THE CARNOT-CARATHÉODORY DISTANCE AND THE INFINITE LAPLACIAN

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Abstract. In $\mathbb{R}^n$ equipped with the Euclidean metric, the distance from the origin is smooth and infinite harmonic everywhere except the origin. Using geodesics, we find a geometric characterization for when the distance from the origin in an arbitrary Carnot-Carathéodory space is a viscosity infinite harmonic function at a point outside the origin. We show that at points in the Heisenberg group and Grushin plane where this condition fails, the distance from the origin is not a viscosity infinite harmonic subsolution. In addition, the distance function is not a viscosity infinite harmonic supersolution at the origin.

1. Introduction

In the Euclidean setting, the distance from the origin is (smoothly) infinite harmonic away from the origin. Such a result can not be obtained in Carnot-Carathéodory spaces because the distance function is not necessarily smooth off the origin [10, 15, 16]. We examine the infinite Laplace equation from the viewpoint of viscosity solutions. We show that the distance from the origin does not even satisfy the infinite Laplace equation in the viscosity sense at all points. Using geodesics, we find a geometric characterization for when the distance from the origin is indeed a viscosity solution. The Heisenberg group and Grushin plane will be examined in more detail, showing that when this condition fails at a point, the distance need not be a viscosity solution. At the origin, the distance function is a viscosity infinite harmonic subsolution to the infinite Laplace equation, but not a viscosity infinite harmonic supersolution.

We divide the paper up as follows: Section 2 discusses Carnot-Carathéodory spaces while Section 3 concerns viscosity infinite harmonic functions and their key properties. Section 4 is the main section, presenting the infinite Laplace material discussed above.

2. Carnot-Carathéodory spaces

In this section, we will briefly discuss the spaces under consideration. We first note that a general Carnot-Carathéodory space is a manifold of topological dimension n. The tangent space is generated by linearly independent vector fields $X_1, X_2, \ldots, X_m$, with $m \leq n$, that satisfy Hörmander's condition. That is, the vector fields and their Lie brackets span the tangent space at each point. The length of a curve is defined by fixing an inner product so that the X_i are orthonormal. By Chow's Theorem [2] any two points can be joined by a curve whose tangent vector lies in $\operatorname{span}\{X_i\}_{i=1}^m$. The natural distance

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between points x and y, denoted $d(x,y)$, is the infimum of lengths of such curves joining x and y and so Carnot-Carathéodory spaces are length spaces. [?] The horizontal gradient of a smooth function u is given by

$$\mathfrak{X}u = (X_1u, X_2u, \ldots, X_mu)$$

and the symmetrized second order horizontal derivative matrix $(\mathfrak{X}^2u)^\star$ has entries

$$(\mathfrak{X}^2u)^\star_{ij} = \frac{1}{2}\big(X_iuX_ju + X_juX_iu\big)$$

for $i,j = 1,2,\ldots,m$. Using these derivatives, the main operator we consider is the infinite Laplace operator, defined by

$$\Delta_\infty u = \langle (\mathfrak{X}^2u)^\star\ \mathfrak{X}u, \mathfrak{X}u\rangle = \sum_{i,j=1}^{m} X_iuX_juX_iX_ju.$$

We consider three main types of Carnot-Carathéodory spaces: Carnot groups, Grushin-type spaces and Riemannian vector fields. Carnot groups, denoted $\mathcal{G}$, are Carnot-Carathéodory spaces with a non-abelian algebraic group law. The tangent space is a stratified nilpotent Lie algebra, denoted $\mathfrak{g}$, with decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

for appropriate vector spaces that satisfy the Lie bracket relation $[V_1, V_j] = V_{1+j}$. The natural number k is called the step of the group. The exponential map is used to define natural dilations δ_r for $r > 0$ on $\mathcal{G}$ via the dilations on $\mathfrak{g}$, also denoted δ_r, given by $\delta_r(V_i) = r^iV_i$.

The Heisenberg algebra h can be identified with $\mathbb{R}^3$ in coordinates (x_1,x_2,x_3) spanned by a basis consisting of vector fields X_1, X_2, and X_3 given by

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3},\ X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3},\ \text{and}\ X_3 = \frac{\partial}{\partial x_3}.$$

By taking Lie brackets, it is easy to see that $X_3 = [X_1, X_2]$ and hence $k = 2$ with $V_1 = \operatorname{span}\{X_1, X_2\}, V_2 = \operatorname{span}\{X_3\}$ and $\delta_r(x_1,x_2,x_3) = (rx_1, rx_2, r^2x_3)$. The corresponding Lie group is the Heisenberg group, denoted $\mathbb{H}$. It has a smooth gauge bi-Lipschitz equivalent to the Carnot-Carathéodory distance given by

$$g(x,0) = \Big((x_1^2 + x_2^2)^2 + 16x_3^2\Big)^{\frac{1}{4}}.$$

For further details concerning Carnot groups and the Heisenberg group see [2], [3], [12], and the references therein.

The second class of spaces under consideration, Grushin-type spaces, lack an algebraic group structure. Their tangent space is constructed by considering $\mathbb{R}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$ and the vector fields

$$X_i = \rho_i(x_1, x_2, \ldots, x_{i-1})\frac{\partial}{\partial x_i}$$

for $i = 2, 3, \ldots, n$ where ρ_i is a (possibly constant, but not identically zero) polynomial. We decree that $\rho_1 \equiv 1$ so that

$$X_1 = \frac{\partial}{\partial x_1}.$$

Points in the Grushin-type space are also denoted $(x_1, x_2, \ldots, x_n)$. Global dilations do not, in general, exist. A special Grushin-type space under consideration is the Grushin plane, denoted $\mathbb{G}$, which has $n = 2$ and $\rho_2 = x_1$. For further results on Grushin-type spaces, see [2], [4], [5] and the references therein.

Note that when $m = n$ and the vectors vanish nowhere, we are in the Riemannian case. See [1] and [6] for further discussion.

3. Viscosity infinite harmonic functions and comparison with cones

As discussed above, our main equation under consideration is the infinite Laplace equation given by

$$-\Delta_\infty u = 0.$$

We now define appropriate weak solutions to this equation. Namely,

Definition 1. An infinite harmonic function u is a continuous function that is a viscosity solution to the infinite Laplace equation. That is, u satisfies the following:

SUB For any point x_0 and function ϕ with $X_iX_j\phi$ continuous for all i, j such that $u(x_0) = \phi(x_0)$ and $u(x) \leq \phi(x)$ near x_0, we have $-\Delta_\infty\phi(x_0) \leq 0$ (u is a viscosity infinite harmonic subsolution).

SUPER For any point x_0 and function ψ with $X_iX_j\psi$ continuous for all i, j such that $u(x_0) = \psi(x_0)$ and $u(x) \geq \psi(x)$ near x_0, we have $-\Delta_\infty\psi(x_0) \geq 0$ (u is a viscosity infinite harmonic supersolution).

It is an open problem if infinite harmonic functions are uniquely determined by their boundary values in general Carnot-Carathéodory spaces. The uniqueness theorems in Carnot groups, Grushin-type spaces and Riemannian vector fields motivated our focus on these spaces. Jensen in $\mathbb{R}^n$ [13], Wang in Carnot groups [18], and Bieske in the Heisenberg group [3], Grushin-type spaces [5] and Riemannian case [6] proved the following theorem.

Theorem A. *Let Ω be a bounded domain and let $\theta\colon \partial\Omega \to \mathbb{R}$ be a continuous function. Then the Dirichlet problem*

$$\begin{cases} -\Delta_\infty u = 0 & \text{on } \Omega \\ u = \theta & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

has a unique viscosity solution u.

There are two kinds of cones when using the Carnot-Carathéodory distance function. The first kind are called *infinite harmonic cones* and are defined using viscosity solutions of the infinite Laplacian. That is,

Definition 2. Let $a, b \in \mathbb{R}$. Given a point x and an open set U, we define the function $D : \partial(U \setminus \{x\}) \to \mathbb{R}$ by

$$D(y) = a + b\, d(x, y).$$

The *infinite harmonic cone* based on (U,x) is the unique viscosity infinite harmonic function $\omega_{U,x}^{a,b}$ in $U \setminus \{x\}$ such that

$$\omega_{U,x}^{a,b} = D \quad \text{on} \quad \partial(U \setminus \{x\}).$$

The second kind are called *metric cones* and are defined by extending the function D in the definition above to all of $\overline{U}$. In Euclidean space these two definitions coincide. However, we will show below that in Carnot groups, Grushin-type spaces and Riemannian vector fields, these definitions may produce different cones when $b > 0$.

The first use of these cones is to define the *comparison with cones* property. Namely,

Definition 3. Let U be an open set, and let $u : U \to \mathbb{R}$. Then u *enjoys comparison with infinite harmonic cones from above in* U if for every open $V \subset U$ and $a, b \in \mathbb{R}$ for which

$$u(y) \leq \omega_{U,x}^{a,b}(y)$$

holds on $\partial(V \setminus \{x\})$, then we have

$$u(y) \leq \omega_{U,x}^{a,b}(y)$$

in V. A similar definition holds for when the function u *enjoys comparison with infinite harmonic cones from below in* U. The function u *enjoys comparison with infinite harmonic cones in* U exactly when it enjoys comparison with infinite harmonic cones from above and below.

The function u *enjoys comparison with metric cones from above in* U if for every open $V \subset U$ and $a, b \in \mathbb{R}$ with $b \geq 0$ for which

$$u(y) \leq D^+(y) \stackrel{\text{def}}{=} a + b\, d(x,y)$$

holds on $\partial(V \setminus \{x\})$, then we have

$$u(y) \leq D^+(y)$$

in V. The function u *enjoys comparison with metric cones from below in* U if for every open $V \subset U$ and $a, b \in \mathbb{R}$ with $b \geq 0$ for which

$$u(y) \geq D^-(y) \stackrel{\text{def}}{=} a - b\, d(x,y)$$

holds on $\partial(V \setminus \{x\})$, then we have

$$u(y) \geq D^-(y)$$

in V. The function u *enjoys comparison with metric cones in* U exactly when it enjoys comparison with metric cones from above and below.

The uniqueness of infinite harmonic functions yields the following result: In spaces where a comparison principle for infinite harmonic functions holds, a viscosity infinite harmonic supersolution in U enjoys comparison with infinite harmonic cones from below in U. Similarly, a viscosity infinite harmonic subsolution enjoys comparison with infinite harmonic cones from above in U and an infinite harmonic function enjoys comparison with infinite harmonic cones in U.

Before discussing comparison with metric cones, we recall the definition of a Lipschitz function and an absolute minimizing Lipschitz extension (AMLE) on a metric space.

Definition 4. Let (X, d) be a metric space and let Y be a proper subset of X. An L-Lipschitz function $F : Y \to \mathbb{R}$ is a function with

$$\mathrm{Lip}(F, Y) \stackrel{\text{def}}{=} \sup_{\substack{x,y \in Y \\ x \neq y}} \frac{|F(x) - F(y)|}{d(x, y)} \leq L < \infty. \tag{3.2}$$

We say that $u : X \to \mathbb{R}$ is an AMLE of F on X exactly when

(i) $\mathrm{Lip}(u, X) = \mathrm{Lip}(F, Y)$.

(ii) for any open set $U \subset\subset X$, we have

$$\mathrm{Lip}(u, U) = \mathrm{Lip}(u, \partial U).$$

Champion and De Pascale [7] proved the following theorem in length spaces. Recall that a length space is a metric space, such as a Carnot-Carathéodory space, in which the distance between two points is the infimum of the lengths of paths connecting the points.

Theorem B. *Let (X, d) be a length space and Y proper open subset of X, then $u : \overline{Y} \to \mathbb{R}$ is an AMLE if and only if u satisfies comparison with metric cones.*

In addition, the tug-of-war games approach in [17] gives the following theorem.

Theorem C. *Let (X, d) be a length space and Y a proper subset of X. For any given Lipschitz function $F : Y \to \mathbb{R}$, there exists a unique AMLE of F on X.*

Because both types of cones are used to characterize different mathematical concepts, it is natural to attempt to establish a relationship between metric cones and infinite harmonic cones. We first need to recall a result of Monti and Serra-Cassano [16].

Theorem D. *Given a point x in a Carnot group, Grushin-type space or Riemannian vector fields, for almost every y, we have*

$$\| \mathfrak{X} d(x, y) \| \leq 1.$$

Using this result, we then have the following proposition.

Proposition 3.1. *([5, 6]) Given a pair (U, x) and $a, b \in \mathbb{R}$, the cone $\omega^{a,b}_{U,x}$ satisfies*

$$\begin{aligned} \omega^{a,b}_{U,x}(y) &\leq a + |b|\, d(x, y) \\ \omega^{a,b}_{U,x}(y) &\geq a - |b|\, d(x, y) \end{aligned}$$

for $y \in U$.

We then have the following corollary.

Corollary 3.2. *A function that enjoys comparison with infinite harmonic cones from above enjoys comparison with metric cones from above. A function that enjoys comparison with infinite harmonic cones from below enjoys comparison with metric cones from below. Thus, infinite harmonic functions enjoy comparison with metric cones.*

Combining the results in this section, we have the following lemma.

Lemma 3.3. *Let X be a Carnot group, Grushin-type space or Riemannian vector fields, let Y be a bounded domain in X and let $F : \partial Y \to \mathbb{R}$ be a Lipschitz function. The function u is the unique AMLE of F into Y if and only if u is infinite harmonic in Y and $u = F$ on ∂Y.*

Proof. Let u be the unique AMLE. If v is the unique infinite harmonic function on Y with boundary data F, v enjoys comparison with infinite harmonic cones and thus comparison with metric cones. By uniqueness of AMLE's, v must equal u. □

4. The distance function and the infinite Laplacian

In the previous section, we showed that the infinite harmonic functions enjoy comparison with metric cones. The interesting question is whether the distance function itself is infinite harmonic off the origin, as in Euclidean space. The answer to this question depends on the geometry of the space.

We begin with two geometric definitions concerning points in a domain U.

Definition 5. Let U be a bounded domain, and x an arbitrary point.

1. A point $y \in U$ is **geodesically accessible from the point** x if
$$y \in \Lambda = \bigcup_{z \in \partial U} \Gamma_{xz}$$
where Γ_{xz} is the union of all geodesics from x to z.
2. A point $y \in U$ that is not geodesically accessible is **geodesically inaccessible from the point x**. That is, $y \notin \Lambda$.
3. A point $y \in U$ is **boundary near to the point x** if there exists $z \in \partial U$ so that
$$d(x, y) < d(x, z).$$
4. A point $y \in U$ that is not boundary near is **boundary far from the point x**. That is, for all $z \in \partial U$, we have
$$d(x, y) \geq d(x, z).$$

We first note that because geodesics need not be unique, the set Λ actually includes *all* geodesics between points x and z. Points that are geodesically accessible from x lie on *some* geodesic from x to the boundary point z. Additionally, it is clear that geodesically accessible implies boundary near, or equivalently, boundary far implies geodesically inaccessible. We next note through the following examples that, unlike the Euclidean case, interior points need not be geodesically accessible.

Example 6. Boundary far points. Constant boundary data with $b > 0$. Consider the Riemann sphere with the spherical metric so that the geodesics are arcs of great circles. Let the domain U be the southern hemisphere and fix the point x as the north pole. Then, on the boundary of U, $a + b\, d(x, y)$ equals a fixed constant D for all $a, b \in \mathbb{R}$. Having constant boundary data, the corresponding infinite harmonic cone is the constant D. Clearly, we have $D < a + b\, d(x, y)$ in U and $D = a + b\, d(x, y)$ on ∂U. We note that the interior points are both boundary far and geodesically inaccessible.

Example 7. Boundary near does not imply geodesically accessible. Let x be the origin in the Heisenberg group $\mathbb{H}$ or the Grushin plane $\mathbb{G}$ and consider the ball of radius R centered at the origin, denoted $B_R(0)$. All points in $B_R(0)$ are boundary near. However, not all points are geodesically accessible. In particular, the points in A_R are geodesically inaccessible. See the figure below for the Heisenberg case. The same picture applies for the Grushin case.

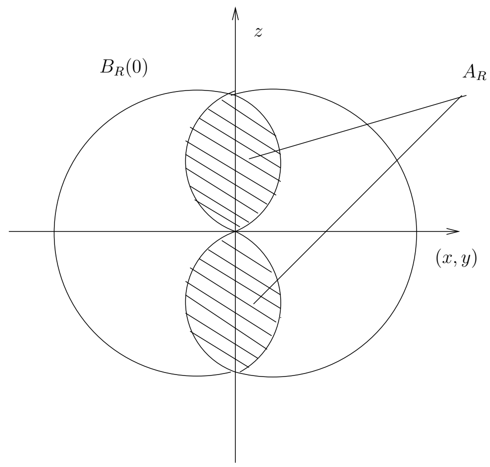

FIGURE 1. Heisenberg ball: set of points geodesically inaccessible from the origin.

We next fix $a, b \in \mathbb{R}$ with $b \geq 0$, a bounded domain U and a point x. Write ω_D for the infinite harmonic cone in U with boundary data $D(y) \stackrel{\text{def}}{=} a + b\, d(x, y)$ on $\partial(U \setminus \{x\})$. We begin by considering cones with constant boundary data. In the case when $b = 0$, we have $\omega_D(y) = D(y) = a$ for all points y in $\overline{U}$. In the case when $b > 0$, we have the following proposition motivated by Example 6.

Proposition 4.1. *Let $D(y)$ be defined as above. If $D(y)$ is constant with $b > 0$ then $x \notin \overline{U}$.*

Proof. Suppose $x \in \overline{U}$. Then $x \in \partial(U \setminus \{x\})$ and $D(x) = a$. Thus, $D(y) = a$ for all $y \in \partial(U \setminus \{x\})$. Choose $x \neq z \in \partial(U \setminus \{x\})$. Then

$$a = D(z) = a + b\, d(x, z)$$

and since $b > 0$ we arrive at a contradiction. □

Because the boundary data is constant, the uniqueness of the infinite harmonic cones produces the constant infinite harmonic cone ω_D. We have the following theorem.

Theorem 4.2. *Let U be a bounded domain in a Carnot-Carathéodory space where infinite harmonic functions are unique, i.e., Theorem A holds. Let $a, b \in \mathbb{R}$ with $b > 0$. Define $D(y) = a + b\, d(x, y)$ as above. Suppose $D(z) = K$ for $z \in \partial(U \setminus \{x\}) = \partial U$ for some constant K. Let ω_D be the (constant) infinite harmonic cone with boundary data K. Then the point $y \in U$ is boundary far from x exactly when $\omega_D(y) < D(y)$.*

Proof. Suppose that y is boundary far from x. Because y is an interior point to $U \setminus \{x\}$, there is an $r > 0$ so that the ball $B(y, r) \subset\subset (U \setminus \{x\})$. Let γ be a geodesic from x to y. Then, there is a point $\hat{x} \in (B(y, r) \setminus \{y\}) \cap \gamma$ with the property

$$d(x, y) = d(x, \hat{x}) + d(\hat{x}, y).$$

Using this property, we see that $D(y) > D(\hat{x})$. We then have

$$\omega_D(y) = K = \omega_D(\hat{x}) \leq D(\hat{x}) < D(y).$$

We note that the penultimate inequality is a consequence of Proposition 3.1.

Suppose next that $\omega_D(y) < D(y)$. Then by Proposition 3.1, we have

$$K = \omega_D(y) < D(y).$$

That is, for any $z \in \partial(U \setminus \{x\})$,

$$a + b\, d(x,z) < a + b\, d(x,y).$$

Because $b > 0$, we conclude that y is boundary far from x. □

The case of non-constant cones is more involved. We have the following one-sided result that parallels the constant case.

Theorem 4.3. *Let U, x, a, b be as in Theorem 4.2. Suppose that $D(z)$ is non-constant on $\partial(U \setminus \{x\})$ and let ω_D be the (non-constant) infinite harmonic cone with boundary data $D(z)$. Then we have the implications*

$$\begin{gathered} y \text{ is boundary far from } x \Rightarrow \\ \omega_D(y) < D(y) \Rightarrow y \text{ is geodesically inaccessible from } x. \end{gathered}$$

Proof. We first observe that as a non-constant (continuous) infinite harmonic function on a compact set, we have that ω_D achieves its maximum on $\overline{U}$. By the strong maximum principle, which follows from the Harnack inequality [14], this maximum can occur only on the boundary.

Now assume that y is boundary far. Suppose $\omega_D(y) = D(y)$. Because y is boundary far and $b > 0$, for all $z \in \partial(U \setminus \{x\})$ we have $D(y) \geq D(z)$. That is,

$$\omega_D(y) \geq \omega_D(z)$$

for all $z \in \partial(U \setminus \{x\})$. This contradicts the fact that the maximum of ω_D occurs only on the boundary. We conclude that $\omega_D(y) < D(y)$.

Next, let $\omega_D(y) < D(y)$ at a point $y \in U$ that is geodesically accessible from x. Let γ be the geodesic from x to a point $z \in \partial U$ that passes through y. Note that $z \neq x$. Because $\omega_D(z) = D(z)$ we have

$$\omega_D(z) - \omega_D(y) > D(z) - D(y) = b\Big(d(x,z) - d(x,y)\Big).$$

Because y lies on γ, we have

$$\omega_D(z) - \omega_D(y) > b\, d(y,z)$$

so that $\text{Lip}\, \omega_D > b$.

However, $\omega_D(\cdot)$ is the AMLE of $D(\cdot)$ and $\text{Lip}\ D \leq b$. By the definition of AMLE (Definition 4),

$$b < \text{Lip}\, \omega_D \leq \text{Lip}\ D \leq b.$$

We arrive at a contradiction and conclude that y must be geodesically inaccessable from x. □

Unlike the constant cone case, the converse implications can not both be true, as this would imply that all geodesically inaccessible points are boundary far, which is not necessarily the case as Example 7 shows. We addresses the first reverse implication by modifying Example 7.

Lemma 4.4. *Let x be the north pole of the Riemann sphere and let U be the northern and eastern hemispheres. Let $a, b \in \mathbb{R}$ with $b > 0$. Then there exists a point $y \in U$ that is boundary near with $\omega_D(y) < D(y)$. Thus, $\omega_D(y) < D(y)$ does not necessarily imply that y is boundary far.*

Proof. Suppose that y is boundary near implies $\omega_D(y) = D(y)$. Fix a point $y \in U$ such that y lies on the equator and in the eastern hemisphere. Then y is boundary far from x. Choose a sequence $\{y_n\}_{n\in\mathbb{N}}$ of points in U from the southern hemisphere that are boundary near to x and converge to the point y. By our assumption, we have $\omega_D(y_n) = D(y_n)$. By continuity of the cone functions, this implies $\omega_D(y) = D(y)$. However, y is boundary far, and so Theorem 4.3, which showed that $\omega_D(y) < D(y)$, is contradicted. □

We recall the following definition.

Definition 8. Given a geodesic space (X, d), let $\gamma : [0, 1] \to X$ be a minimizing geodesic from $x \in X$ to $y \in X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then γ is *extendable* if there is some $\varepsilon > 0$ so that the curve $\hat{\gamma} : [0, 1 + \varepsilon] \to X$ is a minimizing geodesic from x to $\hat{\gamma}(1 + \varepsilon)$ and $\hat{\gamma}|_{[0,1]} = \gamma$.

We note some important examples and non-examples of extendable geodesics.

Example 9. Let (X, d) be the Riemann sphere with the spherical metric. Geodesics from the north pole to the south pole are not extendable. Geodesics from the north pole to any other point are extendable. (Cf. Example 6.)

Example 10. Let (X, d) be the Heisenberg group $\mathbb{H}$. Then geodesics from the origin to all points off the x_3-axis are extendable, while geodesics terminating on the x_3-axis are not extendable [2].

Example 11. Let (X, d) be the Grushin plane $\mathbb{G}$. Geodesics from the origin ending at the x_2-axis are not extendable, while those ending off the x_2-axis are extendable [2].

We now relate points at which a geodesic is extendable to geodesically accessible points.

Proposition 4.5. *Fix a point x in a Carnot-Carathéodory space. Let y be an arbitrary point. Then there exists a bounded domain U with $y \in U$ so that y is geodesically accessible from x if and only if there exists some geodesic γ from x to y that is extendable.*

Proof. Fix the point x. First, let U be a bounded domain so that y is geodesically accessible from x. By definition, there is a geodesic from x to a point $z \in \partial(U \setminus \{x\})$ that meets y. The restriction is also a geodesic from x to y and is extendable to z.

Next, let γ be an extendable geodesic from x to y with extension $\hat{\gamma}$. Let $B(y, r)$ be the open ball centered at y with radius $r << 1$ so that there exists a point $z \in \hat{\gamma} \cap \partial B(y, r)$. Let U be a bounded domain containing y and having $z \in \partial(U \setminus \{x\})$. Then y lies on a geodesic from x to z and is therefore geodesically accessible from x. □

Theorem 4.3 leads to the following corollary.

Corollary 4.6. *Let x be a point in a Carnot-Carathéodory space where infinite harmonic functions are unique, i.e., Theorem A holds. Then the metric cones are infinite harmonic at points y where a geodesic from x to y is extendable. In particular, if x is the origin* *of the Heisenberg group $\mathbb{H}$ then the metric cones are infinite harmonic everywhere except possibly the x_3-axis and if x is the origin of the Grushin plane $\mathbb{G}$ then the metric cones are infinite harmonic everywhere except possibly the x_2-axis.*

Proof. Let y be a point where a geodesic from x to y is extendable. By Proposition 4.5, there is a domain U so that $y \in U$ is geodesically accessible from x. By Theorem 4.3, $\omega_D(y) = D(y)$. □

Our next goal is to remove the word "possibly" from the above two specific examples. We have the following theorem.

Theorem 4.7. *Let $x_0 \in \mathbb{H}$ be a point of the form $(0, 0, x_3^0)$ with $x_3^0 \neq 0$ or $x_0 \in \mathbb{G}$ a point of the form $(0, x_2^0)$ with $x_2^0 \neq 0$. Then there is a function ϕ with $X_iX_j\phi$ continuous for all i, j such that $\phi(x_0) = d(x_0, 0)$ and $d(x, 0) < \phi(x)$ near x_0 but $-\Delta_\infty\phi(x_0) > 0$. Thus, the distance from the origin is not a viscosity infinite harmonic subsolution at these points.*

Proof. We begin with the Grushin plane $\mathbb{G}$ and recall the Grushin vector fields are $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = x_1\frac{\partial}{\partial x_2}$. Let $x_0 = (0, x_2^0)$ with $x_2^0 \neq 0$ and let $x = (x_1, x_2)$ be near x_0. Note that the vector X_2 is the zero vector at x_0. Let $\phi : \mathbb{G} \to \mathbb{R}$ be the function

$$\phi(x) = \phi(x_1, x_2) = \sqrt{\pi}(x_1^4 + 4x_2^2)^{\frac{1}{4}} + \frac{1}{2}x_1 - 2x_1^2 + (x_2 - x_2^0)^4.$$

Then ϕ is smooth near x_0 with $X_1\phi(x_0) = \frac{1}{2}$ and $X_1X_1\phi(x_0) = -4$. We therefore have

$$-\Delta_\infty\phi(x_0) = -\langle (D^2\phi(x_0))^\star \mathfrak{X}\phi(x_0), \mathfrak{X}\phi(x_0)\rangle = -X_1X_1\phi(x_0)(X_1\phi(x_0))^2 = 1 > 0.$$

Thus, if $\phi(x) > d(x, 0)$ near x_0, then $d(x, 0)$ is not a viscosity infinite harmonic subsolution at x_0 and is therefore not an infinite harmonic function at x_0. (Condition SUB would not hold.)

Note that the explicit geodesic formulas in [2] give $\phi(x_0) = \sqrt{2\pi|x_2^0|} = d(x_0, 0)$, and so we only need to show that $d(x, 0) < \phi(x)$ near x_0. If x is of the form $(0, x_2)$, then

$$d(x, 0) = \sqrt{2\pi|x_2|} \leq \sqrt{2\pi|x_2|} + (x_2 - x_2^0)^4 = \phi(x)$$

with equality occurring only when $x = x_0$. At other points, we can see via a computer algebraic program that $\phi(x) - d(x, 0) > 0$ in a neighborhood of x_0.

Similarly, in the Heisenberg group, we let $x_0 = (0, 0, x_3^0)$ with $x_3^0 \neq 0$ and let $\phi : \mathbb{H} \to \mathbb{R}$ be the function

$$\phi(x) = \phi(x_1, x_2, x_3) = \sqrt{\pi}((x_1^2 + x_2^2)^2 + 16x_3^2)^{\frac{1}{4}} + \frac{1}{2}(x_1 + x_2) - 2(x_1^2 + x_2^2) + (x_3 - x_3^0)^4.$$

Then ϕ is smooth near x_0 with $X_1\phi(x_0) = X_2\phi(x_0) = \frac{1}{2}$ and

$$(D^2\phi(x_0))^\star = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

so that, as above,

$$-\Delta_\infty\phi(x_0) = -\langle (D^2\phi(x_0))^\star \mathfrak{X}\phi(x_0), \mathfrak{X}\phi(x_0)\rangle = \left(2 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{2}\right) = 2 > 0.$$

We again note that using [2], $\phi(x_0) = 2\sqrt{\pi|x_3^0|} = d(x_0, 0)$ and so we only need to show that $d(x, 0) < \phi(x)$ near x_0. For points x of the form $(0, 0, x_3)$, we have

$$d(x, 0) = 2\sqrt{\pi|x_3|} \leq 2\sqrt{\pi|x_3|} + (x_3 - x_3^0)^4 = \phi(x)$$

with equality only when $x_3 = x_3^0$. At other points, we proceed as in the Grushin plane case. $\square$

Having shown that the distance function is not a viscosity infinite harmonic subsolution at these points, it is natural to ask if it is a viscosity infinite harmonic supersolution there. We answer in the affirmative with the following theorem and corollary.

Theorem 4.8. *Let $x_0 \in \mathbb{H}$ be a point of the form $(0, 0, x_3^0)$ with $x_3^0 \neq 0$. Let $\overline{x} = (x_1, x_2)$. For real numbers η_1, η_2, η_3 and a 2×2 symmetric matrix X, consider the following inequalities based on the Taylor series* [3]*:*

$$d(x, 0) \geq d(x_0, 0) + x_1\eta_1 + x_2\eta_2 + o(d(x, x_0)) \text{ as } x \to x_0. \tag{4.1}$$

$$\begin{aligned} d(x, 0) \geq\ & d(x_0, 0) + x_1\eta_1 + x_2\eta_2 + (x_3 - x_3^0)\eta_3 \\ & + \frac{1}{2}\langle X\overline{x}, \overline{x}\rangle + o(d^2(x, x_0)) \text{ as } x \to x_0. \end{aligned} \tag{4.2}$$

If η_1, η_2, η_3 and X satisfy these inequalities, then $\eta_1 = \eta_2 = 0$.

Similarly, let $x_0 \in \mathbb{G}$ be a point of the form $(0, x_2^0)$ with $x_2^0 \neq 0$. For real numbers ν_1, ν_2, ν_3, consider the following inequalities: based on the Taylor series [4]*:*

$$d(x, 0) \geq d(x_0, 0) + x_1\nu_1 + o(d(x, x_0)) \text{ as } x \to x_0. \tag{4.3}$$

$$\begin{aligned} d(x, 0) \geq\ & d(x_0, 0) + x_1\nu_1 + 2(x_2 - x_2^0)\nu_2 \\ & + \frac{1}{2}(x_1)^2\nu_3 + o(d^2(x, x_0)) \text{ as } x \to x_0. \end{aligned} \tag{4.4}$$

If ν_1, ν_2 and ν_3 satisfy these inequalities, then $\nu_1 = 0$.

Proof. We shall prove only the Heisenberg case, the Grushin case is similar and omitted. If Equation (4.1) holds, it will hold for the points $x = (x_1, 0, x_3^0)$ as they approach x_0. Using the fact that $x \in B(0, d(x_0, 0))$ [2], we then have

$$0 \geq d(x, 0) - d(x_0, 0) \geq x_1\eta_1 + o(|x_1|) \text{ as } x \to x_0.$$

Dividing by $|x_1|$, we obtain

$$0 \geq \operatorname{sgn}(x_1)\eta_1 + o(1).$$

If η_1 is strictly negative, then choosing $x_1 < 0$ produces a contradiction and if η_1 is strictly positive, then choosing $x_1 > 0$ also produces a contradiction. We conclude that $\eta_1 = 0$. Similarly, $\eta_2 = 0$. If η_1, η_2, η_3 and X satisfy Equation (4.2) then η_1 and η_2 satisfy Equation (4.1). $\square$

The following corollary gives the desired result.

Corollary 4.9. *Let $x_0 \in \mathbb{H}$ be a point of the form $(0, 0, x_3^0)$ with $x_3^0 \neq 0$ or $x_0 \in \mathbb{G}$ a point of the form $(0, x_2^0)$ with $x_2^0 \neq 0$. Then the distance from the origin is a viscosity infinite harmonic supersolution to the infinite Laplace equation at these points.*

Proof. We only prove the Heisenberg case, the Grushin case is similar and omitted. If Condition SUPER is vacuous, we are done. If a function ψ satisfies the hypotheses of Condition SUPER, then by setting $X_1\psi(x_0) = \eta_1, X_2\psi(x_0) = \eta_2, X_3\psi(x_0) = \eta_3$ and $(\mathfrak{X}^2\psi(x_0))^\star = X$, we have a solution to Equation (4.2)[3]. By the Theorem, $X_1\psi(x_0) = X_2\psi(x_0) = 0$ and so

$$-\Delta_\infty\psi(x_0) = 0 \geq 0.$$

Condition SUPER holds and thus the distance is a viscosity infinite harmonic supersolution. ☐

We now consider the distance function at the origin. We recall from Section 2 that a Carnot-Carathéodory space is defined as an n-dimensional manifold whose tangent space is generated by m vectors. We also recall that for a vector v, $\overline{v}$ is the projection of v onto the space V_1. We then begin with the following theorem.

Theorem 4.10. *Given a Carnot-Carathéodory space, let the point x have coordinates $(x_1, x_2, \ldots, x_n)$. Recall that $\overline{x} = (x_1, x_2, \ldots, x_m)$. For real numbers $\eta_1, \eta_2, \ldots, \eta_{m+m_2}$ and a $m \times m$ symmetric matrix X, consider the following inequalities:*

$$d(x,0) \leq \sum_{i=1}^{m} x_i\eta_i + o(d(x,0)) \text{ as } x \to 0. \tag{4.5}$$

$$d(x,0) \leq \sum_{i=1}^{m+m_2} x_i\eta_i + \frac{1}{2}\langle X\overline{x}, \overline{x}\rangle + o(d^2(x,0)) \text{ as } x \to 0. \tag{4.6}$$

Then these inequalities hold for no choice of $\eta_1, \eta_2, \ldots, \eta_{m+m_2}$ or X.

Proof. Suppose Equation (4.5) held for some $\eta_1, \eta_2, \ldots, \eta_m$ and for all points x near the origin. In particular, it would hold for $x = (x_1, 0, \ldots, 0)$, so that Equation (4.5) becomes

$$|x_1| \leq x_1\eta_1 + o(|x_1|).$$

Dividing by $|x_1|$ we have

$$1 \leq (\operatorname{sgn} x_1)\eta_1 + o(1).$$

For $x_1 > 0$ and letting $x_1 \to 0^+$, we see that $1 \leq \eta_1$. For $x_1 < 0$ and letting $x_1 \to 0^-$, we see that $1 \leq -\eta_1$. We then have

$$1 \leq \eta_1 \leq -1$$

and conclude no such η_1 can exist. If there are values $\eta_1, \eta_2, \ldots, \eta_{m+m_2}$ and X that satisfy Equation (4.6) then $\eta_1, \eta_2, \ldots, \eta_m$ satisfy Equation (4.5). ☐

The inability to satisfy these equations produces the following corollary.

Corollary 4.11. *In any Carnot-Carathéodory space, the distance from the origin is a viscosity infinite harmonic subsolution to the infinite Laplace equation at the origin.*

Proof. Let ϕ be a function meeting the requirements of Definition 1. Then, $X_1\phi(0)$, $X_2\phi(0), \ldots X_m\phi(0)$ and $(D^2\phi(0))^\star$ would satisfy Equation (4.6) [3, 4]. Thus, Condition SUB is vacuous. ☐

We now will show that the distance from the origin need not be a viscosity infinite harmonic supersolution at the origin.

Theorem 4.12. *In the Heisenberg group $\mathbb{H}$ and Grushin plane $\mathbb{G}$, the distance from the origin is not a viscosity infinite harmonic supersolution to the infinite Laplace equation at the origin.*

Proof. Consider the function $h : \mathbb{H} \to \mathbb{R}$ given by

$$h(x) = h(x_1, x_2, x_3) = \frac{1}{2}(x_1 + x_2) + 2(x_1^2 + x_2^2) + x_3^4$$

and the function $w : \mathbb{G} \to \mathbb{R}$ given by

$$w(x) = w(x_1, x_2) = \frac{1}{2}x_1 + 2x_1^2 + x_2^4.$$

We consider first the Grushin case. First, we have $w(0) = 0 = d(0, 0)$ and we can compute $X_1w(0) = \frac{1}{2}$ and $X_1X_1w(0) = 4$. Thus, as in Theorem 4.7, we have $-\Delta_\infty w(0) = -1 < 0$. We only need to show that $d(x, 0) \geq w(x)$ near the origin. Any point x of the form $(0, x_2)$, we have $d(x, 0) = \sqrt{2\pi|x_2|}$ and $w(x) = x_2^4$. Thus for small x_2, we have $d(x, 0) > w(x)$. For other points, a graph of $w(x)$ versus $d(x, 0)$ shows that $d(x, 0) > w(x)$ with equality only at the origin.

The Heisenberg case is similar. We have $h(0) = 0 = d(0, 0)$ and

$$-\Delta_\infty h(0) = -\big(2 \times \frac{1}{2}\big) + \big(2 \times \frac{1}{2}\big) = -2 < 0.$$

We note that when a point x is of the form $(0, 0, x_3)$, we have $d(x, 0) = \sqrt{4\pi|x_3|}$ while $h(x) = x_3^4$ and so for x_3 near 0, we have $h(x) < d(x, 0)$. As in the Grushin case, it is easy to see that $h(x) < w(x)$ near the origin. $\square$

In summary, the Carnot-Carathéodory distance is an infinite harmonic function in the Heisenberg group and Grushin plane only at points where the geodesic is extendable. At points away from the origin where the geodesics are not extendable, the distance function is a viscosity infinite harmonic supersolution, but not a viscosity infinite harmonic subsolution. At the origin, the opposite is true; the distance function is a viscosity infinite harmonic subsolution, but not a viscosity infinite harmonic supersolution. This situation can be better visualized in the Heisenberg group $\mathbb{H}$ and Grushin plane $\mathbb{G}$ through the following pictures.

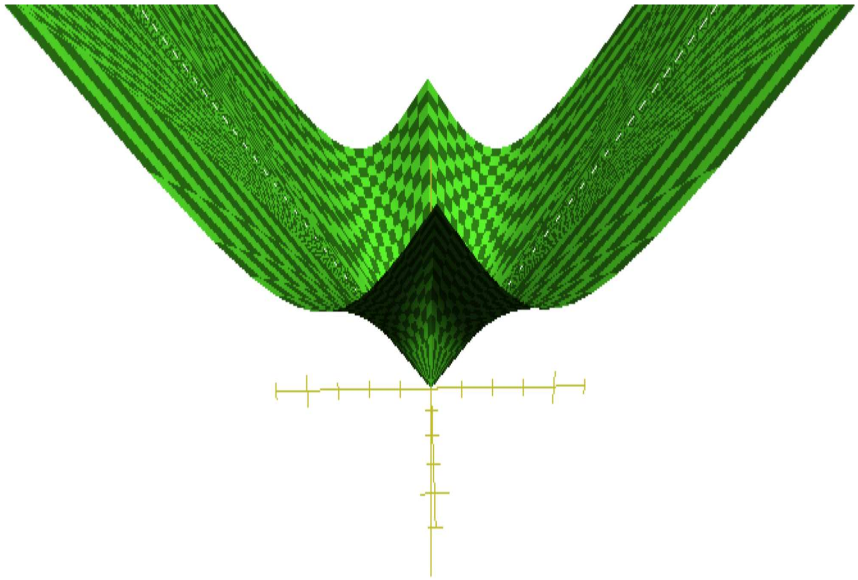

Figure 2. Carnot-Carathéodory distance from the origin in $\mathbb{H}$.

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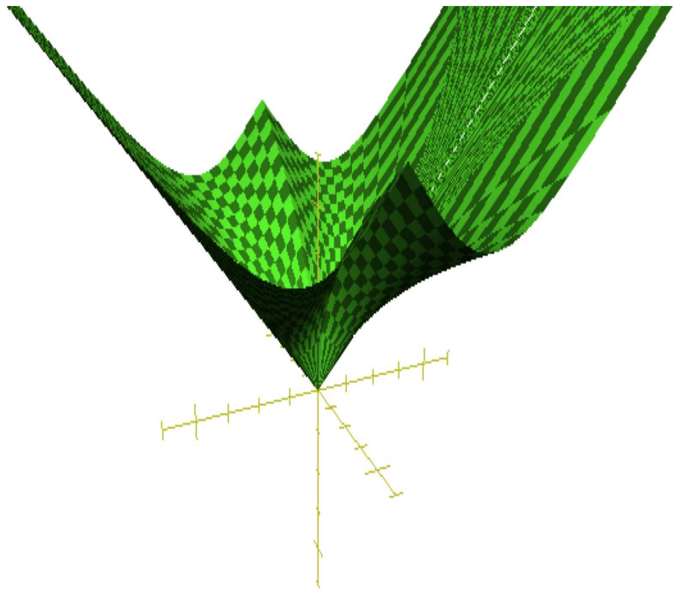

FIGURE 3. Carnot-Carathéodory distance from the origin in $\mathbb{H}$.

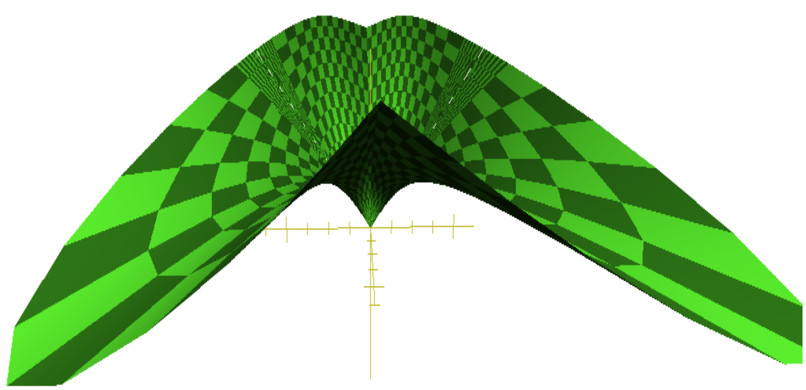

FIGURE 4. Carnot-Carathéodory distance from the origin in $\mathbb{G}$.

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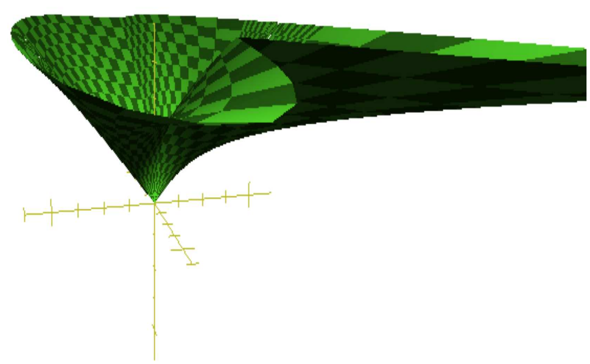

FIGURE 5. Carnot-Carathéodory distance from the origin in $\mathbb{G}$.

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