D_{\sigma_i}$, $\bar{D}'_i = D'_{\sigma_i}$, $\bar{D}''_i = D''_{\sigma_i}$, then we will call always D_i, D'_i, and D''_i, by abuse of notation. Put, also $F_i = F_{\sigma(i),r_{i+1}}$. Rename i by j. From this construction we argue that passing possibly to a subsequence one can build a minimizing sequence D_j with the following properties

(i): $\left|Area(\partial D'_j) + Area(\partial D''_j) - Area(\partial D_j)\right| \leq \frac{v}{j}$, for all j,

(ii): $\lim_{j\to+\infty} Area_g(\partial D'_j) = \lim_{j\to+\infty} Area_{g_\infty}(\partial \tilde{D}^c_j)$,

(iii): $Vol(\tilde{D}^c_j) \to Vol(D_\infty) = v_\infty$,

(iv): $Area(\partial D_\infty) \leq \liminf Area(\partial \tilde{D}^c_j)$,

(v): $v \geq v_\infty \geq (1-\varepsilon)v > 0$,

(vi): $\frac{w_\infty}{v_\infty} \leq \gamma$ with $w_\infty = v - v_\infty$,

(vii): $I_{M_\infty}(v_\infty) = Area(\partial D_\infty)$,

(viii): $Area(\partial D_\infty) = \liminf Area(\partial \tilde{D}^c_j)$.

(i) follows directly by the construction of the sequences (D'_j). (ii) is an easy consequences of the fact that the diffeomorphisms given by $C^{4,\beta}$ convergence are $(1+\varepsilon_j)$-isometry for some sequence $0 \leq \varepsilon_j \to 0$. To prove (iii) observe

$$\begin{aligned} |Vol(\tilde{D}^c_j) - Vol(D_\infty)| &\leq |Vol(\tilde{D}^c_j) - Vol(D_\infty \cap B_{r_{j+1}})| + Vol(D_\infty - B_{r_{j+1}}) \\ &\leq Vol((\tilde{D}^c_j \Delta D_\infty) \cap B_{r_{j+1}}) + Vol(D_\infty - B_{r_{j+1}}), \end{aligned}$$

and so $\lim_{j\to\infty} Vol(\tilde{D}^c_j) = Vol(D_\infty)$ by (28). On the other hand, the definition of the sets $\tilde{D}^c_j$ gives us $\{D^c_j\} \to D$ in $\mathcal{F}_{loc}(M)$. Hence $Area(\partial D) \leq \liminf_{j\to\infty} Area(\partial \tilde{D}^c_j)$ by the lower semicontinuity of boundary area with respect to flat norm in $\mathcal{F}_{loc}(M)$ which actually proves (iv). In (v) the first inequality is true because every $D_{\infty,i}$ is a limit in flat norm of a sequence of currents having volume less than v, the second beacuse the radii r_i are greater than R so $Vol(D_{\infty,i}) \geq (1-\varepsilon)v$. (vi) follows easily by (v). To show (vii) we proceed by contradiction. Suppose that there exists a domain $\tilde{E} \in \tau_{M_\infty}$ having $Vol(\tilde{E}) = v_\infty$, $Area(\partial \tilde{E}) < Area(\partial D_\infty)$. Take the sequence of radii $s_j \in]t_j, t_{j+1}[$ and cut $\tilde{E}$ by coarea obtaining $\tilde{E}_j := \tilde{E} \cap B(p_\infty, s_j)$ in such a manner that

$$Area_{g_\infty}(\tilde{E}_j \cap \partial B(p_\infty, s_j)) \leq \frac{v_\infty}{j}, \tag{30}$$

Of course, $Vol_{g_\infty}(\tilde{E}_j) \to v_\infty$, since $s_j \nearrow +\infty$. Now, fix a point $x_0 \in \partial\tilde{E}$ and a small neighborhood $\mathcal{U}$ of x_0. For j large enough $\mathcal{U} \subseteq B(p_\infty, r_j)$. Push forward $\tilde{E}_j$ in M getting $E_j := F_j(\tilde{E}_j) \subseteq B(p_j, r_{j+1})$ so readjusting volumes by modifying slightly E_i in $F_i(\mathcal{U})$ contained in $B(p_j, t_{j+1})$, we obtain domains $E'_j \subseteq B(p_j, r_{j+1})$ with the properties

$$E'_j \cap D''_j = \emptyset, \tag{31}$$

$$Vol_g(E'_j \cup D''_j) = v, \tag{32}$$

$$Area(\partial E'_j) \leq Area(\partial E_j) + c\Delta v_j, \tag{33}$$

with $\Delta v_j = Vol_g(E'_j) - Vol_g(E_j)$, satisfying $\Delta v_j \to 0$ as $j \to +\infty$, by virtue of $Vol(\tilde{E}_j) \to v_\infty$ (i.e. $Vol(D'_j) \to v_\infty$) and $Vol(D''_j) \to v - v_\infty$. Note that $c = c(n, Q)$ is a constant independent of j. Define $D^*_j := E'_j \cup D''_j$.

$$\begin{aligned} Area(\partial D^*_j) &\leq Area(\partial E'_j) + Area(D''_j) \\ &\leq (1+\varepsilon_j)^{n-1} Area(\partial \tilde{E}_j) + c\Delta v_j + Area(\partial D''_j) \\ &\leq (1+\varepsilon_j)^{n-1} (Area(\partial \tilde{E}) + \frac{v_\infty}{j}) + c\Delta v_j + Area(\partial D''_j), \end{aligned}$$

hence we get

$$\begin{aligned} \liminf_{j\to+\infty} Area(\partial D^*_j) &\leq Area(\partial \tilde{E}) + \liminf_{j\to+\infty} Area(\partial D''_j) \\ &< Area(D_\infty) + \liminf_{j\to+\infty} Area(\partial D''_j) \\ &\leq \liminf_{j\to+\infty} Area(\partial D'_j) + \liminf_{j\to+\infty} Area(\partial D''_j) \\ &\leq I_M(v). \end{aligned}$$

This means that the sequence of domains D^*_j do better than the minimizing sequence D_j, which is a contradiction that proves (vii). The proof of (viii) is similar; in fact we only have to work with D_∞ instead of $\tilde{E}$. We must remark that this can be done since the set of regular points in $\partial D_\infty \cap M_\infty$ is open.

Letting $i \to +\infty$ in (i), taking into account (ii), (iv) and (vii), and Berard-Meyer inequality yields

$$I_{M_\infty}(v_\infty) + \delta c_n w_\infty^{\frac{n-1}{n}} \leq I_M(v). \tag{34}$$

It remains to prove that v_∞ cannot be strictly less than v, by contradiction. We know that $v \leq v_4 \leq v_2$ then D_∞ is a pseudo-bubble as it is easy to check by corollary 3.4. This allow one to have as a direct consequence of lemma 3.6, the following estimate

$$I_{M_\infty}(v) = I_{M_\infty}(v_\infty + w_\infty) \leq I_{M_\infty}(v_\infty) + C_1 v_\infty^{-\frac{1}{n}} w_\infty. \tag{35}$$

Assume $w_\infty > 0$. From (34), (35) and lemma 3.4 one deduce

$$I_{M_\infty}(v_\infty) + \delta c_n w_\infty^{\frac{n-1}{n}} \leq I_M(v) \leq I_{M_\infty}(v) \leq I_{M_\infty}(v_\infty) + C_1 v_\infty^{-\frac{1}{n}} w_\infty. \tag{36}$$

$$\delta c_n w_\infty^{\frac{n-1}{n}} \leq C_1 v_\infty^{-\frac{1}{n}} w_\infty. \tag{37}$$

Dividing the above inequalities by $w_\infty^{\frac{n-1}{n}}$ and combining with (vi) we obtain

$$\gamma(\varepsilon)^{\frac{1}{n}} \geq \delta \frac{c_n}{C_1}, \tag{38}$$

which by our choice of $\varepsilon > 0$ contradicts (23). So $w_\infty = 0$, which means $v_\infty = v$ and clearly $I_{M_\infty}(v) = I_{M_\infty}(v_\infty)$ which proves (II) and (III). To finish the proof, we need of a last argument that give us (IV). In fact

$$\begin{aligned} I_M(v) &= \liminf Area(\partial D'_j) + \liminf Area(\partial D''_j) \\ &= I_{M_\infty}(v_\infty) + \liminf Area(\partial D''_j) \\ &= I_{M_\infty}(v) + \liminf Area(\partial D''_j) \\ &\geq I_{M_\infty}(v). \end{aligned}$$

Which combined with $I_M(v) \leq I_{M_\infty}(v)$ gives $I_M(v) = I_{M_\infty}(v)$ that is exactly (IV).

Remark: It is easy to check that $\liminf Area(\partial D''_j) = 0$. ■

End of the proof of theorem 1. Proof:

Take $v* \leq v_6$. Suppose $0 < v < v^*$.

In first we show (Ia) implies (Ib). Let p_0 be a point where $p \mapsto f(p, v)$ attains its minimum. We show by contradiction that $\beta(p_0, v)$ is a solution of the isoperimetric problem. Assume that there is no isoperimetric domain having volume v. Let D_j be a minimizing sequence, $Vol(D_j) = v$,

$$Area(\partial D_j) \to I_M(v) < f_M(p_0, v) \tag{39}$$

and the isoperimetric profile is not achieved. The choice of v^* ensures the existence of a pseudo-bubble $D_\infty \subseteq M_\infty$, and points p_j satisfying (I)-(IV) of lemma 3.7. Hence $I_M(v) = I_{M_\infty}(v) = Area(\partial D_\infty) = f_{M_\infty}(p_\infty, v)$. A continuity argument with respect to $C^{4,\beta}$ convergence applies, giving $f_{M_\infty}(p_\infty, v) = \lim f_M(p_j, v)$. Furthermore, since p_0 is a minimum point implies that $\forall j\ \ f_M(p_j, v) \geq f_M(p_0, v)$ from this one can argue finally that $f_{M_\infty}(p_\infty, v) \geq f_M(p_0, v)$ which contradicts (39).

In second we show (Ib) implies (Ia). Let D be an isoperimetric domain of sufficiently small volume, it follows from theorem 3.4 that $D = \beta(p_0, v)$ for some point p and small real v. This suffices to ensure that $p \mapsto f(p, v)$ attains its minimum at p_0.

Finally, (II) is a straightforward consequence of lemma 3.7, noticing that for small volumes $I_M(v) = I_{M_\infty}(v)$ for some limit manifold $(M, \tilde{p}_\infty, g_\infty)$ obtained as the limit of the sequence $(M, \tilde{p}_j, g)$ for some sequence of points $\tilde{p}_j$. Furthermore, $I_{M_\infty}(v) = f_{M_\infty}(p_\infty, v)$ for some point p_∞ possibly different from $\tilde{p}_\infty$. Now adjust the sequence of points $\tilde{p}_j$ to get a sequence of points $p_j \in M$ such that $(M, p_j, g) \to (M_\infty, p_\infty, g_\infty)$ with the same M_∞ as above. This goal could be achieved by taking as p_j the points $p_j = F_j(p_\infty) = F_{B_{M_\infty}(\tilde{p}_\infty, R), j}(p_\infty)$ for large j, where $R = d_{M_\infty}(\tilde{p}_\infty, p_\infty) + 1$ and the F_j's are the diffeomorphisms given by the $C^{m,\alpha}$ convergence. ■

4 Asymptotic expansion of the isoperimetric profile

We prove, now, theorem 2 stated in the introduction.

Proof: Let us just recall here the definition of $S = Sup_{p\in M}\{Sc(p)\}$. Let $(p_j)_j$ such that $Sc(p_j) \nearrow S$, take the sequence (M, p_j, g) and apply theorem 3.1 then we get the existence of $(M'_\infty, p'_\infty, g)$ such that passing to a subsequence, if needed, $(M, p_j, g) \to (M'_\infty, p'_\infty, g)$ in $C^{m,\beta}$ topology for $0 < \beta < \alpha$. It is easy to check by a continuity argument that

$$Sc_{M_\infty}(p'_\infty) = S. \tag{40}$$

From the definition of isoperimetric profile and lemma 3.4 follows

$$f_{M'_\infty}(p'_\infty, v) \geq I_{M_\infty}(v) \geq I_M(v). \tag{41}$$

Consider an arbitrary sequence of volumes $v_k \to 0$ and look at the corresponding D_{v_k} we conclude that

$$I_M(v_k) = I_{M_{\infty,k}}(v_k) = f_{M_{\infty,k}}(p_{\infty,k}, v_k).$$

The sequence $(M_{\infty,k})$ belongs again to $\mathcal{M}^{4,\alpha}(n,Q,r)$ and an application of the fundamental theorem of convergence of manifolds to this sequence of manifolds produces a subsequence noted always with v_k, a limit manifold (M_∞, p_∞) with $(M_{\infty,k}, p_{\infty,k}) \to (M_\infty, p_\infty)$ in $C^{4,\beta}$ topology for every $0 < \beta < \alpha$. From the latter construction it follows that

$$I_M(v_k) \sim f_{M_\infty}(p_\infty, v_k), k \to +\infty. \tag{42}$$

Combining (40), (41), (42), (1) yields

$$\frac{f_{M'_\infty}(p'_\infty, v_k) - c_n v_k^{\frac{n-1}{n}}}{v_k^{\frac{n+1}{n}}} \leq \frac{I_M(v_k) - c_n v_k^{\frac{n-1}{n}}}{v_k^{\frac{n+1}{n}}} \tag{43}$$

From the asymptotic relation (42) letting $k \to +\infty$ we conclude that

$$-Sc_{M'_\infty}(p'_\infty) \geq -Sc_{M_\infty}(p_\infty), \tag{44}$$

that immediately gives

$$S \leq Sc_{M_\infty}(p_\infty). \tag{45}$$

Since the construction of M_∞ permits us to have a sequence of points $p''_j \in M$ with $Sc_M(p''_j) \to Sc_{M_\infty}(p_\infty)$ we obtain

$$Sc_{M_\infty}(p_\infty) \leq S. \tag{46}$$

(45), (46), and the arbitrarity of the sequence v_k, finally, give (2). ∎

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