Optimal measures for nonlinear cost functionals

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Abstract

We study an optimization problem for an elliptic PDE where the unknown is a measure. The state equation is a relaxed form of the classical Dirichlet problem and the functional to minimize is nonlinear. We prove a regularity result for the optimal measure, give a description of it and present some numerical examples.

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1 Introduction

We study in this paper a problem of optimal measure for a relaxed elliptic equation. Given a bounded domain Ω in $I\!R^N$, and two functions $f \in L^2(\Omega)$ and $j : \Omega, I\!R \mapsto I\!R$, we consider the following optimization problem

$$\inf_{\mu \in \mathcal{M}_0(\Omega)} \{J(\mu) = \int_\Omega j(x, u_\mu)dx + \alpha\mu(\Omega)\} \tag{1}$$

where $\mathcal{M}_0(\Omega)$ is the set of Borel measures absolutely continuous with respect the capacity of sets, α is a nonnegative parameter, and u_μ is the solution of the state equation

$$\begin{cases} -\Delta u + u\mu &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

In the particular case $j(x, s) = f(x)s$, Problem (1) is the minimization of the so called *elastic compliance* under a constraint of mass. It has been studied by Henrot and Mailllot in [6] who have updated a non expected property of regularity for the minimum. Instead of $\mathcal{M}_0(\Omega)$ as admissible set, they considered the set of measures with a density uniformly bounded by one. They proved that for α small enough, the solution is of "non-relaxed" type: it is a characteristic function. In [3], we treated the minimization of the *elastic compliance* in the whole set of measures $\mathcal{M}_0(\Omega)$. We also found that the minimum is reached for a regular measure, and that this measure can be expressed in an explicit way where the only unknown is its support. Precisely, the result was the following

Theorem 1.1 *For every nonnegative function $f \in L^2(\Omega)$, the optimal measure μ is absolutely continuous with respect to the Lesbegue measure and moreover there exists a set $K \subset \Omega$ such that*

$$\mu = \frac{f}{\sqrt{\alpha}}.\mathcal{L}^N \llcorner K.$$

In the present work, we investigate on properties of the solution of a generalized form of compliance problem. The cost functional is given through the function j supposed differentiable. As regard of the elastic compliance, Problem (1) is said non self adjoint since appears in the derivation of the functional J, the adjoint state defined as the weak solution of the following equation

$$\left\{ \begin{array}{rcll} -\Delta v + v\mu & = & j_s(x,u) & \text{in} \quad \Omega, \\ v & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{3}$$

The main result of this paper concerns the regularity of the optimal measure and its characterization. It is the following

Theorem 1.2 *Let f be a nonnegative function in $L^p(\Omega)$ for $p > N/2$. Then each solution of Problem (1) is absolutely continuous with respect the Lebesgue measure and verifies*

$$\mu = \Big(\frac{1}{2}\Big[\frac{f}{u} + \frac{j_s(x,u)u}{\alpha}\Big] + \Big|\frac{\nabla u}{u}\Big|^2\Big).\mathcal{L}\llcorner\{uv = \alpha\}.$$

Where u and v are respectively solution of (2) and (3) for the measure μ.

Compare to the self adjoint case, is that the *elastic compliance*, the difficulties comes essentially from the apparition of the adjoint state in the optimality conditions. To pass over it, we see each solution of Problem (1) as a solution of a self adjoint problem. The source of this annexe problem is not necessary a nonnegative element of $L^2(\Omega)$. The main part of the paper (section 4) is devoted to generalize Theorem 1.1 in order to treat this new problem. We point out that we have no result about uniqueness of the optimal measure.

This paper is organized as follows. Section 2 is devoted to the setting of the optimization problem, the assumptions on j and the notations. Section 3 deals with the existence of the solution and some necessary optimality conditions. In Section 4 we generalize the result concerning the *elastic compliance* problem on view of study Problem (1). The main result is proved Section 5. In Section 6, we do some remarks and present some numerical results.

2 Setting of the problem and notation

Let Ω be a bounded open set. The admissible set $\mathcal{M}_0(\Omega)$ is the class of all nonnegative Borel measures on Ω such that $\mu(B) = 0$ for every Borel subset B of Ω such that $\text{Cap}(B) = 0$, where the capacity of a subset A is defined by

$$\text{Cap}(A) = \inf\{\int_{I\!R^N} |\nabla u|^2 + u^2 dx : u \in \mathcal{U}_A\}.$$

where $\mathcal{U}_A$ is the set of all functions u of the Sobolev space $H^1(I\!R^N)$ such that $u \geq 1$ almost everywhere in a neighborhood of A.

If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subset E$ with $\text{Cap}(Z) = 0$, we say that $P(x)$ holds quasi-everywhere on E (shortly q.e on E). The expression almost everywhere (shortly a.e.) refers, as usual, to the Lebesgue measure.

The function f is called a source and is generally an element of $L^2(\Omega)$. A solution of the equation (2) is a function which belongs to the Hilbert space $X_\mu(\Omega) = H_0^1(\Omega) \cap L^2(\Omega, \mu)$ endowed with the norm

$$\Big(\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 d\mu\Big)^{1/2}.$$

and which solves the following variational problem

$$\int_\Omega \nabla u \nabla v dx + \int_\Omega uv d\mu = \int_\Omega f v dx \qquad \forall v \in X_\mu(\Omega). \tag{4}$$

It is well known that for all $\mu \in \mathcal{M}_0(\Omega)$ and all $f \in L^2(\Omega)$, there exists an unique u_μ solution of (2) (see for instance [2]).

We assume the following assumptions on the function j:

- j is a Caratheodory function which verified

$$|j(x,s)| \le a(x) + c|s|^r, \tag{5}$$

  where $a \in L^1(\Omega)$, $c \in \mathbb{R}$, $r > 0$ and $r < \frac{2n}{n-4}$ if $N > 4$.

- For all $x \in \Omega$, $j(x,.)$ belongs to $C^1([0,+\infty))$ with

$$|j_s(x,s)| \le a_1(x) + c|s| \tag{6}$$

  where $a_1 \in L^p(\Omega)$ for $p > N/2$.

The parameter α is a positive number. It plays the role of the Lagrangian in the minimization.

We denote by $\mathcal{L}$ the $N-$dimensional Lebesgue measure and by $B(x,r)$ the ball of radius r and centered at the point x. The average of a function u in the ball $B(x,r)$ is defined by

$$\bar{u}_r(x) = \frac{1}{\mathcal{L}(B(x,r)} \int_{B(x,r)} u(y)dy \tag{7}$$

We recall the classical result about Sobolev functions: if $u \in H^1(\Omega)$, then there exists a Borel set $E \subset \Omega$ such that $\mathrm{Cap}(E) = 0$ and for all $x \in \Omega \setminus E$, the following limit exists

$$\tilde{u}(x) = \lim_{r \to 0^+} \bar{u}_r(x);$$

the function $\tilde{u}$ is called the quasi-continuous representative of u. The following lemma is proved in paper [3].

Lemma 2.1 *For every $u \in H_0^1(\Omega)$ the average function $\bar{u}_r$ defined by (7) is continuously differentiable with respect to r in $\mathbb{R}_+^*$. Moreover, suppose that there exists $c \in \mathbb{R}$ such that $u \le c$ $\mathcal{L}-a.e.$; then q.e. in $\{\tilde{u} = c\}$*

$$\liminf_{r \to 0^+} \frac{1}{r}\frac{d\bar{u}_r}{dr}(x) \le 0. \tag{8}$$

Let $x \in \Omega$, we define the family of functions $(\varphi_\varepsilon)_{\varepsilon>0}$ by

$$\varphi_\varepsilon(y) = \begin{cases} 1 - \frac{|y-x|^2}{\varepsilon^2} & \text{if } |y-x| < \varepsilon \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

For all $\varepsilon > 0$, the function φ_ε belongs to $H_0^1(B(x,\varepsilon)) \cap H^2(B(x,\varepsilon))$ and, by Green formula, we have for all $u \in H_0^1(\Omega)$,

$$\frac{1}{w_N \varepsilon^N} \int_\Omega \nabla u \nabla \varphi_\varepsilon dx = -\frac{2}{\varepsilon} \frac{d\bar{u}_\varepsilon}{d\varepsilon}(x). \tag{10}$$

where $\bar{u}_\varepsilon$ is defined by (7). And if x is a Lebesgue point of a function $f \in L^1(\Omega)$, then

$$\lim_{\varepsilon \to 0} \frac{1}{w_N \varepsilon^N} \int_\Omega f \varphi_\varepsilon dx = C_N f(x). \tag{11}$$

where $C_N = 2/(N+2)$.

3 Existence and properties of the optimal measure

The formulation (1) previously stated leads to the following existence result of optimal measures in the set $\mathcal{M}_0(\Omega)$.

Theorem 3.1 *The problem (1) has at least one solution.*

Proof It comes directly from the assumption (5), the fact that the space $\mathcal{M}_0(\Omega)$ endowed with the topology of the γ-convergence is a compact metric space and the fact that the functional $J(\mu)$ is sequentially lower semi-continuous for the γ-convergence (see proof of Theorem 3.1 in [3]). □

The next lemma gives some necessary conditions of optimality for the solution of the problem of minimization (1).

Lemma 3.2 *Let μ be a solution of (1) and u and v the weak solution of respectively of (2) and (3) associated to the measure μ. Then*

$$uv \leq \alpha \;\; q.e. \quad and \quad uv = \alpha \;\; \mu - a.e.$$

Proof The proof is almost similar to the proof of the optimality conditions in [2], we add the linear term $\alpha\mu(\Omega)$ to the functional to derive. □

In order to treat the optimization problem, we begin by study the self adjoint case.

4 The self adjoint problem

In the whole section, we consider Problem (1) with $j(x,s) = f(x)s$. The adjoint state is equal to the solution u of the state equation (2) and the optimality conditions given by the Lemma 3.2 are written as the two following conditions

$$(C_1) \quad |u| \leq \sqrt{\alpha} \quad \text{q.e} \quad \text{in} \quad \Omega,$$

$$(C_2) \quad |u| = \sqrt{\alpha} \quad \mu\text{−a.e} \quad \text{in} \quad \Omega.$$

We denote those two statements (C_1) and (C_2) for convenience of the reader. We begin by generalize Theorem 1.1 to every functions in $L^2(\Omega)$.

Theorem 4.1 *For every $f \in L^2(\Omega)$, the measure μ solution of (1) is absolutely continuous with respect to the Lesbegue measure and there exists two disjoints subset K^+ and K^- of Ω such that:*

$$(\mathrm{D}_1) \qquad \mu = \tfrac{1}{\sqrt{\alpha}}\big(f^+.\mathcal{L}\llcorner K^+ + f^-.\mathcal{L}\llcorner K^-\big),$$

$$(\mathrm{D}_2) \qquad K^+ \subset \{u = \sqrt{\alpha}\}, \;\; K^- \subset \{u = -\sqrt{\alpha}\}.$$

Here u is the solution of (2) associated to the measure μ.

Remark 4.2 It is implicitly contained in the result that f is non positive in K^- and non-negative in K^+.

Remark 4.3 In the characterization of the optimal measure μ, the sets K^+ and K^- are defined up to a subset included in the complementary of the support of f. In the following we choose the most convenient representative.

Proof Let p be a nonnegative integer and $\varphi \in \mathcal{D}(\Omega)$ nonnegative. Let $v = u/\sqrt{\alpha}$, since $u \in X_\mu(\Omega)$ and $u \in L^\infty(\Omega)$, φv^p belongs to $X_\mu(\Omega)$. By taking φv^{2p+1} as test function in (4) we obtain that for every $p \geq 0$

$$\int_\Omega \nabla v \nabla(\varphi v^{2p+1})dx + \sqrt{\alpha}\int_\Omega \varphi d\mu = \int_\Omega f\varphi v^{2p+1}dx. \tag{12}$$

The first term at left of the equation (12) can be write as

$$\int_\Omega \nabla v \nabla \varphi v^{2p+1} dx = \frac{1}{2p+2}\int_\Omega \nabla(v^{2p+2})\nabla\varphi dx + (2p+1)\int_\Omega \varphi v^{2p}|\nabla v|^2 dx. \tag{13}$$

The sequence of function $(\frac{1}{p}v^p)_{p\in I\!N}$ weakly converges in $H_0^1(\Omega)$ to zero as p goes to $+\infty$ hence

$$\liminf_{p\to+\infty}\int_\Omega \nabla v\nabla(\varphi v^{2p+1})dx \geq 0. \tag{14}$$

At right of the equation (12), the sequence of function $(v^{2p+1})_{p\in I\!N}$ strongly converges in $L^2(\Omega)$ to the function $f\chi_{\{u=\sqrt{\alpha}\}} - f\chi_{\{u=-\sqrt{\alpha}\}}$ as p goes to $+\infty$, hence

$$\lim_{p\to+\infty}\int_\Omega f\varphi v^{2p+1}dx = \int_\Omega f\varphi(\chi_{\{u=\sqrt{\alpha}\}} - \chi_{\{u=-\sqrt{\alpha}\}})dx. \tag{15}$$

Passing to the limit in (12) by using the statements (14) and (15), we obtain that for all $\varphi \in \mathcal{D}(\Omega)$ nonnegative

$$\int_\Omega \varphi d\mu \le \frac{1}{\sqrt{\alpha}}\int_\Omega (f\chi_{\{u=\sqrt{\alpha}\}} - f\chi_{\{u=-\sqrt{\alpha}\}})\varphi dx. \tag{16}$$

Since μ is a nonnegative measure, the function $f\chi_{\{u=\sqrt{\alpha}\}} - f\chi_{\{u=-\sqrt{\alpha}\}}$ is also nonnegative and equal to $f^+\chi_{\{u=\sqrt{\alpha}\}} + f^-\chi_{\{u=-\sqrt{\alpha}\}}$. The inequality (16) implies that μ is absolutely continuous with respect to the Lebesgue measure. We finished the proof as in the paper [3]. Since μ is regular, u belongs to $H^2_{\text{loc}}(\Omega)$ and if we defined $K^+ = \{u = \sqrt{\alpha}\}$ and $K^- = \{u = -\sqrt{\alpha}\}$, the function u and the measure μ verify $u.\mu = f$ a.e in $K^+ \cup K^-$ which is nothing else than (D_1). $\square$

Now, we consider a function $f \in L^1(\Omega)$ and we give an extension to the previous result

Theorem 4.4 *For every $f \in L^1(\Omega)$, there exists an unique measure $\mu \in \mathcal{M}_0(\Omega)$ and an unique function $u \in X_\mu(\Omega)$ such that the couple (u, μ) satisfies the optimality conditions (C_1) and (C_2) and is solution of the following variational equation*

$$\int_\Omega \nabla u \nabla\varphi dx + \int_\Omega u\varphi d\mu = \int_\Omega f\varphi dx, \quad \forall \varphi \in X_\mu(\Omega)\cap L^\infty(\Omega) \tag{17}$$

Moreover there exists two disjoints subsets K^+ and K^- of Ω such that (u, μ, K^+, K^-) verifies the statements (D_1) and (D_2) of Theorem 4.1.

To prove this result we will need the following lemma

Lemma 4.5 *Suppose that Theorem 4.4 is verified for two functions f_1, f_2 such that $f_1 \le f_2$. Let (μ_i, u_i) be the couple associated to the function f_i $(i = 1, 2)$ and K_i^+, K_i^- such that $\mu_i = \frac{1}{\sqrt{\alpha}}\left(f_i^+.\mathcal{L}\llcorner K_i^+ + f_i^-.\mathcal{L}\llcorner K_i^-\right)$. Then*

- $u_1 \le u_2$,
- $\chi_{K_1^+} \le \chi_{K_2^+}$ *and* $\chi_{K_2^-} \le \chi_{K_1^-}$

Here we consider the representative of K_i^+ and K_i^- which are included in the support of f_i.

Proof Since $u_i \in L^\infty(\Omega)$, its quasi-continuous representative belongs to $X_{\mu_j}(\Omega)$ for $i \ne j$. By doing the difference of the variational equation (17) verifies by u_i for $i = 1, 2$, and taking $(u_2 - u_1)^-$ as test function, we obtain

$$-\int_\Omega |\nabla(u_2 - u_1)^-|^2 dx + \int_\Omega u_2(u_2 - u_1)^- d\mu_2 - \int_\Omega u_1(u_2 - u_1)^- d\mu_1 \ge 0 \tag{18}$$

Since $|u_2| = \sqrt{\alpha}\ \mu_2-$a.e and $|u_1| \le \sqrt{\alpha}$ q.e, we have $u_2(u_2-u_1)^- = -\sqrt{\alpha}(u_1+\sqrt{\alpha})\ \mu_2-$a.e, and the second term at left of the inequality (18) is non positive. In the same way, we have that $-u_1(u_2-u_1)^- = -\sqrt{\alpha}(\sqrt{\alpha}-u_2)\ \mu_1-$a.e and the last term at left is also non positive. The first statement is proved. Let x be a Lebesgue point for each functions f_i, μ_i, u_i and such that $x \in K_1^+ \cup K_2^-$ (here we identify the measure μ_i with its density). By taking φ_ε defined in (9) as test function in the variational equation verified by $u_2 - u_1$ and using (10), we obtain that

$$-\frac{2}{\varepsilon}\frac{d\overline{(u_2-u_1)}_\varepsilon}{d\varepsilon} + \int_\Omega u_2\varphi_\varepsilon d\mu_2 - \int_\Omega u_1\varphi_\varepsilon d\mu_2 = \int_\Omega (f_2-f_2)\varphi_\varepsilon ddx. \tag{19}$$

Where $\overline{(u_2-u_1)}_\varepsilon$ is the average function defined in (7). Using the first step of the proof and Lemma 2.1, we have

$$\limsup_{\varepsilon\to 0} \frac{2}{\varepsilon}\frac{d\overline{(u_2-u_1)}_\varepsilon}{d\varepsilon} \ge 0.$$

Passing to the limit in (19) and using (11), we obtain that

$$u_2(x)\mu_2(x) - u_1(x)\mu_1(x) \ge f_2(x) - f_1(x),$$

which is nothing else than $K_1^+ \subset K_2^+$ and $K_2^- \subset K_1^-$ in the sense a.e. □

Proof of Theorem 4.4

- **Uniqueness**

Consider two couples (μ_i, u_i) $(i = 1, 2)$, by using the same argument than in the proof of the Lemma 4.5 — in which for the first step we do not use the characterization (D_1) of the measure — we find that $u_1 = u_2$. Let u be a quasi-continuous representative of u_1, then by doing again the difference of the variational equations verified by u respectively for the measure μ_1 and μ_2, we obtain that for all $\varphi \in X_{\mu_1}(\Omega) \cap X_{\mu_2}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega u\varphi d\mu_1 = \int_\Omega u\varphi d\mu_2. \tag{20}$$

By taking $u\varphi$ with $\varphi \in \mathcal{D}(\Omega)$ as test function in (20) and using (C_1), we conclude to the uniqueness of the measure.

- **Existence**

Thanks to Theorem 4.1, Theorem 4.4 is proved for $f \in L^2(\Omega)$. Consider $f \in L^1(\Omega)$, in a first step, we suppose that the function f is lower bounded, and we generalize the result in a second step.

First step. Let $f_n \in L^\infty(\Omega)$ defined by $f_n = f \wedge n$. Let (μ_n, u_n) be the couple optimal measure and solution associated to f_n. Let K_n^+ and K_n^- included in the support of f_n such that

$$\mu_n = \frac{1}{\sqrt{\alpha}}(f_n^+.\mathcal{L}\llcorner K_n^+ + f_n^-.\mathcal{L}\llcorner K_n^-).$$

Thanks to Lemma 4.5, the sequence $(\chi_{K_n^+})_{n\in I\!N}$ is nondecreasing and so converges a.e and in $L^1(\Omega)$ to χ_{K^+} where $K^+ = \cup_n K_n^+$, and, in the same way, $(\chi_{K_n^-})_{n\in I\!N}$ converges a.e and in $L^1(\Omega)$ to χ_{K^-} where $K^- = \cap_n K_n^-$. Combining with the dominated convergence Theorem, we deduce that the sequence of density of measures μ_n converges a.e and in $L^1(\Omega)$ to the density of $\mu \in \mathcal{M}_0(\Omega)$ where μ is given by

$$\mu = \frac{1}{\sqrt{\alpha}}\big(f^+.\mathcal{L}\llcorner K^+ + f^-.\mathcal{L}\llcorner K^-\big). \tag{21}$$

Recall that u_n is solution of the variational equation

$$\int_\Omega \nabla u_n \nabla\varphi dx + \int_\Omega u_n\varphi d\mu_n = \int_\Omega \varphi f_n dx, \quad \forall\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega). \tag{22}$$

Here we can take the test function in $H_0^1(\Omega)\cap L^\infty(\Omega)$ since μ_n is regular. By taking u_n as test function in this equation, we obtain that

$$\int_\Omega |\nabla u_n|^2 dx \leq \sqrt{\alpha}\int_\Omega |f| dx. \tag{23}$$

Hence, up to a subsequence, $u_n \longrightarrow u$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. We deduce that $|u| \leq \sqrt{\alpha}$ q.e in Ω and by the dominated convergence Theorem,

$$\lim_{n\to\infty}\int_\Omega f_n u_n dx = \int_\Omega f u dx.$$

Passing to the limit in (22), the function u is the solution of the equation (17) for the function f and the measure μ given by (21). It only stays to prove that $u = \sqrt{\alpha}$ a.e in K^+, and $u = -\sqrt{\alpha}$ a.e in K^-. But since (u_n) is a nondecreasing sequence, for all $n \in I\!N$, $u = \sqrt{\alpha}$ a.e in K_n^+ and $u_n = \sqrt{\alpha}$ a.e in K^-, the result follows.

Second step. Consider now that f is not lower bounded. Let $(f_n)_{n\in I\!N}$ be the sequence of functions defined by $f_n = -n \vee f$ and (μ_n, u_n) the measure and the function associated to f_n given by the previous step. We follow the same argument than in the previous step: by Lemma 4.5, the sequence (K_n^+) is non increasing, and the sequence (K_n^-) is nondecreasing, we define μ as limit in $L^1(\Omega)$ of $(\mu_n)_{n\in I\!N}$, the measure μ belongs to $\mathcal{M}_0(\Omega)$ and verifies (21) with $K^+ = \cap_n K_n^+$ and $K^- = \cup_n K_n^-$. The rest of the proof is similar. □

Remark 4.6 (On the support of the optimal measure) In the case $f \in L^2(\Omega)$, the characterization of the support of the optimal measure is more precise. Instead of (D_2) we have, up to a subset included in the complementary of the support of f, $K^+ = \{u = \sqrt{\alpha}\}$ and $K^- = \{u = -\sqrt{\alpha}\}$ (see the proof of Theorem 4.1). In the case $f \in L^1(\Omega)$, we only proved the inclusion. In fact, thanks to the uniqueness, we also have the equalities.

Proposition 4.7 *In the Theorem (4.4), we can choose*

$$K^+ = \{u = \sqrt{\alpha}\} \quad \text{and} \quad K^- = \{u = -\sqrt{\alpha}\}.$$

Proof Let (u,μ) be the couple given by Theorem 4.4. Suppose for instance that K^+ is strictly contained in $\tilde{K}^+ = \{u = \sqrt{\alpha}\}$ — the proof will be similar for K^- — and let us define a new measure $\tilde{\mu}$ by

$$\tilde{\mu} = \frac{1}{\sqrt{\alpha}}(f^+.\mathcal{L}\llcorner\tilde{K}^+ + f^-.\mathcal{L}\llcorner K^-).$$

Then u is solution of

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega u\varphi d\tilde{\mu} = \int_\Omega (f + f^+\chi_{\tilde{K}^+\setminus K^+})\varphi dx, \quad \varphi \in X_{\tilde{\mu}}(\Omega) \cap L^\infty(\Omega).$$

Since $|u| = \sqrt{\alpha}\ \tilde{\mu}-$a.e, the couple $(u,\tilde{\mu})$ verified the optimality condition (C_1) and (C_2), and the variational equation (17) for the source $g = f + f^+\chi_{\tilde{K}^+\setminus K^+}$. By Theorem 4.4 there exists two subsets K_g^+ and K_g^- of Ω such that

$$\tilde{\mu} = \frac{1}{\sqrt{\alpha}}(g^+.\mathcal{L}\llcorner K_g^+ + g^-.\mathcal{L}\llcorner K_g^-).$$

Since $g^- = f^-$, it comes that $K_g^- = K^-$. On the other hand, since $g^+ = f^+ + f^+\chi_{\tilde{K}^+\setminus K^+}$, we have

$$(f^+ + f^+\chi_{\tilde{K}^+\setminus K^+})\chi_{K_g^+} = f^+\chi_{\tilde{K}^+}.$$

This last equality implies that modulo a set included in the complementary of the support of f, $\tilde{K}^+$ is included in K^+. It ends the proof. $\square$

We are now in position to treat the general case.

5 Proof of the main result

In this section, we consider the optimization problem (1) and prove Theorem 1.2. Since we have no smoothness assumption on the boundary of Ω, we need of the following lemma which is a result of regularity for the solution of the elliptic problems. It is only an extension of a well-known result for smooth domain.

Lemma 5.1 *Let Ω be a bounded open set of $\mathbb{R}^N$, μ a measure in $\mathcal{M}_0(\Omega)$ and f a nonnegative function in $L^p(\Omega)$ with $p > N/2$. Then the solutions of (2) and (3) belong to $L^\infty(\Omega)$.*

Proof of Lemma 5.1 Since $0 \leq u_\mu \leq u_\Omega$ a.e in Ω (here u_Ω is the solution of (2) for the null measure), it is sufficient to prove the lemma for u_Ω. Let B be a ball such that $\Omega \subset B$. Suppose that Ω has a sufficiently smooth boundary, says C^2, the function $u_B - u_\Omega$ is harmonic inside Ω and has a nonnegative trace on $\partial\Omega$, then $u_B \geq u_\Omega$ inside Ω. Since $u_B \in C_0(\overline{B})$, the result follows. Let us now consider any bounded open set $\Omega \subset B$. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of smooth domain such that $\Omega = \cup_n \Omega_n$, then u_Ω is the strong limit in $L^2(B)$ of $(u_{\Omega_n})_{n\in\mathbb{N}}$. Since $u_{\Omega_n} \leq u_B$, the result follows for the solution of the equation (2). Thanks to condition (6), the same argument is valid for the solution of (3) . $\square$

Proof of Theorem 1.2 Let φ be a function of $X_\mu(\Omega) \cap L^\infty(\Omega)$. Since u and v belongs to $L^\infty(\Omega)$ (Lemma 5.1), we can take $v\varphi$ and $u\varphi$ as test functions respectively in the equation (2) and (3). Then we obtain

$$\int_\Omega v\nabla u \nabla\varphi dx + \int_\Omega \varphi\nabla u\nabla v dx + \int_\Omega uv\varphi d\mu = \int_\Omega fv\varphi dx, \text{ and}$$
$$\int_\Omega u\nabla v \nabla\varphi dx + \int_\Omega \varphi\nabla v\nabla u dx + \int_\Omega vu\varphi d\mu = \int_\Omega j_s(x,u)u\varphi dx.$$

Let $w := uv$, then $w \in X_\mu(\Omega) \cap L^\infty(\Omega)$ and, by summing the both previous equations, w is solution of

$$\int_\Omega \nabla w\nabla\varphi dx + 2\int_\Omega w\varphi d\mu = \int_\Omega g\varphi dx, \;\; \forall\varphi \in X_\mu(\Omega)\cap L^\infty(\Omega). \tag{24}$$

Were $g = fv + j_s(x,u)u - 2\nabla v\nabla u$. It is clear that g belongs to $L^1(\Omega)$. Let w^+ be the positive part of w. By taking $w^+\varphi$ as test function in (24) we obtain that w^+ which belongs to $X_\mu(\Omega)\cap L^\infty(\Omega)$ is solution of

$$\int_\Omega \nabla(w^+)^2\nabla\varphi dx + 4\int_\Omega (w^+)^2\varphi d\mu = 2\int_\Omega gw^+\varphi dx - \int_\Omega |\nabla w^+|^2\varphi dx, \;\; \forall\varphi \in X_\mu(\Omega)\cap L^\infty(\Omega).$$

By the optimality conditions (Lemma 3.2) we have $(w^+)^2 \leq \alpha^2$ q.e in Ω and $(w^+)^2 = \alpha^2$ $\mu-$a.e. in Ω. Then applying Theorem 4.4 and Proposition 4.7, we obtain that

$$4\mu = \frac{1}{\alpha^2}(gw^+ - |\nabla w^+|^2)^+\chi_K.$$

where $K = \{w^+ = \alpha\}$. Since $w = \alpha$ a.e in K, we have that $|\nabla w^+| = 0$ and $\nabla u\nabla v = -|\nabla u/u|^2$ a.e in K. It ends the proof. □

6 Remarks and numerical examples

6.1 Control problems

Considering the control problem

$$\inf_{\mu\in\mathcal{M}_0(\Omega)}\{\int_\Omega |u_\mu - u_0|^2 dx + \alpha\mu(\Omega)\} \tag{25}$$

where u_0 is given, we know that for $\alpha = 0$ the solution can be not regular. It is sufficient to consider u_0 the solution of the state equation (2) for a singular measure μ_0. In this case, μ_0 is the solution of (25). There is no contradiction with Theorem 1.2 which says that if $\alpha > 0$, a regular measure is more economical. Indeed, the Lagrangian parameter strictly positive corresponds to a mass constraint not respected by μ_0. It is easy to see that the family of solution obtained for $\alpha > 0$ represents a continuous perturbation in the sense of the γ-convergence of the optimal measure at $\alpha = 0$.

6.2 Bounded measures

In this part, we solve Problem (1) under the conditions considered in [6]. We identify the measures with their density and we consider the minimization of $J(\mu)$ in the admissible set

$$\mathcal{M}_0(\Omega) = \{\mu \in L^\infty(\Omega), \|\mu\|_{L^\infty(\Omega)} \leq 1\}. \tag{26}$$

The notation $d\mu$ is understood as μdx. We find a generalization of the *elastic compliance* case.

Theorem 6.1 *Let $f \in L^p(\Omega)$ be a nonnegative function. Then Problem (1) has at least one solution and each solution verifies*

$$\mu = \mathcal{L}\llcorner\{uv > \alpha\} + \Big(\frac{1}{2}\Big[\frac{f}{u} + \frac{j_s(x,u)u}{\alpha}\Big] + \Big|\frac{\nabla u}{u}\Big|^2\Big).\mathcal{L}\llcorner\{uv = \alpha\}.$$

Where u and v are respectively solution of (2) and (3) for the measure μ.

Proof The existence of a solution comes directly from the lower semi-continuity of $J(\mu)$ under the *weak—** convergence in $L^\infty(\Omega)$. Let μ be a solution, by taking two different perturbations of the optimal measure: $\mu_\varepsilon = \mu - \varepsilon\varphi\mu$, and $\mu_\varepsilon = \mu + \varepsilon\varphi\mu(1-\mu)$ for ε small enough and $\varphi \in \mathcal{D}(\Omega)$ nonnegative and bounded by one, we obtain the following necessary optimality conditions:

$$uv \geq \alpha \ \ \mu-\text{a.e}, \quad \text{and} \ \ uv = \alpha \ \ \text{a.e in} \ \ \{0 < \mu < 1\}.$$

Following the same argument than in the proof of Theorem 1.2, the produce $w := uv$ is solution of (24). Since $\mu \in L^\infty(\Omega)$, $w \in H^2_{\text{loc}}(\Omega)$ and so $\Delta w = 0$ a.e in $\{0 < \mu < 1\}$ and the result follows. □

Remark 6.2 *Applying Theorem 6.1 to the elastic compliance problem, we see that if $f \geq \sqrt{\alpha}$ then the optimal measure is a characteristic function.*

6.3 Parameter α

Here we give some properties of the optimum with respect to the parameter α. For all $\alpha > 0$ and all measures $\mu \in \mathcal{M}_0(\Omega)$, we denote $G(\alpha)$ the minimum of the problem (1) and

$$E(\alpha, \mu) = \int_\Omega j(x, u_\mu) + \alpha\mu(\Omega).$$

we define the two following real functions

$$\bar{t}(\alpha) = \max\{\mu(\Omega) : G(\alpha) = E(\alpha,\mu)\}, \quad \text{and} \quad \underline{t}(\alpha) = \min\{\mu(\Omega) : G(\alpha) = E(\alpha,\mu)\}.$$

Those functions are well defined since the space $\mathcal{M}_0(\Omega)$ endowed with γ-convergence is compact.

Proposition 6.3 *The function $\overline{t}(\alpha)$ is right continuous and the function $\underline{t}(\alpha)$ is left continuous in $I\!R_+^*$ and the both are non increasing. Moreover for a.e α, $\overline{t}(\alpha) = \underline{t}(\alpha)$. The function G is continuous, increasing, concave and derivable almost everywhere in $I\!R_+$. The limit of G is equal to $\int_\Omega j(x, u_\Omega)$ as α goes to $+\infty$ (where u_Ω is the solution when $\mu = 0$). Moreover, for all $\alpha > 0$, ,we have*

$$G'_g(\alpha) = \overline{t}(\alpha) \quad \text{and} \quad G'_d(\alpha) = \underline{t}(\alpha).$$

Proof It is clear that for any $\alpha > 0$ and $\beta > \alpha$ we have

$$\overline{t}(\beta) \leq \frac{G(\beta) - G(\alpha)}{\beta - \alpha} \leq \underline{t}(\alpha). \tag{27}$$

The function G is then continuous and since G is nondecreasing, the functions $\underline{t}$ and $\overline{t}$ are non increasing, then continuous a.e and $\overline{t}(\alpha) = \underline{t}(\alpha)$ a.e in $I\!R^+$. Suppose that $(\alpha_n)_{n\in I\!N}$ is a decreasing sequence which converges to α and let μ_{α_n} be one optimal measure such that $\mu_{\alpha_n}(\Omega) = \overline{t}(\alpha_n)$. Up to a subsequence $(\mu_{\alpha_n})_{n\in I\!N}$ $\gamma-$converges to a measure μ which is optimal for the Lagrangian α. By the continuity of G, we have $\lim_{n\to\infty} \mu_{\alpha_n}(\Omega) = \mu(\Omega)$ then, with (27) it comes that $\lim_{n\to\infty} \overline{t}(\alpha_n) = \overline{t}(\alpha)$. By a similar argument, we have the left continuity of $\underline{t}(\alpha)$.

□

In the self adjoint case, since the solution of (1) is unique, the function G is derivable everywhere in $I\!R_+^*$ and

$$G'(\alpha) = \mu_\alpha(\Omega).$$

6.4 Numerical examples

We define an algorithm based on that developed by Bendsøe and Sigmund for homogenization problems (see the book [1] or the web site *www.topopt.dtu.dk/*). For the numerical computations, we consider all the measures through their density: the nonnegative function ρ. The functional to minimize is given by

$$J(\rho) = \int_\Omega j(x, u_\rho)dx + \alpha \int_\Omega \rho dx,$$

where u_ρ is the solution of

$$\begin{cases} -\Delta u + \rho u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The technique we use is a gradient descent method that we can accelerate by a Newton procedure. We follow the direction given by the derivative of J with respect to ρ

$$\frac{dJ(\rho)}{d\rho} = \int_\Omega [\alpha - u_\rho v_\rho]dx.$$

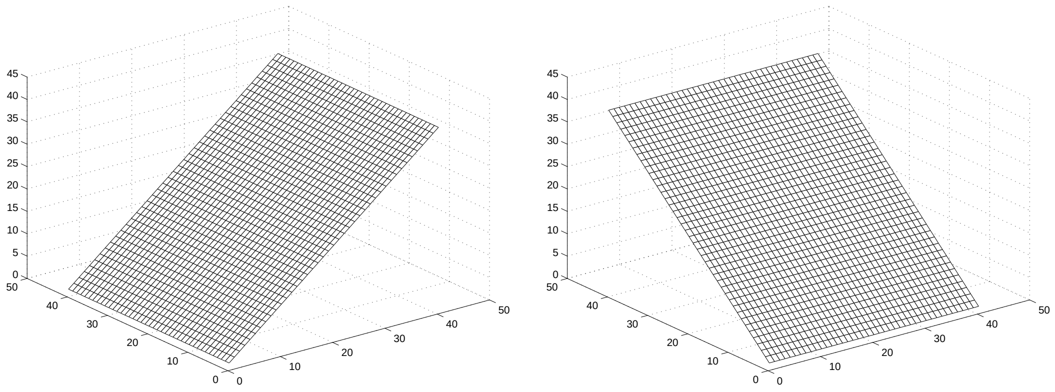

Figure 1: functions f and g

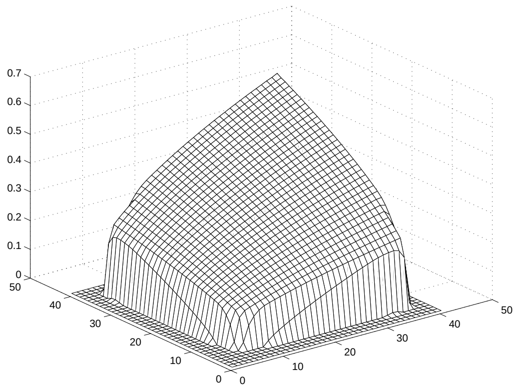

Figure 2: optimal measure

For the numerical examples, we consider the square as the design region Ω. The first resolution concerns the "linear" problem: $j(x, s) = g(x)s$. We consider a source f and a function g represented respectively at left and right in Figure 1. The solution of the optimization problem is then represented in Figure 2 while in Figure 3 we give at left the solution of the state equation and at right the produce of this solution with the adjoint state.

In the second example, we choose $f = 1$ and $j(x, s) = s\chi_\omega$ where ω is the right half part of the square Ω. The optimal measure is represented at left of Figure 4 while the solution of the state equation at right. The adjoint state and the produce of it with the solution are represented respectively at left and right of Figure 5.

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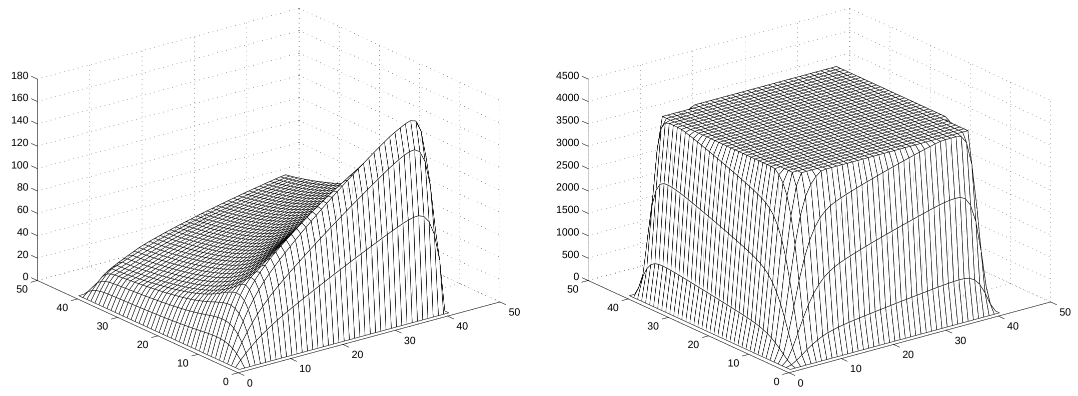

Figure 3: solution and produce with adjoint state

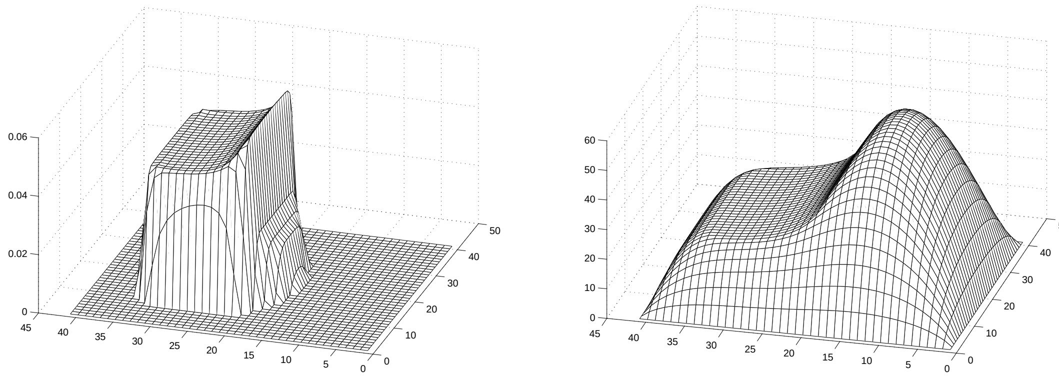

Figure 4: optimal measure and solution

References

[1] BENDSOE, M. P.; SIGMUND, O. *Topology optimization. Theory, methods and applications*, Springer-Verlag, Berlin, 2003. xiv+370 pp.

[2] BUTTAZZO, G.; DAL MASO, G. *Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions*, Appl. Math. Optim. **23** (1991), no. 1, 17–49.

[3] G. BUTTAZZO, N. VARCHON, H. ZOUBAIRI: *Optimal measures for elliptic problems.* Annali Mat. Pur. Appl. (2004), (accepted)

[4] DAL MASO, G.; MOSCO, U. *Wiener's criterion and* Γ*-convergence* , Appl. Math. Optim. **15** (1987), no. 1, 15–63.

[5] EVANS, L.C.; GARIEPY, R.F. *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, 1992.

[6] HENROT, A.; MAILLOT, H. *Optimization of the shape and the location of the actuators in an internal control problem* , Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **4** (2001), no. 3, 737–757.

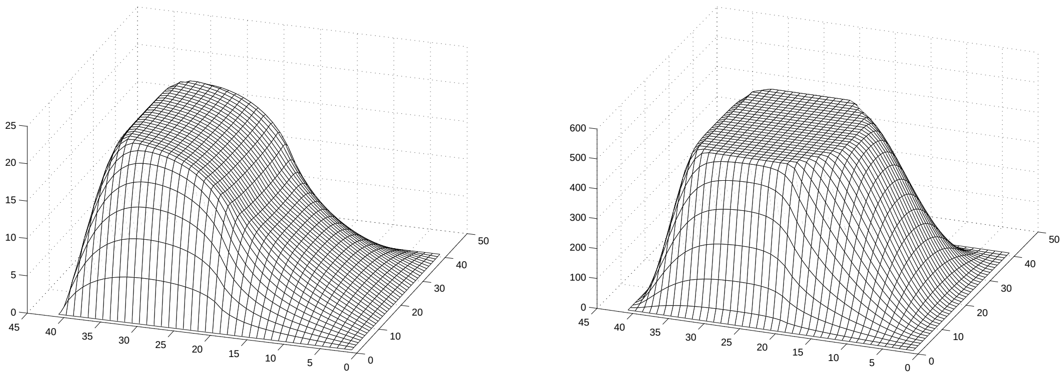

Figure 5: adjoint state and produce

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