Relaxation and Gamma-convergence of supremal functionals

Francesca Prinari
Dipartimento di Matematica
Universitá di Pisa
Via Buonarroti 2, 56127 Pisa, Italy

Sunto. Si prova che il Γ-limite in L^∞_μ di una successione di funzionali supremali della forma $F_k(u) = \mu\text{-ess}\sup_\Omega f_k(x,u)$ é un funzionale supremale. In un controesempio si mostra che la funzione che rappresenta il Γ-limite $F(\cdot, B)$ di una successione di funzionali supremali della forma $F_k(u,B) = \mu\text{-ess}\sup_B f_k(x,u)$ puó dipendere dall'insieme B e si stabilisce una condizione necessaria e sufficiente al fine di rappresentare F nella forma supremale $F(u,B) = \mu\text{-ess}\sup_B f(x,u)$. Come corollario, si dimostra che se f rappresenta un funzionale supremale F, allora l'inviluppo *level convex* di f rappresenta l'inviluppo semicontinuo inferiormente di F rispetto alla topologia debole* di L^∞_μ.

Abstract. – We prove that the Γ-limit in L^∞_μ of a sequence of supremal functionals of the form $F_k(u) = \mu\text{-ess}\sup_\Omega f_k(x,u)$ is itself a supremal functional. We show by a counterexample that, in general, the function which represents the Γ-lim $F(\cdot, B)$ of a sequence of functionals $F_k(u,B) = \mu\text{-ess}\sup_B f_k(x,u)$ can depend on the set B and we give a necessary and sufficient condition to represent F in the supremal form $F(u,B) = \mu\text{-ess}\sup_B f(x,u)$. As a corollary, if f represents a supremal functional, then the level convex envelope of f represents its weak* lower semicontinuous envelope.

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1. – Introduction

Until a few years ago, the main problems of Calculus of Variations were formulated through the minimization of an integral functional. This had even brought to a definition of "variational functionals" based on the characteristic properties of this class (see [14]). Among the large number of papers in which they were studied, we underline the contribution of Buttazzo and Dal Maso (see [7], [8], [9]) on the characterization of functionals which admit an integral representation and their results about the representation of the relaxed functionals. The same authors and others (as Marcellini in [18]) also studied the behavior, with respect to the Γ-convergence, of sequences of integral functionals.

In the last years, a new class of functionals appeared in the study of minimization problems. In fact, in many physical contexts, one would often like to minimize a quantity which cannot be expressed as an integral: for example, a quantity which does not express a mean property of a body or whose value can be relevant on sets of arbitrarily small measure. In these cases, the natural setting in which the problem is formulated is the space L^∞_μ and the natural form of representing the functional is the so called "supremal form":

$$F(u, B) = \mu\text{-ess sup}\,\{f(x, u(x)) \ : \ x \in B\}. \tag{1.1}$$

Given a complete measure space $(\Omega, \mathcal{F}, \mu)$, in a previous paper [1] we completely characterized the class of all lower semicontinuous functionals $F : L^\infty_\mu(\Omega) \times \mathcal{F} \to \overline{\mathbf{R}}$ which can be represented in the supremal form (1.1). The key tool in the proof is a result of Barron, Cardaliaguet and Jensen (see [3]) analogous to the Radon-Nikodym theorem for measures. Moreover we showed that in the nonatomic case a supremal functional of the form (1.1) is weak* lower semicontinuous on $L^\infty_\mu(\Omega)$ if and only if the function $f(x, \cdot)$ is *level convex* for a.e. x, that is for every $t \in \mathbf{R}$ the level set $\{z \in \mathbf{R}^N \colon f(x, z) \le t\}$ is convex. As a corollary of this results, one can deduce a characterization of all weak* lower semicontinuous functionals $F : L^\infty_\mu \to \overline{\mathbf{R}}$ which can be represented in the form

$$F(u) = \mu\text{-ess sup}\,\{f(x, u(x)) \ : \ x \in \Omega\} \tag{1.2}$$

for a suitable level convex function f(see [19]).

The main difference between an integral functional $G(u, B) = \int_B f(x, u(x))d\mu(x)$ and a supremal functional of the form (1.1) consists in their behavior with respect to the union of sets. While the first is additive on disjoint sets, the satisfies a *countable supremality* property, i.e.

$$F(u, \bigcup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} F(u, A_n). \tag{1.3}$$

In the study of relaxation and Γ-convergence problems of supremal functionals of the form (1.1), this is the most difficult point to clarify and to face. In fact, this property is necessary in order to obtain the supremal representation (1.1) and cannot be weakened by assuming a property of *finite supremality* even if we add a lower semicontinuity assumption (see Example 3.1). Moreover Example 3.3 shows that in general the Γ-limit of a sequence of supremal functionals on L^∞_μ of the form (1.1) and the Γ-limit of a sequence of supremal functionals on $W^{1,\infty}$ of the form

$$F(u, A) = \operatorname*{ess\,sup}_{x \in A} f(x, Du(x)) \tag{1.4}$$

do not satisfy property (1.3). In particular this means that the class of supremal functionals on L^∞_μ of the form (1.1) (and the class of supremal functionals on $W^{1,\infty}$ of the form (1.4)) is not closed with respect to Γ-convergence.

Instead, in order to represent a functional $F : L^\infty_\mu \to \overline{\mathbf{R}}$ in the supremal form (1.2), the required property is a *quasi locality* behavior with respect to a "piecewise" function:

$$F(u\mathbf{1}_A + v\mathbf{1}_{\Omega\setminus A}) \leq F(u) \vee F(v).$$

This property turns out to be stable under Γ-convergence. Under an equicoercivity assumption (see hypothesis $(\mathbf{Hc_k})$), we show a compactness result with respect to Γ-convergence for sequences of supremal functionals $F_k : L^\infty_\mu \to \overline{\mathbf{R}}$ of the form (1.2) and we prove that the Γ-limit of every converging subsequence can be represented in the same form.
If we consider a sequence of supremal functionals $F_k : L^\infty_\mu \times \mathcal{F} \to \overline{\mathbf{R}}$, in order to obtain a compactness theorem and a supremal representation of the Γ-limit in the form (1.1), we have to add a further hypothesis on the behavior of the sequence of the minimum values of the functionals F_k (see assumption $(\mathbf{H_1})$ and its weakened formulation $(\mathbf{H_2})$). Under this condition we show that, up to a subsequence, there exists a functional F of the form (1.1) such that $F_k(\cdot, A)$ Γ-converges to $F(\cdot, A)$ for every open set A. Moreover, from $(\mathbf{H_2})$ we can deduce a necessary condition: if we drop it, we can exhibit sequences of supremal functionals as in Example 3.1 whose Γ-limit cannot be written in the supremal form (1.1)
As a corollary of the Γ-convergence Theorem 2.4, we give an explicit representation formula for the relaxed functional of a supremal functional. By using a level convex conjugation introduced by Volle in [22], we will prove that the weak* lower semicontinuous envelope of a supremal functional of the form (1.1) (and of the form (1.2)) is itself a supremal functional represented by the level convex envelope of f. Moreover, by using a Jensen's inequality for level convex functions, we give a relaxation theorem for supremal functionals through Young measures. This result is analogous to that one stated in [17] in the integral case.
We observe that if $\Omega \subset \mathbf{R}$, we can apply our relaxation theorem to represent the relaxed functional (with respect to the weak* topology of $W^{1,\infty}$) of the supremal functional

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u'(x)).$$

Barron and Jensen studied this problem in [4] and obtained an analogous result. Their technique is different: it is based on an L^p approximation of the functional and requires the continuity of f with respect to the first variable. Instead, our proof does not need this assumption and we can work just with measurable functions, which seems to be a more natural framework. Finally, the same representation results (relaxation and Γ-convergence) for the functional

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, Du(x)),$$

with $\Omega \subset \mathbf{R}^N$ and $N > 1$, are still open.
The paper is organized in the following way. In Section 2 we will state our representation results for Γ-limit of supremal functionals (see Theorems 2.3 and 2.6) and, as application, we will give the relaxation Theorem 2.5. Moreover we recall some properties of the level convex functions and, following Volle, we introduce the level convex envelope of a given function. In Section 3 we give the proofs of Theorems

2.3 and 2.4 and we produce an example in which we drop assumption ($\mathbf{H_2}$) and the Γ-limit does not satisfy the property (1.3). The last two sections are devoted to the relaxation problem. In Section 4 we prove Theorem 2.5 while in Section 5 we deduce a representation for the relaxed functional (see Theorems 5.3) by using Young measures.

2. – Preliminaries and main results.

The necessary and sufficient conditions for the lower semicontinuity of integral functionals involve notions of convexity in some form. The study of weak* lower semicontinuity for supremal functionals on L^∞ has led to the concept of level convex functions. In order to state the main results of this paper, we give some definitions, we recall some properties of this class of functions and we introduce the largest lower semicontinuous and level convex function less than or equal to a given function.
Let $(\Omega, \mathcal{F}, \mu)$ a complete measure space with μ non-negative, σ-finite, non atomic and complete measure. We denote for brevity by L^∞_d (respectively $L^1_d(\Omega)$) the space $L^\infty(\Omega; \mathbf{R}^d)$ (respectively the space $L^1(\Omega; \mathbf{R}^d)$), by $\mathcal{B}_d$ the Borel σ-field of $\mathbf{R}^d$ and by μ-sup the μ-essential supremum.

Definition **2.1** *A function $f : \Omega \times \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ is said to be*

(a) a supremand if f is $\mathcal{F} \otimes \mathcal{B}_d$-measurable;

(b) a normal supremand if f is $\mathcal{F} \otimes \mathcal{B}_d$-measurable and $f(x, \cdot)$ is lower semicontinuous (l.s.c. for short) on $\mathbf{R}^d$ for μ-a. e. $x \in \Omega$;

(c) a level convex normal supremand if f is a normal supremand such that, for μ-a. e. $x \in \Omega$, $f(x, \cdot)$ is level convex, i.e. for every $t \in \mathbf{R}$ the level set $\{z \in \mathbf{R}^d \ : \ f(x, z) \leq t\}$ is convex.

We prove the following version of Jensen's inequality for level convex functions.

Theorem **2.1** *Let $f : \mathbf{R}^d \to \overline{\mathbf{R}}$ be a lower semicontinuous and level convex function and let μ be a probability measure on $\mathbf{R}^d$. Then for every function $u \in L^1_\mu$ we have*

$$f\Big(\int u d\mu\Big) \leq \mu\text{-}\sup(f \circ u).$$

Proof. Let us define $\gamma := \mu\text{-}\sup(f \circ u)$ and $E_\gamma := \{z \in \mathbf{R}^d : \ f(z) \leq \gamma\}$. Then $u(z) \in E_\gamma$ for μ-a.e. $z \in \mathbf{R}^d$. Since f is lower semicontinuous and level convex, E_γ is a closed convex set. As μ is a probability measure, $(\int u(z) d\mu) \in E_\gamma$, which proves the assertion. □

If f is not level convex, we use a level convex conjugation introduced by Volle in [22] in order to obtain the largest lower semicontinuous level convex minorant f.

Definition **2.2**

(i) Given $f : \mathbf{R}^d \to \overline{\mathbf{R}}$*, we set* $f^c(\eta, r) := \sup\{z\cdot\eta : \ f(z) < r\}$ *for any* $(\eta, r) \in \mathbf{R}^d \times \mathbf{R}$*;*

(ii) given $\phi : \mathbf{R}^d \times \mathbf{R} \to \overline{\mathbf{R}}$*, we set* $\phi^\gamma(z) := \sup_\eta \sup\{r : \ \ \phi(\eta, r) < \eta \cdot z\}$ *for any* $z \in \mathbf{R}^d$*.*

In the following, we refer to f^c as the conjugate function of f and to ϕ^γ as the conjugate function of ϕ; $f^{c\gamma}$ is said the biconjugate function (or the level convex envelope) of f. Let us observe that they are slightly different from that ones introduced by Barron and Liu in [5]. The next theorem describes the class of functions which coincide with their biconjugate; for a proof, see Theorem 3.4 in [22].

THEOREM **2.2** *Let* $f : \mathbf{R}^d \to \overline{\mathbf{R}}$*. Then* $f = f^{c\gamma}$ *if and only if* f *is lower semicontinuous and level convex. In particular, it follows that*

$$\begin{aligned} f^{c\gamma} &= \sup\{h : \mathbf{R}^d \to \overline{\mathbf{R}} : \ h^{c\gamma} = h, \ h \le f\} \\ &= \sup\{h : \mathbf{R}^d \to \overline{\mathbf{R}} : \ h \text{ l.s.c. and level convex}, \ h \le f\} \end{aligned}$$

i.e. $f^{c\gamma}$ *is the largest lower semicontinuous and level convex function less than or equal to* f*.*

Now we can state the main results of this paper. Assume that $L^1_d(\Omega)$ is a separable space.

THEOREM **2.3** *Let* $F_k : L^\infty_d \to \overline{\mathbf{R}}$ *be a sequence of supremal functionals defined by*

$$F_k(u) = \mu\text{-}\sup\{f_k(x, u(x)) : \ x \in \Omega\},$$

where $f_k : \Omega \times \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ *are normal supremands satisfying the following assumption:*

$(\mathbf{Hc_k})$ *there exists a Borel function* $\phi : \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ *such that*

$$\lim_{|z|\to\infty} \phi(z) = +\infty \tag{2.5}$$

and

$$f_k(x, z) \ge \phi(z) \tag{2.6}$$

for every $k \in \mathbf{N}$*, for* μ*-a.e.* $x \in \Omega$ *and for every* $z \in \mathbf{R}^d$*.*
Then there exists a subsequence $(F_{n_k})_k$ *of* $(F_k)_k$ *such that* F_{n_k} Γ*-converges to a functional* $F : L^\infty_d \to \overline{\mathbf{R}}$ *with respect to the weak* topology of* L^∞_d*. Moreover there exists a level convex normal supremand* f *such that the representation formula*

$$F(u) = \mu\text{-}\sup\{f(x, u(x)) : \ x \in \Omega\}$$

holds for every $u \in L^\infty_d$*.*

THEOREM **2.4** *Let Ω be a topological space satisfying the second axiom of countability and let μ be the completion of a nonnegative, σ-finite, non atomic Borel measure on Ω. Let $F_k : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ be a sequence of supremal functionals defined by*

$$F_k(u,B) = \mu\text{-}\sup\{f_k(x,u(x)) : x \in B\} \tag{2.7}$$

where $f_k : \Omega \times \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ are normal supremands satisfying the assumption $(\mathbf{Hc_k})$ *and the following*

$(\mathbf{H_1})$ *the sequence $g_k(x) = \inf\{f_k(x,z) : z \in \mathbf{R}^d\}$ converges strongly in L^∞_d to a function $g : \Omega \to \mathbf{R} \cup \{+\infty\}$.*
Then, there exists a subsequence $(F_{n_k})_k$ of $(F_k)_k$ such that $F_{n_k}(\cdot, A)$ Γ-converges to $F(\cdot, A)$ with respect to the weak$*$ topology of L^∞_d for every open set $A \subset \Omega$. Moreover, there exists a level convex normal supremand f such that the representation formula

$$F(u,A) = \mu\text{-}\sup\{f(x,u(x)) : x \in A\} \tag{2.8}$$

holds for every open set $A \subset \Omega$.

As a consequence of Theorem 2.4, if $F : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ is a supremal functional of the form

$$F(u,B) = \mu\text{-}\sup\{f(x,u(x)) : x \in B\} \tag{2.9}$$

then its weak* lower semicontinuous envelope, defined by

$$\overline{F}(u,B) := \sup\{G(v) : G : L^\infty_d(B) \to \overline{\mathbf{R}},\ G\ w^*\text{l.s.c.},\ G(\cdot) \le F(\cdot,B) \text{ on } L^\infty_d(B)\}$$

is a supremal functional represented by the level convex envelope of f. In fact the following result holds for relaxed supremal functional:

THEOREM **2.5** *Let $f : \Omega \times \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ is a normal supremand satisfying the following assumption:*

$(\mathbf{Hc_0})$ *there exists a Borel function $\phi : \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ satisfying (2.5) and such that*

$$f(x,z) \ge \phi(z) \tag{2.10}$$

for μ-a.e. $x \in \Omega$ and for every $z \in \mathbf{R}^d$. Let $F : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ be defined by (2.8). Then

$$\overline{F}(u,B) = \mu\text{-}\sup\{f^{c\gamma}(x,u(x)) : x \in B\} \tag{2.11}$$

for every $B \in \mathcal{F}$ and for every $u \in L^\infty_d$.

With the aim to show the representation formula (2.11), we devote the second part of this section to prove the $\mathcal{F} \otimes \mathcal{B}_d$-measurability of $f^{c\gamma}$ when f is a normal supremand. We begin with the following proposition (see Proposition 3.3 in [22]) which makes precise some useful properties of the operator $f \mapsto f^c$.

PROPOSITION **2.1** *Let $f : \mathbf{R}^d \to \overline{\mathbf{R}}$ be a function such that $f \not\equiv -\infty$. Then the conjugate function f^c satisfies the following properties:*

(a) for every $r \in \mathbf{R}$, $f^c(\cdot, r)$ is lower semicontinuous, convex, proper or identically $-\infty$;

(b) for every $y \in \mathbf{R}^d$ $f^c(y, \cdot)$ is lower semicontinuous, increasing. Moreover,

$$(\inf_{i \in I} f_i)^c = \sup_{i \in I} f_i^c$$

for every family $(f_i)_{i \in I}$ of functions defined on $\mathbf{R}^d$.

In the sequel we exclude the not interesting case $f \equiv -\infty$. By using Proposition 2.1, we can establish the following property of measurability for f^c.

PROPOSITION **2.2** *Let $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ be a normal supremand. Let f^c be the conjugate of f with respect to z, i.e.*

$$f^c(x, \eta, r) := \sup\{z \cdot \eta : \; f(x, z) < r\}$$

for every $(x, \eta, r) \in \Omega \times \mathbf{R}^d \times \mathbf{R}$. Then $f^c(\cdot, \eta, r)$ is $\mathcal{F}$-measurable for every $(\eta, r) \in \mathbf{R}^d \times \mathbf{R}$.

PROOF. Fix $(\eta, r) \in \mathbf{R}^d \times \mathbf{R}$. Since

$$f^c(x, \eta, r) = \sup_{\substack{\rho \in \mathbf{Q} \\ \rho < r}} \sup_{n \in \mathbf{N}} \sup\{z \cdot \eta : z \in \mathbf{R}^d, \; |z| \leq n, \; f(x, z) \leq \rho\},$$

it is sufficient to prove that

$$\phi_\rho(x) := \sup\{z \cdot \eta : z \in \mathbf{R}^d, \; |z| \leq n, \; f(x, z) \leq \rho\}$$

is $\mathcal{F}$-measurable for every $\rho \in \mathbf{Q}$. For every $t \in \mathbf{R}$ the set $\{x \in \Omega : \phi_\rho(x) > t\}$ is the projection on Ω of the set

$$\{(x, z) \in \Omega \times \mathbf{R}^d : z \cdot \eta > t, \; |z| \leq n, \; f(x, z) \leq \rho\}$$

which belongs to $\mathcal{F} \otimes \mathcal{B}_d$. Therefore the $\mathcal{F}$-measurability of ϕ_ρ follows from the Projection Theorem (see, e.g., [11], Theorem XIII.3). □

Finally, we can obtain the following measurability property for the biconjugate $f^{c\gamma}$ of f.

THEOREM **2.6** *Let $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ be a normal supremand. Then $f^{c\gamma}$ is a normal supremand.*

PROOF.
Suppose first that

$$\lim_{|z| \to \infty} f(x, z) = +\infty \tag{2.12}$$

for every $x \in \Omega$. We shall remove later this assumption by an approximation argument.

In order to obtain the $\mathcal{F}\otimes\mathcal{B}_d$- measurability of $f^{c\gamma}$, it is sufficient to prove that

$$A_t := \{(x,z)\in\Omega\times\mathbf{R}^d:\ f^{c\gamma}(x,z)>t\}$$

is measurable for every $t\in\mathbf{R}$. By the definition of $f^{c\gamma}$ (see also Proposition 3.4 of [22]), we have

$$\begin{aligned}A_t &= \Omega\times\mathbf{R}^d\setminus\Big(\bigcap_{\eta\in\mathbf{R}^d}\bigcap_{r>t}\{(x,z)\in\Omega\times\mathbf{R}^d:\ f^c(x,\eta,r)\geq\eta\cdot z\}\Big)\\ &= \bigcup_{\eta\in\mathbf{R}^d}\bigcup_{r>t}\{(x,z)\in\Omega\times\mathbf{R}^d:\ f^c(x,\eta,r)<\eta\cdot z\}.\end{aligned}$$

By Proposition 2.1, $f^c(x,\eta,\cdot)$ is increasing. Thus we have

$$\begin{aligned}A_t &= \bigcup_{\eta\in\mathbf{R}^d}\bigcup_{r>t,\ r\in\mathbf{Q}}\{(x,z)\in\Omega\times\mathbf{R}^d:\ f^c(x,\eta,r)<\eta\cdot z\}\\ &= \bigcup_{r>t,\ r\in\mathbf{Q}}\bigcup_{\eta\in\mathbf{R}^d}\{(x,z)\in\Omega\times\mathbf{R}^d:\ f^c(x,\eta,r)<\eta\cdot z\}.\end{aligned}$$

Let us fix $\bar r\in\mathbf{Q}$ and, for every $\eta\in\mathbf{R}^d$, let us define

$$E(\eta,\bar r) := \{(x,z)\in\Omega\times\mathbf{R}^d:\ f^c(x,\eta,\bar r)<\eta\cdot z\}.$$

By Proposition 2.2 for every $\eta\in\mathbf{R}^d$ $f^c(\cdot,\eta,\bar r)$ is $\mathcal{F}$-measurable and hence $E(\eta,\bar r)$ is $\mathcal{F}\otimes\mathcal{B}_d$-measurable. If we prove that

$$\bigcup_{\eta\in\mathbf{R}^d}E(\eta,\bar r)=\bigcup_{\eta\in\mathbf{Q}^d}E(\eta,\bar r),$$

then A_t is a countable union of $\mathcal{F}\otimes\mathcal{B}_d$-measurable sets and so A_t is $\mathcal{F}\otimes\mathcal{B}_d$-measurable.
Let $(x,z)\in\bigcup_{\eta\in\mathbf{R}^d}E(\eta,\bar r)$. Then there exists $\bar\eta$ such that $(x,z)\in E(\bar\eta,\bar r)$, that means

$$f^c(x,\bar\eta,\bar r)<\bar\eta\cdot z \tag{2.13}$$

Now we have to consider two cases:

(i) $f^c(x,\bar\eta,\bar r)=-\infty$. This implies that

$$\{(x,\xi)\in\Omega\times\mathbf{R}^d:\ f(x,\xi)<\bar r\}=\emptyset.$$

Then, we have $f^c(x,\eta,\bar r)=-\infty$ for every $\eta\in\mathbf{Q}^d$, i.e. $(x,z)\in E(\eta,\bar r)$ for every $\eta\in\mathbf{Q}^d$.

(ii) $f^c(x,\bar\eta,\bar r)\in\mathbf{R}$. By using (2.12), there exists a compact set $K=K(x,\bar r)$ in $\mathbf{R}^d$ such that $\xi\in K$ if $f(x,\xi)<\bar r$. So, if $U\subset\mathbf{R}^d$ is a neighborhood of $\bar\eta$, then there exists $M>0$ such that

$$f^c(x,\eta,\bar r)=\sup_{\{\xi\in K, f(x,\xi)<\bar r\}}\{\xi\cdot\eta\}\leq M$$

for every $\eta\in U$. Since $f^c(x,\cdot,\bar r)$ is convex and bounded in U, $f^c(x,\cdot,\bar r)$ is continuous in $\bar\eta$ and so, by (2.13), there exists $\bar\xi\in\mathbf{Q}^d$ such that $f^c(x,\bar\xi,\bar r)<\bar\xi\cdot z$, i.e. $(x,z)\in E(\bar\xi,\bar r)$.

Now we remove assumption (2.12). For every $n \in \mathbf{N}$ let us define

$$f_n(x,z) := \begin{cases} f(x,z) & \text{if } |z| \le n \\ +\infty & \text{if } |z| > n. \end{cases}$$

Since f_n satisfies (2.12) for every $n \in \mathbf{N}$, by the first part of this proof, $(f_n)^{c\gamma}$ is $\mathcal{F} \otimes \mathcal{B}_d$- measurable and level convex for every $x \in \Omega$. We observe that

$$(f_n)^{c\gamma}(x,z) = +\infty \tag{2.14}$$

for every $n \in \mathbf{N}$, for every $t \in \mathbf{R}^d$ with $|z| > n$ and for every $x \in \Omega$. In fact, let us define

$$\phi_n(x,z) := \begin{cases} (f_n)^{c\gamma}(x,z) & \text{if } |z| \le n \\ +\infty & \text{if } |z| > n. \end{cases}$$

Then $\phi_n(x,\cdot)$ is level convex and $\phi_n(x,z) \le f_n(x,z)$ for every $x \in \Omega$ and for every $z \in \mathbf{R}^d$. This implies $\phi_n(x,z) \le (f_n)^{c\gamma}(x,z)$ for every $x \in \Omega$ and for every $z \in \mathbf{R}^d$. In particular, (2.14) follows and setting

$$g(x,z) := \inf_{n \in \mathbf{N}} (f_n)^{c\gamma}(x,z),$$

we have that g is $\mathcal{F} \otimes \mathcal{B}_d$-measurable, $g(x,\cdot)$ is level convex for every $x \in \Omega$ and

$$f^{c\gamma}(x,z) \le g(x,z) = \inf_{n \in \mathbf{N}, n \ge |t|} (f_n)^{c\gamma}(x,z) \le f(x,z)$$

for every $z \in \mathbf{R}^d$ and for every $x \in \Omega$. Finally, let us define

$$\Gamma g(x,z) := \sup\{h(z) \ : \ h : \mathbf{R}^d \to \overline{\mathbf{R}},\ h \ \text{ l.s.c.},\ h(z) \le g(x,z) \text{ for every } z \in \mathbf{R}^d\},$$

i.e. the l.s.c. envelope of g with respect to the second variable. By Proposition 2.6.3. in [10], Γg is $\mathcal{F} \otimes \mathcal{B}_d$-measurable. Since the level set

$$\{z \in \mathbf{R}^d \ : \ \Gamma g(x,z) \le \lambda\} = \bigcap_{\rho > \lambda} \overline{\{(z \in \mathbf{R}^d : \ g(x,z) \le \rho\}}$$

is convex for every $\lambda \in \mathbf{R}$ and for every $x \in \Omega$, Γg is a level convex normal supremand. Moreover, $\Gamma g(x,z) \le f(x,z)$ for every $z \in \mathbf{R}^d$ and $x \in \Omega$ and so

$$\Gamma g(x,z) \le f^{c\gamma}(x,z) \le g(x,z) \tag{2.15}$$

for every $z \in \mathbf{R}^d$ and $x \in \Omega$. Since $f^{c\gamma}(x,\cdot)$ is a lower semicontinuous function, from (2.15) and from the definition of Γg, we deduce that

$$f^{c\gamma}(x,z) = \Gamma g(x,z)$$

for every $z \in \mathbf{R}^d$ and for every $x \in \Omega$. Therefore $f^{c\gamma}$ is a normal supremand. □

3. – Gamma-convergence Theorem

In this section we show the Gamma-convergence Theorems 2.3 and 2.4. Before coming to their proofs, we recall the following results: the first is a sequential characterization of Γ-limits with respect to the weak* topology, while the second is a general abstract compactness result that assures the existence of Γ-converging subsequences. The proofs of Propositions 3.1 and 3.2 are analogous to those ones of Theorem 8.10 and Corollary 8.12 in [13]. In the sequel $X = L^1_d$ and thus X' will be the space L^∞_d.

PROPOSITION **3.1** *Let X be a separable Banach space and let X' be its dual space. Let $\Phi : X' \to \overline{\mathbf{R}}$ be a function such that*

$$\lim_{||x'|| \to +\infty} \Phi(x') = +\infty,$$

where $||\cdot||$ is the norm of X' and let (F_k) be a sequence of functionals from X' into $\overline{\mathbf{R}}$. Suppose that $F_k \geq \Phi$ for every $n \in \mathbf{N}$. Then the functional

$$F^-(x') = \Gamma\text{-}\liminf_{k\to+\infty} F_k(x') \quad [\textit{respectively the functional } F^+(x) = \Gamma\text{-}\limsup_{k\to+\infty} F_k(x')]$$

denoted as the Γ-lower limit *or more shortly as the* Γ-liminf *[respectively, denoted as the* Γ-upper limit *or more shortly as the* Γ-limsup*] is characterized by the following properties:*

(a) For every $x' \in X'$ and for every sequence (x'_k) converging to x' in X' it is

$$F^-(x') \leq \liminf_{k\to+\infty} F_k(x'_k) \quad [\text{respectively } F^+(x') \leq \limsup_{k\to+\infty} F_k(x'_k)];$$

(b) For every $x' \in X'$ there exists a sequence (x'_k) converging to x' in x' such that

$$F^-(x') = \liminf_{k\to+\infty} F_k(x'_k) \quad [\text{respectively } F^+(x') = \limsup_{k\to+\infty} F_k(x'_k)];$$

In particular (F_k) Γ-converges to F in the weak topology of X' if and only if*

(i) for every $x' \in X'$ and for every sequence (x'_k) converging weakly to x', it is*

$$F(x') \leq \liminf_{k\to\infty} F_k(x'_k);$$

(ii) for every $x' \in X'$ there exists a sequence (x'_k) converging weakly to $x' \in X'$ such that*

$$F(x') = \lim_{k\to\infty} F_k(x'_k).$$

PROPOSITION **3.2** *Under the hypothesis of Proposition 3.1, there exists a subsequence of $(F_k)_k$ which Γ-converges in the weak* topology of X'.*

In order to prove Theorems 2.3 and 2.4, the fundamental tools we will use in the followings are the representation results shown in [1] (see Theorem 3.2, Theorem 4.1 and Remark 4.3): they give a characterization of all lower semicontinuous functionals $F : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ which can be written in a supremal form

$$F(u,B) = \mu\text{-}\sup_{x\in B} f(x,u(x)). \tag{3.16}$$

THEOREM **3.1** *Let $F : L^\infty_d \times \mathcal{B}_d \to \overline{\mathbf{R}}$ be a mapping which satisfies the assumptions:*

- $(\mathcal{P}_1)$ *(locality): $F(u,A) = F(v,B)$ whenever $u = v$ μ-a.e. on B and $|(A\triangle B)| = 0$, $u, v \in L^\infty_d$ and $A, B \in \mathcal{F}$;*
- $(\mathcal{P}_2)$ *(contable supremality): $F(u, \bigcup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} F(u, A_n)$ whenever $u \in L^\infty_d$ and $B_n \in \mathcal{F}$ for every $n \in \mathbf{N}$;*
- $(\mathcal{P}_3)$ *(strong lower semicontinuity): $F(\cdot, B)$ is strongly lower semicontinuous for every $B \in \mathcal{F}$.*

Then there exists a normal supremand f such that the representation formula (3.16) holds. Moreover, this supremand f is unique up to μ-equivalence.

THEOREM **3.2** *Let $F : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ be a mapping which satisfies assumptions $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and the following*

- $(\mathcal{P}_4)$ *(weak* lower semicontinuity) for every $B \in \mathcal{F}$ the mapping $F(\cdot, B)$ is weakly* lower semicontinuous in $L^\infty_d(B)$.*

Then there exists a level convex normal supremand f such that the representation formula (3.16) holds. Moreover, this supremand f is unique up to μ-equivalence.

REMARK **3.1** It is easy to see that the supremality condition $(\mathcal{P}_2)$ is equivalent to the following one:

$$(\mathcal{P}_5)\begin{cases} i)\ (\textit{monotonicity})\ F(u,A) \le F(u,B) \text{ for all } u \in L^\infty_d \text{ and } A, B \in \mathcal{F} \text{ with } A \subset B; \\ ii)\ (\textit{countable supremality on pairwise disjoint sets})\ F(u, \bigcup_{n=1}^{\infty} E_n) \le \bigvee_{n=1}^{\infty} F(u, E_n) \\ \quad \text{whenever } u \in L^\infty_d \text{ and } E_n \in \mathcal{F} \text{ with } E_n \cap E_m = \emptyset \text{ when } n \ne m. \end{cases}$$

Indeed, the implication $(\mathcal{P}_2) \Rightarrow (\mathcal{P}_5)$ is straightforward.
For the opposite implication, let us observe that, given $u \in L^\infty_d$, $A_i \in \mathcal{F}$ for every $i \in \mathbf{N}$ and defined

$$E_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j,$$

thanks to $(\mathcal{P}_5)$ ii), we obtain

$$F(u, \bigcup_{i=1}^{\infty} A_i) = F(u, \bigcup_{i=1}^{\infty} E_i) \le \bigvee_{i=1}^{\infty} F(u, E_i)$$

and so, by $(\mathcal{P}_5)$ i),

$$F(u, \bigcup_{i=1}^{\infty} A_i) \le \bigvee_{i=1}^{\infty} F(u, A_i) \le F(u, \bigcup_{i=1}^{\infty} A_i).$$

As a corollary of Theorem 3.2, it follows the following characterization of all lower semicontinuous functionals $F : L^\infty_d \to \overline{\mathbf{R}}$ which can be represented in the supremal form

$$F(u) = \mu\text{-}\sup_{x \in \Omega} f(x, u(x)) \tag{3.17}$$

for a suitable supremand f. By using this result, the proof of Theorem 2.3 follows very easily.

COROLLARY **3.1** *Let $F : L^\infty_d \to \overline{\mathbf{R}}$ be a weakly* lower semicontinuous mapping which satisfies the following assumption*

($\mathcal{P}_\mathbf{6}$) *(finite quasi locality) $F(u\mathbf{1}_A + v\mathbf{1}_{\Omega\setminus A}) \leq F(u) \vee F(v)$ for every $A \in \mathcal{F}$ and for every $u, v \in L^\infty_d$.*

Then there is some level convex normal supremand f such that the representation formula (3.17) holds.

We give only the sketch of its proof. For the details, see the proof of Theorem 3.4.2 in [19].

PROOF OF COROLLARY 3.1. As in the proof of Theorem 3.1 in [1], without loss of generality we can suppose that F is L-Lipschitz continuous with respect to the L^∞_d norm. Let $G_m : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ be defined by

$$G_m(u, A) := \begin{cases} \inf\{F(u\mathbf{1}_A + v\mathbf{1}_{\Omega\setminus A}) : v \in L^{\infty,m}\} & \text{if } u \in L^{\infty,m} \\ +\infty & \text{otherwise} \end{cases}$$

where $L^{\infty,m} = \{u \in L^\infty_d : ||u||_{L^\infty_d} \leq m\}$. One can show that G_m satisfies all the hypotheses of Theorem 3.2. Thus there exists a level convex normal supremand f_m (L-Lipschitz continuous with respect to z) such that

$$G_m(u, A) = \mu\text{-}\sup_{A} f_m(x, u(x)) \tag{3.18}$$

for every $u \in L^\infty_d$ and for every $A \in \mathcal{F}$. In particular,

$$F(u) = \mu\text{-}\sup_{A} f_m(x, u(x))$$

for every $u \in L^{\infty,m}$. Since

$$G_m(u, A) = G_{m+1}(u, A) \vee \left(\inf_{v \in L^{\infty,m}} F(v)\right)$$

for every $m \in \mathbf{N}$, for every $u \in L^{\infty,m}$ and for every $A \in \mathcal{F}$, by Proposition 2.3 in [1] there exists a μ-negligible set $N \subseteq \Omega$ such that

$$f_m(x, z) = f_{m+1}(x, z) \vee \left(\inf_{v \in L^{\infty,m}} F(v)\right)$$

for every $m \in \mathbf{N}$, for every $x \in \Omega \setminus N$ and for every $z \in \mathbf{R}^d$, $|z| \leq m$. In particular for every $m \geq |z|$ and for every $x \in \Omega \setminus N$ the sequence $(f_m(x,z))_{m\in\mathbf{N}}$ is nonincreasing and for every $(x,z) \in (\Omega \setminus N) \times \mathbf{R}^d$, we can define

$$f(x,z) := \inf_{m \geq |z|} f_m(x,z) = \lim_{m\to\infty} f_m(x,z).$$

Since for every $m \geq |z|$ the function $f_m(x,z)$ is L-Lipschitz continuous and level convex in z, we have that f is a normal level convex supremand and from (3.18) it follows that

$$F(u) = \mu\text{-}\sup_{\Omega} f(x,u(x))$$

for every $u \in L^\infty(\Omega)$. □

Proof of Theorem 2.3. By Proposition 3.2 (applied to $X = L^1_d(\Omega)$, so that $X' = L^\infty_d(\Omega)$), there exists a subsequence of $(F_k)_k$ which Γ-converges in the weak* topology of L^∞_d to a functional F. Since it is l.s.c. with respect to the weak* topology of L^∞_d, it is sufficient to prove that F satisfies property $(\mathcal{P}_\mathbf{6})$ and then to apply Corollary 3.1. Let $u, v \in L^\infty_d$ and let A be a measurable set with $\mu(A) > 0$ (otherwise it is trivial). Let $(u_k)_{k\in\mathbf{N}}, (v_k)_{k\in\mathbf{N}} \subseteq L^\infty_d$, $u_k \to u$ weakly* in L^∞_d, $v_k \to v$ weakly* in L^∞_d such that $F(u) = \lim_{k\to\infty} F_k(u_k)\ andF(v) = \lim_{k\to\infty} F_k(v_k)$. Since $F_k(u_k\mathbf{1}_A + v_k\mathbf{1}_{\Omega\setminus A}) \leq F_k(u_k) \vee F_k(v_k)$, we have

$$\begin{aligned} F(u\mathbf{1}_A + v\mathbf{1}_{\Omega\setminus A}) &\leq \liminf_{k\to\infty} F_k(u_k\mathbf{1}_A + v_k\mathbf{1}_{\Omega\setminus A}) \\ &\leq \liminf_{k\to\infty} (F_k(u_k) \vee F_k(v_k)) \\ &= \lim_{k\to\infty} F_k(u_k) \vee \lim_{k\to\infty} F_k(v_k) \\ &= F(u) \vee F(v). \end{aligned}$$

□

In the proof of Theorem 2.4, the crucial point will be to check that the Γ-limit satisfies property $(\mathcal{P}_\mathbf{2})$. Let us observe that, in general, under the only assumption $(\mathbf{Hc_k})$, the Γ-limit of a sequence of supremal functionals (2.7) satisfies only the following property

$(\mathcal{P}_\mathbf{7})$ *(finite supremality)* $F(u, \bigcup_{n=1}^{k} A_n) = \bigvee_{n=1}^{k} F(u, A_n)$ *whenever* $u \in L^\infty_d$ *and* $A_n \in \mathcal{F}$ *for every* $n \in \{1, 2, \cdots, k\}$.

This condition does not imply the countable supremality $(\mathcal{P}_\mathbf{2})$, even if the functional is lower semicontinuous, as the next example shows:

Example **3.1** Consider $\Omega = (a,b) \subset \mathbf{R}$, $\mathcal{F}$ the σ-field of Lebesgue measurable sets and μ the Lebesgue measure. Denote by $\widehat{A}$ the set of the Lebesgue points of density of A. Fix $c \in (a,b)$. The functional $F : L^\infty \times \mathcal{F} \to \overline{\mathbf{R}}$ defined by

$$F(u,A) = \begin{cases} 1 & \text{if } c \in \widehat{A} \\ 0 & \text{otherwise} \end{cases}$$

satisfies $(\mathcal{P}_\mathbf{1})$, $(\mathcal{P}_\mathbf{4})$ and $(\mathcal{P}_\mathbf{7})$, but not $(\mathcal{P}_\mathbf{2})$ and it does not admit a supremal form.

The following proposition establishes when a functional that satisfies $(\mathcal{P}_\mathbf{1})$, $(\mathcal{P}_\mathbf{4})$ and $(\mathcal{P}_\mathbf{7})$ satisfies also the countable supremality $(\mathcal{P}_\mathbf{2})$.

LEMMA **3.1** *Assume that F satisfies $(\mathcal{P}_\mathbf{1})$, $(\mathcal{P}_\mathbf{4})$ and $(\mathcal{P}_\mathbf{7})$. Then the supremality condition $(\mathcal{P}_\mathbf{2})$ is equivalent to the following:*

$(\mathcal{P}_\mathbf{8})$ *for every $u \in L^\infty_d$ and for every sequence $(A_n) \subset \mathcal{F}$, there exists $m \in L^\infty_d$ such that*

$$\begin{cases} i)\ F(u, A_n) \geq F(m, A_n) \text{ for every } n \in \mathbf{N} \\ ii)\ F(m, \bigcup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} F(m, A_n). \end{cases}$$

PROOF. If F satisfies $(\mathcal{P}_\mathbf{2})$, then $(\mathcal{P}_\mathbf{8})$ holds with $m = u$.
For the opposite implication, we observe that, for every $u \in L^\infty_d$, thanks to $(\mathcal{P}_\mathbf{7})$, $F(u, \cdot)$ is increasing with respect to the inclusion of sets and so, by Remark 3.1, we need only to prove the supremality on sequences of pairwise disjoint sets. Let $u \in L^\infty_d$, let $E_n \in \mathcal{F}$ be a sequence of pairwise disjoint sets and let $A_n := \bigcup_{k=1}^{n} E_k$ and $A := \bigcup_{n=1}^{\infty} E_n$. Let m be as in $(\mathcal{P}_\mathbf{8})$. For every $B \in \mathcal{F}$ set

$$\mathbf{1}_B(x) := \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Since $u\mathbf{1}_{A_n} + m\mathbf{1}_{A \setminus A_n} \to u$ weakly* in $\in L^\infty_d(A)$, by passing to a suitable subsequence and by using properties $(\mathcal{P}_\mathbf{4})$, $(\mathcal{P}_\mathbf{7})$, $(\mathcal{P}_\mathbf{1})$ and $(\mathcal{P}_\mathbf{8})$ we have that

$$\begin{aligned} F(u, A) &\leq \lim_{n\to\infty} F(u\mathbf{1}_{A_{k_n}} + m\mathbf{1}_{A \setminus A_{k_n}}, A) \\ &= \lim_{n\to\infty} F(u, A_{k_n}) \vee \lim_{n\to\infty} F(m, A \setminus A_{k_n}) \\ &\leq \lim_{n\to\infty} F(u, A_n) \vee F(m, A) \\ &= \lim_{n\to\infty} F(u, A_n) \vee \lim_{n\to\infty} F(m, A_n) \\ &\leq \lim_{n\to\infty} F(u, A_n). \end{aligned}$$

By $(\mathcal{P}_\mathbf{7})$, the last inequality implies

$$\lim_{n\to\infty} F(u, A_n) = \lim_{n\to\infty} \bigvee_{k=1}^{n} F(u, E_k) = \bigvee_{n=1}^{\infty} F(u, E_n).$$

□

Now we can proceed to prove the main results of this paper.

PROOF OF THEOREM 2.4. Under the hypothesis $(\mathbf{Hc_k})$, let us observe that for every $k \in \mathbf{N}$ there exists $m_k \in L^\infty_d$ such that $f(x, m_k(x)) = \min\{f_k(x, z) \ : \ z \in \mathbf{R}^d\}$ for μ- a.e. $x \in \Omega$. In fact, by coercivity assumptions (2.5) and (2.6), for every $k \in \mathbf{N}$ there exists $M_k > 0$ such that $\inf\{f_k(x, z) \ : \ z \in \mathbf{R}^d\} = \min\{f_k(x, z) \ : \ z \in \mathbf{R}^d, |z| \leq M_k\}$ for μ- a.e. $x \in \Omega$. By Theorem 1.2 of [15] (applied to every normal supremand f_k), for every $k \in \mathbf{N}$ there exists a measurable map $m_k : \Omega \to \mathbf{R}^d$, $|m_k(x)| \le M_k$ for μ- a.e. $x \in \Omega$ such that

$$f_k(x, m_k(x)) \le f_k(x, z)$$

for μ- a.e. $x \in \Omega$ and for every $z \in \mathbf{R}^d$. In particular, $g_k(x) = f(x, m_k(x))$ is a measurable function. More generally, instead of $(\mathbf{H_1})$, we can suppose the weaker assumption:

($\mathbf{H_2}$) *for every* $B \in \mathcal{F}$, $F_k(m_k, B)$ *converges to* $\mu\text{-}\sup\{g(x) : x \in B\}$ *where* $g : \Omega \to \mathbf{R} \cup \{+\infty\}$ *is a measurable function.*

Let $\mathcal{A}$ be a countable basis for Ω which is stable for finite union. For every $B \in \mathcal{A}$, by Proposition 3.2 (applied to $X = L^1_d(\Omega)$, so that $X' = L^\infty_d(\Omega)$), there exists a subsequence of $(F_k(\cdot, B))_k$ which Γ-converges in the weak* topology of L^∞_d. By using a diagonal procedure, it is possible to select a subsequence (which we still denote by $(F_k)_k$) such that, for every $B \in \mathcal{A}$, $(F_k(\cdot, B))_k$ Γ-converges in the weak* topology of L^∞_d. Let $F(\cdot, B)$ be its Γ-limit. According to Proposition 3.1, for every $A \in \mathcal{F}$ set

$$F^+(u, A) = \Gamma\text{-}\limsup_{k\to\infty} F_k(u, A) = \min\{\limsup_{k\to\infty} F_k(u_k, A) \; : \; u_k \to u \text{ weakly* in } L^\infty_d\}$$

and

$$F^-(u, A) = \Gamma\text{-}\liminf_{k} F_k(u, A) = \min\{\liminf_{k} F_k(u_k, A) \; : \; u_k \to u \text{ weakly* in } L^\infty_d\}.$$

Since F_k are local functionals, observe explicitly that in the definition of $F^+(u, A)$ (and of $F^-(u, A)$) we can consider sequences which converge to u weakly* in $L^\infty_d(A)$. Up to a subsequence, we shall prove that F^+ can be represented in a supremal form and that

$$F^+(u, A) = F^-(u, A) = \bigvee_{n=1}^{\infty} F(u, A_n) \tag{3.19}$$

for every open set $A \subset \Omega$ such that $A = \bigcup_{n=1}^\infty A_n$, with $A_n \in \mathcal{A}$. First of all, we observe that F^+ and F^- satisfy the following properties:

(a) for every $u \in L^\infty_d$, $F^+(u, \cdot)$ and $F^-(u, \cdot)$ are increasing with respect to inclusion;

(b) for every $A, B \in \mathcal{F}$ and for every $u \in L^\infty_d$

$$F^+(u, A \cup B) = F^+(u, A) \vee F^+(u, B);$$

(c) for every $A, B \in \mathcal{F}$ and for every $u \in L^\infty_d$

$$F^-(u, A \cup B) \le F^-(u, A) \vee F^+(u, B).$$

In fact, if $A, B \in \mathcal{F}$, $A \subset B$ and if $u_k \to u$ weakly* in L^∞_d such that

$$F^+(u, B) = \limsup_{k\to\infty} F_k(u_k, B),$$

then

$$F^+(u,A) \le \limsup_{k\to\infty} F_k(u_k,A) \le \limsup_{k\to\infty} F_k(u_k,B) = F^+(u,B).$$

Analogous arguments hold for F^- and give (a).
To show (b), if $u_k \to u$ weakly* in L^∞_d such that $F^+(u,A) = \limsup_{k\to\infty} F_k(u_k,A)$ and $v_k \to u$ weakly* in L^∞_d such that $F^+(u,B) = \limsup_{k\to\infty} F_k(v_k,B)$, then

$$\begin{aligned} F^+(u,A\cup B) &\le \limsup_{k\to\infty} F_k(u_k 1_A + v_k 1_{B\setminus A}, A\cup B) \\ &\le \limsup_{k\to\infty} F_k(u_k,A) \vee \limsup_{k\to\infty} F_k(v_k,B) \\ &= F^+(u,A) \vee F^+(u,B). \end{aligned}$$

The opposite inequality follows by property (a) and proves (b). Analogous arguments hold for F^- and proves (c).
Now let us prove that F^+ satisfies the hypothesis of Theorem 3.2.
In order to prove property $(\mathcal{P}_\mathbf{1})$, let $u,v \in L^\infty_d$ and $A,B \in \mathcal{F}$ such that $u=v$ μ-a.e. $x \in B$ and $\mu(A\triangle B)=0$. Then there exists a sequence $(u_k)_k \subset L^\infty_d$, u_k converging to u weakly* in L^∞_d such that $F^+(u,A) = \limsup_{k\to\infty} F_k(u_k,A)$. Let $v_k := u_k 1_A + v 1_{\Omega\setminus A}$. Then $v_k \to v$ weakly* in L^∞_d. From the locality of F_k and by definition of F^+,

$$F^+(u,A) = \limsup_{k\to\infty} F_k(u_k,A) = \limsup_{k\to\infty} F_k(v_k,B) \ge F^+(v,B).$$

The proof of the inequality $F^+(u,A) \le F^+(v,B)$ is similar.
Property $(\mathcal{P}_\mathbf{4})$ is a consequence of the definition of F^+. In order to apply Theorem 3.2, it remains to prove only property $(\mathcal{P}_\mathbf{2})$. We distinguish two cases. Suppose, first, that

$$\exists u \in L^\infty_d \text{ s. t. } F^+(u,\Omega) < +\infty. \tag{3.20}$$

Thus, if $(u_k)_k \in L^\infty_d$ is such that $\limsup_{k\to\infty} F_k(u_k,\Omega) = F^+(u,\Omega)$, by (2.6), we have

$$\begin{aligned} \limsup_{k\to\infty} \mu\text{-}\sup_{x\in\Omega} \phi(m_k(x)) &\le \limsup_{k\to\infty} \mu\text{-}\sup_{x\in\Omega} f_k(x,m_k(x)) \\ &\le \limsup_{k\to\infty} F_k(u_k,\Omega) < +\infty. \end{aligned} \tag{3.21}$$

By (2.5) and by (3.21), we obtain that the sequence $(m_k)_k$ is bounded in L^∞_d. Without loss of generality, extracting a further subsequence, we can suppose that there exists $m \in L^\infty_d$ such that $m_k \to m$ weakly* in L^∞_d. From now on F^+ and F^- refer to this new subsequence. By property (b) and thanks to Lemma 3.1, in order to obtain the countable supremality of F^+ it is sufficient to prove that m satisfies i) and ii) of property $(\mathcal{P}_\mathbf{8})$. First of all we observe that if $u \in L^\infty_d$, then

$$F^+(m,B) \le F^+(u,B) \tag{3.22}$$

for every $B \in \mathcal{F}$. In fact, if $u_k \to u$ weakly* in L^∞_d such that

$$F^+(u,B) = \limsup_{k\to\infty} F_k(u_k,B),$$

then, by definition of m_k,

$$F^+(u,B) = \limsup_{k\to\infty} F_k(u_k,B) \geq \limsup_{k\to\infty} F_k(m_k,B) \geq F^+(m,B). \tag{3.23}$$

Moreover, by (3.23) with m in the place of u and by assumption $(\mathbf{H_2})$, we obtain

$$F^+(m,B) = \limsup_{k\to\infty} F_k(m_k,B) = \mu\text{-}\sup_{x\in B} g(x) \tag{3.24}$$

for every $B \in \mathcal{F}$. In particular m satisfies property $(\mathcal{P}_\mathbf{8})$ for every $u \in L^\infty_d$ and $(A_i)_{i\in\mathbf{N}} \subset \mathcal{F}$. Since F^+ satisfies all the hypothesis of Theorem 3.2, there exists a level convex normal supremand f such that

$$F^+(u,B) = \mu\text{-}\sup_{x\in B} f(x,u(x)) \tag{3.25}$$

for every $B \in \mathcal{F}$ and for every $u \in L^\infty_d$. In particular, since for every open set $A \subset \Omega$, there exists $(A_n)_{n\in\mathbf{N}} \subset \mathcal{A}$ such that $A = \bigcup_{n=1}^\infty A_n$ and since F^+ satisfies property $(\mathcal{P}_\mathbf{2})$, we have

$$F^+(u,A) = \bigvee_{n=1}^{\infty} F^+(u,A_n) = \bigvee_{n=1}^{\infty} F^-(u,A_n) \leq F^-(u,A) \leq F^+(u,A). \tag{3.26}$$

Therefore, for every open set $A \subset \Omega$ and for every $u \in L^\infty_d$ we can define (without ambiguity)

$$F(u,A) := \bigvee_{n=1}^{\infty} F(u,A_n)$$

and (3.26) implies that $(F_k(\cdot,A))_k$ Γ-converges to $F(\cdot,A)$ for every open set $A \subset \Omega$. In particular, by (3.25),

$$F(u,A) = \mu\text{-}\sup_{x\in A} f(x,u(x))$$

for every open set $A \subset \Omega$ for every $u \in L^\infty_d$.
Now we consider the other case. Assume that $F^+(u,\Omega) = +\infty$ for every $u \in L^\infty_d$. Set

$$\mathcal{A}^* := \{A \in \mathcal{A} \ :\ \text{there exists } u \in L^\infty_d \text{ such that } F^+(u,A) < +\infty\} \tag{3.27}$$

and $\Omega' := \bigcup_{A\in\mathcal{A}^*} A$. Observe that, if $A \subset \Omega$ is a open set such that $\mu(A \setminus \Omega') > 0$, then there exists $B \in \mathcal{A} \setminus \mathcal{A}^*$ with $B \subset A$. This implies

$$F^+(u,A) \geq F^+(u,B) = +\infty \tag{3.28}$$

for every $u \in L^\infty_d$. If we prove that F^+ satisfies property $(\mathcal{P}_\mathbf{2})$ for every $u \in L^\infty_d(\Omega')$ and $(A_n)_{n\in\mathbf{N}} \subset \mathcal{F}$, $A_n \subset \Omega'$, then, by applying Theorem 3.2, there exists a level convex normal supremand $f : \Omega' \times \mathbf{R}^d \to \overline{\mathbf{R}}$ such that (3.25) holds for every $u \in L^\infty_d(\Omega')$ and for every $\mathcal{F}$-measurable set $B \subset \Omega'$. Then, setting

$$f^*(x,z) := \begin{cases} f(x,z) & \text{if } x \in \Omega' \\ +\infty & \text{otherwise} \end{cases},$$

it results a level convex normal supremand and by (3.28), we can easily conclude that $F(u,A) = \mu\text{-}\sup_{x\in A} f^*(x,u(x))$ for every open set $A \subset \Omega$ and for every $u \in L^\infty_d$.
If $\inf\{F^+(u,\Omega') \ : \ u \in L^\infty_d\} < +\infty$, then it is sufficient to repeat the the first part of this proof with Ω' instead of Ω . Otherwise, assume that $F^+(u,\Omega') = +\infty$ for every $u \in L^\infty_d(\Omega')$. By definition of $\mathcal{A}^*$, for every $B_n \in \mathcal{A}^*$ we have that (3.20) holds with B_n instead of Ω. Therefore for every $B_n \in \mathcal{A}^*$ there exists a subsequence of m_k which converges in $L^\infty_d(B_n)$. By using a diagonal procedure, it is possible to select a subsequence (which we still denote $(m_k)_k$) such that, for every $B_n \in \mathcal{A}^*$, m_k converges to some m^n in the weak* topology of $L^\infty_d(B_n)$. From now on F^+ and F^- refer to this new subsequence. If $x \in \Omega'$, then there exists $B_n \in \mathcal{A}^*$ such that $x \in B_n \subset \Omega'$. Thus, without ambiguity, we can define $\overline{m} : \Omega' \to \overline{\mathbf{R}}$ in the following way:

(3.29) $$\overline{m}(x) := m^n(x) \text{ if } \ x \in B_n.$$

Observe that, for every $C \subset \Omega'$, $C \in \mathcal{F}$ we have that

(3.30) $$\mu\text{-}\sup_{x\in C} g(x) < +\infty \ \Rightarrow \ m_k \to \overline{m} \ \text{ weakly* in } L^\infty_d(C)$$

In fact, by using hypothesis $(\mathbf{H_2})$ and (2.6), we

$$\limsup_{k\to\infty} \mu\text{-}\sup_{x\in C} \phi(m_k(x)) \leq \limsup_{k\to\infty} \mu\text{-}\sup_{x\in C} f_k(x, m_k(x) \leq \limsup_{k\to\infty} \mu\text{-}\sup_{x\in C} g(x) < +\infty$$

which implies by (2.5), that $(m_k)_k$ is bounded in $L^\infty_d(C)$. Thus there exists a subsequence $(m_{n_k})_k$ which converges weakly* to m_C in $L^\infty_d(C)$. In particular $m_C = \overline{m}^n$ on $C \cap B_n$ and thus $m_C = \overline{m}$ on C. For the arbitrariness of the subsequence, (3.30) follows. Now let $u \in L^\infty_d$ and $C_n \in \mathcal{F}$ such that $C_n \subset C_{n+1} \subset \Omega'$ and define $C := \bigcup_{n=1}^\infty C_n$. If $u_k \to u$ in the weak* topology of L^∞_d such that $F^+(u,C_n) = \limsup_{k\to\infty} F_k(u_k, C_n)$, then

(3.31) $$F^+(u,C_n) \geq \lim_{k\to\infty} F_k(m_k, C_n) = \mu\text{-}\sup_{x\in C_n} g(x).$$

In particular if $\mu\text{-}\sup_{x\in C} g(x) = +\infty$, we can deduce that

$$F^+(u,C) = \lim_{n\to\infty} F^+(u,C_n) = +\infty.$$

Instead, if $\mu\text{-}\sup_{x\in C} g(x) < +\infty$, by using (3.30) and by proceeding as in the proof of (3.23) and (3.24) with $\overline{m}$ instead m, we obtain that for every $n \in \mathbf{N}$

$$F^+(u,C_n) \geq F^+(\overline{m},C_n) = \lim_{k\to\infty} F_k(m_k,C_n) = \mu\text{-}\sup_{x\in C_n} g(x)$$

and

$$F^+(\overline{m},C) = \mu\text{-}\sup_{x\in C} g(x).$$

In particular, $\overline{m}$ satisfies property (i) and (ii) of $(\mathcal{P}_\mathbf{8})$ and, by applying Lemma 3.1, we can state that F^+ satisfies property $(\mathcal{P}_\mathbf{2})$ on Ω'. This concludes the proof of Theorem 2.4 . □

REMARK **3.2** In the proof of Theorem 2.4 we have shown that

$$\Gamma\text{-}\limsup_{k\to\infty} F_k(\cdot, B) = \mu\text{-}\sup\{f(x,u(x)) \ : \ x \in B\}$$

for every $B \in \mathcal{F}$ without using any topological assumption on Ω. In particular, in a general measure space, under the hypothesis ($\mathbf{Hc_k}$) and ($\mathbf{H_2}$), if there exists the Γ-limit of $(F_k(\cdot, B))_k$ for every $B \in \mathcal{F}$, then it coincides with the $\Gamma\limsup_{k\to\infty} F_k(\cdot, B)$ and thus the supremal representation of Γ-limit holds every $B \in \mathcal{F}$.

In the following example, we produce a wide class of functionals F_k which satisfy the assumption ($\mathbf{H_2}$).

EXAMPLE **3.2** Let $(f_k)_{k\in\mathbf{N}}$ be a sequence of non negative normal supremands such that $\inf_{\mathbf{R}^d} f_k(x, \cdot) = 0$ for a.e. $x \in \Omega$ and for every $k \in \mathbf{N}$. Then $(f_k)_{k\in\mathbf{N}}$ satisfies the assumption ($\mathbf{H_2}$). In fact, in this case, $g_k(x) = 0$.

The representation result of Theorem 2.4 may fail if we drop the assumption ($\mathbf{H_2}$). In fact we can exhibit the following example.

EXAMPLE **3.3** Let $\Omega := (0, 1)$, $d = 1$, $\mathcal{F}$ the σ-field of Lebesgue measurable sets and μ the Lebesgue measure. Let $\phi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ such that $\lim_{|z|\to\infty} \phi(z) = +\infty$ and $\phi(z) \geq \phi(0) = 0$. Let us define

$$f_k(x, z) := x^k + \phi(z).$$

Then we have that f_k satisfies the condition ($\mathbf{Hc_k}$) and $g_k(x) := \inf\{f_k(x, z) : \ z \in \mathbf{R}\} = x^k$ converges weakly* in L^∞ to $g(x) = 0$, but does not converge strongly in L^∞. Moreover, setting $F_k(u, B) := \operatorname{ess\,sup}_{x\in B} f_k(x, u(x))$ for every $B \in \mathcal{F}$ and for every $u \in L^\infty$, we obtain that the sequence

$$F_k(0, (a, b)) = \operatorname*{ess\,sup}_{x\in(a,b)} x^k = b^k$$

converges to

$$F(0, (a, b)) = \begin{cases} 0 & \text{if } b < 1 \\ 1 & \text{if } b = 1 \end{cases}$$

and thus it does not satisfy hypothesis ($\mathbf{H_2}$). Moreover, if we define

$$G_k(u, (a, b)) := \operatorname*{ess\,sup}_{x\in(a,b)} f_k(x, u'(x))$$

with $0 \leq a < b \leq 1$, $u \in W^{1,\infty}(0, 1)$, it is easy to prove that the $\Gamma\text{-}\lim_{k\to\infty} G_k(0, (a, b))$ (with respect to the weak* topology of $W^{1,\infty}$) is equal to

$$G(0, (a, b)) = \begin{cases} 0 & \text{if } b < 1 \\ 1 & \text{if } b = 1 \end{cases}$$

This means that the class of supremal functionals on $W^{1,\infty}$ is not closed with respect to the Γ-convergence.

In the next section (see Remark 4.3), we prove that for every $(a,b) \subset \Omega$, there exists

$$F(\cdot,(a,b)) := \Gamma\text{-}\lim_{k\to\infty} F_k(\cdot,(a,b))$$

with respect to the weak* topology of L^∞ and

$$F(u,(a,b)) = \begin{cases} \underset{x\in(a,b)}{\text{ess sup}}\ \phi^{c\gamma}(u(x)) & \text{if } b<1 \\ \underset{x\in(a,b)}{\text{ess sup}}\ \phi^{c\gamma}(u(x)) \vee 1 & \text{if } b=1 \end{cases} \tag{3.32}$$

which is not a supremal functional.

We conclude this section with the following proposition: it states that the assumption $(\mathbf{H_2})$, restricted on the open set, is a necessary condition in order to obtain a representation result for the Γ-limit.

PROPOSITION **3.3** *Let $F_k : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ be a sequence of supremal functionals defined by*

$$F_k(u,B) = \mu\text{-}\sup\{f_k(x,u(x)) : x \in B\},$$

where $f_k : \Omega \times \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ are normal supremands satisfying assumption $(\mathbf{Hc_k})$. Let $m_k \in L^\infty_d$ such that $f(x,m_k(x)) = \min\{f_k(x,z) : z \in \mathbf{R}^d\}$ for μ- a.e. $x \in \Omega$. If $F_k(\cdot,A)$ Γ-converges to $F(\cdot,A)$ for every open set A and if F satisfies the supremal representation $F(u,A) = \mu\text{-}\sup\{f(x,u(x)) : x \in A\}$ for every $u \in L^\infty_d$, then there exists a measurable function $g : \Omega \to \mathbf{R} \cup \{+\infty\}$ such that $F_k(m_k,A)$ converges to $\mu\text{-}\sup\{g(x) : x \in A\}$ for every open set A .

PROOF. If there exists $u \in L^\infty_d$ such that $F(u,\Omega) < +\infty$, then, following the proof of Theorem 2.4, we have that $m_k \to m$ weakly* in L^∞_d and $\lim_{k\to\infty} F_k(m_k,A) = F(m,A)$ for every open set A. Thus it is sufficient to choose $g(x) := f(x,m(x))$. Instead, if $F(u,\Omega) = +\infty$ for every $u \in L^\infty_d$, we consider $\mathcal{A}^*$ and $\overline{m}$ given, respectively, by definition (3.27) and (3.28). For fixed an open set $A \subset \Omega'$, we can consider two cases:

(I) $\min_{L^\infty_d(A)} F(\cdot,A) < +\infty$. Then, reasoning as in the proof of Theorem 2.4, we have that $m_k \to \overline{m}$ weakly * in $L^\infty_d(A)$ and

$$\mu\text{-}\sup_A f(x,\overline{m}(x)) = \lim_{k\to\infty} F_k(m_k,A).$$

(II) $\min_{L^\infty_d(A)} F(\cdot,A) = +\infty$. This implies that $\mu\text{-}\sup_A f(x,\overline{m}(x)) = +\infty$ (otherwise, if $A = \bigcup_{n=1}^\infty B_n$, with $B_n \in \mathcal{A}^*$, by the definition of $\overline{m}$ we have $F(m^n,B_n) \le \mu\text{-}\sup_A f(x,\overline{m}(x) \in \mathbf{R}$ for every $n \in \mathbf{N}$ and thus, by the coercivity of F (see Proposition 6.7 in [13]), we obtain that $\overline{m} \in L^\infty_d(A)$ and that $\min_{L^\infty_d(A)} F(\cdot,A) \le F(\overline{m},A) = \mu\text{-}\sup_A f(x,\overline{m}(x)) < +\infty$, in contradiction with respect to the assumption). Therefore, by applying Theorem 7.8 in [13], we have

$$\begin{aligned}\lim_{k\to\infty} F_k(m_k, A) &= \lim_{k\to\infty} \inf_{u\in L^\infty_d} F_k(u, A)\\ &= \min_{u\in L^\infty_d} F(u, A)\\ &= +\infty = \mu\text{-}\sup_{x\in A} f(x, \overline{m}(x)).\end{aligned}$$

In every case, by setting

$$g(x) := \begin{cases} f(x, \overline{m}(x)) & \text{if } x \in \Omega' \\ +\infty & \text{otherwise,}\end{cases}$$

we have

$$\lim_{k\to\infty} F_k(m_k, A) = \mu\text{-}\sup_{x\in A} g(x).$$

□

4. – Relaxation theorem

In this section we give the proof of the representation formula for the weak* lower semicontinuous envelope of a supremal functional.

Proof of Theorem 2.5. Under the hypothesis ($\mathbf{Hc_0}$) and proceeding as in the first part of the proof of Theorem 2.4, we can state that there exists $m \in L^\infty_d$ such that $f(x, m(x)) = \min\{f(x, z) \ : \ z \in \mathbf{R}^d\}$ for μ- a.e. $x \in \Omega$. In particular

$$\overline{F}(m, B) = \mu\text{-}\sup\{f(x, m(x)) \ : \ x \in B\} = \min_{L^\infty_d} F(u, B)$$

for every $B \in \mathcal{F}$. Thus the constant sequence $G_n = F$ satisfies hypotheses ($\mathbf{Hc_0}$) and ($\mathbf{H_2}$) of Theorem 2.4. By applying Remark 3.2, we obtain that $\overline{F} = \Gamma$ $hbox{-}\lim_{n\to\infty} G_n = \Gamma \liminf_{n\to\infty} G_n$ is a supremal functional and there exists a level convex normal supremand g such that the representation formula

$$\overline{F}(u, B) = \mu\text{-}\sup\{g(x, u(x)) \ : \ x \in B\}$$

holds for every $u \in L^\infty_d$ and for every $B \in \mathcal{F}$. To obtain (2.11), it is sufficient to prove that there exists $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $g(x, z) = f^{c\gamma}(x, z)$ for every $(x, z) \in (\Omega \setminus N) \times \mathbf{R}^d$. By Proposition 2.3 of [1], applied to F and $\overline{F}$, there exists $N_1 \in \mathcal{F}$ such that $\mu(N_1) = 0$ and

$$g(x, z) \le f(x, z)$$

for every $(x, z) \in (\Omega \setminus N_1) \times \mathbf{R}^d$. So

$$g(x, z) \le f^{c\gamma}(x, z) \tag{4.33}$$

for every $(x, z) \in (\Omega \setminus N_1) \times \mathbf{R}^d$. Moreover, by Theorem 2.2, $f^{c\gamma}$ is a level convex normal supremand and thus the functional $G : L^\infty_d \times \mathcal{F} \to \overline{\mathbf{R}}$ given by

$$G(u, B) := \mu\text{-}\sup \{f^{c\gamma}(x, u(x)) \ : \ x \in B\}$$

is weakly* lower semicontinuous in L^∞_d (see Remark 4.4 of [1]). This implies that

$$G(u, B) \leq \overline{F}(u, B)$$

for every $u \in L^\infty$ and for every $B \in \mathcal{F}$ and so, by applying again Proposition 2.3 of [1], there exists $N_2 \in \mathcal{F}$ such that $\mu(N_2) = 0$

$$f^{c\gamma}(x, z) \leq g(x, z) \tag{4.34}$$

for every $(x, z) \in (\Omega \setminus N_2) \times \mathbf{R}^d$. Inequalities (4.33) and (4.34) now imply

$$f^{c\gamma}(x, z) = g(x, z)$$

for every $(x, z) \in (\Omega \setminus (N_2 \cap N_1)) \times \mathbf{R}^d$. □

COROLLARY **4.1** *Let $f : \Omega \times \mathbf{R}^d \to]-\infty, +\infty]$ be a normal supremand satisfying assumption* $(\mathbf{Hc_0})$ *and let $F : L^\infty_d \to \mathbf{R}$ be defined by*

$$F(u) = \mu\text{-}\sup \{f(x, u(x)) \ : \ x \in \Omega\}.$$

Then the weak lower semicontinuous envelope of F is given by*

$$\overline{F}(u) = \mu\text{-}\sup \{f^{c\gamma}(x, u(x)) \ : \ x \in \Omega\}$$

for every $u \in L^\infty_d$.

REMARK **4.1** In Theorem 2.5, the coercivity assumption on f ensures the property $(\mathcal{P}_\mathbf{2})$ for the functional $\overline{F}$ when F is supremal functional. Now, if a general functional $F : L^\infty_d \times \mathcal{B}_d \to \overline{\mathbf{R}}$ satisfies properties $(\mathcal{P}_\mathbf{1})$ and $(\mathcal{P}_\mathbf{7})$, then, proceeding as in the first part of the proof of Theorem 2.4, we can obtain that also $\overline{F}$ satisfies properties $(\mathcal{P}_\mathbf{1})$, $(\mathcal{P}_\mathbf{4})$ and $(\mathcal{P}_\mathbf{7})$. So, by Lemma 3.1, $\overline{F}$ is a supremal functional if and only if it satisfies property $(\mathcal{P}_\mathbf{8})$. In the example 4.1 we represent $\overline{F}$ without any coercivity condition on F.

REMARK **4.2** Under the notation of Example 3.3, we prove that for every $(a, b) \subset \Omega$, there exists $F(\cdot, (a, b)) := \Gamma\text{-}\lim F_k(\cdot, (a, b))$ with respect to the weak* topology of L^∞ and it is given by (3.32). By using the relaxation Theorem 2.5, for every $(a, b) \subset \Omega$ there exists $u_k \to u$ weakly* in $L^\infty((a, b))$ such that

$$\operatorname*{ess\,sup}_{x \in (a,b)} \phi^{c\gamma}(u(x)) = \lim_{k \to \infty} \Big(\operatorname*{ess\,sup}_{x \in (a,b)} \phi(u_k(x)) \Big).$$

Then, if $b < 1$, we have

$$F^+(u, (a, b)) \leq \limsup_{k \to \infty} \left\{ \operatorname*{ess\,sup}_{x \in (a,b)} \phi(u_k(x)) + b^k \right\} = \operatorname*{ess\,sup}_{x \in (a,b)} \phi^{c\gamma}(u(x)) \tag{4.35}$$

while, if $b=1$, since $u_k \cdot 1_{(a,1-k^{-1/2})} \to u$ weakly* in $L^\infty((a,1))$ and since $\phi(0)=0$, we have

$$
\begin{aligned}
F^+(u,(a,1)) &\le \limsup_{k\to\infty} \Big\{ F_k(u_k,(a,1-k^{-1/2})) \vee F_k(0,(1-k^{-1/2},1)) \Big\} \\
&= \limsup_{k\to\infty} \Big\{ \operatorname{ess}\sup_{x\in(a,1-k^{-1/2})} (\phi(u_k(x))+x^k) \vee \operatorname{ess}\sup_{x\in(1-k^{-1/2},1)} (\phi(0)+x^k) \Big\} \\
&\le \limsup_{k\to\infty} \Big\{ \operatorname{ess}\sup_{x\in(a,1)} \phi(u_k(x)) + (1-k^{-1/2})^k \Big\} \vee 1
\end{aligned}
$$

and thus

$$
F^+(u,(a,1)) \le \operatorname{ess}\sup_{x\in(a,1)} \phi^{c\gamma}(u(x)) \vee 1. \tag{4.36}
$$

On the other hand, if $u_k \to u$ weakly* in $L^\infty(0,1)$ is such that

$$
\liminf_{k\to\infty} F_k(u_k,(a,b)) = F^-(u,(a,b)),
$$

we have

$$
F^-(u,(a,b)) \ge \liminf_{k\to\infty} (\operatorname{ess}\sup_{x\in(a,b)} \phi(u_k(x))+a^k) \ge \operatorname{ess}\sup_{x\in(a,b)} \phi^{c\gamma}(u(x)). \tag{4.37}
$$

Therefore if $b<1$, (4.36) and (4.37) give $F(u,(a,b)) = \operatorname{ess}\sup_{x\in(a,1)} \phi^{c\gamma}(u(x))$, while, since

$$
F^-(u,(a,1)) \ge 1,
$$

by using (4.35) and (4.36), we can conclude that

$$
F(u,(a,1)) = \operatorname{ess}\sup_{x\in(a,1)} \phi^{c\gamma}(u(x)) \vee 1.
$$

In particular $F(0,(a,1))=1$ and $F(0,(a,b))=0$ for every $b<1$.

Example **4.1** Let $f:\Omega\times\mathbf{R}^d \to]-\infty,+\infty]$ be a normal supremand. Assume that there exists $m\in L^\infty_d$ such that

$$
f(x,m(x)) \le f(x,z) \tag{4.38}
$$

for every $z\in\mathbf{R}^d$ and for μ- a.e. $x\in\mathbf{R}^d$. Then

$$
\overline{F}(u,B) = \mu\text{-}\sup\{f^{c\gamma}(x,u(x)) \ :\ x\in B\}
$$

for every $B\in\mathcal{F}$ and for every $u\in L^\infty_d$.
In fact for every $B\in\mathcal{F}$ $\mu\text{-}\sup\{f(x,m(x)) \ :\ x\in B\} = \overline{F}(m,B)$ and so $\overline{F}$ satisfies $(\mathcal{P}_{\mathbf{7}})$. As in the last part of the proof of Theorem 4.2, it follows that the supremand function which represents $\overline{F}$ is $f^{c\gamma}$.

REMARK **4.3** If $F : L^\infty_d \times \mathcal{B}_d \to \overline{\mathbf{R}}$ is a supremal functional

$$F(u,B) = \mu\text{-}\sup\{f(x,u(x)) \ : \ x \in B\}$$

where f is only a supremand, then, the strong lower semicontinuous envelope

$$\Gamma F(u,B) := \sup\{G(v) \ : \ G : L^\infty_d \to \overline{\mathbf{R}}, G \text{ strongly l.s.c. in } L^\infty_d, G(\cdot) \leq F(\cdot, B) \text{ on } L^\infty_d\},$$

is a supremal functional and it is represented by

$$\Gamma f(x,z) = \sup\{g(z) \ : \ g : \mathbf{R}^d \to \overline{\mathbf{R}}, \ g \ \text{ l.s.c., } g(z) \leq f(x,z) \text{ for every } z \in \mathbf{R}^d\},$$

i.e. the l.s.c. envelope of f respect to the second variable. In fact, if we set

$$F_\lambda(u,B) := \inf\{F(v,B) \vee \lambda||u - v||_{L^\infty(B)} \ : \ v \in L^\infty(\Omega)\},$$

for every $\lambda > 0$, F_λ turns out to be λ-Lipschitz continuous whenever F is finite in at least one point. Thus

$$F_\lambda(u,B) \leq \Gamma F(u,B)$$

for every $\lambda > 0$. Moreover, from Proposition 2.4 of [1] applied to ΓF ,

$$\Gamma F(u,B) = \sup\{(\Gamma F)_\lambda(u,B) \ : \ \lambda > 0\} \leq \sup\{F_\lambda(u,B) \ : \ \lambda > 0\}$$

for every $B \in \mathcal{F}$ and for every $u \in L^\infty_d$. Therefore

$$\Gamma F(u,B) = \sup\{F_\lambda(u,B) \ : \ \lambda > 0\}$$

for every $B \in \mathcal{F}$ and for every $u \in L^\infty_d$. From this representation, it is easy to prove that ΓF satisfies property $(\mathcal{P}_\mathbf{2})$. Moreover, ΓF satisfies $(\mathcal{P}_\mathbf{1})$ and $(\mathcal{P}_\mathbf{3})$ of Theorem 3.1. So it can be represented in a supremal form by a unique normal supremand g. Applying Proposition 2.2 of [1], there exists $N \in \mathcal{F}$, $\mu(N) = 0$, such that

$$g(x,z) \leq \Gamma f(x,z)$$

for every $x \in \Omega \setminus N$ and for every $z \in \mathbf{R}^d$. On the other hand,

$$H(u,B) := \mu\text{-}\sup\{\Gamma f(x,u(x)) \ : \ x \in B\}$$

is a strongly l.s.c. functional such that $H(u,B) \leq F(u,B)$ for every $u \in L^\infty_d$ and for every $B \in \mathcal{F}$. Therefore, applying again Proposition 2.2 of [1], there exists $M \in \mathcal{F}$, $\mu(M) = 0$, such that

$$\Gamma f(x,z) \leq g(x,z)$$

for every $x \in \Omega \setminus M$ and for every $s \in \mathbf{R}^d$. So

$$\Gamma F(u,B) = \mu\text{-}\sup\{\Gamma f(x,u(x)) \ : \ x \in B\}$$

for every $u \in L^\infty_d$ and for every $B \in \mathcal{F}$.

REMARK **4.4** If $f : \Omega \times \mathbf{R}^d \to]-\infty, +\infty]$ is a supremand which satisfies the hypothesis $(\mathbf{Hc_0})$ or (4.38), then $\overline{F} = \overline{\Gamma F}$ and it is represented by $(\Gamma f)^{c\gamma}$. In fact, it is sufficient to observe that Γf satisfies, respectively, $(\mathbf{Hc_0})$ or (4.38) and to apply Theorem 2.5 or Example 4.1.

5. – Relaxation through Young measures

In analogy to the relaxation theorem for integral functionals of [17], at the end of this paper we state a relaxation theorem for supremal functionals by using the Young measures. First of all, we give their definition and their main properties, following [20] and [21] (see also [6] for the characterization of narrow convergence). In this section, $(\Omega, \mathcal{F}, \mu)$ is a measure space where μ is a nonnegative Radon measure. We suppose that it is a complete, nonatomic and finite measure and that Ω is a locally compact, metrizable and separable space, l.c.s. for short.

Definition **5.1** *(a) A Young measure on $\Omega \times \mathbf{R}^d$ is a nonnegative measure τ on $\Omega \times \mathbf{R}^d$ such that $\tau(B \times \mathbf{R}^d) = \mu(B)$ for any Borel set $B \subset \Omega$, i.e. μ is the image of τ by the projection map $(x, z) \to x$.*

(b) For any $\mathcal{F}$-measurable function $u : \Omega \to \mathbf{R}^d$, the Young measure associated to u is the image of μ by the map $x \to (x, u(x))$, that is $\nu(A \times B) = \mu(A \cap u^{-1}(B))$ for any Borel sets $A \subset \Omega$ and $B \subset \mathbf{R}^d$.

(c)Let $(\tau_x)_{x\in\Omega}$ be a family of probability measures on $\mathbf{R}^d$ such that $x \mapsto \tau_x(B)$ is $\mathcal{F}$-measurable on Ω for every $B \in \mathcal{B}_d$. A Radon measure τ is defined by the formula

$$\tau(C) := \int_\Omega \tau_x(C_x) d\mu(x)$$

where $C_x =: \{z \ : \ (x, z) \in C\}$. We write in this case $\tau = \mu \otimes \tau_x$.

Example **5.1** (1) When ν is a Young measure associated to the function u , then $\nu = \mu \otimes \delta_{u(x)}$.

(2) When τ is a Young measure on $\Omega \times \mathbf{R}^d$, there exists a family $(\tau_x)_{x\in\Omega}$ of probability measures on $\mathbf{R}^d$ such that $\tau = \mu \otimes \tau_x$. This decomposition is known as disintegration of τ.

We denote by $\mathcal{Y}(\Omega, \mu, \mathbf{R}^d)$ the set of all Young measures on $\Omega \times \mathbf{R}^d$ and on it we consider the following topology.

Definition **5.2** *The narrow topology on $\mathcal{Y}(\Omega, \mu, \mathbf{R}^d)$ is the weakest topology for which the maps $\tau \mapsto \int_{\Omega\times\mathbf{R}^d} \phi d\tau(x)$ are continuous, where $\phi \in C_c(\Omega \times \mathbf{R}^d)$.*

When $(z_k)_k$ is a sequence of $\mathcal{F}$-measurable functions $z_k : \Omega \to \mathbf{R}^d$ such that the sequence of the associated Young measures $(\nu_k)_k$ narrowly converges to τ, where τ is some Young measure, we say that the sequence $(z_k)_k$ generates the Young measure τ.

The following proposition (see [21]) contains a lower semicontinuity result.

Proposition **5.1** *Let $(z_k)_k : \Omega \to \mathbf{R}^d$ be a sequence of $\mathcal{F}$-measurable functions and suppose that it generates the Young measure ν. Let $f : \Omega \times \mathbf{R}^d \to]-\infty, +\infty]$ be a normal supremand. Assume that the negative part $f^-(x, z_k(x))$ is weakly relatively compact in $L^1(\Omega, \mathbf{R}^d)$. Then*

$$\liminf_{k\to\infty} \int_\Omega f(x, z_k(x))dx \geq \int_\Omega \int_{\mathbf{R}^d} f(x, z)d\nu_x(z)dx.$$

By using Proposition 5.1, we can prove the following theorem which generalizes the result of Lemma 3.2 in [4].

THEOREM **5.1** *Let $(z_k)_k$ be a bounded sequence in $L^\infty(\Omega, \mathbf{R}^d)$ and suppose that $(z_k)_k$ generates the Young measure ν. Let $f : \Omega\times\mathbf{R}^d \to]-\infty, +\infty]$ be a normal supremand. Then*

$$\liminf_{k\to\infty}(\mu \otimes \delta_{z_k(x)}\text{-}\sup_{\Omega\times\mathbf{R}^d} f(x, z)) \geq \mu \otimes \nu_x\text{-}\sup_{\Omega\times\mathbf{R}^d} f(x, z)$$

where $\delta_{z_k(x)}$ is the Dirac measure concentrated in $z_k(x)$.

PROOF. If this is not true, there exists $\epsilon > 0$, there exists a subsequence of $(z_k)_k$ (which we still denote by $(z_k)_k$) and for every $k \in \mathbf{N}$ there exists $E_k \subset \Omega \times \mathbf{R}^d$ such that $\mu \otimes \delta_{z_k(x)}(\Omega \times \mathbf{R}^d \setminus E_k) = 0$ and

$$\sup_{E_k} f(x, z) \leq \mu \otimes \nu_x\text{-}\sup_{\Omega\times\mathbf{R}^d} f(x, z) - \epsilon. \tag{5.39}$$

Let $M > 0$ such that $||z_k||_{L^\infty} \leq M$ for every $k \in \mathbf{N}$ and let

$$\Lambda := \mu \otimes \nu_x\text{-}\sup_{\Omega\times\mathbf{R}^d} f(x, z) - \epsilon.$$

If we set $F := \{(x, z) \in \Omega\times\mathbf{R}^d : \ |z| \leq M, f(x, z) \leq \Lambda\}$, then F is $\mathcal{F}\otimes\mathcal{B}_d$-measurable. Moreover, by (5.39), $F^c := (\Omega\times\mathbf{R}^d\setminus F) \subset (\Omega\times\mathbf{R}^d\setminus E_k)\cup(\Omega\times\{z \in \mathbf{R}^d : \ |z| > M\})$ for every $k \in \mathbf{N}$. This implies that $\mu \otimes \delta_{z_k(x)}(F^c) \leq \mu \otimes \delta_{z_k(x)}(\Omega \times \mathbf{R}^d \setminus E_k) + \mu \otimes \delta_{z_k(x)}(\Omega \times \{z \in \mathbf{R}^d : \ |z| > M\}) = 0$. Setting

$$g(x, z) = \mathbf{1}_{F^c}(x, z) = \begin{cases} 1 & \text{if } (x, y) \in F^c \\ 0 & \text{otherwise,} \end{cases}$$

we have that g is a $\mathcal{F} \otimes \mathcal{B}_d$- measurable function and l.s.c. with respect to y. So, by Proposition 5.1,

$$\begin{aligned} \liminf_{k\to\infty} \int_\Omega \Big(\int_{\mathbf{R}^d} \mathbf{1}_{F^c}(x, z) d\delta_{z_k(x)}(z)\Big) d\mu(x) &= \liminf_{k\to\infty} \int_\Omega g(x, z_k(x)) d\mu(x) \\ &\geq \int_\Omega \Big(\int_{\mathbf{R}^d} g(x, z) d\nu_x(z)\Big) dx \\ &= (\mu \otimes \nu_x)(F^c) \end{aligned}$$

i.e.

$$0 = \liminf_{k\to\infty} \mu \otimes \delta_{z_k(x)}(F^c) \geq (\mu \otimes \nu_x)(F^c).$$

Therefore

$$\mu \otimes \nu_x\text{-}\sup_{\Omega\times\mathbf{R}^d} f(x, z) \leq \mu \otimes \nu_x\text{-}\sup_{F} f(x, z) \leq \Lambda$$

which is a contradiction. □

The next theorem will be useful in the followings. For a proof, see [21]:

THEOREM **5.2** *(1) (Prohorov compactness with parameter) Let $(z_k)_k$ be a bounded sequence of $L^1(\Omega, \mathbf{R}^d)$ and let $(\tau_k)_k$ be the associated Young measures. Then there exist a subsequence $(\tau_{k_n})_n$ and a Young measure ν such that $(\tau_{k_n})_n$ narrowly converges to ν.*

(2) If $(z_k)_k$ converges to z weakly in $L^1(\Omega, \mathbf{R}^d)$, then, μ-a.e. the disintegration τ_x has a barycenter $bar(\tau_x) = \int_{\mathbf{R}^d} z d\tau_x(z) = z(x)$.

Finally, we give a representation theorem of $\overline{F}$ by using Young measures.

THEOREM **5.3** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where μ is a nonnegative, finite, Radon measure and Ω is a locally compact, metrizable and separable space. Assume that the hypotheses of Theorem 2.4 are satisfied. Then, for every $u \in L^\infty_d$ and for every $B \in \mathcal{F}$ it holds:*

$$\overline{F}(u, B) = \min\left\{\mu\text{-}\sup_{x\in B}\{\sigma_x\text{-}\sup_{z\in\mathbf{R}^d} f(x,z)\} \;:\; \sigma \in \mathcal{B}(u)\right\} \tag{5.40}$$

where

$$\mathcal{B}(u) = \left\{\sigma \;:\; \sigma \textit{ Young measure, } u(x) = \int_{\mathbf{R}^d} z d\sigma_x(z) \textit{ for } \mu\textit{-a.e.} x \in \Omega\right\} \tag{5.41}$$

PROOF. Let $u \in L^\infty_d$, $B \in \mathcal{F}$ and $\sigma \in \mathcal{B}(u)$. By applying Theorem 2.1 to the level convex function $f^{c\gamma}$, we have that

$$\begin{aligned}\mu\text{-}\sup_B f^{c\gamma}(x, u(x)) &= \mu\text{-}\sup_{x\in B} f^{c\gamma}(x, \int_{\mathbf{R}^d} z d\sigma_x(z)) \\ &\le \mu\text{-}\sup_{x\in B}\{\sigma_x\text{-}\sup_{z\in\mathbf{R}^d} f^{c\gamma}(x,z)\} \\ &\le \mu\text{-}\sup_{x\in B}\{\sigma_x\text{-}\sup_{z\in\mathbf{R}^d} f(x,z)\},\end{aligned}$$

which implies, thanks to Theorem 4.2,

$$\overline{F}(u, B) \le \inf\{\mu\text{-}\sup_{x\in B}\{\sigma_x\text{-}\sup_{z\in\mathbf{R}^d} f(x,z)\} \;:\; \sigma \in \mathcal{B}(u)\}. \tag{5.42}$$

Then, let $(u_k)_{k\in\mathbf{N}} \subset L^\infty_d$ such that

$$u_k \to u \text{ weakly* in } L^\infty_d(\Omega),$$

and

$$\overline{F}(u, B) = \liminf_{k\to\infty} F(u_k, B).$$

Let

$$\sigma_{k,x} := \delta_{u_k(x)}.$$

Thanks to Theorem 5.3, there exists a subsequence (that, without loss of generality, we denote again by $(u_k)_{k\in\mathbf{N}}$) and a Young measure $\sigma \in \mathcal{B}(u)$ such that $(u_k)_{k\in\mathbf{N}}$ generates σ. If we define

$$\mathcal{F}(\sigma, B) := \mu\text{-}\sup_{x\in B}\{\sigma_x\text{-}\sup_{z\in\mathbf{R}^d} f(x,z)\},$$

by applying Theorem 5.2, we obtain

$$\begin{aligned}\overline{F}(u,B) &= \liminf_{k\to\infty} F(u_k,B)\\ &= \liminf_{k\to\infty} \mu\text{-}\sup_{x\in B}\{\sigma_{k,x}\text{-}\sup_{z\in\mathbf{R}^d} f(x,z)\}\\ &= \liminf_{k\to\infty} \mathcal{F}(\sigma_k,B)\\ &\geq \mu\text{-}\sup_{x\in B}\{\sigma_x\text{-}\sup_{z\in\mathbf{R}^d} f(x,z)\}.\end{aligned}$$

Together with (5.42), this implies (5.40). □

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