BACH-FLAT GRADIENT STEADY RICCI SOLITONS

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Abstract. In this paper we prove that any n-dimensional ($n \geq 4$) complete Bach-flat gradient steady Ricci soliton with positive Ricci curvature is isometric to the Bryant soliton. We also show that a three-dimensional gradient steady Ricci soliton with divergence-free Bach tensor is either flat or isometric to the Bryant soliton. In particular, these results improve the corresponding classification theorems for complete locally conformally flat gradient steady Ricci solitons in [8, 10].

1. The results

A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *gradient Ricci soliton* if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho\, g_{ij}$$

for some constant ρ. For $\rho = 0$ the Ricci soliton is *steady*, for $\rho > 0$ it is *shrinking* and for $\rho < 0$ *expanding*. The function f is called a *potential function* of the gradient Ricci soliton. Clearly, when f is a constant a gradient Ricci soliton is simply a Einstein manifold. Thus Ricci solitons are natural extensions of Einstein metrics. Gradient Ricci solitons play an important role in Hamilton's Ricci flow as they correspond to self-similar solutions, and often arise as singularity models. Therefore it is important to classify gradient Ricci solitons or understand their geometry.

In this paper we shall focus our attention on gradient steady Ricci solitons (M^n, g_{ij}, f), which are possible Type II singularity models in the Ricci flow, satisfying the steady soliton equation

$$R_{ij} + \nabla_i \nabla_j f = 0. \tag{1.1}$$

It is now well-known that compact gradient steady solitons must be Ricci flat. In dimension $n = 2$, Hamilton [14] discovered the first example of a complete noncompact gradient steady soliton, defined on $\mathbb{R}^2$ and called the *cigar soliton*, where the metric is given explicitly by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

The cigar soliton has positive curvature and is asymptotic to a cylinder of finite circumference at infinity. Furthermore, Hamilton [14] showed that *the only complete steady soliton on a two-dimensional manifold with bounded (scalar) curvature R which assumes its maximum at an origin is, up to scaling, the cigar soliton*. For $n \geq 3$, Robert Bryant proved that *there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci soliton on $\mathbb{R}^n$* (see, e.g., Chow et al. [13] for details). The Bryant soliton has positive sectional curvature, linear curvature decay, and volume growth of geodesic balls of radius r on the

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order of $r^{(n+1)/2}$. In the Kähler case, the first author [5] constructed a complete gradient steady Kähler-Ricci soliton on $\mathbb{C}^m$, for $m \geq 2$, with positive sectional curvature and $U(m)$ symmetry. For additional examples, we refer the readers to the survey paper [6] by the first author and the references therein.

A well-known conjecture of Perelman [16], concerning gradient steady Ricci solitons, states that in dimension $n = 3$ the Bryant soliton is the only complete noncompact (κ-noncollapsed) gradient steady soliton with positive curvature. Despite some recent important progresses, it remains a big challenge to prove this conjecture of Perelman [1]. For $n \geq 4$, such a uniqueness result is not expected to hold, and it is desirable to find geometrically interesting conditions under which the uniqueness would hold. In [8], the first and third author proved that a complete noncompact n-dimensional ($n \geq 3$) locally conformally flat gradient steady Ricci soliton with positive sectional curvature is isometric to the Bryant soliton. Moreover, they showed that a complete noncompact n-dimensional locally conformally flat gradient steady Ricci soliton is either flat or isometric to the Bryant soliton. The same results for $n \geq 4$ were proved independently by the second and fourth author [10] by using different method. More recently, Brendle [3] (see also Proposition 5.2 below) showed that for an 3-dimensional gradient steady soliton (M^3, g_{ij}, f) if the scalar curvature R is positive and tends to zero at infinity, and that (M^3, g_{ij}, f) is asymptotic to the Bryant soliton in some suitable sense, then (M^3, g_{ij}, f) is locally conformally flat, hence isometric to the Bryant soliton. When $n = 4$, X. Chen and Y. Wang [12] have proved that any 4-dimensional complete half-conformally flat gradient steady Ricci soliton is either Ricci flat, or locally conformally flat (hence isometric to the Bryant Soliton by [8] and [10]).

In this paper, motivated by the very recent work [7] on Bach-flat shrinking Ricci solitons, we study complete Bach-flat steady Ricci solitons. A well-known fact is that if a n-dimensional manifold ($n \geq 4$) is either Einstein or locally conformally flat, then it is Bach-flat. In addition, in dimension $n = 4$, if a 4-manifold is *half-conformally flat* or *locally conformal to an Einstein* 4-manifold, then it is also Bach-flat.

Let us recall that on any n-dimensional manifold (M^n, g_{ij}) ($n \geq 4$) the Bach tensor, introduced by R. Bach [1] in early 1920s' to study conformal relativity, is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R_{kl}W_i{}^k{}_j{}^l. \tag{1.2}$$

Here W_{ikjl} is the Weyl tensor. In terms of the Cotton tensor

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R)\,,$$

we also have

$$B_{ij} = \frac{1}{n-2}(\nabla_k C_{kij} + R_{kl}W_i{}^k{}_j{}^l)\,. \tag{1.3}$$

Our first main result concerns the classification of Bach-flat gradient steady Ricci solitons:

Theorem 1.1. *For $n \geq 4$, let (M^n, g_{ij}, f) be a complete gradient steady Ricci soliton with positive Ricci curvature such that the scalar curvature R attains its maximum at some interior point. If in addition (M^n, g_{ij}, f) is Bach-flat, then it is isometric to the Bryant soliton up to a scaling factor.*

[1]Added in the proof: most recently S. Brendle [4] has proved this conjecture.

In dimension three we can prove a stronger result. To describe it, note that when $n = 3$, while the expression of B_{ij} in (1.2) is not well defined, the expression in (1.3) makes perfect sense, so we can use it to define the Bach tensor in 3-D as

$$B_{ij} = \nabla_k C_{kij}\,. \tag{1.4}$$

Theorem 1.2. *Let (M^3, g_{ij}, f) be a three-dimensional complete gradient Ricci solitons with divergence-free Bach tensor (i.e., $\operatorname{div} B = 0$). Then (M^3, g, f) is either Einstein or locally conformally flat.*

Using the 3-D classification of locally conformally flat gradient steady Ricci solitons (see [8]), we have:

Corollary 1.3. *A complete three-dimensional gradient steady Ricci soliton (M^3, g_{ij}, f) with divergence-free Bach tensor is either flat or isometric to the Bryant soliton (up to a scaling factor).*

Remark 1.4. The assumption of Bach-flat or divergence-free Bach is, at least a priori, weaker than that of locally conformally flat. Thus, Corollary 1.3 could be very helpful in proving Perelman's conjecture stated before.

Finally, in Section 5, we present some applications and discuss Bach-flat gradient expanding solitons.

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2. Background material

In this section, we recall some background material needed in the proof of our main theorems.

Recall that on any n-dimensional Riemannian manifold (M^n, g_{ij}) $(n \geq 3)$, the Weyl curvature tensor is given by

$$\begin{aligned} W_{ijkl} =& R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}), \end{aligned}$$

and the Cotton tensor by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R).$$

In terms of the Schouten tensor

$$A_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}, \tag{2.1}$$

we have

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}A_{jl} - g_{il}A_{jk} - g_{jk}A_{il} + g_{jl}A_{ik}), \tag{2.2}$$

and

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}. \tag{2.3}$$

It is well known that, for $n = 3$, W_{ijkl} vanishes identically, while $C_{ijk} = 0$ if and only if (M^3, g_{ij}) is locally conformally flat; for $n \geq 4$, $W_{ijkl} = 0$ if and only if (M^n, g_{ij}) is locally conformally flat. Moreover, for $n \geq 4$, the Cotton tensor C_{ijk} is, up to a constant factor, the divergence of the Weyl tensor:

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}. \tag{2.4}$$

We remark that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices:

$$C_{ijk} = -C_{jik} \quad \text{and} \quad g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0\,. \tag{2.5}$$

Moreover, for $n \geq 4$, the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R_{kl}W_i{}^k{}_j{}^l.$$

By (2.4), we have an equivalent expression of the Bach tensor:

$$B_{ij} = \frac{1}{n-2}(\nabla_k C_{kij} + R_{kl}W_i{}^k{}_j{}^l). \tag{2.6}$$

Next we recall some basic facts about complete gradient steady Ricci solitons satisfying Eq. (1.1) .

Lemma 2.1. (Hamilton [15]**)** *Let (M^n, g_{ij}, f) be a complete gradient steady Ricci soliton satisfying Eq. (1.1). Then we have*

$$\nabla_i R = 2R_{ij}\nabla_j f, \tag{2.7}$$

and

$$R + |\nabla f|^2 = C_0$$

for some constant C_0. Here R denotes the scalar curvature.

Lemma 2.2. *Let (M^n, g_{ij}, f) be a complete gradient steady soliton. Then it has nonnegative scalar curvature $R \geq 0$.*

Lemma 2.2 is a special case of a more general result of B.-L. Chen [11] which states that $R \geq 0$ for any ancient solution to the Ricci flow.

Note that, by Lemma 2.2, the constant C_0 in Lemma 2.1 must be positive for any non-trivial gradient steady soliton. Hence, by scaling the metric g, we can normalize it to be one so that

$$R + |\nabla f|^2 = 1. \tag{2.8}$$

Lemma 2.3. (Cao-Chen [8]**)** *Let (M^n, g_{ij}, f) be a complete noncompact gradient steady soliton with positive Ricci curvature $Rc > 0$. Assume the scalar curvature R attains its maximum at some origin O. Then, there exist some constants $0 < c_1 \leq 1$ and $c_2 > 0$ such that the potential function f satisfies the estimates*

$$c_1 r(x) - c_2 \leq -f(x) \leq r(x) + |f(O)|, \tag{2.9}$$

where $r(x) = d(O, x)$ is the distance function from O. In particular, f is a strictly concave exhaustion function achieving its maximum at the only critical point O, and the underlying manifold M^n is diffeomorphic to $\mathbb{R}^n$.

Finally, in the spirit of [7], we recall the covariant 3-tensor D_{ijk},

$$\begin{aligned} D_{ijk} =& \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) \\ &+ \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f - g_{jk}\nabla_i f)\,, \end{aligned}$$

which was first introduced in [8] and played the key role in classifying locally conformally flat gradient steady Ricci solitons [8] and Bach-flat gradient shrinking Ricci solitons [7]. Note that, D_{ijk} has the same symmetry properties as the Cotton tensor:

$$D_{ijk} = -D_{jik} \quad \text{and} \quad g^{ij}D_{ijk} = g^{ik}D_{ijk} = 0\,. \tag{2.10}$$

Lemma 2.4. (Cao-Chen [8, 7]**)** *Let (M^n, g_{ij}, f) $(n \geq 3)$ be a complete gradient steady soliton. Then D_{ijk} is related to the Cotton tensor C_{ijk} and the Weyl tensor W_{ijkl} by*

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla_l f\,. \tag{2.11}$$

On the other hand, for any gradient Ricci soliton, it turns out that the Bach tensor B_{ij} can be expressed in terms of D_{ijk} and the Cotton tensor C_{ijk} (for the proof see [7]):

$$B_{ij} = -\frac{1}{n-2}(\nabla_k D_{ikj} + \frac{n-3}{n-2}C_{jli}\nabla_l f)\,. \tag{2.12}$$

Moreover, we recall that the norm of D_{ijk} is linked to the geometry of level surfaces of the potential function f by the following:

Lemma 2.5. (Cao-Chen [8, 7]**)** *Let (M^n, g_{ij}, f) $(n \geq 3)$ be an n-dimensional gradient steady Ricci soliton. Then, at any point $p \in M^n$ where $\nabla f(p) \neq 0$, we have*

$$|D_{ijk}|^2 = \frac{2|\nabla f|^4}{(n-2)^2}|h_{ab} - \frac{H}{n-1}g_{ab}|^2 + \frac{1}{2(n-1)(n-2)}|\nabla_a R|^2 \tag{2.13}$$

where h_{ab} and H are the second fundamental form and the mean curvature of the level surface $\Sigma = \{f = f(p)\}$, and g_{ab} is the induced metric on Σ.

Thus, the vanishing of D_{ijk} implies the umbilicity of the level surfaces of the potential function as well as the constancy of the scalar curvature on them (see also Proposition 3.2 below). For further details on the tensor D_{ijk} we refer the interested reader to [7, Section 3].

3. Proof of Theorem 1.1

As in [7], the first step in proving Theorem 1.1 is to show that, for noncompact steady gradient Ricci solitons with positive Ricci curvature, the Bach-flatness implies the vanishing of the 3-tensor D_{ijk}.

Lemma 3.1. *Let (M^n, g_{ij}, f) $(n \geq 3)$ be a complete Bach-flat gradient steady Ricci soliton with positive Ricci curvature such that the scalar curvature R attains its maximum at some interior point. Then, $D_{ijk} = 0$.*

Proof. Since (M^n, g_{ij}, f) $(n \geq 4)$ has positive Ricci curvature and that the scalar curvature R attains its maximum at some interior point O, by Lemma 2.3, there exist constants $c_1, c_2 > 0$ such that

$$f(x) \leq -c_1 r(x) + c_2 \,, \tag{3.1}$$

where $r(x)$ is the distance to the origin O. Moreover, since $R_{ij} > 0$, from the well-known Bishop volume comparison theorem we know that (M^n, g_{ij}, f) has at most Euclidean volume growth, i.e., there exists a positive constant $C > 0$, such that

$$Vol(B_s(O)) \leq C\, s^n \,, \tag{3.2}$$

for any geodesic ball $B_s(O)$. By the definition of D_{ijk} and using the identities (2.12), (2.10) and (2.5), it follows from the same argument as in [7] that

$$\begin{aligned}\int_{B_s(O)} B_{ij}\nabla_i f\nabla_j f e^f \, dV_g &= -\frac{1}{n-2}\int_{B_s(O)} \nabla_k D_{ikj}\nabla_i f\nabla_j f e^f \, dV_g \\ &= -\frac{1}{2}\int_{B_s(O)} |D_{ijk}|^2 e^f \, dV_g - \frac{1}{n-2}\int_{\partial B_s(O)} D_{ikj}\nabla_i f\nabla_j f e^f \nu_k \, d\sigma \,,\end{aligned}$$

where ν denotes the outward unit normal to $\partial B_s(O)$. Again, from the definition of D_{ijk}, it is easy to check that, for sufficiently large s, we have

$$\begin{aligned}\Big|\int_{\partial B_s(O)} D_{ikj}\nabla_i f\nabla_j f e^f \nu_k \, d\sigma\Big| &\leq C\int_{\partial B_s(O)} (|R_{ij}| + |R|)|\nabla f|^3 e^f \, d\sigma \\ &\leq 2C\int_{\partial B_s(O)} e^f \, d\sigma \,,\end{aligned}$$

where we have used identity (2.8) and the fact that $|R_{ij}| \leq R$ (since g has positive Ricci curvature). By letting $s \to +\infty$ and using (3.1)-(3.2), we obtain

$$0 = \int_M B_{ij}\nabla_i f\nabla_j f e^f \, dV_g = -\frac{1}{2}\int_M |D_{ijk}|^2 e^f \, dV_g \,,$$

implying $D_{ijk} = 0$.

□

By Lemma 2.5, the vanishing of the tensor D_{ijk} implies many rigidity properties about the geometry of the level surfaces of the potential function f:

Proposition 3.2 (Proposition 3.2 in [7])**.** *Let (M^n, g, f) $(n \geq 3)$ be any complete gradient Ricci soliton with $D_{ijk} = 0$, and let c be a regular value of f and $\Sigma_c = \{f = c\}$ be the level surface of f. Set $e_1 = \nabla f/|\nabla f|$ and pick any orthonormal frame $e_2, \cdots, e_n$ tangent to the level surface Σ_c, Then:*

(a) $|\nabla f|^2$ and the scalar curvature R of (M^n, g_{ij}, f) are constant on Σ_c;

(b) $R_{1a} = 0$ for any $a \geq 2$, hence $e_1 = \nabla f/|\nabla f|$ is an eigenvector of Rc;

(c) the second fundamental form h_{ab} of Σ_c is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$;

(d) the mean curvature H is constant on Σ_c;

(e) on Σ_c, the Ricci tensor of (M^n, g_{ij}, f) either has a unique eigenvalue λ, or has two distinct eigenvalues λ and μ of multiplicity 1 and $n-1$ respectively. In either case, $e_1 = \nabla f/|\nabla f|$ is an eigenvector of λ. Both λ and μ are constant on Σ_c.

Now we are in the position to complete the proof of Theorem 1.1:

Proof of Theorem 1.1. Let (M^n, g, f), $n \geq 4$, be a complete Bach-flat gradient steady Ricci solitons with positive Ricci curvature such that the scalar curvature R attains its maximum at some interior point $O \in M$. Then, by Lemma 2.3 we know that f is proper, strictly concave, has a unique critical point at O, and that M^n is diffeomorphic to $\mathbb{R}^n$. On the other hand, by Lemma 3.1, we have that $D_{ijk} = 0$. Therefore for $n = 4$, from [7, Theorem 1.4] and the assumption of positive Ricci curvature, we conclude that (M^4, g_{ij}, f) is isometric to the Bryant soliton up to a scaling factor.

From now on let us consider $n \geq 5$. First of all, on $M \setminus \{O\}$, the soliton metric g_{ij} can be expressed as

$$ds^2 = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta^a d\theta^b,$$

where $(\theta^2, \cdots, \theta^n)$ is any local coordinates system on the lever surface $\Sigma = \{f = f(p)\}$ at $p \in M \setminus \{O\}$. Note that, since $D_{ijk} = 0$, $|\nabla f|^2$ depends only on f by Proposition 3.2 (a). Hence, by a suitable change of variable, we can further express g_{ij} as

$$ds^2 = dr^2 + g_{ab}(r, \theta) d\theta^a d\theta^b \ , \quad 0 < r < \infty \, .$$

Here $r(x)$ is the distance function from O. We remark that, by Lemma 2.3, $|f|(x)$ is proportional to $r(x)$.

Claim 1: For $r > 0$, the induced metric $\bar{g}_{\Sigma_r} = g_{ab}(r, \theta) d\theta^a d\theta^b$ on each level surface Σ_r is Einstein.

Indeed, we have the following more general fact:

Lemma 3.3. *Let (M^n, g, f) $(n \geq 4)$ be a complete gradient Ricci soliton with $D_{ijk} = 0$. Then each regular level surface Σ, with the induced metric $\bar{g}_\Sigma$, is an Einstein manifold.*

Proof. Let $\{e_1, e_2, \cdots, e_n\}$ be any orthonormal frame, with $e_1 = \nabla f / |\nabla f|$ and $e_2, \cdots, e_n$ tangent to Σ. Then, by the Gauss equation and Proposition 3.2 (c), the sectional curvatures of (Σ, g_{ab}) are given by

$$R^{\Sigma}_{abab} = R_{abab} + h_{aa}h_{bb} - h^2_{ab} = R_{abab} + \frac{H^2}{(n-1)^2} \, .$$

Thus, the Ricci curvatures of (Σ, g_{ab}) are

$$R^{\Sigma}_{aa} = R_{aa} - R_{1a1a} + \frac{H^2}{n-1}.$$

On the other hand, by Theorem 5.2 (b) in [7], we know that $W_{1a1a} = 0$. Thus,

$$\begin{aligned} R_{1a1a} = & \frac{1}{n-2}(R_{aa} + R_{11}) - \frac{R}{(n-1)(n-2)} \\ = & - R_{aa} + \frac{R}{n-1} \, . \end{aligned}$$

Hence, it follows that

$$R^{\Sigma}_{aa} = 2R_{aa} + \frac{H^2 - R}{n-1} \, .$$

But, by Proposition 3.2, R, H and $\mu = R_{aa}$ are constant along Σ. This proves that (Σ, g_{ab}) has constant Ricci curvature.

□

Claim 2: On $M \setminus \{O\}$, the metric g takes the form of a warped product metric:

$$ds^2 = dr^2 + w(r)^2 \bar{g}_E\,, \quad r \in (0, +\infty)\,, \tag{3.3}$$

where w is some nonnegative smooth function on M^n vanishing only at O, and $\bar{g}_E = \bar{g}_{\Sigma_1}$ is the Einstein metric defined on the level surface Σ_1.

Indeed, by identity (4.6) in [7] and Propositon 3.2, we have

$$\frac{\partial}{\partial r} g_{ab} = -2h_{ab} = \phi(r) g_{ab}\ ,$$

where $\phi(r) = -2H(r)/(n-1)$. Thus, it follows easily that

$$g_{ab}(r, \theta) = e^{\Phi(r)} g_{ab}(1, \theta),$$

where

$$\Phi(r) = \int_1^r \phi(r)\, dr.$$

This proves Claim 2.

By scaling, we can assume that

$$\mathrm{Ric}_{\bar{g}_E} = (n-2)k\, \bar{g}_E, \quad \text{with} \quad k = -1, 0, 1. \tag{3.4}$$

We shall see below that in fact $k = 1$, as we expected.

Claim 3: We have

$$\lim_{r \to 0} \frac{w(r)}{r} = 1\,.$$

Clearly, $w(r) \to 0$ as $r \to 0$. On the other hand, on $M \setminus \{O\}$, the Ricci tensor and the scalar curvature of the metric g in (3.3) take the form (see [2, Proposition 9.106])

$$\mathrm{Ric}_g = -(n-1)\frac{w''}{w} dr \otimes dr + \big((n-2)(k - (w')^2) - w\, w''\big)\, \bar{g}_E\,,$$

and

$$R_g = -2(n-1)\frac{w''}{w} + \frac{(n-1)(n-2)}{w^2}\big(k - (w')^2\big)$$

respectively. Here we have used the Claim 1 and the normalization (3.4).

From the expression of the Ricci tensor above and the fact that $|Rc| \leq 1$ on M^n, it is immediate to see that w''/w must be bounded as $r \to 0$. Hence, from the above scalar curvature expression, it is easy to deduce the claim. In particular, we can conclude that the Einstein constant $k = 1$ for the metric $\bar{g}_E$.

Claim 4: $\bar{g}_E$ is equal to the standard round metric $\bar{g}_{\mathbb{S}^{n-1}}$ on the unit sphere $\mathbb{S}^{n-1}$.

This essentially follows from the previous claims and the elementary fact that infinitesimally the metric g is approximately Euclidean near O. In fact, the standard expansion of the metric g around O, written in any normal coordinates $(x^1, \cdots, x^n)$, gives

$$\begin{aligned} g &= (\delta_{ij} + \sigma_{ij}(x))\, dx^i \otimes dx^j \\ &= g_{\mathbb{R}^n} + \sigma_{ij}\, dx^i \otimes dx^j \,, \end{aligned}$$

where $\sigma_{ij} = \mathcal{O}(|x|^2)$. To pass to polar coordinates, we write $x^i = r\phi^i(\theta^1, \ldots, \theta^{n-1)})$, with $r \in (0, +\infty)$ and $(\theta^1, \ldots, \theta^{n-1})$ being local coordinates on $\mathbb{S}^{n-1}$. Notice that $|\phi^1|^2 + \cdots + |\phi^n|^2 = 1$ and $|x| = r$. Thus, one has

$$\begin{aligned} g &= (1 + \sigma_{ij}\phi^i\phi^j) dr \otimes dr + r\, \sigma_{ij} \frac{\partial \phi^i}{\partial \theta^\alpha} \phi^j dr \otimes d\theta^\alpha + r\, \sigma_{ij} \frac{\partial \phi^j}{\partial \theta^\alpha} \phi^i d\theta^\alpha \otimes dr + \\ &\quad + \big(r^2 \bar{g}^{\mathbb{S}^{n-1}}_{\alpha\beta} + r^2 \sigma_{ij} \frac{\partial \phi^i}{\partial \theta^\alpha} \frac{\partial \phi^j}{\partial \theta^\beta} \big)\, d\theta^\alpha \otimes d\theta^\beta \,, \end{aligned}$$

with $\sigma_{ij} = \mathcal{O}(r^2)$. Comparing with (3.3), we see that $\sigma_{ij}\phi^j = 0$ and

$$w^2(r)\bar{g}_E \,=\, r^2 \bar{g}_{\mathbb{S}^{n-1}} + r^2 \sigma_{ij} \frac{\partial \phi^i}{\partial \theta^\alpha} \frac{\partial \phi^j}{\partial \theta^\beta}\, d\theta^\alpha \otimes d\theta^\beta \,, \quad r \in (0, +\infty) \,.$$

Now using the fact that $\sigma_{ij} = \mathcal{O}(r^2)$ and Claim 3, and taking the limit as $r \to 0$, we obtain

$$\bar{g}_E \,=\, \bar{g}_{\mathbb{S}^{n-1}} \,.$$

Therefore, on $M \setminus \{O\}$, we have

$$ds^2 = dr^2 + w(r)^2 \bar{g}_{\mathbb{S}^{n-1}} \,, \quad r \in (0, +\infty) \,,$$

proving that the soliton metric g is rotationally symmetric. Therefore, it follows that (M^n, g, f) is the Bryant soliton, because we know that M^n is diffeomorphic to $\mathbb{R}^n$ and the Bryant soliton is the only non-flat rotationally symmetric gradient steady soliton on $\mathbb{R}^n$ up to scaling. This completes the proof of Theorem 1.1.

□

4. Proof of Theorem 1.2

In the special case $n = 3$, we can show that divergence-free Bach tensor implies the vanishing of the Cotton tensor for all gradient Ricci solitons by a pointwise argument, which allows us to remove the assumptions on the positivity of the Ricci curvature and the scalar curvature achieving its interior maximum.

Proof of Theorem 1.2. Let (M^3, g, f) be a three-dimensional complete gradient Ricci soliton with divergence-free Bach tensor. We recall that in dimension three we have defined the Bach tensor as

$$B_{ij} = \nabla_k C_{kij} \,. \tag{4.1}$$

We claim that

$$\nabla_j B_{ij} = -C_{ijk} R_{jk} \,. \tag{4.2}$$

Indeed, in terms of the Schouten tensor

$$A_{ij} = R_{ij} - \frac{R}{4} g_{ij} \,,$$

and the Cotton tensor

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik} \,,$$

we have

$$B_{ij} = \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) \,.$$

Hence

$$\begin{aligned}\nabla_i B_{ij} =&\nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj})\\ =&(\nabla_i \nabla_k - \nabla_k \nabla_i)\nabla_k A_{ij}\\ =&-R_{il}\nabla_l A_{ij} + R_{kl}\nabla_k A_{lj} + R_{ikjl}\nabla_k A_{il}\\ =&R_{ikjl}\nabla_k A_{il}\,.\end{aligned}$$

On the other hand, since the Weyl tensor $W = 0$ in dimension three, (2.2) becomes

$$R_{ijkl} = g_{ik}A_{jl} - g_{il}A_{jk} - g_{jk}A_{il} + g_{jl}A_{ik}\,.$$

Therefore,

$$\nabla_i B_{ij} = (A_{jk}g_{il}C_{lki} + A_{ik}C_{kji}) = -R_{ki}C_{jki}\,,$$

proving the claim.

Now assume (M^3, g, f) is any three-dimensional gradient Ricci soliton. Recall that, for $n = 3$, we have

$$\begin{aligned}C_{ijk} =&D_{ijk}\\ =&R_{jk}\nabla_i f - R_{ik}\nabla_j f + \frac{1}{4}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) - \frac{R}{2}(g_{jk}\nabla_i f - g_{ik}\nabla_j f)\,.\end{aligned}$$

Thus, using (4.2) and (2.5), we get

$$\operatorname{div}(B)\cdot\nabla f = -C_{ijk}R_{jk}\nabla_i f = -\frac{1}{2}|C_{ijk}|^2.$$

Therefore, $\operatorname{div}(B) = 0$ implies the Cotton tensor $C_{ijk} = 0$, which is equivalent to that (M^3, g, f) is locally conformally flat.

□

Consequently, by combining Theorem 1.2 and the classification theorem in [8] for three-dimensional complete locally conformally flat gradient steady Ricci solitons, we have

Corollary 4.1. *Let (M^3, g, f) be a complete gradient steady Ricci soliton with divergence-free Bach tensor, then it is either flat or isometric to the Bryant soliton.*

Remark 4.2. For $n \geq 4$, it is known among experts that the divergence of the Bach tensor is given by

$$\nabla_j B_{ij} = \frac{n-4}{(n-2)^2}C_{ijk}R_{jk}\,.$$

5. Further remarks

It was proved in [7, Theorem 1.4] that any 4-dimensional complete gradient steady Ricci soliton with $D_{ijk} = 0$ is either Ricci flat, or locally conformally flat but non-flat (hence isometric to the Bryant soliton by [8] and [10]). In the proof of Theorem 1.1, we have actually shown the following

Proposition 5.1. *Let (M^n, g_{ij}, f), $n \geq 4$, be a complete gradient steady Ricci soliton with $D_{ijk} = 0$. If, in addition, the Ricci curvature is positive and the scalar curvature R attains its maximum at some interior point, then (M^n, g_{ij}, f) is isometric to the Bryant soliton up to a scaling factor.*

On the other hand, Brendle [3] proved the following result[2]:

Proposition 5.2 (Brendle [3]). *Let (M^n, g, f) $(n \geq 3)$ be a n-dimensional gradient steady Ricci soliton. Suppose that the scalar curvature R of (M^n, g) is positive and approaches zero at infinity. Denote by $\psi : (0,1) \to \mathbb{R}$ the smooth function such that the vector field*

$$X =: \nabla R + \psi(R)\nabla f = 0$$

on the Bryant soliton, and define $u : (0,1) \to \mathbb{R}$ by

$$u(s) = \log \psi(s) + \frac{1}{n-1}\int_{1/2}^{s} \left(\frac{n}{1-t} - \frac{n-1-(n-3)t}{(1-t)\psi(t)}\right)dt.$$

Moreover, assume that there exists an exhaustion of M^n by bounded domains Ω_l such that

$$\lim_{l\to\infty}\int_{\partial\Omega_l} e^{u(R)} < \nabla R + \psi(R)\nabla f, \nu >= 0. \tag{5.1}$$

Then $X = 0$ and $D_{ijk} = 0$. In particular, for $n = 3$, (M^3, g, f) is isometric to the Bryant soliton.

As an immediate consequence of Proposition 5.1 and Proposition 5.2, we obtain

Corollary 5.3. *Let (M^n, g_{ij}, f) $(n \geq 4)$ be a complete gradient steady Ricci soliton with positive Ricci curvature such that the scalar curvature R approaches zero at infinity. Moreover, assume that condition (5.1) in Proposition 5.2 is satisfied for some exhaustion of M^n by bounded domains Ω_l. Then (M^n, g, f) is isometric to the Bryant soliton.*

Remark 5.4. By Lemma 2.3, f is an exhaustion function on M^n.

Finally, the techniques used in the proof of Theorem 1.1 can be easily adapted to the case of complete gradient expanding Ricci solitons with nonnegative Ricci curvature which are solutions of the equation

$$R_{ij} + \nabla_i\nabla_j f = -\tfrac{1}{2}\, g_{ij}\,. \tag{5.2}$$

We also normalize the potential function f, up to an additive constant, by

$$R + |\nabla f|^2 + f = 0, \tag{5.3}$$

which is a well-known identity for expanding Ricci solitons (see [15]).

The need ingredient is the following lemma, which should be known to experts in the field:

Lemma 5.5. *Let (M^n, g_{ij}, f) $(n \geq 3)$ be a complete noncompact gradient expanding soliton with nonnegative Ricci curvature $Rc \geq 0$. Then, there exist some constants $c_1 > 0$ and $c_2 > 0$ such that the potential function f satisfies the estimates*

$$\frac{1}{4}\big(r(x) - c_1\big)^2 - c_2 \leq -f(x) \leq \frac{1}{4}\big(r(x) + 2\sqrt{-f(O)}\,\big)^2, \tag{5.4}$$

where $r(x)$ is the distance function from any fixed base point in M^n. In particular, f is a strictly concave exhaustion function achieving its maximum at some interior point O, which we take as the base point, and the underlying manifold M^n is diffeomorphic to $\mathbb{R}^n$.

[2]Although Brendle only stated this result for $n = 3$ in [3], the same argument, as shown by him in the preprint arXiv:1010.3684v1, works for all dimensions $n \geq 4$.

Proof. The upper bound follows from (5.3) and the assumption of $R \geq 0$ which together imply $|\nabla(-f)|^2 \leq (-f)$. The lower bound is an easy consequence of the second variation of arc length argument as in, e.g., [9, p.179], applied to the equation

$$\nabla_i \nabla_j (-f) = R_{ij} + \frac{1}{2} g_{ij} \geq \frac{1}{2} g_{ij}.$$

Moreover, since $|\nabla(-f)| \leq \sqrt{-f} \leq \frac{1}{2} r(x) + \sqrt{-f(O)}$, $-f$ is clearly proper and hence an exhaustion function. Therefore M^n is diffeomorphic to $\mathbb{R}^n$.

□

Remark 5.6. Clearly, in Lemma 5.5 and the results below, we can replace the assumption of nonnegative Ricci curvature $Rc \geq 0$ by $Rc \geq -(\frac{1}{2} - \epsilon) g$ for any small $\epsilon > 0$. Of course, the normalizing of f and the coefficients in (5.4) has to be adjusted accordingly.

Taking advantage of this growth estimates on the potential function f, it is immediate to deduce the analogous of Lemma 3.1 for expanding solitons, namely

Lemma 5.7. *Let (M^n, g_{ij}, f) $(n \geq 3)$ be a complete Bach-flat gradient expanding Ricci soliton with nonnegative Ricci curvature. Then, $D_{ijk} = 0$.*

Having this at hand, it is sufficient to follow the *proof of Theorem 1.1* in Section 3 to obtain the rotational symmetry. More precisely, we have

Theorem 5.8. *For $n \geq 4$, let (M^n, g_{ij}, f) be a complete Bach-flat gradient expanding Ricci soliton with nonnegative Ricci curvature, then it is rotationally symmetric.*

For $n = 3$, by using Theorem 1.2, we have

Theorem 5.9. *Let (M^3, g_{ij}, f) be a three-dimensional complete expanding gradient Ricci solitons with divergence-free Bach tensor and nonnegative Ricci curvature. Then (M^3, g, f) is rotationally symmetric.*

For a discussion of the expanding Ricci solitons which are rotationally symmetric, see [13, Chapter 1, Section 5], where the authors provide the existence of solutions with positive Ricci curvature (analogous to the Bryant soliton).

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