A STABILITY RESULT FOR NEUMANN PROBLEMS IN DIMENSION $N \geq 3$

ALESSANDRO GIACOMINI

Abstract. We give a sufficient condition in dimension $N \geq 3$ in order to obtain the stability of a sequence of Neumann problems on fractured domains.

1. Introduction

Given Ω open and bounded in $\mathbb{R}^N$, (K_n) a sequence of compact sets in $\mathbb{R}^N$, consider the following Neumann problems

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \setminus K_n \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cup (\partial K_n \cap \Omega) \end{cases} \tag{1.1}$$

with $f \in L^2(\Omega)$: we intend (1.1) satisfied in the usual weak sense of Sobolev spaces, that is $u \in H^1(\Omega \setminus K_n)$ and

$$\int_{\Omega\setminus K_n} \nabla u \nabla \varphi + \int_{\Omega\setminus K_n} u\varphi = \int_{\Omega\setminus K_n} f\varphi$$

for all $\varphi \in H^1(\Omega \setminus K_n)$. If (K_n) converges to a compact set K in the Hausdorff metric , we look for conditions on the sequence (K_n) such that, considered the problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \setminus K \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cup (\partial K \cap \Omega), \end{cases} \tag{1.2}$$

the solutions u_n of (1.1) (extended to 0 on $K_n \cap \Omega$) converge to the solution u of (1.2) (extended to 0 on $K \cap \Omega$). If this is the case, we say that the Neumann problems (1.1) are stable.

The problem of stability for elliptic problems under Neumann boundary conditions has been widely investigated. Usually, since in general the domains $\Omega \setminus K_n$ are not regular, it is not possible to deal with the problem using extension operators (see for example [17], [18]).

In dimension $N = 2$, Chambolle and Doveri [9] in 1997 proved a stability result under a uniform limitation of $\mathcal{H}^1(K_n)$ and of the number of the connected components of K_n; Bucur and Varchon [3] in 2000 proved that if K_n has at most m connected components ($m \in \mathbb{N}$), the stability of the problems is equivalent to the condition $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$.

In dimension $N \geq 3$, the bound on the number of the connected components of K_n is not a relevant feature and a condition similar to that of Bucur and Varchon doesn't hold: in fact, problems (1.1) could be not stable even if the sets K_n are connected. In 1997, Cortesani [11] proved that in general, if K is contained in a C^1 submanifold of $\mathbb{R}^N$, the limit of solutions of (1.1) satisfies a transmission condition on K. Several results on this transmission condition are known under additional assumptions on (K_n). In the case in which K_n is contained in a hyperplane M and is the complement in M of a periodic grid of $(N-1)$ dimensional balls, the problem is treated in [21]. In [7], a continuity result is obtained in the case $K_n \subseteq M$ and K_n satisfies appropriate capacitary conditions on the boundary. In Murat [19] and Del Vecchio [14] (see also [23],[24]), the case of a sieve (Neumann sieve) is considered: the transmission conditions that occur in the limit are determined in relation to capacitary properties of the holes of the sieve.

In this paper, we suppose that the sets K_n, locally, are sufficiently regular subsets of $(N-1)$-dimensional Lipschitz submanifolds of $\mathbb{R}^N$ in such a way that homogenization effects due to the possible holes cannot occur.

Let π be the hyperplane $x_N = 0$ in $\mathbb{R}^N$ and let C be an $(N-1)$-dimensional finite closed cone with nonempty relative interior. We say that the sequence (K_n) satisfies the C-condition if there exist constants $\delta, L_1, L_2 > 0$ such that, for all n and for all $x \in K_n$, there exists $\Phi_x : B_\delta(x) \to \mathbb{R}^N$ with

(a) for all $z_1, z_2 \in B_\delta(x)$:
$$L_1|z_1 - z_2| \leq |\Phi_x(z_1) - \Phi_x(z_2)| \leq L_2|z_1 - z_2|;$$

(b) $\Phi_x(x) = 0$ and $\Phi_x(B_\delta(x) \cap K_n) \subseteq \pi$;

(c) for all $y \in B_{\frac{\delta}{2}}(x) \cap K_n$,
$$\Phi_x(y) \in C_y \subseteq \Phi_x(B_\delta(x) \cap K_n)$$

for some finite closed cone C_y in π congruent to C. Conditions (a), (b) imply that, near x, K_n is a subset of an $(N-1)$-dimensional Lipschitz submanifold $M_{n,x}$ of $\mathbb{R}^N$ and condition (c) implies that K_n is sufficiently regular in $M_{n,x}$, essentially a finite union of Lipschitz subsets.

The main result of the paper is that, if the sequence (K_n) satisfies the C-condition and $K_n \to K$ in the Hausdorff metric, then the spaces $W^{1,p}(\Omega \setminus K_n)$ converge in the sense of Mosco (see Section 2) to the space $W^{1,p}(\Omega \setminus K)$ for $1 < p \leq 2$. As a consequence for the case $p = 2$, the problems (1.1) are stable, that is transmission conditions in the limit are avoided.

The hypotheses above are not sufficient to cover the case $p > 2$; moreover, point (b) in C-condition cannot be omitted: in fact a sort of "curvilinear" cone condition given only by points (a) and (c) does not provide the Mosco convergence. We will see these facts through explicit examples.

The paper is organized as follows: in Section 2, we introduce the basic notation; after some preliminaries, we prove the main stability result in Section 4. In Section 5, we give the above mentioned examples of non-stability which require some basic techniques of Γ-convergence.

2. Notation and preliminaries

In this section, we introduce the basic notation and the tools employed in the rest of the paper.

The Mosco convergence. Let X be a reflexive Banach space, (Y_n) a sequence of closed subspaces of X. Let us pose

$$Y' := \{x \in X \,:\, x = w\text{-}\lim y_{n_k}\,,\, y_{n_k} \in Y_{n_k},\, n_k \to +\infty\} \tag{2.1}$$

and

$$Y'' := \{x \in X \,:\, x = s\text{-}\lim y_n\,,\, y_n \in Y_n \text{ for } n \text{ large}\}; \tag{2.2}$$

Y' and Y'' are called, respectively, the *weak-limsup* and the *strong-liminf* of the sequence (Y_n) in the sense of Mosco. We say that the sequence (Y_n) converges in the sense of Mosco if $Y' = Y'' = Y$ and we call Y the Mosco limit of (Y_n). Clearly $Y'' \subseteq Y'$: as a consequence, in order to prove that $Y_n \to Y$ in the sense of Mosco, it is sufficient to prove that $Y' \subseteq Y$ (*weak-limsup condition*) and $Y \subseteq Y''$ (*strong-liminf condition*). Since Y'' is closed, the strong-liminf condition can be established proving the inclusion $D \subseteq Y''$, D being a dense subset of Y.

Let Ω' be open and bounded in $\mathbb{R}^N$, Ω_n, Ω open subsets of Ω', $p \in [1, +\infty]$. We can identify the Sobolev space $W^{1,p}(\Omega_n)$ with a closed subspace of $L^p(\Omega'; \mathbb{R}^{N+1})$ through the map

$$\begin{array}{ccc} W^{1,p}(\Omega_n) & \longrightarrow & L^p(\Omega'; \mathbb{R}^{N+1}) \\ u & \longmapsto & (u, D_1 u, \dots, D_N u) \end{array} \tag{2.3}$$

with the convention of extending u and ∇u to zero on $\Omega' \setminus \Omega_n$.

Let Y and Y_n be the closed subspaces of $L^p(\Omega'; \mathbb{R}^{N+1})$ corresponding to $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega_n)$ respectively. We say that $W^{1,p}(\Omega_n)$ converges to $W^{1,p}(\Omega)$ in the sense of Mosco if Y is the Mosco limit of the sequence (Y_n) in the space $L^p(\Omega'; \mathbb{R}^{N+1})$.

Stability of Neumann problems. Let Ω' be open and bounded in $\mathbb{R}^N$; consider the Neumann problems

$$\begin{cases} -\Delta u_n + u_n = f \\ u \in H^1(\Omega_n) \end{cases} \tag{2.4}$$

and

$$\begin{cases} -\Delta u + u = f \\ u \in H^1(\Omega) \end{cases} \tag{2.5}$$

with $f \in L^2(\Omega')$, Ω, Ω_n open subsets of Ω'; we intend (2.4) and (2.5) in the usual weak sense, that is

$$u \in H^1(\Omega_n), \quad \int_{\Omega_n} \nabla u_n \nabla \varphi + \int_{\Omega_n} u\varphi = \int_{\Omega_n} f\varphi \quad \forall \varphi \in H^1(\Omega_n)$$

and

$$u \in H^1(\Omega), \quad \int_{\Omega} \nabla u \nabla \varphi + \int_{\Omega} u\varphi = \int_{\Omega} f\varphi \quad \forall \varphi \in H^1(\Omega).$$

We say that the problems (2.4) converge to the problem (2.5) if $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^2(\Omega'; \mathbb{R}^{N+1})$ under the identification (2.3).

Hausdorff metric on compact sets. Let Ω be open and bounded in $\mathbb{R}^N$. We indicate the set of all compact subsets of $\overline{\Omega}$ by $\mathcal{K}(\overline{\Omega})$. $\mathcal{K}(\overline{\Omega})$ can be endowed with the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \max\left\{ \sup_{x \in K_1} \operatorname{dist}(x, K_2), \sup_{y \in K_2} \operatorname{dist}(y, K_1) \right\}$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = \operatorname{diam}(\Omega)$ if $K \neq \emptyset$. It turns out that $\mathcal{K}(\overline{\Omega})$ endowed with the Hausdorff metric is a compact space (see e.g. [22]).

3. Some auxiliary results

In this section, we prove some results that are used in the proof of the main theorem of the paper. We begin recalling some properties of sets which satisfy the cone condition.

Consider a closed ball $B \subseteq \mathbb{R}^N$ not containing 0 and $x \in \mathbb{R}^N$. The set

$$C := x + \{\lambda y \, : \, y \in B, 0 \leq \lambda \leq 1\}$$

is called a *finite closed cone* in $\mathbb{R}^N$ with vertex at x.

A *parallelepiped* with a vertex at the origin is a set of the form

$$P := \left\{ \sum_{j=1}^{N} \lambda_j y_j \, : \, 0 \leq \lambda_j \leq 1, \, 1 \leq j \leq N \right\}$$

where $y_1, \dots, y_N$ are N linearly independent vectors in $\mathbb{R}^N$.

Definition 3.1. *Let C be a finite closed cone in $\mathbb{R}^N$ with vertex at the origin. We say that a compact set $K \subseteq \mathbb{R}^N$ satisfies the cone condition with respect to C if for all $x \in K$ there exists a finite closed cone C_x congruent to C such that $x \in C_x \subseteq K$.*

If K satisfies the cone condition with respect to a cone C, it turns out that it is the union of the closure of a finite number of Lipschitz open sets. In fact, the following result holds.

Proposition 3.2. *Let C be a finite closed cone in $\mathbb{R}^N$ with vertex at the origin and let $K \subseteq \mathbb{R}^N$ be a compact set with $\operatorname{diam}(K) \leq M$ which satisfies the cone condition with respect to C. Then for every $\rho > 0$, there exist a finite number $A_1, A_2, \cdots, A_m$ of compact subsets of K with $\operatorname{diam}(A_j) \leq \rho$ and a finite number $P_1, P_2, \cdots, P_m$ of congruent parallelepipeds with a vertex at the origin such that:*

(a) *for all $x \in K$ there exists $1 \leq i \leq m$ with $P_i \subseteq C_x$;*

(b) $K = \bigcup_{i=1}^{m} K_i$ *where* $K_i = \bigcup_{x \in A_i} (x + P_i)$.

The number m and the parallelepipeds $P_1, \ldots, P_m$ depend only on C, M, ρ, and not on the particular set K.

Moreover there exists $\overline{\rho} > 0$, depending only on C, such that for $\rho < \overline{\rho}$, the following facts hold for all $i = 1, \ldots, m$:

(c) *for every $y \in \partial K_i$, there exists $\eta > 0$, an orthogonal coordinate system $(\xi_1, \ldots, \xi_n)$ and a Lipschitz function f such that $B_\eta(y) \cap K_i = B_\eta(y) \cap \{\xi = (\xi_1, \ldots, \xi_n) : \xi_n \leq f(\xi_1, \ldots, \xi_{n-1})\}$;*

(d) $\operatorname{int}(K_i) = \bigcup_{x \in A_i} (x + \operatorname{int}(P_i))$.

Proof. Properties (a), (b) and (c) can be obtained as in the Gagliardo theorem on the decomposition of open sets with the cone property (see [1], Thm. 4.8). In particular, $\overline{\rho}$ can be chosen as the distance of the center of P_i from ∂P_i; with this choice of $\overline{\rho}$, it turns out that, if a ball B of radius $r < \frac{\overline{\rho}}{2}$ is such that $B \cap (x_1 + P_i) \neq \emptyset$ and $B \cap (x_2 + P_i) \neq \emptyset$ for some $x_1, x_2 \in A_i$, then B cannot intersect relative opposite faces of $x_1 + P_i$ and $x_2 + P_i$ respectively.

Let us turn to the proof of point (d). The inclusion

$$\bigcup_{x \in A_i} (x + \operatorname{int}(P_i)) \subseteq \operatorname{int}(K_i)$$

is immediate. Let $y \in \operatorname{int}(K_i)$ and let $r < \frac{\overline{\rho}}{2}$ be such that $B_r(y) \subseteq K_i$. There exists $x \in A_i$ such that $y \in x + P_i$. If $y \in x + \operatorname{int}(P_i)$ for some z, the result is obtained. Let us suppose that $y \in x + \partial P_i$. For every $z \in B_r(y)$, there exists $x_z \in A_i$ with $z \in x_z + P_i$. If $y \in x_z + \operatorname{int}(P_i)$, the proof is concluded; let us assume by contradiction that $y \in x_z + \partial P_i$ for all $z \in B_r(y)$. Clearly $y - x_z$ cannot belong to the same face of P_i as z varies in $B_r(y)$ because this would contradict $z \in x_z + P_i$ for all $z \in B_r(y)$. Since $B_r(y)$ cannot intersect relative opposite faces of the parallelepipeds $x + P_i$ with $x \in A_i$, we conclude that there exists a vertex v_j of P_i such that $y - x_z$ belongs to a face passing through v_j for all $z \in B_r(y)$. Let $Q_j := \{\lambda(x - v_j) : x \in P_i, \lambda > 0\}$ and let $y_n \to y$ be such that $y - y_n \in \operatorname{int}(Q_j)$. For n large enough, since $y \in x_{y_n} + \partial P_i$, we obtain $y_n \notin x_{y_n} + P_i$ which is absurd. This concludes the proof of point (d). □

Let now consider a sequence (K_n) of compact subsets of $\mathbb{R}^N$ satisfying the cone condition with respect to a given finite closed cone C with vertex at the origin. If K_n converges to a compact set K in the Hausdorff metric, clearly K satisfies the cone condition with respect to C. Let $\mathcal{P}(K_n)$ be the family of all parallelepipeds contained in K_n and congruent to the parallelepipeds $P_1, \dots, P_m$ which appear in the decomposition (b) of Proposition 3.2 and let $\mathcal{P}(K)$ be the analogous family for K. Define $\mathcal{P}_r(K)$ as the subset of $\mathcal{P}(K)$ consisting of parallelepipeds P such that there exists $n_k \to \infty$ and $P^k \in \mathcal{P}(K_{n_k})$ with $P^k \to P$ in the Hausdorff metric. Let us pose

$$K_r := \{x \in K \,:\, x \in \operatorname{int}(P'), P' \in \mathcal{P}_r(K)\}, \tag{3.1}$$

and

$$K_s := K \setminus K_r. \tag{3.2}$$

We call the elements of K_r *regular points* of K (relative to the approximation given by (K_n)) and the elements of K_s *singular points* of K: K_r is clearly an open set.

Proposition 3.3. *Let C be a finite closed cone in $\mathbb{R}^N$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$ satisfying the cone condition with respect to C and converging to a compact set K in the Hausdorff metric. Then $\mathcal{H}^{N-1}(K_s) < +\infty$.*

Proof. Let us fix ρ smaller than the constant $\overline{\rho}$ given by Proposition 3.2 (which does not depend on n). By point (b) of the same proposition, we can write

$$K_n = \bigcup_{i=1}^m K_n^i \qquad \text{with} \qquad K_n^i := \bigcup_{x \in A_n^i} (x + P_i)$$

where $A_n^1, \dots, A_n^m$ are compact subsets of K_n with $\operatorname{diam}(A_n^i) \leq \rho$ and $P_1, \dots, P_m$ are parallelepipeds with a vertex at the origin. There exists $n_k \to \infty$ such that $A_{n_k}^i \to A^i$ in the Hausdorff metric for $i = 1, \dots, m$: clearly $K_{n_k}^i$ converges to $K^i := \bigcup_{x \in A^i}(x + P^i)$ in the Hausdorff metric. Let us prove that $\operatorname{int}(K^i) \subseteq K_r$ for $i = 1, \dots, m$. Since $\operatorname{diam}(A^i) \leq \rho$, by point (d) of Proposition 3.2, we have $\operatorname{int}(K^i) = \bigcup_{x \in A^i}(x + \operatorname{int}(P^i))$; given $x_0 \in A^i$ and $x_{n_k} \in A_{n_k}^i$ with $x_{n_k} \to x_0$, we have that $x_0 + P^i$ is the Hausdorff limit of $x_{n_k} + P^i$. Since $\operatorname{int}(x_0 + P^i) = x_0 + \operatorname{int}(P^i)$, we conclude that $\operatorname{int}(K^i) \subseteq K_r$ and so $\bigcup_{i=1}^m \operatorname{int}(K^i) \subseteq K_r$.

By point (c) of Proposition 3.2, we have that K^i has Lipschitz boundary; we conclude that

$$\mathcal{H}^{N-1}(K_s) = \mathcal{H}^{N-1}(K \setminus K_r) \leq \sum_{i=1}^m \mathcal{H}^{N-1}(\partial K^i) < +\infty.$$

The proof is now complete. □

4. The main result

We now recall the main regularity assumption on the sequence (K_n) of compact subsets of $\mathbb{R}^N$ in order to obtain the stability result mentioned in the Introduction. We assume $N \geq 3$.

Let π be the hyperplane $x_N = 0$ in $\mathbb{R}^N$.

Definition 4.1. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$. We say that (K_n) satisfies the C-condition if there exist constants $\delta, L_1, L_2 > 0$ such that, for all n and for all $x \in K_n$, there exists $\Phi_x : B_\delta(x) \to \mathbb{R}^N$ with:*

(a) *for all $z_1, z_2 \in B_\delta(x)$:*

$$L_1|z_1 - z_2| \leq |\Phi_x(z_1) - \Phi_x(z_2)| \leq L_2|z_1 - z_2|;$$

(b) *$\Phi_x(x) = 0$ and $\Phi_x(B_\delta(x) \cap K_n) \subseteq \pi$;*

(c) *for all $y \in B_{\frac{\delta}{2}}(x) \cap K_n$,*

$$\Phi_x(y) \in C_y \subseteq \Phi_x(B_\delta(x) \cap K_n)$$

for some finite closed cone C_y in π congruent to C.

For technical reasons, we assume that $L_1 \mathrm{diam}(C) < \frac{1}{8}\delta$: this is clearly not restrictive up to reducing C.

We can now state the main result of the paper.

Theorem 4.2. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$, Ω a bounded open subset of $\mathbb{R}^N$, $1 < p \leq 2$, (K_n) a sequence of compact subsets of $\mathbb{R}^N$ satisfying the C-condition and converging to a compact set K in the Hausdorff metric. Then the spaces $W^{1,p}(\Omega \setminus K_n)$ converge to $W^{1,p}(\Omega \setminus K)$ in the sense of Mosco.*

In order to prove the main theorem, we need to analyze the structure of the sets K_n and K. This is done in the following lemmas.

Lemma 4.3. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$ converging to K in the Hausdorff metric. Suppose that (K_n) satisfies the C-condition. Then there exist $m \geq 1$ such that, for n large enough,*

$$K_n = \bigcup_{i=1}^{m} K_n^i$$

with K_n^i compact, $B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$ for some $x_n^i \in K_n$ such that $x_n^i \to x^i \in K$ for all $i = 1, \ldots, m$ and $K \subseteq \bigcup_{i=1}^m B_{\frac{\delta}{3}}(x^i)$; moreover $\Phi_{x_n^i}(K_n^i)$ satisfies the cone condition with respect to C for all $i = 1, \ldots, m$.

Proof. Since K is compact, there exists a finite number of points $x^1, \ldots, x^m \in K$ such that

$$K \subseteq \bigcup_{i=1}^{m} B_{\frac{\delta}{4}}(x^i). \tag{4.1}$$

As $K_n \to K$ in the Hausdorff metric, there exist $x_n^i \in K_n$ such that $x_n^i \to x^i$ for $i = 1, \ldots, m$. For n large enough, we clearly have

$$K_n \subseteq \bigcup_{i=1}^{m} B_{\frac{\delta}{3}}(x_n^i). \tag{4.2}$$

In order to conclude the proof, it is sufficient to take K_n^i as the preimage under $\Phi_{x_n^i}$ of the union of all cones $C' \subseteq \pi$ congruent to C such that $C' \subseteq \Phi_{x_n^i}(B_\delta(x_n^i) \cap K_n)$ and $C' \cap \Phi_{x_n^i}(\overline{B}_{\frac{\delta}{3}}(x_n^i) \cap K_n) \neq \emptyset$. In fact, K_n^i is compact and the inclusion $B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i$ comes directly from the definition of K_n^i and the fact that (K_n) satisfies the C-condition; moreover, the inclusion $K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$ comes from the assumption $L_1 \mathrm{diam}(C) < \frac{1}{8}\delta$, and by (4.2) we have $K_n = \bigcup_{i=1}^m K_n^i$. Finally, by construction, $\Phi_{x_n^i}(K_n^i)$ satisfies the cone condition with respect to C for all n and $i = 1, \ldots, m$, and by (4.1) we have $K \subseteq \bigcup_{i=1}^m B_{\frac{\delta}{3}}(x^i)$ which concludes the proof. □

Lemma 4.4. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$ converging to K in the Hausdorff metric. Let (K_n) satisfy the C-condition and let $K_n = \bigcup_{i=1}^m K_n^i$ according to the decomposition given by Lemma 4.3. Then, up to a subsequence, for $i = 1, \ldots, m$, $x_n^i \to x^i \in K$, $K_n^i \to K^i \subseteq K$ in the Hausdorff metric, $\Phi_{x_n^i} \to \Phi_i$ uniformly on $B_{\frac{3}{4}\delta}(x^i)$ with*

(a)
$$K \subseteq \bigcup_{i=1}^{m} B_{\frac{\delta}{3}}(x^i);$$

(b)
$$B_{\frac{\delta}{3}}(x^i) \cap K \subseteq K^i \subseteq B_{\frac{3}{4}\delta}(x^i);$$

(c)
$$K = \bigcup_{i=1}^{m} K^i;$$

(d) *for all* $z_1, z_2 \in B_{\frac{3}{4}\delta}(x^i)$*:*

$$L_1|z_1 - z_2| \leq |\Phi_i(z_1) - \Phi_i(z_2)| \leq L_2|z_1 - z_2|;$$

(e)
$$\Phi_i(K \cap B_{\frac{3}{4}\delta}(x^i)) \subseteq \pi.$$

Moreover, $\Phi_i(K^i)$ *satisfies the cone condition with respect to* C *for all* $i = 1, \ldots, m$*.*

Proof. By Lemma 4.3, $x_n^i \to x^i \in K$ for all $i = 1, \ldots, m$ and $K \subseteq \bigcup_{i=1}^m B_{\frac{\delta}{3}}(x^i)$; this proves point (a). Since $K_n \to K$ in the Hausdorff metric, up to a subsequence, $K_n^i \to K^i \subseteq K$ in the Hausdorff metric for $i = 1, \ldots, m$. Fix $i \in \{1, \ldots, m\}$. Note that, for n large enough, $\overline{B}_{\frac{3}{4}\delta}(x^i) \subseteq B_\delta(x_n^i)$. We deduce that $\Phi_{x_n^i}$ are well defined on $B_{\frac{3}{4}\delta}(x^i)$; since they are equicontinuous and equibounded, we may assume that $\Phi_{x_n^i} \to \Phi_i$ uniformly on $B_{\frac{3}{4}\delta}(x^i)$ with

$$L_1|z_1 - z_2| \leq |\Phi_i(z_1) - \Phi_i(z_2)| \leq L_2|z_1 - z_2|$$

for all $z_1, z_2 \in B_{\frac{3}{4}\delta}(x^i)$. This proves point (d).

Passing to the limit in the relations

$$B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$$

$$K_n = \bigcup_{i=1}^m K_n^i$$

$$\Phi_{x_n^i}(K_n \cap B_{\frac{3}{4}\delta}(x_n^i)) \subseteq \pi,$$

we obtain points (b), (c) and (e).

Finally, it is easy to see that $\Phi_i(K^i)$ satisfies the cone condition with respect to C. In fact, fix $y \in K^i$; since $K_n^i \to K^i$ in the Hausdorff metric, there exists $y_n \in K_n^i$ with $y_n \to y$. As $\Phi_{x_n^i}(K_n^i)$ satisfies the cone condition with respect to C, there exists C_n finite closed cone in π congruent to C such that $\Phi_{x_n^i}(y_n) \in C_n \subseteq \Phi_{x_n^i}(K_n^i)$. Up to a subsequence, $C_n \to C'$ in the Hausdorff metric with C' congruent to C. Then $\Phi_i(y) \in C' \subseteq \Phi_i(K^i)$ since $\Phi_{x_n^i}(K_n^i) \to \Phi_i(K^i)$ in the Hausdorff metric. □

We can now pass to the proof of the main theorem.

Proof of Theorem 4.2. Let Y' and Y'' be the weak-limsup and the strong-liminf of the sequence $W^{1,p}(\Omega \setminus K_n)$ respectively. We have to prove that $Y' = Y'' = W^{1,p}(\Omega \setminus K)$.

Let us start with the inclusion

$$Y' \subseteq W^{1,p}(\Omega \setminus K). \tag{4.3}$$

Let (u_k) be a sequence in $W^{1,p}(\Omega \setminus K_{n_k})$ $(n_k \to +\infty)$, and let $v, w_1, \cdots, w_N \in L^p(\Omega)$ be such that $u_k \to v$ and $D_i u_k \to w_i$ weakly in $L^p(\Omega)$ for $i = 1, \ldots, N$ with the identification (2.3). Since $K_{n_k} \to K$ in the Hausdorff metric, it is readily seen that for $i = 1, \ldots, N$, $w_i = D_i v$ in the sense of distributions in $\Omega \setminus K$. Since (K_n) satisfies the C-condition, we have $\mathcal{L}^N(K) = 0$; as a consequence, we get $v = 0$ and $w_1, \ldots, w_N = 0$ a.e. on K, and so we conclude that $(v, w_1, \ldots, w_N)$ is the element of $L^p(\Omega; \mathbb{R}^{N+1})$ associated to a function of $W^{1,p}(\Omega \setminus K)$ according to (2.3).

We can thus pass to the inclusion

$$W^{1,p}(\Omega \setminus K) \subseteq Y''; \tag{4.4}$$

we have to prove that, given $u \in W^{1,p}(\Omega \setminus K)$, there exists $u_n \in W^{1,p}(\Omega \setminus K_n)$ such that $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^p(\Omega; \mathbb{R}^{N+1})$. By standard arguments on Mosco Convergence, it is sufficient to prove that, given any subsequence n_j, there exists a further subsequence n_{j_k} and a sequence $u_k \in W^{1,p}(\Omega \setminus K_{n_{j_k}})$ such that $(u_k, \nabla u_k) \to (u, \nabla u)$ strongly in $L^p(\Omega; \mathbb{R}^{N+1})$. Thus we deduce that, in order to prove (4.4), we can reason up to subsequences.

Using the decomposition given by Lemma 4.3, there exists $m \geq 1$ such that

$$K_n = \bigcup_{i=1}^{m} K_n^i$$

with K_n^i compact, $B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$ for some $x_n^i \in K_n$, and $\Phi_{x_n^i}(K_n^i)$ satisfying the cone condition with respect to C for all $i = 1, \ldots, m$. By Lemma 4.4, up to a subsequence, $x_n^i \to x^i \in K$ for all $i = 1, \ldots, m$, with $K \subseteq \bigcup_{i=1}^m B_{\frac{\delta}{3}}(x^i)$, and $\Phi_{x_n^i} \to \Phi_i$ uniformly on $B_{\frac{3}{4}\delta}(x^i)$ such that, for all $z_1, z_2 \in B_{\frac{3}{4}\delta}(x^i)$

$$L_1|z_1 - z_2| \leq |\Phi_i(z_1) - \Phi_i(z_2)| \leq L_2|z_1 - z_2|.$$

Moreover, $K_n^i \to K^i$ in the Hausdorff metric with

$$K = \bigcup_{i=1}^{m} K^i,$$

$B_{\frac{\delta}{3}}(x^i) \cap K \subseteq K^i \subseteq B_{\frac{3}{4}\delta}(x^i)$ and $\Phi_i(K^i)$ satisfies the cone condition with respect to C for all $i = 1, \ldots, m$. Finally, we have that

$$\Phi_{x_n^i}(K_n^i) \to \Phi_i(K^i) \tag{4.5}$$

in the Hausdorff metric for $i = 1, \ldots, m$.

We begin proving the strong-liminf condition in the particular case in which $u \in W^{1,p}(\Omega \setminus K)$, $\mathrm{supp}(u) \subset\subset B_{\frac{\delta}{3}}(x^i)$ and

$$\mathrm{supp}(u \circ \Phi_i^{-1}) \cap \pi \subseteq [\Phi_i(K^i)]_r, \tag{4.6}$$

where, according to (3.1), $[\Phi_i(K^i)]_r$ denotes the set of regular points of $\Phi_i(K^i)$ relative to the approximation (4.5). Pose $w := u \circ \Phi_i^{-1}$; we have $w \in W^{1,p}(\Phi_i(B_{\frac{\delta}{3}}(x)) \setminus \Phi_i(K^i))$. As in Section 3, let $\mathcal{P}_r(\Phi_i(K^i))$ denote the family of parallelepipeds contained in $\Phi_i(K^i)$ and congruent to the parallelepipeds $P_1, \ldots, P_m$ given by Proposition 3.2, that are limit in the Hausdorff metric of parallelepipeds P^n congruent to $P_1, \ldots, P_m$ and contained in $\Phi_{x_n^i}(K_n^i)$. By (3.1) and (4.6) there exist $D_1, \ldots, D_t \in \mathcal{P}_r(\Phi_i(K^i))$ such that

$$\mathrm{supp}(w) \cap \pi \subseteq \bigcup_{j=1}^{t} \mathrm{int}_\pi(D_j)$$

where $\mathrm{int}_\pi(\cdot)$ denotes the interior relative to π. Let $Q_j \subseteq \mathrm{int}_\pi(D_j)$ be a parallelepiped in π such that $\mathrm{supp}(w) \cap \pi \subseteq \mathrm{int}_\pi(Q_j)$ and let $\varepsilon > 0$ be such that, posed $U_j := \mathrm{int}_\pi(Q_j) \times]-\varepsilon, \varepsilon[$, $(j = 1, \ldots, t)$,

$$\bigcup_{j=1}^{t} U_j \subseteq \Phi_i(B_{\frac{\delta}{3}}(x^i)).$$

Through a partition of unity associated to $\{U_1, \ldots, U_t, U_0\}$ with $U_0 := \mathbb{R}^N \setminus \Phi_i(K^i)$, we may write

$$w = \sum_{j=0}^{t} \psi_j w,$$

with $\psi_j \in C^\infty(U_j)$, $\mathrm{supp}(\psi_j) \subset\subset U_j$, so that

$$u = \sum_{j=0}^{t} (\psi_j \circ \Phi_i) u.$$

Note that $\mathrm{supp}((\psi_0 \circ \Phi_i)u) \cap K = \emptyset$ so that

$$(\psi_0 \circ \Phi_i)u \in W^{1,p}(\Omega \setminus K_n)$$

for n large enough, that is $(\psi_0 \circ \Phi_i)u \in Y''$. In order to conclude, it is thus sufficient to deal with the case $\mathrm{supp}(w) \subset\subset U_j$ for $j = 1, \dots, t$.

Let us fix $j \in \{1, \dots, t\}$. Set $U_j^+ := U_j \cap (\mathbb{R}^{N-1} \times]0, \varepsilon[)$, $U_j^- := U_j \cap (\mathbb{R}^{N-1} \times]-\varepsilon, 0[)$, and let $w^\pm := w_{|U_j^\pm}$. We have $w^\pm \in W^{1,p}(U_j^\pm)$: let $\widetilde{w}^\pm$ be the extension by reflection of $w^\pm$ on U_j. Note that $\mathrm{supp}(\widetilde{w}^\pm) \subset\subset U_j$. Up to a subsequence, $Q_j \subseteq \Phi_{x_n^i}(K_n^i)$ because $D_j \in \mathcal{P}_r(\Phi_i(K^i))$ and $Q_j \subseteq \mathrm{int}_\pi(D_j)$; we deduce that $U_j \setminus \Phi_{x_n^i}(K_n^i)$ has exactly two connected components that we indicate by B^+ and B^- (note that they do not depend on n for n large). As a consequence $\Phi_{x_n^i}^{-1}(U_j) \setminus K_n$ has exactly two connected components given by $\Phi_{x_n^i}^{-1}(B^+)$ and $\Phi_{x_n^i}^{-1}(B^-)$ respectively. Consider

$$v_n := \begin{cases} \widetilde{w}^+ \circ \Phi_i & \text{on } \Phi_{x_n^i}^{-1}(B^+) \\ \\ \widetilde{w}^- \circ \Phi_i & \text{on } \Phi_{x_n^i}^{-1}(B^-). \end{cases}$$

Since $\widetilde{w}^\pm$ has compact support in U_j, we deduce that for n large enough

$$v_n \in W^{1,p}(\Omega \setminus K_n).$$

Since $K_n^i \to K^i$ in the Hausdorff metric and $\widetilde{w}^\pm \circ \Phi_i$ does not depend on n, $v_n \to u$ and $\nabla v_n \to \nabla u$ a.e. in Ω. By the Dominated Convergence Theorem, we deduce that $(v_n, \nabla v_n) \to (u, \nabla u)$ in $L^p(\Omega; \mathbb{R}^{N+1})$ under the identification (2.3). This proves $u \in Y''$ in the case u satisfies (4.6).

In order to complete the proof of the theorem, we have to see that the assumption (4.6) is not restrictive. Consider $u \in W^{1,p}(\Omega \setminus K)$. Let $\{\varphi_1, \dots, \varphi_m, \varphi_0\}$ be a C^∞ partition of unity associated to $B_{\frac{\delta}{3}}(x^1), \dots, B_{\frac{\delta}{3}}(x^m), \mathbb{R}^N \setminus K$. We can write

$$u = \sum_{i=0}^m \varphi_i u.$$

Since $\mathrm{supp}(\varphi_0 u) \cap K = \emptyset$, we have that $\mathrm{supp}(\varphi_0 u) \cap K_n = \emptyset$ for n large enough and so $\varphi_0 u \in W^{1,p}(\Omega \setminus K_n)$. This implies $\varphi_0 u \in Y''$. We deduce that it is not restrictive to assume $\mathrm{supp}(u) \subset\subset B_{\frac{\delta}{3}}(x^i)$ for some $i = 1, \dots, m$.

Let us consider

$$K_s := \bigcup_{i=1}^m \Phi_i^{-1}\left([\Phi_i(K^i)]_s\right)$$

where, according to (3.2), $[\Phi_i(K^i)]_s$ denotes the set of singular points of $\Phi_i(K^i)$ under the approximation (4.5). By Lemma 3.2, we obtain

$$\mathcal{H}^{N-2}(K_s) < +\infty; \tag{4.7}$$

by Theorem 3 in section 4.7.2 of [15], since $1 < p \leq 2$, we deduce that $c_p(K_s, \Omega) = 0$, where

$$c_p(K_s, \Omega) := \inf \left\{ \int_\Omega |\nabla u|^p \,:\, u \in W_0^{1,p}(\Omega),\, u \geq 1 \,\text{in a neighborhood of } \mathrm{K_s} \right\}.$$

By standard properties of capacity, there exists a sequence (ψ_k) in $C_c^\infty(\mathbb{R}^N)$ with $\psi_k \to 0$ in $W^{1,p}(\mathbb{R}^N)$ and $\psi_k \geq 1$ on a neighborhood of K_s. Since

$$u = \psi_k u + (1 - \psi_k)u,$$

we deduce that the set

$$\mathcal{D} := \left\{ v \in W^{1,p}(\Omega \setminus K) : \mathrm{supp}(v) \cap K_s = \emptyset \right\}$$

is dense in $W^{1,p}(\Omega \setminus K) \cap L^\infty(\Omega \setminus K)$ and hence in $W^{1,p}(\Omega \setminus K)$. As observed in Section 2, in order to prove (4.4), it is sufficient to check the inclusion $\mathcal{D} \subseteq Y''$. If $u \in \mathcal{D}$, we have that

$$\operatorname{supp}(u \circ \Phi_i^{-1}) \cap \Phi_i(K^i) \subseteq [\Phi_i(K^i)]_r.$$

Consider $V_1, V_2 \subseteq \pi$ open in the relative topology of π and such that

$$\operatorname{supp}(u \circ \Phi_i^{-1}) \cap \Phi_i(K^i) \subset\subset V_1 \subset\subset V_2 \subset\subset [\Phi_i(K^i)]_r;$$

let $\varepsilon > 0$ with $U_2 := V_2 \times]-\varepsilon, \varepsilon[\subseteq \Phi_i(B_{\frac{\delta}{3}}(x^i))$ and set $U_1 := V_1 \times]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. Consider $\varphi \in C_c^\infty(\Phi_i^{-1}(U_2))$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $\Phi_i^{-1}(U_1)$. Since $u \in \mathcal{D}$, we deduce $\operatorname{supp}((1-\varphi)u) \cap K = \emptyset$ that is $(1-\varphi)u \in W^{1,p}(\Omega \setminus K_n)$ for n large enough and so $(1-\varphi)u \in Y''$. Moreover, since

$$\operatorname{supp}((\varphi u) \circ \Phi_i^{-1}) \cap \pi \subseteq [\Phi_i(K^i)]_r,$$

we deduce by the previous step that $\varphi u \in Y''$. We conclude $u = \varphi u + (1-\varphi)u \in Y''$ and the theorem is proved. ☐

From Theorem 4.2 in the case $p = 2$, we may deduce the stability of the Neumann problems mentioned in the Introduction.

Corollary 4.5. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$, (K_n) a sequence of compact subsets of $\mathbb{R}^N$ satisfying the C-condition and converging to a compact set K in the Hausdorff metric. Let Ω be an open and bounded subset of $\mathbb{R}^N$, $f \in L^2(\Omega)$, and let u_n and u be the solutions of the following Neumann problems*

$$\begin{cases} -\Delta u_n + u_n = f \\ u \in H^1(\Omega \setminus K_n), \end{cases} \tag{4.8}$$

$$\begin{cases} -\Delta u + u = f \\ u \in H^1(\Omega \setminus K). \end{cases} \tag{4.9}$$

Pose $u_n = 0$, $\nabla u_n = 0$ on $K_n \cap \Omega$, and $u = 0$, $\nabla u = 0$ on $K \cap \Omega$.

Then we have $u_n \to u$ strongly in $L^2(\Omega)$ and $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^N)$, so that the problems (4.8) are stable.

Proof. Let u_n be the solution of (4.8) and u the solution of (4.9). We assume the identification (2.3). From the equation (4.8), we have that $(u_n, \nabla u_n)$ is bounded in $L^2(\Omega; \mathbb{R}^{N+1})$. There exists $v \in L^2(\Omega; \mathbb{R}^{N+1})$ such that up to a subsequence, $(u_n, \nabla u_n) \to v$ weakly in $L^2(\Omega; \mathbb{R}^{N+1})$. By Theorem 4.2, we have that $H^1(\Omega \setminus K_n)$ converges to $H^1(\Omega \setminus K)$ in the sense of Mosco. Thus we deduce $v \in H^1(\Omega \setminus K)$; moreover, taking $\varphi \in H^1(\Omega \setminus K)$, there exists $\varphi_n \in H^1(\Omega \setminus K_n)$ with $(\varphi_n, \nabla \varphi_n) \to (\varphi, \nabla \varphi)$ strongly in $L^2(\Omega; \mathbb{R}^{N+1})$. We conclude that

$$\begin{aligned} \int_{\Omega \setminus K} \nabla v \nabla \varphi + \int_{\Omega \setminus K} v\varphi &= \lim_n \int_{\Omega \setminus K_n} \nabla u_n \nabla \varphi_n + \int_{\Omega \setminus K_n} u_n \varphi_n = \\ &= \lim_n \int_{\Omega \setminus K_n} f \varphi_n = \\ &= \int_{\Omega \setminus K} f\varphi, \end{aligned} \tag{4.10}$$

that is $v = u$. Finally, taking $\varphi_n = u_n$ and using again (4.10), we have that

$$||u_n||_{L^2(\Omega; \mathbb{R}^{N+1})} \to ||u||_{L^2(\Omega; \mathbb{R}^{N+1})}.$$

We conclude that $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^2(\Omega; \mathbb{R}^{N+1})$ and so the proof is complete. ☐

Remark 4.6. *Similarly, under the same hypotheses of Theorem 4.2, we can prove that the Neumann problems*

$$\left\{ \begin{array}{l} -\Delta_p u_n + |u_n|^{p-2} u_n = f \\ u_n \in W^{1,p}(\Omega \setminus K_n) \end{array} \right. \tag{4.11}$$

where $1 < p \leq 2$, Ω is open and bounded in $\mathbb{R}^N$, $f \in L^p(\Omega)$ and $\Delta_p u_n := \operatorname{div}(|\nabla u_n|^{p-2} \nabla u_n)$, converge to the Neumann problem

$$\left\{ \begin{array}{l} -\Delta_p u + |u|^{p-2} u = f \\ u \in W^{1,p}(\Omega \setminus K), \end{array} \right. \tag{4.12}$$

that is $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^p(\Omega; \mathbb{R}^{N+1})$ under the identification (2.3).

Remark 4.7. The Mosco convergence proved in Theorem 4.2 is the key point in order to prove the stability of more general problems. We now breafly sketch an application to fracture mechanics in linearly elastic bodies.

For every open and bounded set $A \subseteq \mathbb{R}^N$, let us pose

$$LD^{1,2}(A) := \left\{ u \in H^1_{\mathrm{loc}}(A; \mathbb{R}^N) \,:\, E(u) \in L^2(A, \mathrm{M}^{n\times n}_{\mathrm{sym}}) \right\},$$

where $\mathrm{M}^{n\times n}_{\mathrm{sym}}$ denotes the set of symmetric matrices of order N and $E(u)$ denotes the symmetric part of the gradient of u. Let $|M| := [\mathrm{tr}(M^2)]^{\frac{1}{2}}$ denote the standard norm in $\mathrm{M}^{n\times n}_{\mathrm{sym}}$.

Consider (K_n) a sequence of compact subsets of $\mathbb{R}^N$ satisfying the C-condition with respect to a given $(N-1)$-dimensional finite closed cone C and converging to K in the Hausdorff metric. Let Ω be open and bounded in $\mathbb{R}^N$ and let $\partial_D \Omega$ be a Lipschitz part of $\partial\Omega$. Consider $g_n, g \in H^1(\Omega; \mathbb{R}^N)$ with $g_n \to g$ strongly and let

$$\Gamma_n := \left\{ u \in LD^{1,2}(\Omega \setminus K_n) \,:\, u = g_n \text{ on } \partial_D \Omega \setminus K_n \right\}$$

and

$$\Gamma := \left\{ u \in LD^{1,2}(\Omega \setminus K) \,:\, u = g \text{ on } \partial_D \Omega \setminus K \right\}.$$

Given the *Lamé coefficients* μ, λ, let $u_n \in LD^{1,2}(\Omega \setminus K_n)$ be the minimum of

$$\min_{v \in \Gamma_n} \int_{\Omega \setminus K_n} \mu |E(v)|^2 + \frac{\lambda}{2} |\mathrm{tr}\, Ev|^2 \, d\mathcal{L}^N$$

and let $u \in LD^{1,2}(\Omega \setminus K)$ be the minimum of

$$\min_{v \in \Gamma} \int_{\Omega \setminus K} \mu |E(v)|^2 + \frac{\lambda}{2} |\mathrm{tr}\, Ev|^2 \, d\mathcal{L}^N.$$

Using the Mosco convergence given by Theorem 4.2 and the density result by Chambolle [8] (adapted to $\Omega \setminus K \subseteq \mathbb{R}^N$), it can be proved that $E(u_n) \to E(u)$ strongly in $L^2(\Omega; \mathrm{M}^{n\times n}_{\mathrm{sym}})$ with the convention of considering $E(u_n) = 0$ and $E(u) = 0$ on $\Omega \cap K_n$ and $\Omega \cap K$ respectively. This can be interpreted as the convergence of the equilibrium deformations for the elastic body Ω with fractures K_n and boundary displacements g_n to the equilibrium deformation relative to the fracture K and the boundary displacement g.

5. Non-stability examples

In this section, we give two explicit examples of non-stability when the conditions of Theorem 4.2 are violated. In Example 1, we see that the C-condition is not sufficient in the case $p > 2$: in fact some problems related to capacity can occur which in the case $1 < p \leq 2$ were avoided thank to (4.7). In Example 2, we see that a sort of uniform "curvilinear" cone condition for the sequence (K_n) given only by points (a) and (c) in the C-condition does not guarantee the Mosco convergence of the spaces $W^{1,p}(\Omega \setminus K_n)$ even in the case $1 < p \leq 2$.

EXAMPLE 1. Let Q, Q', Q'' be the open unit cube in $\mathbb{R}^N$, $\mathbb{R}^{N-1}$, and $\mathbb{R}^{N-2}$ respectively. For every $n \geq 1$, let us pose

$$K_n := \left\{ \left[0, \frac{1}{2} - \frac{1}{n}\right] \cup \left[\frac{1}{2} + \frac{1}{n}, 1\right] \right\} \times \overline{Q''} \times \left\{ \frac{1}{2} \right\}.$$

(K_n) is a sequence of compact sets in $\mathbb{R}^N$ whose limit in the Hausdorff metric is

$$K = \overline{Q'} \times \left\{ \frac{1}{2} \right\}.$$

Let us pose $L := \{\frac{1}{2}\} \times \overline{Q''} \times \{\frac{1}{2}\}$, $S_1 := Q' \times]0, \frac{1}{2}[$ and $S_2 := Q' \times]\frac{1}{2}, 1[$.

Let C be the finite closed cone in $\mathbb{R}^{N-1}$ determined by $B_{\frac{1}{8}}(P)$ with $P := (\frac{1}{8}, \frac{1}{8}, \dots, \frac{1}{8})$. Clearly (K_n) satisfies the C-condition.

We claim that, if $p > 2$, then the spaces $W^{1,p}(\Omega \setminus K_n)$ do not converge to $W^{1,p}(\Omega \setminus K)$ in the sense of Mosco. In fact, assuming the Mosco convergence, by Remark 4.6, we deduce that the Neumann problems

$$\begin{cases} -\Delta_p v + |v|^{p-2} v = f \\ v \in W^{1,p}(Q \setminus K_n) \end{cases} \tag{5.1}$$

with $f \in L^p(Q)$ converge to the problem

$$\begin{cases} -\Delta_p v + |v|^{p-2} v = f \\ v \in W^{1,p}(Q \setminus K). \end{cases} \tag{5.2}$$

Let $f = \chi_{S_2}$ and let u_n, u be the solutions of (5.1) and (5.2) respectively. We readily deduce that $u = \chi_{S_2}$; since $(u_n, \nabla u_n) \to (u, \nabla u)$ in $L^p(Q; \mathbb{R}^{N+1})$ under the identification (2.3), we obtain that $u_n \to u$ strongly in $W^{1,p}(S_i)$ for $i = 1, 2$. By strong convergence in $W^{1,p}(S_1)$, we get $u_n \to 0$ c_p-q.e. on L, while from strong convergence in $W^{1,p}(S_2)$, we deduce $u_n \to 1$ c_p-q.e. on L. Since $c_p(L, Q) \neq 0$ as $p > 2$, we get a contradiction: we conclude that the Mosco convergence does not hold.

EXAMPLE 2. Let Q, Q', Q'' be the open unit cube in $\mathbb{R}^N$, $\mathbb{R}^{N-1}$, and $\mathbb{R}^{N-2}$ respectively. Let us write $Q = Q' \times]0, 1[$. For every $n \geq 1$ let us pose

$$K_n := \bigcup_{i=1}^{n-1} \left[\frac{1}{3}, \frac{2}{3}\right] \times \overline{Q''} \times \left\{ \frac{i}{n} \right\}.$$

(K_n) is a sequence of compact sets in $\mathbb{R}^N$ whose limit in the Hausdorff metric is

$$K = \left[\frac{1}{3}, \frac{2}{3}\right] \times \overline{Q''} \times [0, 1].$$

Let us pose $S_1 := \left]0, \frac{1}{3}\right[\times Q'' \times]0, 1[$ and $S_2 := \left]\frac{2}{3}, 1\right[\times Q'' \times]0, 1[$.

Let C be the finite close cone in $\mathbb{R}^{N-1}$ determined by $B_{\frac{1}{6}}(P)$ with $P := (\frac{1}{6}, \frac{1}{6}, \dots, \frac{1}{6})$. Clearly there exists $\delta > 0$ such that, for all n and for all $x \in K_n$, posed

$$\Phi_x(y) := y - x,$$

$\Phi_x : B_\delta(x) \to \mathbb{R}^N$ satisfies conditions (a) and (c) of Definition 4.1 with respect to C. Observe that condition (b) is not satisfied: in particular, $\Phi_x(B_\delta(x) \cap K_n) \not\subseteq \pi$.

Let $1 < p \leq 2$ and let us consider the Neumann problems

$$\begin{cases} -\Delta_p v + |v|^{p-2} v = f \\ v \in W^{1,p}(Q \setminus K_n) \end{cases} \tag{5.3}$$

with $f \in L^p(Q)$. We claim that the problems (5.3) do not converge to the Neumann problem

$$\begin{cases} -\Delta_p v + |v|^{p-2} v = f \\ v \in W^{1,p}(Q \setminus K) \end{cases} \tag{5.4}$$

in the sense given in Remark 4.6, that is $(u_n, \nabla u_n) \not\to (u, \nabla u)$ strongly in $L^p(Q;\mathbb{R}^{N+1})$ where u_n and u are the solutions of problems (5.3) and (5.4) respectively and the identification (2.3) is assumed. This implies that $W^{1,p}(Q \setminus K_n)$ does not converge to $W^{1,p}(Q \setminus K)$ in the sense of Mosco and so it proves that point (b) in the C-condition cannot be omitted.

We employ a Γ-convergence technique. Let us consider the following functionals $F_n : L^p(Q) \to [0, \infty]$ defined by

$$F_n(z) := \begin{cases} \dfrac{1}{p} \displaystyle\int_Q |\nabla z|^p & \text{if } z \in W^{1,p}(Q \setminus K_n) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.5}$$

We will prove that, up to a subsequence, (F_n) Γ-converges with respect to the strong topology of $L^p(Q)$ to a functional F such that if $z \in L^p(Q)$ and $F(z) < +\infty$, then

$$z_{|S_i} \in W^{1,p}(S_i) \quad \text{for } i = 1, 2, \tag{5.6}$$

$$z(\cdot, x_N) \in W^{1,p}(Q') \quad \text{for a.e } x_N \in]0,1[. \tag{5.7}$$

Let us assume for the moment (5.6) and (5.7). Given $f \in L^p(Q)$, the functional

$$G(u) := \frac{1}{p} \int_Q |u|^p - \int_Q fu$$

is a continuous perturbation of F_n: as a consequence,

$$\Gamma - \lim_n (F_n + G) = F + G.$$

Note that the solution u_n of problem (5.3) is precisely the minimum of $F_n + G$: from this, we derive that for all n

$$F_n(u_n) + G(u_n) \leq 0. \tag{5.8}$$

Suppose that the problems (5.3) converge to the problem (5.4): then in particular, $u_n \to u$ strongly in $L^p(Q)$ where, as usual, u is extended to 0 on K. Note that $F(u) < +\infty$ because of (5.8) and the Γ-liminf inequality. If we choose

$$f(x) := \chi_{S_1}$$

we conclude that u is equal to 1 on S_1 and equal to 0 on S_2. With the identification (2.3), we get $u = f$. Clearly $f(\cdot, x_N) \notin W^{1,p}(Q')$ for $x_N \in]0,1[$ and so we get a contradiction. This proves that the problems (5.3) do not converge to problem (5.4).

In order to perform the previous argument by contradiction, we have to prove (5.6) and (5.7). This can be done in the following way. Let $z_n \to z$ strongly in $L^p(Q)$ with

$$F_n(z_n) \leq C < +\infty. \tag{5.9}$$

Since

$$\frac{1}{p} \int_{S_1} |\nabla z_n|^p + \frac{1}{p} \int_{S_2} |\nabla z_n|^p \leq C,$$

we deduce that $z_{|S_i} \in W^{1,p}(S_i)$ for $i = 1, 2$ and so we get (5.6). For a.e. $x_N \in]0,1[$, we have that $z_n(\cdot, x_N) \to z(\cdot, x_N)$ strongly in $L^p(Q')$; by (5.9) and Fatou's lemma, we have

$$\frac{1}{p} \int_0^1 \left(\liminf_n \int_{Q'} |\nabla z_n(y, x_N)|^p \, dy \right) dx_N \leq C,$$

so that for a.e. $x_N \in]0,1[$, there exists $C_{x_N} > 0$ and a subsequence n_k such that

$$\frac{1}{p} \int_{Q'} \sum_{i=1}^{N-1} |D_i z_{n_k}(y, x_N)|^p \, dy \leq C_{x_N}.$$

We conclude that for a.e. $x_N \in [0,1]$, $z(\cdot, x_N) \in W^{1,p}(Q')$ so that (5.7) is proved and the proof is complete.

ACKNOWLEDGEMENTS

The author wishes to thank Gianni Dal Maso for having proposed him the problem, and for many helpful and interesting discussions.

References

[1] Adams R.A.: *Sobolev Spaces*, Academic Press, New York (1975).

[2] Attouch H., Picard C.: Comportement limite de problèmes de transmission unilatéraux à travers des grilles de forme quelconque, *Rend. Sem. Mat. Univ. Politec. Torino* **45** (1987), 71-85.

[3] Bucur D., Varchon N.: Stability of the Neumann problem for variations of boundary, *C. R. Acad. Sci., Paris, Ser I, Math.* **331** No. 5 (2000), 371-374.

[4] Bucur D., Varchon N.: A duality approach for the boundary variation of Neumann problems. Preprint Univ. France-Comté, 2000.

[5] Bucur D., Zolesio J.P.: Shape Optimization for elliptic problems under Neumann boundary conditions, in *Calculus of Variations, Homogenization and Continuum Mechanics (CIRM - Luminy, Marseille, 1993)*, ed. by Bouchitté G., Buttazzo G., Suquet P., 117-129, Series on Advanced in Mathematics for Applied Sciences **18**, World Scientific, Singapore, 1994.

[6] Bucur D., Zolesio J.P.: Optimisation de forme sous contrainte capacitaire, *C. R. Acad. Sci. Paris, Série I* **318** (1994), 795-800.

[7] Bucur D., Zolesio J.P.: Continuité par rapport au domaine dans le problème de Neumann, *C. R. Acad. Sci. Paris, Série I* **319** (1994), 57-60.

[8] Chambolle: A density result in two-dimensional linearized elasticity and applications. Preprint CEMERADE, Université de Paris-Dauphine, 2001.

[9] Chambolle A., Doveri F.: Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets. *Comm. Partial Differential Equations* **22** (1997), 811-840.

[10] Cioranescu D., Murat F.: Un terme étrange venu d'ailleurs , *Collège de France Seminar*, Reserch Notes in Mathematics **60**, 98-138 and **70**, 154-178, Pitman, London, 1982.

[11] Cortesani, G.: Asymptotic behavior of a sequence of Neumann problems. *Comm. Partial Differential Equations* **22** (1997), 1691-1729

[12] Dal Maso G.: *An Introduction to* Γ*-Convergence*, Birkhäuser, Boston (1993).

[13] Damlamian A.: Le problème de la passoire de Neumann, *Rend. Sem. Mat. Univ. Politec. Torino* **43** (1985), 427-450.

[14] Del Vecchio, T.: The thick Neumann's sieve. *Ann. Mat. Pura Appl.* **147** (1987), 363-402

[15] Evans L.C., Gariepy R. F.: *Measure Theory and Fine Properties of Function* CRC Press, Boca Raton, 1992.

[16] Krushlov E.Ya.: On the Neumann boundary problem in a domain with complicated boundary, *Math. USSR Sbornik* **12** (1970), 553-571.

[17] Krushlov E.Ya.: The asymptotic behavior of solutions of the second boundary value problem under fragmentation of the boundary of the domain, *Math. USSR Sbornik* **35** (1979), 266-282.

[18] Marchenko A.V., Krushlov E.Ya.: *Boundary value problems in domains with finely-granulated boundaries* (in Russian), Naukova Dumka, Kiev, 1974.

[19] Murat, F.: The Neumann sieve, in *Non-linear variational problems (Isola D'Elba, 1983)*, ed. by A. Marino, 24-32, Research Notes in Mathematics **127**, Pitman, London, 1985.

[20] Nguetseng G.: Problèmes d'écrans perforés pour l'équation de Laplace, *RAIRO Modél. Math. Anal. Numér.* **19** (1985), 33-63.

[21] Picard C.: Analyse limite d'equations variationelles dans un domaine contenant un grille, *RAIRO Modél. Math. Anal. Numér.* **21** (1987), 293-326.

[22] Rogers C.A.:*Hausdorff Measures.* Cambridge University Press, Cambridge, 1970.

[23] Sanchez-Palencia E.: Boundary value problems in domains containing perforated walls, *Collège de France Seminar*, 309-325, Research Notes in Mathematics **70**, Pitman, London, 1982.

[24] Sanchez-Palencia E.: Problèmes mathématiques liés à l'écoulement d'un fluide visquex à travers une grille, *E. De Giorgi Colloquium*, Research Notes in Mathematics **125**, Pitman, London, 1985.

(Alessandro Giacomini) S.I.S.S.A., Via Beirut 2-4, 34014, Trieste, Italy
E-mail address, A. Giacomini: `giacomin@sissa.it`