Fixed point theorems for middle point linear operators in L^1

Milena Chermisi * , Anna Martellotti †

Abstract

In this paper we introduce the notion of middle point linear operators. We prove a fixed point result for middle point linear operators in L^1. We then present some examples and, as an application, we derive a Markov-Kakutani type fixed point result for commuting family of α-nonexpansive and middle point linear operators in L^1.

1 Introduction

Furi and Vignoli [11] proved that any α-nonexpansive map $T : K \to K$ on a nonempty, bounded, closed, convex subset K of a Banach space X satisfies

$$\inf_{x \in K} \|T(x) - x\| = 0,$$

where α is the Kuratowski measure of non compactness on X. It is of great importance to obtain the existence of fixed points for such mappings in many applications such as eigenvalue problems as well as boundary value problems, including approximation theory, variational inequalities, and complementarity problems. Such results are used in applied mathematics, engineering, and economics.

In this paper we give optimal sufficient conditions for T to have a fixed point on K in case that $X = L^1(\mu)$, where μ is a σ-finite measure, and as a minor application, in case that X is a reflexive Banach space.

The study of fixed point theory has been pursued by many authors and many results are known in literature. In order to have an overview of the problem, we present a brief survey of most relevant fixed point theorems. Darbo [7] showed that any α-contraction $T : K \to K$ has at least one fixed point on every nonempty, bounded, closed, convex subset K of a Banach space. Later, Sadovskiĭ [15] extended the Darbo's result for α-condensing mappings. Belluce and Kirk [3] obtained fixed point results for nonlinear mappings T, defined on a convex and weakly compact subset K of a Banach space, for which $V := I - T$ satisfies

$$\left\| V\left(\frac{x+y}{2}\right)\right\| \leq \frac{1}{2}\left(\|V(x)\| + \|V(y)\|\right), \qquad \text{for any } x, y \in K. \tag{1.1}$$

Lennard [13] proved that any nonexpansive maps $T : K \to K$ has at least one fixed point on every nonempty, $\|\cdot\|_{L^1}$-bounded, ρ-compact, convex subset of $L^1(\mu)$, where ρ is the metric of the convergence locally in measure. For other results we refer to [14, 10, 17, 12, 8, 16].

The purpose of this paper is two-fold:

*Fachbereich Mathematik, Universität Duisburg-Essen, Lotharstraße 65, 47057, Duisburg, Germany. E-mail: milena.chermisi@uni-duisburg-essen.de

†Dipartimento di Matematica, Università di Perugia, via Vanvitelli 1, 06123, Perugia, Italy. E-mail: amart@dipmat.unipg.it

- to introduce the notion of middle point linear operator, which extends the notion of convexity in the sense of (1.1);
- to show that any continuous operator $T : (K, \rho) \to (K, \rho)$ has at least one fixed point in K, whenever K is a nonempty, $\|\cdot\|_{L^1}$-bounded, ρ-closed and convex subset of $L^1(\mu)$ and T is middle point linear and α-nonexpansive.

The class of middle point linear operators, which are defined in Definition 3.1, comprises not only convex operators in the sense of (1.1) but also affine operators. However, as shown in Example 3.1, middle point linear operators are not necessarily affine. We provide a characterization of middle point linear operators and present some useful properties, such as the stability under pointwise convergence, the convexity of the set of fixed points and the fact that it suffices to test middle point linearity on a dense subsets of the domain.
The fixed point theorem, which is the main result of the paper, is stated in Theorem 4.3. The idea is to prove, exploiting a result of Bukhvalov [6], that the functional $f : x \mapsto \|T(x) - x\|_{L^1}$ attains its minimum value on every nonempty, $\|\cdot\|_{L^1}$-bounded, ρ-closed and convex subset of $L^1(\mu)$ (see Lemma 4.2); the conclusion simply follows as a consequence of Furi-Vignoli's Theorem [11].
Using a slightly different argument, it is also possible (see Remark 4.1) to prove a fixed point theorem for middle point linear operators defined on a convex and weakly compact subset of an arbitrary Banach space, generalizing some previuos results of Belluce and Kirk [3, Theorem 4.1, Theorem 4.2]. In particular, this implies that any α-nonexpansive and middle point linear operator $T : K \to K$ has at least one fixed point on every nonempty, bounded, closed and convex subset K of a reflexive Banach space.
We remark that Theorem 4.3 is optimal as Example 4.1 shows: the assumption of middle point linearity on T cannot be avoided, even when K is assumed to be weakly compact. We also present several examples, namely, Examples 4.2–4.4, which show that Theorem 4.3 applies in situation where neither Sadovskiĭ's Theorem nor Lennard's Theorem do.
A first application of Theorem 4.3 leads to a fixed point result for uniform limits of middle point linear operators (see Proposition 4.1). As second application of Theorem 4.3, we derive a generalization of Markov-Kakutani Theorem (see [12, 9]); more precisely, we show that any commuting family $\mathcal{F}$ of α-nonexpansive and middle point linear operators has a common fixed point on K, whenever $T : (K, \rho) \to (K, \rho)$ is continuous for any $T \in \mathcal{F}$ and K is a nonempty, $\|\cdot\|_{L^1}$-bounded, ρ-closed, convex subset of $L^1(\mu)$ (see Theorem 4.4).
The paper is organized as follows. In Section 2 we give a review of basic notions and we fix notations. In Section 3 we introduce and characterise middle point operators, and present some useful properties. In Section 4 we show all the above mentioned fixed point results for middle point linear operators in $L^1(\mu)$ (and in Banach spaces) and we present the examples.

2 Preliminaries

Let (Ω, Σ, μ) be a σ-finite measure space, and let $(\Omega_m)_{m=1}^{\infty}$ be a μ-partition of Ω with $\mu(\Omega_m) < +\infty$. We denote by $M(\mu)$ the collection of all equivalence classes of functions $x : \Omega \to \widetilde{\mathbb{R}}$ which are μ-measurable and finite almost everywhere, modulus the μ-a.e. equivalence. $M(\mu)$ can be endowed with the metric

$$\rho(x, y) := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{1}{\mu(\Omega_m)} \int_{\Omega_m} \frac{|x - y|}{1 + |x - y|} d\mu, \qquad x, y \in M(\mu).$$

It is well known that the metric ρ is translation-invariant and induces the topology of convergence locally in measure. If $\mu(\Omega) < +\infty$, then the topology of the convergence locally in measure on

$M(\mu)$ is equivalent to the topology of the convergence in measure which is induced by the metric

$$\rho(x,y) := \int_\Omega \frac{|x-y|}{1+|x-y|} d\mu. \tag{2.1}$$

Throughout the paper, the symbol $\|\cdot\|$ will denote either the norm of a generic normed space or the norm of $L^1(\mu)$. Since $L^1(\mu)$ will be endowed with the norm topology and the topology induced by ρ, we will say that a subset of $L^1(\mu)$ is bounded (respectively, closed and complete) if it is $\|\cdot\|_{L^1}$-bounded (respectively, $\|\cdot\|_{L^1}$-closed and $\|\cdot\|_{L^1}$-complete). We denote by X_1 the closed unit ball of $L^1(\mu)$.

Remark 2.1. Remember that $\big(M(\mu),\rho\big)$ is a Frèchet space. If μ is finite, $\big(M(\mu),\rho\big)$ is exactly the metric completion of the metric linear space $\big(L^1(\mu),\rho\big)$. We recall also that, any subset A of $L^1(\mu)$ is closed (and hence complete) whenever A is ρ-closed.

Given a metric space X and a bounded set $A \subset X$, we denote by $\alpha(A)$ the *Kuratowski measures of non compactness* of A, i.e.

$$\alpha(A) := \inf\{\varepsilon > 0 : A \text{ can be covered by finitely many sets of diameter } \leqslant \varepsilon\}.$$

For the properties and examples, we refer to [2] or [16].
A map $T : X \to X$ on a normed space $(X, \|\cdot\|)$ is called

nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for every $x, y \in X$;

α-nonexpansive if it is continuous and $\alpha\big(T(A)\big) \leq \alpha(A)$ for every $A \subset X$;

In the sequel we shall simply write *nonexpansive* for maps in $L^1(\mu)$ that are $\|\cdot\|_{L^1}$-nonexpansive.

3 Middle point linear operators

We introduce a new class of operators in normed spaces.

Definition 3.1. Let X_1 be the closed unit ball of a normed space X and K be a convex subset of X (also $K = X$). A continuous operator $T : K \to X$ is said *middle point linear* if for every positive number $r > 0$ the following property holds:

$$T\left(\frac{x+y}{2}\right) \in \frac{x+y}{2} + rX_1, \qquad \text{whenever } x, y \in K,\ T(x) \in x + rX_1 \text{ and } T(y) \in y + rX_1. \tag{3.1}$$

Remark 3.1. Any *affine* operator $T : X \to X$ on a normed space X, i.e. any map such that

$$T(ax + (1-a)y) = aT(x) + (1-a)T(y), \qquad \text{for any } x, y \in X \text{ and } a \in [0,1],$$

is middle point linear.

However, as it is shown in the following example, middle point operators need not to be affine.

Example 3.1. Let $\varphi : [0, +\infty) \to [0,1]$ be a non-increasing continuous function. Define the operator $T : X \to X$ as

$$T(x) := \varphi\big(\|x\|\big)\, x, \tag{3.2}$$

for all $x \in X$. The operator T is middle point linear. Indeed, fix $r > 0$ and choose $x, y \in X$ such that

$$\|x - T(x)\| = \big[1 - \varphi\big(\|x\|\big)\big]\|x\| \leqslant r \qquad \text{and} \qquad \|y - T(y)\| = \big[1 - \varphi\big(\|y\|\big)\big]\|y\| \leqslant r.$$

Without loss of generality, we can assume that $\|x\| \leqslant \|y\|$. This implies that $\left\|\frac{x+y}{2}\right\| \leqslant \|y\|$ and, by the monotonicity of φ, $\varphi(\|y\|) \leqslant \varphi\big(\|x+y\|/2\big)$. Then

$$\left\|\frac{x+y}{2} - T\left(\frac{x+y}{2}\right)\right\| = \left[1-\varphi\big(\|x+y\|/2\big)\right]\left\|\frac{x+y}{2}\right\| \leq \left[1-\varphi\big(\|y\|\big)\right]\|y\| \leqslant r,$$

that is $T\left(\frac{x+y}{2}\right) \in \frac{x+y}{2} + rX_1$.

Remark 3.2. It is rather natural to compare this new definition with the usual convexity in the real line, namely, in case that $X = \mathbb{R}$.
There exist convex middle point linear mappings, as $x \mapsto \exp(x)$ for any $x \in [0,\infty)$. However, convex functions are not necessarily middle point linear; for instance, the map $h : [0,1] \to [0,1]$, defined by $h(x) := x^2$ for all $x \in [0,1]$, is not middle point linear since $|h(0) - 0| \leq r$ and $|h(1) - 1| \leq r$ for all $r < 1/8$, while $|h(1/2) - 1/2| = 1/4 > r$. Anyway, we observe that the map h is middle point linear on $[1, +\infty)$.

Remark 3.3. Condition (1.1) implies (3.1), but the converse is not true. For instance, the operator $T : X_1 \to X_1$, defined as in (3.2) with $\varphi(r) =: (1-r)^2$, $r \in [0,1]$, is middle point linear but (1.1) is not satisfied, since the map $r \mapsto (1-\varphi(r))r$ is not convex in $[0,1]$. Another example is given by the map $S(x) := x - 1 + e^{-x^2}$ for any $x \in [-M, M]$ with M large enough.

It is easy to prove that if T is middle point linear then property (3.1) holds for every convex combination of x and y, namely the following statement holds.

Proposition 3.1. *Let X be a normed space and $T : X \to X$ be a middle point linear operator. For each $r > 0$, if $x, y \in X$ are such that $T(x) \in x + rX_1$ and $T(y) \in y + rX_1$, then*

$$T\big(\lambda x + (1-\lambda)y\big) \in \lambda x + (1-\lambda)y + rX_1,$$

for each $\lambda \in [0,1]$.

Proof. The proof is based on a standard procedure. First, we prove the statement for every dyadic rational in $(0,1)$ and then, by the continuity of T, for every number $\lambda \in (0,1)$.

□

The following characterization of middle point linear operators is a direct consequence of the Definition 3.1 and Proposition 3.1.

Proposition 3.2. *Let X be a normed space and $T : X \to X$ be a continuous operator. Then T is middle point linear if and only if the functional $f : X \to \mathbb{R}$ defined as $f(x) := \|T(x) - x\|$ is quasi-convex, i.e. the set $\{x \in X \,:\, f(x) \leqslant r\}$ is convex in X for any $r > 0$.*

Remark 3.4. The class of affine operators enjoys closedness under convex combination as well as under usual map composition. In general, this is not true for middle point linear operators.
To see this, let $X = \mathbb{R}$ and $S, T : \mathbb{R} \to \mathbb{R}$ be defined by $S(x) := x - 2x^2$ and $T(x) := x + 2x^4$. From Proposition 3.2, S and T are middle point linear since both the mappings $x \mapsto |S(x) - x| = 2x^2$ and $x \mapsto |T(x) - x| = 2x^4$ are convex. However, $R := \frac{1}{2}(T + S)$ is not middle point linear, since $\|R(x) - x\| = |x^4 - x^2|$ is not quasi-convex.
Consider now the operator $U : \mathbb{R} \to \mathbb{R}$, defined by $U(x) := x - x^3$. Clearly, U is middle point linear since $\|U(x) - x\| = |x^3|$ is quasi-convex, but U^2 is not since $\|U^2(x) - x\| = |x^3(x^6 - 3x^4 + 3x^2 - 2)|$ fails to be quasi-convex.

Pointwise convergence respects middle point linearity, namely the following result can easily be checked.

Proposition 3.3. *Let $T_n : X \to X$ be a sequence of middle pont linear operators, and let $T : X \to X$ be its pointwise limit (i.e. $\lim_{n\to+\infty} \|T_n(x) - T(x)\| = 0$ for every $x \in X$). Then T is middle point linear.*

It is also interesting to note that it suffices to test property (3.1) on dense subsets of X as the following proposition shows.

Proposition 3.4. *Let $T : X \to X$ be a continuous operator and D be a dense subset of X. If* (3.1) *holds for any pair $x, y \in D$, then T is middle point linear on X.*

Proof. Let $r > 0$ be fixed, and let $x, y \in X$ be such that $T(x) \in x + rX_1$ and $T(y) \in y + rX_1$. Fix $\varepsilon > 0$, and let $\delta(\varepsilon)$ be determined by the continuity of T in x and y; choose $x_\delta, y_\delta \in D$ such that

$$\|x - x_\delta\| \le \delta \qquad \text{and} \qquad \|y - y_\delta\| \le \delta.$$

Then

$$\|T(x) - T(x_\delta)\| \le \varepsilon \qquad \text{and} \qquad \|T(y) - T(y_\delta)\| \le \varepsilon.$$

Since $\|x_\delta - T(x_\delta)\| \le \|x_\delta - x\| + \|x - T(x)\| + \|T(x) - T(x_\delta)\| \le r + \delta + \varepsilon$, and analogously $\|y_\delta - T(y_\delta)\| \le r + \delta + \varepsilon$ there follows

$$\left\| \frac{x_\delta + y_\delta}{2} - T\left(\frac{x_\delta + y_\delta}{2}\right) \right\| \leqslant r + \delta + \varepsilon,$$

whence, letting first δ and then ε go to 0, we reach the conclusion. □

4 Fixed points for middle point linear operators in $L^1(\mu)$

From now on, (Ω, Σ, μ) will be a σ-finite measure space. In this section we present fixed point theorems for middle point operators in $L^1(\mu)$. For convenience of the reader, we first recall the following result of Bukhvalov, called *optimization without compactness*.

Theorem 4.1. [6] *Let $(C_n)_n$ be a family of bounded, ρ-closed and convex sets having the finite intersection property. Then the intersection $\bigcap_n C_n$ is nonempty.*

We now prove, using Theorem 4.1, the following lemma, which generalizes a result still due to Bukhvalov [6].

Lemma 4.1. *Let K be a nonempty, bounded ρ-closed, convex subset of $L^1(\mu)$. Then any quasi-convex functional $f : K \to \mathbb{R}$ which is lower bounded and lower semi-continuous with respect to ρ, attains its minimum value on K.*

Proof. Let $a := \inf_{x\in K} f(x)$ and $(x_n)_n$ be a minimizing sequence in K such that $f(x_n)$ is decreasing and converges to a. Then

$$a \leqslant f(x_{n+1}) < f(x_n),$$

for all n. Consider the sub-level sets of f defined as

$$F_n := \{x \in K \mid f(x) \leqslant f(x_n)\}.$$

Assume, without loss of generality, that $f(x_n) > a$ for every n. Thus each F_n is nonempty and the sequence $(F_n)_n$ is decreasing. Hence, the intersection of finitely many F_n is nonempty. In view of the lower semi-continuity of f, each F_n is ρ-closed, and in view of the quasi-convexity, convex. Since $F_n \subset K$, each F_n is also bounded. Then, by 4.1, $\bigcap_{n=1}^{\infty} F_n$ is nonempty, i.e there is a point $\xi \in K$ such that $f(\xi) \leqslant f(x_n)$ for every n, whence $f(\xi) = a$. □

Using Lemma 4.1, we obtain the following minimum property for middle point operators in L^1.

Lemma 4.2. *Let K be a nonempty, bounded, ρ-closed, convex subset of $L^1(\mu)$. If $T : (K, \rho) \to (K, \rho)$ is a continuous, middle point linear operator, then the real-valued functional $f(x) := \|T(x) - x\|$ attains its minimum value on K.*

Proof. Clearly, the functional f is lower bounded by 0 and ρ-lower semicontinuous since T is ρ-continuous and also the norm $\|\cdot\|_{L^1} : K \to [0, \infty[$ is ρ-lower semicontinuous. By Proposition 3.2, it is quasi-convex. The conclusion then follows from Lemma 4.1. □

We need now the following result due to Furi and Vignoli.

Theorem 4.2. [11] *Let X be a Banach space, α the Kuratowski measure of non compactness on X and K a nonempty, bounded, closed, convex subset of X. If $T : K \to K$ is α-nonexpansive then* $\inf_{x\in K} \|T(x) - x\| = 0$.

From Lemma 4.2 and Theorem 4.2 we obtain the following fixed point theorem in L^1, which is the main result of our paper.

Theorem 4.3. *Let K be a nonempty, bounded, ρ-closed, convex subset of $L^1(\mu)$. If $T : (K, \rho) \to (K, \rho)$ is a continuous, α-nonexpansive, middle point linear operator, then T has at least one fixed point in K.*

Remark 4.1. Using a slightly different argument, it is possible to obtain the following fixed point results for middle point operators in Banach spaces.

(BS) If K is a nonempty, weakly compact and convex subset of a Banach space and $T : K \to K$ is a middle point linear operator satisfying $\inf_{x\in K} \|Tx - x\| = 0$ then T has a fixed point in K.

(RBS) Any α-nonexpansive and middle point linear operator $T : K \to K$ on a nonempty, bounded, closed, convex subset K of a reflexive Banach space has at least one fixed point in K.

Since any bounded, closed, convex subset of a reflexive Banach space is weakly compact (see [4, Corollary III.19]), we obtain (RBS) as consequence of (BS) and Theorem 4.2.
Now, to prove (BS), let us pick $\xi \in K$ such that $c_0 = \|T(\xi) - \xi\| < +\infty$. The set

$$C := \{x \in K : \|T(x) - x\| \leq c_0\}$$

is weakly compact since it is a closed and convex subset of a weakly compact set. Furthermore, the functional $f(x) := \|T(x) - x\| < +\infty$. is quasi-convex and continuous. This implies that f is lower semicontinuous with respect to the weak topology, since for every $c \in \mathbb{R}$ the set $f^{-1}\big((c, +\infty)\big)$ is weakly open. Thus, f attains its minimum value on C: there exists a point $x_0 \in C$ such that $f(x_0) \leq f(x)$ for any $x \in C$. Clearly, $f(x_0) \leq f(x)$ for any $x \in K$, that is x_0 is a fixed point under T.

We underline that both results (BS) and (RBS) generalize some previous results of Belluce and Kirk [3, Theorem 4.1, Theorem 4.2] which obtained fixed point results for nonlinear mappings T in Banach spaces for which $V := I - T$ satisfies (1.1). Recall that condition (1.1) is stronger than (3.1) (see Remark 3.3).
According to the results in [6], one can presume that ρ-closedness of bounded convex sets in $L^1(\mu)$ is a sufficient surrogate to compactness. The next example shows that the assumption that T is middle point linear cannot be avoided, even when K is assumed to be weakly compact.

Example 4.1. In [1] Alspach has given the following example of fixed-point free map. Let $\Omega = [0,1]$ with the usual Lebesgue measure μ, and let

$$K := \{x \in L^1(\mu), 0 \le x \le 2, \|x\| = 1\}.$$

Then K is convex, closed (actually weakly compact) and ρ-closed, since in K we have dominated convergence. Consider the operator $T : K \to K$ defined by

$$T(x)(t) := \begin{cases} 2x(2t) \wedge 2 & \text{when } 0 \le t \le \dfrac{1}{2} \\ [2x(2t-1) - 2] \vee 0 & \text{when } \dfrac{1}{2} < t \le 1. \end{cases} \tag{4.1}$$

Then, according to [1], T is an isometry (therefore it is non-expansive and norm continuous) but it has no fixed point. Again, since in K every convergence is dominated, T is also continuous as a self map of (K, ρ).
According to Theorem 4.3, T cannot be middle point linear. Indeed for the maps

$$x(t) := \begin{cases} \dfrac{3}{2} & \text{when } 0 \le t \le \dfrac{1}{2} \\ \dfrac{1}{2} & \text{when } \dfrac{1}{2} < t \le 1 \end{cases} \qquad \text{and} \qquad y := 2\,\mathbf{1}_{[\frac{3}{8},\frac{1}{2}] \cup [\frac{5}{8},1]},$$

where $\mathbf{1}_A$ is the characteristic function of the set A, one finds $\|x - T(x)\| = \|y - T(y)\| = \dfrac{1}{2}$, while

$$\xi_0 := \frac{x+y}{2} = \frac{3}{4}\mathbf{1}_{[0,\frac{3}{4}]} + \frac{7}{4}\mathbf{1}_{(\frac{3}{4},\frac{1}{2}]} + \frac{1}{4}\mathbf{1}_{(\frac{1}{2},\frac{5}{4}]} + \frac{5}{4}\mathbf{1}_{(\frac{5}{4},1]}$$

satisfies $\|T(\xi_0) - \xi_0\| = \dfrac{47}{64} > \dfrac{1}{2}$.

Remark 4.2. In [10] the two authors established that an α-Lipschitz map $T : Q \to Q$ with constant $k \ge 1$ defined on a bounded and convex subset Q of a normed space has the property that

$$\eta(T) := \inf_{x \in Q} \|x - T(x)\| \le \left(1 - \frac{1}{k}\right) \chi_0(K),$$

where $\chi_0(K)$ is the infimum of all $\delta > 0$ such that K admits a finite dimensional δ approximation of the identity. Example 4.1 shows that, in general, the above infimum is not a minimum: indeed, T as in (4.1) is an isometry, and hence α-Lipschitz with constant $k = 1$. It follows that $\eta(T) = 0$, but there exists no point x_0 of K satisfying

$$\|x_0 - T(x_0)\| \le \|x - T(x)\|$$

for every $x \in K$, since T is a fixed-point free map on K.

The following result concerns approximation of the fixed points of T.

Proposition 4.1. *Assume μ is a finite measure. Let K be a nonempty, bounded, ρ-closed, convex subset of $L^1(\mu)$, and let $T_n : (K, \rho) \to (K, \rho)$ be a sequence of continuous, middle point linear, α-nonexpansive operator, uniformly norm-converging to some T (namely, $\|T_n(x) - T(x)\| \to 0$ uniformly in K). Then T has a fixed point.*

Proof. By Theorem 4.3, we know that each T_n has at least one fixed point $x_n \in K$. By Theorem 1.4 in [6], there exist an increasing sequence of integers $n_1 < n_2 < \ldots < n_k < \ldots$, an $x_0 \in L^1(\mu)$ and a sequence $(\lambda_n)_n \subset [0, 1]$ such that

$$\sum_{i=n_{j-1}}^{n_j} \lambda_i = 1, \quad \text{and} \quad \xi_j := \sum_{i=n_{j-1}}^{n_j} \lambda_i x_i \xrightarrow{\rho} x_0.$$

Then, from the ρ-continuity of each T_n, $T_n(\xi_j) \xrightarrow{\rho} T_n(x_0)$. Since $T_n(x_n) = x_n$ for every $n \in \mathbb{N}$, and $(T_n)_n$ converges uniformly to T in K we deduce that $\|T(x_n) - x_n\| \leq \varepsilon$ for n suitably large. By Proposition 3.3, T is middle point linear, and hence $\|T(\xi_j) - \xi_j\| \leq \varepsilon$ for j suitably large, in force of Proposition 3.1.
Now we find

$$\rho(T(x_0), x_0) \leq \rho(T(x_0), T(\xi_j)) + \rho(T(\xi_j), \xi_j) + \rho(\xi_j, x_0) \leq$$
$$\rho(T(x_0), T_n(x_0)) + \rho(T_n(x_0), T_n(\xi_j)) + \rho(T_n(\xi_j), T(\xi_j)) + \|T(\xi_j) - \xi_j\| + \rho(\xi_j, x_0) \leq$$
$$\|T(x_0) - T_n(x_0)\| + \rho(T_n(x_0), T_n(\xi_j)) + \|T_n(\xi_j) - T(\xi_j)\| + \|T(\xi_j) - \xi_j\| + \rho(\xi_j, x_0) < \varepsilon$$

for n and j suitably large. Therefore $T(x_0) = x_0$. □

Note that, under the above assumptions, T is middle point linear, by Proposition 3.3 and continuous also with respect to the ρ-continuity, but we do not know if T is α-non expansive; therefore we cannot apply directly Theorem 4.3

We give now some applications of Theorem 4.3 in $(L^1[0, 1], \mu)$ where μ denotes now the Lebesgue measure. The following examples show also that Theorem 4.3 applies in situation where neither Sadovskiĭ's Fixed Point Theorem [15] nor Lennard's Fixed Point Theorem [13] do: in fact in Sadovskiĭ's Theorem, T is required to be α-condensing, while in Lennard's Theorem, K needs to be ρ-compact and T nonexpansive.

Example 4.2. Let $T : X_1 \to X_1$ be defined as in (3.2) for all $x \in X_1$, where $\varphi : [0, 1] \to [0, 1]$ is a non-increasing continuous mapping such that $\varphi(0) = 1$. The operator T is well defined and from Example 3.1 middle point linear.
Clearly, $T : (X_1, \|\cdot\|) \to (X_1, \|\cdot\|)$ is continuous since

$$\|T(x) - T(x_0)\| = \|\varphi(\|x\|)x - \varphi(\|x_0\|)x_0\| \leq \varphi(\|x\|)\|x - x_0\| + |\varphi(\|x\|) - \varphi(\|x_0\|)| \, \|x_0\|$$

Moreover, T is α-nonexpansive. In fact, if $B \subset X_1$, then

$$\alpha(T(B)) \leqslant \alpha\big(\overline{co}\{B, 0\}\big) = \alpha\big(B \cup \{0\}\big) = \max\{\alpha(B), \alpha(\{0\})\} = \alpha(B),$$

since $T(x) = \varphi\big(\|x\|\big)\, x + \big(1 - \varphi\big(\|x\|\big)\big)\, 0 \in \text{co}\{B, 0\}$ for every $x \in B$.
To show that $T : (X_1, \rho) \to (X_1, \rho)$ is continuous, fix $x_0 \in X_1$ and a sequence $(x_n)_n$ of points of X_1 converging locally in measure to x_0, i.e. $\lim\limits_{n \to +\infty} \rho(x_n, x_0) = 0$. Then, since ρ is translation invariant and $\rho(cx, 0) \leq \rho(x, 0)$ for every $x \in L^1(\mu)$ and $c \in [-1, 1]$, we have

$$\begin{aligned} \rho(T(x_n), T(x_0)) &\leq \rho\big(\varphi(\|x_n\|)(x_n - x_0), 0\big) + \rho\big([\varphi(\|x_n\|) - \varphi(\|x_0\|)]x_0, 0\big) \\ &\leq \rho\big(x_n - x_0, 0\big) + \rho\big([\varphi(\|x_n\|) - \varphi(\|x_0\|)]x_0, 0\big). \end{aligned} \tag{4.2}$$

The second summand in (4.2) tends to 0 since φ is continuous and $\lim_{n\to+\infty} \rho(c_n x, 0) = 0$ for every $x \in L^1(\mu)$ and every sequence $(c_n)_n$ of real numbers with $\lim_{n\to+\infty} c_n = 0$. Hence, the conclusion follows.
Therefore, by Theorem 4.3, T has a fixed point in X_1. Notice that $T(0) = 0$ and $T(r\,\partial X_1) = \varphi(r) r\,\partial X_1$ for every $r \in [0,1]$. Note also that both Sadovskiĭ's Theorem and Lennard's Theorem cannot be applied, since T is neither α-condensing nor nonexpansive and X_1 is not ρ-compact.

Example 4.3. Let $T : L^1[0,1] \to L^1[0,1]$ be defined as

$$T(x)(t) := \frac{1}{2}\, x\left(\frac{t}{2}\right), \qquad t \in [0,1].$$

The operator T is linear and hence middle point linear. Since

$$\|T(x)\| = \int_0^1 \frac{1}{2}\left| x\left(\frac{t}{2}\right)\right| d\mu(t) = \int_0^{1/2} |x(s)|\, d\mu(s) \leqslant \|x\|, \qquad \forall\, x \in L^1[0,1],$$

$T : (K, \|\cdot\|) \to (K, \|\cdot\|)$ is continuous and nonexpansive (and hence α-nonexpansive) on X_1.
Moreover, since for all $x_0, x \in X_1$

$$\begin{aligned}\rho(T(x), T(x_0)) &= \int_0^1 \frac{|x(t/2) - x_0(t/2)|}{2 + |x(t/2) - x_0(t/2)|}\, d\lambda(t) \leqslant \int_0^1 \frac{|x(t/2) - x_0(t/2)|}{1 + |x(t/2) - x_0(t/2)|}\, d\lambda(t) = \\ &= 2\int_0^{1/2} \frac{|x(s) - x_0(s)|}{1 + |x(s) - x_0(s)|}\, d\lambda(s) \leqslant 2\,\rho(x, x_0),\end{aligned}$$

$T : (X_1, \rho) \to (X_1, \rho)$ is Lipschitz-continuous on X_1.
Therefore, from Theorem 4.3, T has a fixed point in X_1. In particular, $T(0) = 0$. Notice that we cannot apply Lennard's Theorem since X_1 is not ρ-compact and that it is not easy to understand whether T is α-condensing.

Example 4.4. Consider the multiplication operator $M_f : L^1[0,1] \to L^1[0,1]$ defined for all $x \in L^1[0,1]$ as

$$M_f(x)(t) := f(t) \cdot x(t), \quad \text{for all } t \in [0,1],$$

where $f \in L^\infty([0,1] \times [0,1])$ with $\|f\|_\infty \leq 1$. The operator M_f is linear, bounded with $\|M_f\| = \|f\|_\infty$ (hence $M_f : (K, \|\cdot\|) \to (K, \|\cdot\|)$ continuous), and M_f is nonexpansive.
Moreover, $M_f : (L^1[0,1], \rho) \to (L^1[0,1], \rho)$ is Lipschitz-continuous; indeed, for all $x, y \in L^1[0,1]$

$$\begin{aligned}\rho(M_f(x), M_f(y)) &= \int_0^1 \frac{|f(t)|\,|x - y|}{1 + |f(t)|\,|x - y|}\, d\lambda \leqslant \rho(\|f\|_\infty (x - y), 0) \leqslant \\ &\leqslant \max\{1, \|f\|_\infty\}\, \rho(x - y, 0) \leq \rho(x, y).\end{aligned}$$

It is clear that M_f maps X_1 into X_1 (in general, rX_1 into rX_1). Therefore, by Theorem 4.3, M_f has at least one fixed point on X_1. Notice that we cannot apply Lennard's Theorem since X_1 is not ρ-compact and that T is not necessarily α-condensing (it depends on the choice of f).

In the literature one finds several results concerning common fixed point theorems for families of self maps (see, for instance Chapter 9 in [12]). All these results however require some form of compactness for the domain of the family.
As an application of Theorem 4.3 we shall derive a common fixed point theorem without compactness, that generalizes Markov-Kakutani Fixed Point Theorem (see [12, Theorem 9.1.4] or [9, § V.10 Theorem 6]).
To this aim, we first need the following geometrical property for fixed points of middle point linear operators.

Proposition 4.2. *Let X be a normed space and $T : X \to X$ be a middle point linear operator. If T as two different fixed points x and y then any convex combination of x and y is a fixed point under T.*

Proof. Since $T(x) = x$ and $T(y) = y$, $T(x) \in x + rX_1$ and $T(y) \in y + rX_1$ for any $r > 0$. Thus, by Proposition 3.1,

$$\|T\big(\alpha x + (1-\alpha)y\big) - \alpha x + (1-\alpha)y\| \le r$$

for any $r > 0$ and any $\alpha \in [0, 1]$, that gives the conclusion. □

Theorem 4.4. *Let K be a nonempty, bounded, ρ-closed, convex subset of $L^1(\mu)$ and $\mathcal{F}$ be a commuting family of α-nonexpansive, middle point linear operators such that $T : (K, \rho) \to (K, \rho)$ is continuous for any $T \in \mathcal{F}$. Then $\mathcal{F}$ has a common fixed point, i.e., there exists a point $\xi \in K$ such that $T(\xi) = \xi$ for all $T \in \mathcal{F}$.*

Proof. For any $T \in \mathcal{F}$, define $F(T)$ to be the set of fixed points of T. From Theorem 4.3 and Proposizion 4.2, $F(T)$ is non-empty, ρ-closed and convex.
Moreover, for any $T, S \in \mathcal{F}$ and $x_0 \in F(T)$, we have

$$T[S(x_0)] = S[T(x_0)] = S(x_0),$$

that is, $S(x_0)$ is a fixed point under T. Therefore, $S(F(T)) \subset F(T)$. Consider now the restriction $S : F(T) \to F(T)$ of S. From Theorem 4.3, S has a fixed point on $F(T)$. In other words, S and T have a common fixed point. This argument can be repeated for every finite subfamily of $\mathcal{F}$. Thus, the family $\{F(T), T \in \mathcal{F}\}$ satisfies the Finite Intersection Property, and hence, by Theorem 4.1, there is a point $\eta \in \bigcap_{T \in \mathcal{F}} F(T)$. □

A similar argument had been used in [3] in the framework of symmetric spaces. Since $L^1(\mu)$ is not a symmetric space, Theorem 4.4 cannot be derived from there.
On the other side, by means of Remark 4.1, one can derive a slight extension of Markov-Kakutani Theorem in the framework of reflexive Banach spaces, since linear maps are middle point linear as well.

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