UNIVERSITA' DI PISA

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Dipartimento di Matematica "L. Tonelli"

Dottorato di Ricerca in Matematica - Anno 2005

Some Integral Pinching Results in Riemannian Geometry

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Contents

Introduction

"It is geometers dream to find a canonical metric g_{best} on a given smooth manifold M so that all topology of M will be captured by geometry"

(M. Gromov)[1]

Ever since Poincaré stated his famous conjecture about the three–sphere, mathematicians have been concerned with the problem of capturing the topological properties of a manifold by its metric structure. This has been an extremely fruitful area of mathematics, leading Hamilton to the definition of the Ricci flow which also provided a great stimulus to the study of geometric evolutions. After Perelman's breakthrough and his solution of the Poincaré conjecture, it is natural to look at the issues of geometric analysis with a new perspective.

The aim of this thesis is to present some results in the subject of *curvature pinching*. This concept was introduced in 1951 by Rauch [43] who conjectured that a simply connected, compact, Riemannian manifold M^n whose sectional curvatures all lie in the interval $(1, 4]$ is necessarily diffeomorphic to the standard sphere $\mathbb{S}^n$. A result of this type is usually referred to as a *Sphere Theorem*. After a large amount of research, now we know that the answer is positive, due to the fundamental work of Klingenberg, Berger and Rauch [35] for the topological statement and the recent proof of the original conjecture by Brendle and Schoen [2], based on the results of Böhm and Wilking [3].

We say that a manifold has (pointwise) 1/4–pinched curvature if the ratio of the minimum and the maximum of the sectional curvatures is always larger than a quarter.

[1] M. Gromov, *Spaces and questions*, GAFA, Special Volume (2000), 118–161.

Theorem 1 (The Sphere Theorem) *Let (M, g) be a compact Riemannian manifold with 1/4–pinched curvature. Then M admits a metric of constant curvature, therefore is diffeomorphic to a spherical space form, that is, to a quotient of $\mathbb{S}^n$.*

It is well known that the curvature tensor $Rm_g = R_{ijkl}$ of a Riemannian manifold (M^n, g) can be decomposed into three canonical parts which have the same symmetries as Rm_g, namely, in a coordinate system $\{x^1, \cdots, x^n\}$ we have

$$R_{ijkl} = U_{ijkl} + V_{ijkl} + W_{ijkl} \,,$$

where U is the part related to the scalar curvature $R_g = R_{ijkl}g^{ik}g^{jl}$,

$$U_{ijkl} = \frac{R_g}{n(n-1)} \left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \,,$$

V is related to the traceless Ricci tensor $\dot{Ric}_g = Ric_g - \frac{R_g}{n} g$, where Ric_g is the Ricci tensor $R_{ik} = R_{ijkl}g^{jl}$,

$$V_{ijkl} = \frac{1}{n-2} \left(\dot{R}_{ik}g_{jl} - \dot{R}_{il}g_{jk} - \dot{R}_{jk}g_{il} + \dot{R}_{jl}g_{ik}\right)$$

and the last term W is called Weyl tensor. An important property of this latter is that it is identically zero (for algebraic reasons) in two and three dimensions.
Natural candidates to be “good” metrics on a manifold are the ones such that some of the terms in the decomposition vanish. For example, very relevant are the Einstein metrics which satisfy $\dot{Ric} = 0$. When this condition holds and also the Weyl tensor W is null, the metric have constant curvature (more properly, constant sectional curvature), that is, $R_{ijkl} = U_{ijkl}$. Among the Riemannian manifolds, these latter are the most simple and completely understood, namely, we have the following *Classification Theorem* which can be read as another *pinching result.*

Theorem 2 (Riemann, Cartan) *Let (M^n, g) be a complete Riemannian n–manifold with constant sectional curvature K. Then the universal Riemannian covering of M^n, is isometric to the standard $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$ if $K = 1$, $K = 0$ or $K = -1$, respectively.*

These theorems suggest one of the basic question in global differential geometry: what conditions on the curvature tensor imply that a Riemannian manifold is homeomorphic or diffeomorphic to a *space form* (a manifold of constant sectional curvature)?

In two dimensions, according to the Gauss–Bonnet formula, for every metric g, one has

$$\int_M k_g \, dV_g = 2\pi\chi(M) \, .$$

As a consequence, by the *Uniformization Theorem* for surfaces which asserts that any compact surface M admits a Riemannian metric of constant curvature $k_g = +1, 0, -1$ and the Classification Theorem, any surface belongs to a unique geometric type (spherical, Euclidean or hyperbolic) determined by a topology invariant, namely the Euler–Poincaré characteristic $\chi(M)$. In other words, in two dimensions an *integral* pinching condition as, for example, positivity of the above integral implies a topological conclusion, encoded in the Euler–Poincaré characteristic, that the manifold is diffeomorphic to a spherical space form.

We will present similar kind of results in dimension higher than two, with particular attention to the following question.

Main Question: *Is it possible to characterize spherical space forms by means of integral (or mixed integral–pointwise) pinching conditions instead of pointwise ones?*

The first result in this direction was obtained by Chang, Gursky and Yang in [11]. They showed the existence of metrics with positive Ricci curvature on four dimensional Riemannian manifolds with positive scalar curvature and satisfying an integral pinching condition on the Ricci tensor. Through a conformal deformation of the metric, they were able to pass from positivity of a certain integral quantity to the pointwise positivity of the integrand which in turn implies the positivity of the Ricci tensor and, by a result of Margerin [38], ensure the existence of a metric of constant positive sectional curvature.

Their proof is based on establishing the existence of a solution of a forth order fully nonlinear equation.

Choosing a simpler equation (of second order), Gursky and Viaclovsky in [30] reproved

the same result. A consequence of such theorem is that the sphere in dimension four can be characterized by an integral curvature condition involving the components of the curvature tensor [12].
Our thesis is based on extending the analysis of the equation of Gursky and Viaclovsky to get integral pinching results for other families of manifolds, namely, three–dimensional manifolds and locally conformally flat ones, that is, manifolds with zero Weyl tensor.

In Chapter 1 we focus on three–manifolds. In 1982, Hamilton [31] introduced the Ricci flow in order to study "dynamically" the relationships between the topology and the curvature of three–manifolds. He showed that the metric of any compact three–dimensional Riemannian manifold with positive Ricci curvature can be deformed, via the Ricci flow, to a metric of constant positive curvature.

Theorem 3 (Hamilton) *If (M, g) is a closed three–dimensional Riemannian manifold with positive Ricci curvature, then M is diffeomorphic to a spherical space form.*

In dimension three and in presence of positive scalar curvature, the positivity of the Ricci tensor is implied by a pointwise pinching condition as

$$|Ric_g|^2 \leq \frac{3}{8} R_g^2 \,.$$

This is not the sharp quadratic condition which implies that the Ricci tensor is positive, if we assume that $R_g > 0$. The optimal one would be

$$|Ric_g|^2 < \frac{1}{2} R_g^2 \,.$$

In Chapter 1 we will show that the first of these two pinching conditions can be replaced by its integral version.

Theorem 4 (with Djadli [8]) *Let (M, g) be a closed three–dimensional Riemannian manifold with positive scalar curvature. If*

$$\int_M |Ric_g|^2 \, dV_g \leq \frac{3}{8} \int_M R_g^2 \, dV_g \,,$$

then M is diffeomorphic to a spherical space form.

However, the integral pinching constant 3/8 cannot be enlarged to get 1/2, as we can construct a metric on $\mathbb{S}^2 \times \mathbb{S}^1$ with

$$\int_M |Ric_g|^2 \, dV_g \Bigg/ \int_M R_g^2 \, dV_g < 3/8 + \varepsilon < \frac{1}{2} \, ,$$

for some explicit positive $\varepsilon > 0$ (joint work with Di Cerbo [7]).

It is well known (see for instance Milnor [37]) that any closed, oriented, three–manifold M can be decomposed "uniquely" into prime pieces

$$M = \Sigma_1 \# \cdots \# \Sigma_n \# k(\mathbb{S}^2 \times \mathbb{S}^1) \# K_1 \# \cdots \# K_m \, ,$$

where every $\pi_1(\Sigma_i)$ is finite and every K_j is aspherical, that is, $\pi_l(K_j)$ is trivial for each $l \geq 2$.

With Perelman's proof of Thurston Geometrization Conjecture (see for instance Milnor [37]) we now know that the "spherical" pieces are of the form $\mathbb{S}^3/\Gamma_i$, where $\Gamma_i \subset O(4)$ are isometry subgroups. So the picture looks like

$$M = \mathbb{S}^3/\Gamma_1 \# \cdots \# \mathbb{S}^3/\Gamma_n \# k(\mathbb{S}^2 \times \mathbb{S}^1) \# K_1 \# \cdots \# K_m \, .$$

As Gromov and Lawson in [21] proved that every three–manifold of the form $M = X_0 \# X_1$, where X_1 is aspherical, cannot admit a metric with positive scalar curvature, we conclude that any three–manifold with positive scalar curvature has to be of the form

$$M = \mathbb{S}^3/\Gamma_1 \# \cdots \# \mathbb{S}^3/\Gamma_n \# k(\mathbb{S}^2 \times \mathbb{S}^1) \, .$$

Moreover, they proved also the converse that a manifold of this form always admits a metric with positive scalar curvature.

As a corollary of our result, we have a condition to distinguish spherical space forms $\mathbb{S}^3/\Gamma$ among the family of three–manifolds with positive scalar curvature.

We want to point out that our proof of Theorem 4 is independent of Thurston Geometrization Conjecture since it just relies on the cited Hamilton's theorem about three–manifolds with positive Ricci curvature.

By completeness, in Chapter 2 we present a proof of the sphere theorem of Chang, Gursky and Yang in [12], previously cited, along the line of analysis of Gursky and Viaclovsky.

Theorem 5 (Chang, Gursky, Yang) *Let (M, g) be a closed four–dimensional Riemannian manifold with positive scalar curvature. If the curvature satisfy*

$$\int_M \left(-\frac{1}{4}|W_g|^2 - \frac{1}{2}|\overset{\circ}{Ric}_g|^2 + \frac{1}{24}R_g^2 \right) dV_g > 0\,,$$

then M is diffeomorphic to either $\mathbb{S}^4$ or $\mathbb{RP}^4$.

In Chapter 3 we concentrate on *locally conformally flat* manifolds of even dimension, i.e., those with zero Weyl tensor, if $n \geq 4$. We already mentioned how the Gauss–Bonnet formula is a bridge between topological and geometrical informations in dimension two. The situation in higher dimension is more involved, as the analogous formula due to Chern, is less intuitive and striking, nevertheless one can still try to use the integral information it provides in order to have some classification results. Indeed, in 1994 Gursky [26] was able to classify four and six–dimensional locally conformally flat manifolds with non–negative scalar curvature and positive Euler–Poincaré characteristic $\chi(M)$.

Theorem 6 (Gursky) *Let (M, g) be a closed, locally conformally flat, n–dimensional Riemannian manifold, $n = 4$ or 6, with non–negative scalar curvature, then $\chi(M) \leq 2$. Furthermore, $\chi(M) = 2$ if and only if M is diffeomorphic to the standard sphere, and $\chi(M) = 1$ if and only if M is diffeomorphic to the standard real projective space.*

As in the case of surfaces, for manifolds of even dimension, the positivity of the Euler–Poincaré characteristic gives an integral pinching condition on the curvature of the metric. The result of Gursky is not true in higher dimensions. Consider, for instance, the product of $\mathbb{S}^4$, with the canonical metric, and a four dimensional hyperbolic space form. This manifold has zero scalar curvature and positive Euler–Poincaré characteristic. Hence, in order to extend the above classification result to higher dimensions, one has to add some additional conditions.

In order to state our result we need some definitions. For a Riemannian manifold (M^n, g), we define the tensor

$$A_g^t = \frac{1}{n-2}\left(Ric_g - \frac{t}{2(n-1)}R_g g\right),$$

where t is a real number. For $t = 1$ this is called Schouten tensor.

We say that $A_g^t \in \Gamma^+_{\frac{n}{2}}$ if, for all $1 \leq k \leq \frac{n}{2}$, the k–elementary functions of the eigenvalues of the tensor A_g^t are positive.

The fact that $A_g^1 \in \Gamma^+_{\frac{n}{2}}$ implies that the Ricci tensor is non–negative and the Euler–Poincaré characteristic is less or equal than two, then by the work of Schoen and Yau [44], the possible diffeomorphism types are classified. In particular, if the Euler–Poincaré characteristic is positive, it follows that M^n is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{RP}^n$.

The goal of Chapter 3 is to show the following rigidity result.

Theorem 7 (with Djadli and Ndiaye [9]) *Let (M, g) be a closed, locally conformally flat, n–dimensional Riemannian manifold, $n \geq 8$ even, with positive scalar curvature and with positive Euler–Poincaré characteristic.*

There exists a constant $t_0 = t_0(n, diam(M, g), \|\nabla^2 Rm\|) < 1$ such that, if

$$A_g^t \in \Gamma^+_{\frac{n}{2}},$$

for some $t \in (t_0, 1]$, then M is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{RP}^n$.

We remark that the hypotheses do not imply the non–negativity of the Ricci tensor.

Notation and conventions

We will introduce some notation and conventions which will be used throughout the paper.

We consider a smooth n–dimensional Riemannian manifold (M, g), where M is always a closed manifold, meaning that M is compact and without boundary. For a Riemannian metric g, we denote the Levi–Civita connection by ∇, the Christoffel symbols in some coordinate system by Γ and the curvature tensor by Rm which can be either the $(1, 3)$ or the $(0, 4)$ type, depending of the context. The Ricci curvature is denoted by Ric, the scalar curvature by R and the volume element by dV. If it is important to make clear to which metric these tensors belong, we write Rm_g and so on.

Given a coordinate system $\{x^1 \cdots x^n\}$, we denote with $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$ the covariant derivatives associated to g. Sometimes we will denote the covariant derivative of a tensor (for example a 2–tensor T) as $\nabla_k T_{ij} = T_{ij,k}$. The components of the Hessian ∇^2 of some function u will be $\nabla_i \nabla_j u = \nabla^2_{ij} u = u_{ij}$ and similarly for higher derivatives. The components of the metric g are given by $\{g_{ij}\}$, and the inverse of the metric by $\{g^{ij}\}$. The Laplacian of a function u with respect to g is given by $\Delta_g u = g^{ij} \nabla^2_{ij} u$, where the Einstein summation convention is used. The Riemannian metric induces norms on all the tensor bundles, in coordinates this norm is given, for a tensor $T = T^{j_1 \dots j_l}_{i_1 \dots i_k}$, by

$$|T|^2_g = g^{i_1 m_1} \cdots g^{i_k m_k} g_{j_1 n_1} \dots g_{j_l n_l} T^{j_1 \dots j_l}_{i_1 \dots i_k} T^{n_1 \dots n_l}_{m_1 \dots m_k} .$$

Chapter 1

Three–manifolds

1.1 Integral pinched three–manifolds are space forms

In this chapter, using Hamilton's result, we prove the existence of an Einstein metric of positive curvature on compact, three–dimensional manifolds with positive scalar curvature and satisfying an integral pinching condition involving the second symmetric function of the Schouten tensor (joint work with Djadli, see [8]).

More precisely, we consider (M, g), a closed (i.e. compact without boundary), smooth, three–dimensional Riemannian manifold. Given a section A of the bundle of symmetric 2–tensors, we can use the metric to raise an index and view A as a tensor of type $(1, 1)$, or equivalently as a section of $End(TM)$. This allows us to define $\sigma_2(g^{-1}A)$ the second elementary function of the eigenvalues of $g^{-1}A$, namely, if we denote by λ_1, λ_2 and λ_3 these eigenvalues

$$\sigma_2(g^{-1}A) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \, .$$

See the Appendix for general definitions and results on the k–elementary symmetric curvature.

Consider the tensor (here t is a real number)

$$A_g^t = Ric_g - \frac{t}{4} R_g g \, ,$$

where Ric_g and R_g denote the Ricci and the scalar curvature of g respectively. Note that for $t = 1$, A^1_g is the classical Schouten tensor $A^1_g = Ric_g - \frac{1}{4}R_g g$ (see [1]). Hence, with our notations, $\sigma_2(g^{-1}A^t_g)$ denotes the second elementary symmetric function of the eigenvalues of $g^{-1}A^t_g$.

Our present work is motivated by a recent paper of Gursky and Viaclovsky [29]. Namely, they showed that, giving a closed three–manifold M, a metric g_0 on M (with normalized volume) satisfying $\int_M \sigma_2(g_0^{-1}A^1_{g_0})dV_{g_0} \geq 0$ is critical (over all metrics of normalized volume) for the functional

$$\mathcal{F} \,:\, g \rightarrow \int_M \sigma_2(g^{-1}A^1_g)dV_g$$

if and only if g_0 has constant sectional curvature.

Actually, it is not easy to exhibit a critical metric for this functional. What we prove here (this is a consequence of our main result) is that, assuming that there exists a metric g on M with positive scalar curvature and such that $\int_M \sigma_2(g^{-1}A^1_g)dV_g \geq 0$ then the functional $\mathcal{F}$ admits a critical point (over all metrics of normalized volume) g_0 with $\int_M \sigma_2(g_0^{-1}A^1_{g_0})dV_{g_0} \geq 0$.

We will denote $Y(M, [g])$ the Yamabe invariant associated to (M, g) (here $[g]$ is the conformal class of the metric g, that is $[g] = \{\tilde{g} = e^{-2u}g \text{ for } u \in C^\infty(M)\}$). We recall that

$$Y(M, [g]) = \inf_{\tilde{g}\in[g]} \frac{\int_M R_{\tilde{g}}dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{\frac{1}{3}}} \,.$$

An important fact that will be useful is that if g has positive scalar curvature then $Y(M, [g]) > 0$.

Our main result is the following:

Theorem 1.1.1 *Let (M, g) be a closed three–dimensional Riemannian manifold with positive scalar curvature.*

There exists a positive constant $C = C(diam(M, g), \|\nabla^2 Rm\|)$ *such that if*

$$\int_M \sigma_2(g^{-1}A^1_g)\, dV_g + C\left(\frac{7}{10} - t_0\right) Y(M, [g])^2 > 0\,,$$

for some $t_0 \leq 2/3$*, then there exists a conformal metric* $\tilde{g} = e^{-2u}g$ *with* $R_{\tilde{g}} > 0$ *and* $\sigma_2(g^{-1}A^{t_0}_{\tilde{g}}) > 0$ *pointwise. Moreover, we have the inequalities*

$$(3t_0 - 2)R_{\tilde{g}}\tilde{g} < 6Ric_{\tilde{g}} < 3(2 - t_0)R_{\tilde{g}}\tilde{g}\,. \tag{1.1}$$

As an application, when $t_0 = 2/3$, we obtain

Theorem 1.1.2 *Let* (M, g) *be a closed three–dimensional Riemannian manifold with positive scalar curvature. There exists a positive constant* $C' = C'(diam(M, g), \|\nabla^2 Rm\|)$ *such that if*

$$\int_M \sigma_2(g^{-1}A^1_g)\, dV_g + C'Y(M, [g])^2 > 0\,,$$

then there exists a conformal metric $\tilde{g} = e^{-2u}g$ *with positive Ricci curvature (*$Ric_{\tilde{g}} > 0$*). In particular if* $\int_M \sigma_2(g^{-1}A^1_g)\, dV_g \geq 0$ *then there exists a conformal metric* $\tilde{g} = e^{-2u}g$ *with positive Ricci curvature (*$Ric_{\tilde{g}} > 0$*).*

Using Hamilton's theorem (Theorem 3 in the Introduction), we get:

Corollary 1.1.3 *Let* (M, g) *be a closed three–dimensional Riemannian manifold with positive scalar curvature. There exists a positive constant* $C' = C'(diam(M, g), \|\nabla^2 Rm\|)$ *such that if*

$$\int_M \sigma_2(g^{-1}A^1_g)\, dV_g + C'Y(M, [g])^2 > 0\,,$$

then M *is diffeomorphic to a spherical space form, i.e.* M *admits a metric with constant positive sectional curvature. In particular, if* $\int_M \sigma_2(g^{-1}A^1_g)\, dV_g \geq 0$ *then* M *is diffeomorphic to a spherical space form.*

Remark 1.1.4 *Using the fact that* $\sigma_2(g^{-1}A^1_g) = -\frac{1}{2}\,|Ric_g|^2 + \frac{3}{16}R^2_g$*, the hypothesis*

$$\int_M \sigma_2(g^{-1}A^1_g)\, dV_g \geq 0$$

can be written as

$$\int_M |Ric_g|^2 \, dV_g \leq \frac{3}{8} \int_M R_g^2 dV_g \, .$$

Actually we do not know if the pinching constant $\frac{3}{8}$ is optimal, even if we believe that this is the case. Anyway we can prove that cannot be enlarge too much. Indeed, for the manifold $(\mathbb{S}^2 \times \mathbb{S}^1, g_{prod})$ with the product metric, we have

$$|Ric_g|^2 = \frac{1}{2} R_g^2 \, .$$

Moreover, it is possible to construct a family of metrics on $\mathbb{S}^2 \times \mathbb{S}^1$ with positive scalar curvature and with the pinching constant less than $\frac{1}{2}$ and close to $\frac{3}{8}$ (see Proposition 1.5.1).

Actually all these results are the consequence of the following more general result:

Theorem 1.1.5 *Let (M, g) be a closed three–dimensional Riemannian manifold with positive scalar curvature. There exists a positive constant $C = C(diam(M, g), \|\nabla^2 Rm\|)$ such that if*

$$\int_M \sigma_2(g^{-1} A_g^1) \, dV_g + \frac{1}{24} \left(\frac{7}{10} - t_0 \right) \inf_{g' = e^{-2u} g \, , \, |\nabla_g u|_g \leq C} \left(\int_M R_{g'}^2 e^{-u} dV_{g'} \right) > 0 \, ,$$

for some $t_0 \leq 2/3$, then there exists a conformal metric $\tilde{g} = e^{-2u} g$ with $R_{\tilde{g}} > 0$ and $\sigma_2(g^{-1} A_{\tilde{g}}^{t_0}) > 0$ pointwise. Moreover, we have the inequalities

$$(3t_0 - 2) R_{\tilde{g}} \tilde{g} < 6 Ric_{\tilde{g}} < 3(2 - t_0) R_{\tilde{g}} \tilde{g} \, . \tag{1.2}$$

There is a way to relate these result to the so called Q–curvature (the curvature associated to the Paneitz operator). The Paneitz operator introduced by Paneitz in [41] has demonstrated its importance in dimension 4 (see for example Chang–Gursky–Yang [11, 12] and Djadli–Malchiodi [15]). In dimension three, the Q–curvature is defined by

$$Q_g = -\frac{1}{4} \Delta_g R_g - 2 |Ric_g|_g^2 + \frac{23}{32} R_g^2 \, ,$$

the Paneitz operator being defined (in dimension 3) by

$$P_g = \Delta_g^2 - div_g \left(-\frac{5}{4} R_g g + 4 Ric_g \right) d - \frac{1}{2} Q_g \, .$$

The Paneitz operator satisfies the conformal covariant property, that is, if $\rho \in C^\infty(M)$, $\rho > 0$, then, for all $\varphi \in C^\infty(M)$, $P_{\rho^{-4}g}(\varphi) = \rho^7 P_g(\rho\varphi)$. We can now state the Corollary:

Corollary 1.1.6 *Let (M, g) be a closed three–dimensional Riemannian manifold with non–negative Yamabe invariant. If there exists a metric $g' \in [g]$ such that the Q–curvature of g' satisfies*

$$Q_{g'} \geq \frac{1}{48} R_{g'}^2 \,,$$

then M is diffeomorphic to a quotient of $\mathbb{R}^3$ if $Y(M, [g]) = 0$ or to a spherical space form if $Y(M, [g]) > 0$.

Let us emphasize the fact that, in our results, we do not make any assumption on the positivity of the Ricci tensor, we only assume that its trace is positive and a pinching on its L^2–norm.

During the preparation of the manuscript, we learned that Ge, Lin and Wang [18] showed a weaker version of Corollary 1.1.3, namely they proved that if (M, g) is a closed three–dimensional Riemannian manifold with positive scalar curvature and if $\int_M \sigma_2(g^{-1}A_g^1)\, dV_g > 0$, then M is diffeomorphic to a spherical space form. Their proof is completely different from ours since they use a very specific conformal flow. Moreover, they cannot recover the equality case for the integral pinching assumption.

For the proof of Theorem 1.1.1 and Theorem 1.1.5, we will be concerned with the following equation for a conformal metric $\tilde{g} = e^{-2u} g$:

$$\left(\sigma_2(g^{-1}A_{\tilde{g}}^t)\right)^{1/2} = f e^{2u} \,, \tag{1.3}$$

where f is a positive function on M (see in the Appendix). Let $\sigma_1(g^{-1}A_g^1)$ be the trace of A_g^1 with respect to the metric g. We have the following formula for the transformation of A_g^t under this conformal change of metric:

$$A_{\tilde{g}}^t = A_g^t + \nabla_g^2 u + (1-t)(\Delta_g u) g + du \otimes du - \frac{2-t}{2} |\nabla_g u|_g^2 g \,. \tag{1.4}$$

Since

$$A_g^t = A_g^1 + (1-t)\sigma_1(g^{-1}A_g^1)g\,,$$

this formula follows easily from the standard transformation law of the Schouten tensor (see [45]):

$$A_{\tilde{g}}^1 = A_g^1 + \nabla_g^2 u + du \otimes du - \frac{1}{2}|\nabla_g u|_g^2 g\,. \tag{1.5}$$

Using this formula we may write equation (1.3) with respect to the background metric g

$$\sigma_2\left(g^{-1}\left(A_g^t + \nabla_g^2 u + (1-t)(\Delta_g u)g + du\otimes du - \frac{2-t}{2}|\nabla_g u|_g^2 g\right)\right)^{1/2} = f(x)e^{2u}\,.$$

1.2 Upper bound and higher order estimates

Throughout the sequel, (M,g) will be a closed three–dimensional Riemannian manifold with positive scalar curvature. Since M is compact and $R_g > 0$, there exists $\delta > -\infty$ such that A_g^δ is positive definite (i.e. $Ric_g - \frac{\delta}{4}R_g g > 0$ on M). Note that δ only depends on $\|Rm\|$. For $t \in [\delta, 2/3]$, consider the path of equations (in the sequel we use the notation $A_{u_t}^t = A_{g_t}^t$ for g_t given by $g_t = e^{-2u_t}g$)

$$\sigma_2^{1/2}(g^{-1}A_{u_t}^t) = fe^{2u_t}\,, \tag{1.6}$$

where $f = \sigma_2^{1/2}(g^{-1}A_g^\delta) > 0$. Note that $u \equiv 0$ is a solution of (1.6) for $t = \delta$.

Proposition 1.2.1 (Upper bound) *Let $u_t \in C^2(M)$ be a solution of (1.6) for some $t \in [\delta, 2/3]$, with $A_{u_t}^t \in \Gamma_2^+$. Then $u_t \leq \overline{\delta}$, where $\overline{\delta}$ depends only on $\|Rm\|$.*

Proof. From Newton's inequality $\sqrt{3}\sigma_2^{1/2} \leq \sigma_1$ (see Lemma A.2.3–(iv)), so for all $x \in M$

$$\sqrt{3}fe^{2u_t} \leq \sigma_1(g^{-1}A_{u_t}^t)\,.$$

Let $p \in M$ be a maximum of u_t, then using (1.4), since the gradient terms vanish at p and $(\Delta_g u_t)(p) \leq 0$ (recall that $\Delta_g u = g^{ij}\nabla_i\nabla_j u$), we have

$$\begin{aligned}
\sqrt{3}f(p)e^{2u_t(p)} &\leq \sigma_1(g^{-1}A^t_{u_t})(p) \\
&= \sigma_1(g^{-1}A^t_g)(p) + (4-3t)(\Delta_g u_t)(p) \\
&\leq \sigma_1(g^{-1}A^t_g)(p) \\
&\leq \sigma_1(g^{-1}A^\delta_g)(p) = \frac{4-3\delta}{4}R_g(p)\,.
\end{aligned}$$

Since M is compact, this implies $u_t \leq \overline{\delta}$, for some $\overline{\delta}$ depending only on $\|Rm\|$.

□

Once we have an upper bound for the solutions of equation (1.6), we immediately get C^1 and C^2 estimates. For the proof see the Appendix, Theorem A.4.1. We have

Proposition 1.2.2 (C^1 and C^2 estimates) *Let $u_t \in C^4(M)$ be a solution of* (1.6) *for some $t \in [\delta, 2/3]$, with $A^t_{u_t} \in \Gamma^+_2$. Then*

$$\sup_M \left(|\nabla_g u_t|^2_g + |\nabla^2_g u_t|_g\right) \leq C_1\,,$$

where C_1 depends only on diam(M,g) and $\|\nabla^2 Rm\|$.

1.3 Lower bound

For the lower bound, we need the following lemmas:

Lemma 1.3.1 *For a conformal metric $\tilde{g} = e^{-2u}g$, we have the following integral transformation law*

$$\begin{aligned}
\int_M \sigma_2(\tilde{g}^{-1}A^1_{\tilde{g}})e^{-4u}\,dV_g &= \int_M \sigma_2(g^{-1}A^1_g)\,dV_g + \frac{1}{8}\int_M R_g|\nabla_g u|^2_g\,dV_g - \frac{1}{4}\int_M |\nabla_g u|^4_g\,dV_g \\
&\quad + \frac{1}{2}\int_M \Delta_g u|\nabla_g u|^2_g\,dV_g - \frac{1}{2}\int_M A^1_g(\nabla_g u, \nabla_g u)\,dV_g\,.
\end{aligned}$$

Proof. Denote $\tilde{\sigma_1} = \sigma_1(\tilde{g}^{-1}A^1_{\tilde{g}})$, $\sigma_1 = \sigma_1(g^{-1}A^1_g)$, $\tilde{\sigma_2} = \sigma_2(\tilde{g}^{-1}A^1_{\tilde{g}})$, $\sigma_2 = \sigma_2(g^{-1}A^1_g)$. We have

$$2\tilde{\sigma_2} = \tilde{\sigma_1}^2 - |A^1_{\tilde{g}}|^2_{\tilde{g}}\,.$$

By equation (1.5), we have

$$\tilde{\sigma_1}e^{-2u} = \sigma_1 + \Delta_g u - \frac{1}{2}|\nabla_g u|^2_g\,,$$

so

$$\tilde{\sigma_1}^2 e^{-4u} = \sigma_1^2 + (\Delta_g u)^2 + \frac{1}{4}|\nabla_g u|^4_g + 2\sigma_1\Delta_g u - \Delta_g u|\nabla_g u|^2_g - \sigma_1|\nabla_g u|^2_g\,.$$

After simple computations, we get

$$\begin{aligned}|A^1_{\tilde{g}}|^2_{\tilde{g}}\; e^{-4u} \;&=\; |A^1_g|^2_g + |\nabla^2_g u|^2_g + \frac{3}{4}|\nabla_g u|^4_g - \sigma_1|\nabla_g u|^2_g - \Delta_g u|\nabla_g u|^2_g + \\ &\quad + 2(A^1_g)_{ij}\nabla^{2\,ij}_g u + 2(A^1_g)_{ij}\nabla^i_g u\nabla^j_g u + 2\nabla^2_{g\,ij}u\nabla^i_g u\nabla^j_g u\,.\end{aligned}$$

Putting all together, we obtain

$$\begin{aligned}2\tilde{\sigma_2}e^{-4u} \;&=\; 2\sigma_2 + (\Delta_g u)^2 - |\nabla^2_g u|^2_g - \frac{1}{2}|\nabla_g u|^4_g + 2\sigma_1\Delta_g u \\ &\quad - 2(A^1_g)_{ij}\nabla^{2\,ij}_g u - 2(A^1_g)_{ij}\nabla^i_g u\nabla^j_g u - 2\nabla^2_{g\,ij}u\nabla^i_g u\nabla^j_g u\,.\end{aligned}$$

Now, by simple computation, we have the following identities

$$-2\int_M (A^1_g)_{ij}\nabla^{2\,ij}_g u\,dV_g = -2\int_M \sigma_1\Delta_g u\,dV_g\,,$$

$$-2\int_M \nabla^2_{ij}u\nabla^i_g u\nabla^j_g u\,dV_g = \int_M \Delta_g u|\nabla_g u|^2_g\,dV_g\,,$$

where we integrated by parts and we used the Schur's lemma,

$$2\nabla^j_g(Ric_g)_{ij} = \nabla_i R_g\,,$$

for the first identity. Finally we get

$$\begin{aligned}2\int_M \tilde{\sigma_2}e^{-4u}\,dV_g \;&=\; 2\int_M \sigma_2\,dV_g \\ &\quad + \int_M \left[(\Delta_g u)^2 - |\nabla^2_g u|^2_g - \frac{1}{2}|\nabla_g u|^4_g + \Delta_g u|\nabla_g u|^2_g - 2A^1_g(\nabla_g u, \nabla_g u)\right]\,dV_g\,,\end{aligned}$$

Now using the integral Bochner formula

$$\int_M |\nabla_g^2 u|_g^2 \, dV_g + \int_M Ric_g(\nabla_g u, \nabla_g u) \, dV_g - \int_M (\Delta_g u)^2 \, dV_g = 0 \, ,$$

and the definition of the Schouten tensor A_g^1, we get the final result.

□

Now, in order to get a uniformly lower bound of the solutions, we need to estimate all the integral terms in the previous Lemma. For the last one which contains A_g we can derive some informations by using the hypothesis on the conformal metric $\tilde{g}$. Indeed, we have the following:

Lemma 1.3.2 *Let $\tilde{g} = e^{-2u} g$. If $A_{\tilde{g}}^t \in \Gamma_2^+$, then we have the following estimate*

$$\frac{1}{2} \int_M A_g(\nabla_g u, \nabla_g u) \, dV_g < \frac{3-2t}{8} \int_M R_{\tilde{g}} |\nabla_g u|_g^2 e^{-2u} \, dV_g + \frac{1}{4} \int_M \Delta_g u |\nabla_g u|_g^2 \, dV_g - \frac{1}{4} \int_M |\nabla_g u|_g^4 \, dV_g \, .$$

Proof. Since $A_{\tilde{g}}^t \in \Gamma_2^+$, by Proposition A.6.2, we get

$$-A_{\tilde{g}}^t > -\sigma_1(\tilde{g}^{-1} A_{\tilde{g}}^t) \tilde{g} = -(4-3t)\sigma_1(\tilde{g}^{-1} A_{\tilde{g}}^1) e^{-2u} g \, .$$

Hence we get

$$-A_{\tilde{g}}^1 - (1-t)\sigma_1(\tilde{g}^{-1} A_{\tilde{g}}^1) e^{-2u} g > -(4-3t)\sigma_1(\tilde{g}^{-1} A_{\tilde{g}}^1) e^{-2u} g \, ,$$

which implies that

$$A_{\tilde{g}}^1 < (3-2t)\sigma_1(\tilde{g}^{-1} A_{\tilde{g}}^1) e^{-2u} g \, .$$

Applying this to $\nabla_g u$ we obtain

$$\frac{1}{2} A_{\tilde{g}}^1(\nabla_g u, \nabla_g u) < \frac{3-2t}{8} R_{\tilde{g}} |\nabla_g u|_g^2 e^{-2u} \, .$$

Using the conformal transformation law of the tensor $A_{\tilde{g}}^1$, we have

$$A_{\tilde{g}}^1(\nabla_g u, \nabla_g u) = A_g^1(\nabla_g u, \nabla_g u) + \nabla_{ij}^2 u \nabla_g^i u \nabla_g^j u + \frac{1}{2} |\nabla_g u|_g^4 \, .$$

Now integrating over M and using the identity

$$\int_M \nabla_{ij}^2 u \nabla_g^i u \nabla_g^j u \, dV_g = -\frac{1}{2} \int_M \Delta_g u |\nabla_g u|_g^2 \, dV_g \, ,$$

we get the final result.

□

Now we are able to prove the following lower bound (recall that C_1 is given by Proposition 1.2.2)

Proposition 1.3.3 (Lower Bound) *Assume that for some $t \in [\delta, 2/3]$ the following estimate holds*

$$(1.7)\quad \int_M \sigma_2(g^{-1}A_g^1)\,dV_g + \frac{1}{24}\left(\frac{7}{10}-t\right)\inf_{g'=e^{-2\varphi}g\,,\,|\nabla_g\varphi|_g\leq C_1}\left(\int_M R_{g'}^2 e^{-\varphi}dV_{g'}\right) = \mu_t > 0\,.$$

Then there exists $\underline{\delta}$ depending only on $diam(M,g)$ and $\|\nabla^2 Rm\|$ such that if $u_t \in C^2(M)$ is a solution of (1.6) and if $A_{u_t}^t \in \Gamma_2^+$, then $u_t \geq \underline{\delta}$.

Proof. Since $A_g^t = A_g^1 + (1-t)\sigma_1(g^{-1}A_g^1)g$, we easily have that

$$\sigma_2(A_g^t) = \sigma_2(A_g^1) + (1-t)(5-3t)\sigma_1(g^{-1}A_g^1)^2\,.$$

Letting $\tilde g = e^{-2u_t}g$,

$$\begin{aligned} e^{4u_t}f^2 = \sigma_2(g^{-1}A_{u_t}^t) &= \sigma_2(g^{-1}A_{u_t}^1) + (1-t)(5-3t)\left(\sigma_1(g^{-1}A_{u_t}^1)\right)^2 \\ &= e^{-4u_t}\left(\sigma_2(\tilde g^{-1}A_{u_t}^1) + \frac{1}{16}(1-t)(5-3t)R_{\tilde g}^2\right)\,. \end{aligned}$$

Integrating this with respect to dV_g , we obtain

$$\begin{aligned} C\int_M e^{4u_t}\,dV_g &\geq \int_M f^2 e^{4u_t}\,dV_g \\ &= \int_M \sigma_2(\tilde g^{-1}A_{u_t}^1)e^{-4u_t}\,dV_g + \frac{1}{16}(1-t)(5-3t)\int_M R_{\tilde g}^2 e^{-4u_t}\,dV_g \\ &= \int_M \sigma_2(\tilde g^{-1}A_{u_t}^1)e^{-4u_t}\,dV_g + \frac{1}{16}(1-t)(5-3t)\int_M R_{\tilde g}^2 e^{-u_t}\,dV_{\tilde g}\,, \end{aligned}$$

where $C>0$ is chosen so that $f^2 \leq C$ (recall that, since $f = \sigma_2(g^{-1}A_g^\delta)$, C depends only on $\|Rm\|$). Using the conformal deformation law for the scalar curvature of $\tilde g$

$$R_{\tilde g}e^{-2u_t} = R_g + 4\Delta_g u_t - 2|\nabla_g u_t|_g^2\,,$$

we can rewrite integral identity in Lemma 1.3.1 as

$$\begin{aligned} \int_M \sigma_2(\tilde g^{-1}A_{u_t}^1)e^{-4u_t}\,dV_g &= \int_M \sigma_2(g^{-1}A_g^1)\,dV_g + \frac{1}{8}\int_M R_{\tilde g}|\nabla_g u_t|_g^2 e^{-2u_t}\,dV_g \\ &\quad -\frac{1}{2}\int_M A_g^1(\nabla_g u, \nabla_g u)\,dV_g\,. \end{aligned}$$

Notice that, since $A^t_{u_t} \in \Gamma^+_2$, we have

$$0 < \sigma_1(g^{-1}A^t_{u_t}) = (4-3t)\sigma_1(g^{-1}A^1_{u_t}) = \frac{4-3t}{4} R_{\tilde{g}}\,,$$

and so $R_{\tilde{g}} > 0$. By Lemma 1.3.2, we obtain

$$\begin{aligned}\int_M \sigma_2(\tilde{g}^{-1}A^1_{u_t})e^{-4u_t}\,dV_g \quad &\geq \quad \int_M \sigma_2(g^{-1}A^1_g)\,dV_g - \frac{1-t}{4}\int_M R_{\tilde{g}}|\nabla_g u_t|^2_g e^{-2u_t}\,dV_g \\ &\quad -\frac{1}{4}\int_M \Delta_g u_t|\nabla_g u_t|^2_g\,dV_g + \frac{1}{4}\int_M |\nabla_g u_t|^4_g\,dV_g\,.\end{aligned}$$

By Young's inequality, since $R_{\tilde{g}} > 0$, one clearly has

$$\int_M R^2_{\tilde{g}} e^{-u_t}\,dV_{\tilde{g}} \geq \frac{2}{\varepsilon}\int_M R_{\tilde{g}}|\nabla_g u_t|^2_g e^{-2u_t}\,dV_g - \frac{1}{\varepsilon^2}\int_M |\nabla_g u_t|^4_g\,dV_g\,,$$

for all $\varepsilon > 0$. By an easy computation, we have

$$\frac{1}{16}(1-t)(5-3t) = \frac{1}{24}(\frac{7}{10}-t) + P_2(t)\,,$$

where $P_2(t)$ is a positive, second order, polynomial in t. Putting all together, we obtain (for $C > 0$ depending only on $\|Rm\|$)

$$\begin{aligned}C\int_M e^{4u_t}\,dV_g \quad &\geq \quad \int_M \sigma_2(\tilde{g}^{-1}A^1_{u_t})e^{-4u_t}\,dV_g + \frac{1}{16}(1-t)(5-3t)\int_M R^2_{\tilde{g}}e^{-u_t}\,dV_{\tilde{g}} \\ &= \quad \int_M \sigma_2(\tilde{g}^{-1}A^1_{u_t})e^{-4u_t}\,dV_g + \left(\frac{1}{24}(\frac{7}{10}-t) + P_2(t)\right)\int_M R^2_{\tilde{g}}e^{-u_t}\,dV_{\tilde{g}} \\ &\geq \quad \int_M \sigma_2(g^{-1}A^1_g)\,dV_g + \frac{1}{24}(\frac{7}{10}-t)\int_M R^2_{\tilde{g}}e^{-u_t}\,dV_{\tilde{g}} \\ &\quad + P_2(t)\int_M R^2_{\tilde{g}}e^{-u_t}\,dV_{\tilde{g}} - \frac{1-t}{4}\int_M R_{\tilde{g}}|\nabla_g u_t|^2_g e^{-2u_t}\,dV_g \\ &\quad -\frac{1}{4}\int_M \Delta_g u_t|\nabla_g u_t|^2_g\,dV_g + \frac{1}{4}\int_M |\nabla_g u_t|^4_g\,dV_g\,.\end{aligned}$$

Now using Young's inequality and the conformal change equation of the scalar curvature, we get

$$\begin{aligned}C\int_M e^{4u_t}\,dV_g \quad &\geq \quad \int_M \sigma_2(g^{-1}A^1_g)\,dV_g + \frac{1}{24}(\frac{7}{10}-t)\int_M R^2_{\tilde{g}}e^{-u_t}\,dV_{\tilde{g}} \\ &\quad + \left(\frac{2P_2(t)}{\varepsilon} - \frac{1-t}{4}\right)\int_M R_g|\nabla_g u_t|^2_g\,dV_g \\ &\quad + \left(\frac{8P_2(t)}{\varepsilon} - (1-t) - \frac{1}{4}\right)\int_M \Delta_g u_t|\nabla_g u_t|^2_g\,dV_g \\ &\quad + \left(\frac{3-2t}{4} - \frac{P_2(t)}{\varepsilon^2} - \frac{4P_2(t)}{\varepsilon}\right)\int_M |\nabla_g u_t|^4_g\,dV_g\,.\end{aligned}$$

We choose now $\varepsilon = \varepsilon(t) > 0$, such that $\frac{8P_2(t)}{\varepsilon} - (1-t) - \frac{1}{4} = 0$. One can easily check that, with this choice,

$$\frac{2P_2(t)}{\varepsilon} - \frac{1-t}{4} \geq 0 \quad \text{and} \quad \frac{3-2t}{4} - \frac{P_2(t)}{\varepsilon^2} - \frac{4P_2(t)}{\varepsilon} \geq 0\,.$$

Finally, recalling that according to Proposition 1.2.2, $\max_M |\nabla_g u_t|_g \leq C_1$ with C_1 depending only on $\mathrm{diam}(M,g)$ and $\|\nabla^2 Rm\|$, we obtain the following estimate (for a certain $C > 0$ depending only on $\|Rm\|$)

$$\begin{aligned} C\int_M e^{4u_t}\, dV_g \;&\geq\; \int_M \sigma_2(g^{-1}A^1_g)\, dV_g + \frac{1}{24}\Big(\frac{7}{10} - t\Big)\int_M R^2_{\tilde{g}} e^{-u_t}\, dV_{\tilde{g}} \\ &\geq\; \int_M \sigma_2(g^{-1}A^1_g)\, dV_g + \frac{1}{24}\Big(\frac{7}{10} - t\Big) \inf_{g'=e^{-2\varphi}g\,,\, |\nabla_g\varphi|_g \leq C_1} \left(\int_M R^2_{g'} e^{-\varphi} dV_{g'}\right) = \mu_t > 0\,. \end{aligned}$$

This gives

$$\max_M u_t \geq \log \mu_t - C(\mathrm{diam}(M,g), \|Rm\|)\,.$$

Since $\max_M |\nabla_g u_t|_g \leq C_1$, this implies the Harnack inequality

$$\max_M u_t \leq \min_M u_t + C(\mathrm{diam}(M,g), \|\nabla^2 Rm\|)\,,$$

by simply integrating along a geodesic connecting points at which u_t attains its maximum and minimum. Combining this two inequalities, we obtain

$$u_t \geq \min_M u_t \geq \log \mu_t - C =: \underline{\delta}\,,$$

where C only depends on $\mathrm{diam}(M,g)$ and $\|\nabla^2 Rm\|$. This ends the proof of the Lemma.

□

1.4 Proof of the main result

For the proof of Theorem 1.1.5 we use the continuity method. Our 1-parameter family of equations, for $t \in [\delta, t_0]$, is

$$\sigma_2^{1/2}(g^{-1}A^t_{u_t}) = f(x)e^{2u_t}\,, \tag{1.8}$$

with $f(x) = \sigma_2^{1/2}(g^{-1}A_g^\delta) > 0$, and δ was chosen so that A_g^δ is positive definite. Define the set

$$\mathcal{S} = \left\{ t \in [\delta, t_0] \mid \exists \, \text{a solution} \, u_t \in C^{2,\alpha}(M) \, \text{of} \, \text{ (1.8)} \, \text{with} \, A_{u_t}^t \in \Gamma_2^+ \right\} .$$

Clearly, with our choice of f, $u \equiv 0$ is a solution for $t = \delta$. Since A_g^δ is positive definite, $\delta \in \mathcal{S}$, and $\mathcal{S} \neq \emptyset$. Let $t \in \mathcal{S}$, and u_t be a solution. By Proposition A.3.1, the linearized operator at u_t, $\mathcal{L}^t : C^{2,\alpha}(M) \to C^\alpha(M)$, is invertible. The implicit function theorem tells us that $\mathcal{S}$ is open. By Proposition 1.2.1 we obtain uniform upper bound on the solutions u_t, independent of t. We may then apply Proposition 1.2.2 to obtain a uniform gradient and Hessian bounds on u_t, and by Proposition 1.3.3, we get a uniform lower bound. In order to have compactness for a sequence of solutions in $\mathcal{S}$ we need $C^{2,\alpha}$–estimates, which follow from Theorem A.5.1. Finally, the Ascoli–Arzelà's theorem, implies that $\mathcal{S}$ must be closed, therefore $\mathcal{S} = [\delta, t_0]$. The metric $\tilde{g} = e^{-2u_{t_0}} g$ then satisfies $\sigma_2(A_{\tilde{g}}^{t_0}) > 0$ and $R_{\tilde{g}} > 0$. The inequalities (1.2) for the Ricci curvature of $\tilde{g}$ follow from Proposition A.6.2.

Proof of Theorem 1.1.1

As we showed in the previous section, there exists a constant C_1 depending only on $\mathrm{diam}(M, g)$ and $\|\nabla^2 Rm\|$ such that all solutions of equation (1.6) satisfy $\max_M |\nabla_g u|_g < C_1$.

We consider the following quantity:

$$I(M, g) = \inf_{g' = e^{-2\varphi} g \, , \, |\nabla_g \varphi|_g \leq C_1} \left(\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right) .$$

We let, for $g' = e^{-2\varphi} g$

$$i(g') = \int_M R_{g'}^2 e^{-\varphi} dV_{g'} .$$

As one can easily check, if two metrics g_1 and g_2 are homothetic, then $i(g_1) = i(g_2)$. So, we have

$$I(M, g) = \inf_{g' = e^{-2\varphi} g \, , \, Vol(M, g') = 1 \text{ and } |\nabla_g \varphi|_g \leq C_1} \left(\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right) .$$

We have the following

Lemma 1.4.1 *There exists a positive constant* $C = C(diam(M,g), \|\nabla^2 Rm\|)$ *such that*

$$I(M,g) \geq C\left(Y(M,[g])\right)^2 .$$

Proof. As we have seen

$$I(M,g) = \inf_{g'=e^{-2\varphi}g\,,\,Vol(M,g')=1 \text{ and } |\nabla_g\varphi|_g \leq C_1} \left(\int_M R_{g'}^2 e^{-\varphi} dV_{g'}\right) .$$

Take $\varphi \in C^\infty(M)$ such that, for $g' = e^{-2\varphi}g$, $Vol(M,g') = 1$ and such that $|\nabla_g\varphi|_g \leq C_1$ where C_1 is given by Proposition 1.2.2. Since $Vol(M,g') = 1$, if p is a point where φ attains its minimum we have

$$e^{-3\varphi(p)} Vol(M,g) \geq 1 ,$$

and then, there exists C_0 depending only on $\operatorname{diam}(M,g)$ such that $\varphi(p) \leq C_0$. Now, using the mean value theorem, it follows since $|\nabla_g\varphi|_g$ is controlled by a constant C_1, that $\max \varphi \leq C_0'$ where C_0' depends only on $\operatorname{diam}(M,g)$ and $\|\nabla^2 Rm\|$.
Using this, we clearly have that

$$\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \geq e^{-C_0'} \int_M R_{g'}^2 dV_{g'} .$$

Using Hölder inequality and the definition of the Yamabe invariant, we get (recall that $Vol(M,g') = 1$)

$$\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \geq e^{-C_0'} \left(Y(M,[g])\right)^2 ,$$

and then $I(M,g) \geq e^{-C_0'}\left(Y(M,[g])\right)^2$. This ends the proof.

□

Now, Theorem 1.1.1 is a direct consequence of Theorem 1.1.5 and of Lemma 1.4.1.

Proof of Corollary 1.1.6

Assume that M admits a metric g' such that $Q_{g'} \geq \frac{1}{48} R_{g'}^2$ and $Y(M,[g']) \geq 0$. Recall that

$$Q_{g'} = -\frac{1}{4}\Delta_{g'} R_{g'} - 2|Ric_{g'}|_{g'}^2 + \frac{23}{32} R_{g'}^2 ,$$

Integrating $Q_{g'}$ on M with respect to $dV_{g'}$ we obtain (since $Q_{g'} \geq 0$)

$$\int_M |Rig_{g'}|^2_{g'} dV_{g'} \leq \frac{23}{64} \int_M R^2_{g'} dV_{g'} \,. \tag{1.9}$$

Now if we compute $\int_M \sigma_2(g'^{-1}A^1_{g'})$ using (1.9), we have (recall that $\sigma_2(g'^{-1}A^1_{g'}) = -\frac{1}{2}|Ric_{g'}|^2_{g'} + \frac{3}{16}R^2_{g'}$):

$$\int_M \sigma_2(g'^{-1}A^1_{g'}) \geq \frac{1}{128} \int_M R^2_{g'} dV_{g'} \geq 0\,.$$

Now, consider the conformal Laplacian operator $L_{g'} = \Delta_{g'} - \frac{1}{8}R_{g'}$. We have using the assumption $Q_{g'} \geq \frac{1}{48}R^2_{g'}$

$$L_{g'}R_{g'} = \Delta_{g'}R_{g'} - \frac{1}{8}R^2_{g'} \leq -8|Ric_{g'}|^2_{g'} + \frac{22}{8}R^2_{g'} - \frac{1}{12}R^2_{g'} \leq \left(-\frac{8}{3} + \frac{22}{8} - \frac{1}{12}\right) R^2_{g'} = 0.$$

Applying a Lemma due to Gursky [27], since $Y(M,[g']) \geq 0$ we have either $R_{g'} > 0$ (if $Y(M,[g']) > 0$) or $R_{g'} \equiv 0$ (if $Y(M,[g']) = 0$). If $Y(M,[g']) > 0$ we can apply Theorem 1.1.1 to conclude that M is diffeomorphic to a spherical space form. Otherwise, if $Y(M,[g']) = 0$, since $Q_{g'} \geq \frac{1}{48}R^2_{g'}$ and $R_{g'} \equiv 0$, we deduce, using the expression giving $Q_{g'}$, that $Ric_{g'} \equiv 0$ and then M is diffeomorphic to a quotient of $\mathbb{R}^3$.

This ends the proof of the corollary.

1.5 Optimality of the result

As a corollary of the main theorem, we got that if a closed Riemannian manifold M admits a metric g such that

$$\begin{aligned} R_g &> 0 \\ \int_M |Ric_g|^2_g \, dV_g &\leq \frac{3}{8} \int_M R^2_g \, dV_g \,, \end{aligned}$$

then it must be diffeomorphic to a quotient space of $\mathbb{S}^3$ by a group of fixed point free isometries in the standard metric. In this section we will prove that the pinching constant $\frac{3}{8}$ cannot be enlarge too much, since we can construct a metric on $\mathbb{S}^2 \times \mathbb{S}^1$ with a pinching constant close to $\frac{3}{8}$. Here the result (joint work with Di Cerbo, see [7]).

Proposition 1.5.1 *Let g be a metric on $\mathbb{S}^2 \times \mathbb{S}^1$ of the type*

$$g = e^{-f} g_{\mathbb{S}^2} \oplus e^{-2f} g_{\mathbb{S}^1}\,,$$

where f is a smooth function on $\mathbb{S}^2$ and $g_{\mathbb{S}^1}$, $g_{\mathbb{S}^2}$ denote the canonical metrics on the unit spheres. Then, for every $\varepsilon > \frac{1}{15}$, there exist a function f such that the metric g satisfies

$$\begin{aligned} R_g &> 0 \\ \int_{\mathbb{S}^2\times\mathbb{S}^1} |Ric_g|_g^2\, dV_g &\leq \left(\frac{3}{8}+\varepsilon\right) \int_{\mathbb{S}^2\times\mathbb{S}^1} R_g^2\, dV_g\,. \end{aligned}$$

Proof. For the construction we will follow [6].

Let (M^m, g) and (N^n, h) be two closed Riemannian manifolds of dimension m and n respectively, and let $f : M \to \mathbb{R}$ be a smooth function on M. On the product manifold $\widetilde{M} = M \times N$ consider a metric $\widetilde{g}$ of the form

$$\tilde{g} = e^{-Af} g \oplus e^{-Bf} h\,,$$

where A and B are constants.

As a notation, we will use Latin indices, i, j, etc., for the coordinates on M and Greek indices, α, β, etc., for the coordinates on N. Under this notations, clearly we have $\forall\, i, j = 1 \dots m$ and $\forall\, \alpha, \beta = 1 \dots n$

$$\tilde{g}_{i\alpha} = \tilde{g}^{i\alpha} = 0\,,$$

$$\tilde{g}^{ij} = e^{Af} g^{ij}, \quad \tilde{g}^{\alpha\beta} = e^{Bf} h^{\alpha\beta}\,.$$

Since we are free to choose the constants A and B, following [6] we set $A = 1$ and $B = 2$ in order to simplify the computation and to have good integral terms. Under this we get the following formula for the components of the Ricci curvature of $\tilde{g}$

$$\begin{aligned} \widetilde{R}_{jl} &= R_{jl} + \nabla^2_{jl} f + \frac{1}{2} g_{jl} \left(\Delta f - |\nabla f|^2\right), \\ \widetilde{R}_{\beta\gamma} &= R_{\beta\gamma} + e^{-f} h_{\beta\gamma} \left(\Delta f - |\nabla f|^2\right)\,. \end{aligned}$$

Then, the scalar curvature of $\tilde{g}$ becomes

$$\widetilde{R} = e^f R^M + e^{2f} R^N + e^f \left(3\Delta f - 2|\nabla f|^2\right)\,.$$

Now let $(M, g) = (\mathbb{S}^2, g)$ and $(N, h) = (\mathbb{S}^1, g_{\mathbb{S}^1})$, where g (for the moment) is a generic metric on $\mathbb{S}^2$ and $g_{\mathbb{S}^1}$ is the standard metric on $\mathbb{S}^1$. Of course we have

$$R_{\beta\gamma} = 0, R^N = 0\,.$$

Hence we get the following pointwise formulas

$$\begin{aligned} |Ric_{\tilde{g}}|^2_{\tilde{g}} &= e^{2f}\Big[|Ric_g|^2_g + |\nabla^2_g f|^2_g + \frac{3}{2}(\Delta f - |\nabla_g f|^2_g)^2 + \Delta f(\Delta f - |\nabla_g f|^2_g) + \\ &\quad +2R^{ij}\nabla^2_{ij} f + R^M(\Delta f - |\nabla_g f|^2_g)\Big]\,, \end{aligned}$$

$$R^2_{\tilde{g}} = e^{2f}\Big[(R^M)^2 + 9(\Delta_g f)^2 + 4|\nabla_g f|^2_g - 12\Delta_g f|\nabla_g f|^2_g + 6R^M\Delta_g f - 4R^M|\nabla_g f|^2_g\Big]\,.$$

We fix $\alpha \in \mathbb{R}$ and we integrate over $\mathbb{S}^2 \times \mathbb{S}^1$. After some integration by parts argument, we obtain

$$\begin{aligned} \frac{1}{Vol(\mathbb{S}^1)}\int_{\mathbb{S}^2\times\mathbb{S}^1}\left(-|Ric_{\tilde{g}}|^2_{\tilde{g}} + \alpha R^2_{\tilde{g}}\right)dV_{\tilde{g}} &= \int_{\mathbb{S}^2}\left(-|Ric_g|^2_g + \alpha R^2_g\right)dV_g + \\ &\quad + \int_{\mathbb{S}^2}\Big[-|\nabla^2_g f|^2_g + (9\alpha - \frac{5}{2})(\Delta_g f)^2 \\ &\quad +(4\alpha - \frac{3}{2})|\nabla_g f|^4_g + (4 - 12\alpha)\Delta_g f|\nabla_g f|^2_g \\ &\quad +(6\alpha - 2)R^M\Delta_g f + (1 - 4\alpha)R^M|\nabla_g f|^2_g\Big]\,dV_g\,. \end{aligned}$$

Now we pick on $\mathbb{S}^2$ the standard metric, i.e. $g = g_{\mathbb{S}^2}$. Of course we have

$$R_{ij} = g_{ij}, R = 2\,.$$

Using the integral Bochner formula,

$$\int_{\mathbb{S}^2}|\nabla^2_g f|^2\,dV_g + \int_{\mathbb{S}^2}Ric_g(\nabla_g f, \nabla_g f)\,dV_g - \int_{\mathbb{S}^2}(\Delta_g f)^2\,dV_g = 0\,,$$

finally we obtain

$$\begin{aligned} \frac{1}{Vol(\mathbb{S}^1)}\int_{\mathbb{S}^2\times\mathbb{S}^1}\left(-|Ric_{\tilde{g}}|^2_{\tilde{g}} + \alpha R^2_{\tilde{g}}\right)dV_{\tilde{g}} &= \int_{\mathbb{S}^2}\left(-|Ric_g|^2_g + \alpha R^2_g\right)dV_g + \\ &\quad + \int_{\mathbb{S}^2}\Big[(9\alpha - \frac{7}{2})(\Delta_g f)^2 + (4\alpha - \frac{3}{2})|\nabla_g f|^4_g \\ &\quad +(4 - 12\alpha)\Delta_g f|\nabla_g f|^2_g + (3 - 8\alpha)|\nabla_g f|^2_g\Big]\,dV_g\,. \end{aligned}$$

The standard metric on $\mathbb{S}^2$ can be written in spherical coordinates as

$$g_{\mathbb{S}^2} = dr^2 + \sin(r)^2 d\theta^2 \,,$$

where $0 \leq r \leq \pi$ and $0 \leq \theta \leq 2\pi$. We define the function f on $\mathbb{S}^2$ as follows

$$f(r,\theta) = f(r) := \begin{cases} \frac{1}{a}\, exp\left(-\frac{1}{1-\left(\frac{r}{b}\right)^2}\right), & \text{for } -b < r < b \\ 0 & \text{otherwise} \end{cases}$$

where $a = 0.752$ and $b = 1.95$. We denote with $'$ the derivative with respect to r. Under this choice we have that the scalar curvature of $\tilde{g}$

$$R_{\tilde{g}} = e^{f(r)} \left(3f''(r) + 3\frac{\cos(r)}{\sin(r)} f'(r) - 2f'(r)^2 + 2\right)$$

is positive on $\mathbb{S}^2 \times \mathbb{S}^1$. Using that $dV_g = \sin(r) dr \wedge d\theta$ have the following approximation of the integral terms, namely

$$\int_{\mathbb{S}^2} \Delta_g f^2 \, dV_g = 2\pi \int_0^\pi \left(3f''(r) + 3\frac{\cos(r)}{\sin(r)} f'(r)\right)^2 \sin(r)\, dr \simeq (2\pi)1.2121] \,,$$

$$\int_{\mathbb{S}^2} |\nabla_g f|^2_g \, dV_g = 2\pi \int_0^\pi f'(r)^2 \sin(r)\, dr \simeq (2\pi)0.171 \,,$$

$$\int_{\mathbb{S}^2} |\nabla_g f|^4_g \, dV_g = 2\pi \int_0^\pi f'(r)^4 \sin(r)\, dr \simeq (2\pi)0.0359 \,,$$

$$\int_{\mathbb{S}^2} \Delta_g f |\nabla_g f|^2_g \, dV_g = 2\pi \int_0^\pi \left(3f''(r) + 3\frac{\cos(r)}{\sin(r)} f'(r)\right) f'(r)^2 \sin(r)\, dr \simeq (2\pi)0.0058 \,.$$

Let $\alpha = \frac{3}{8} + \varepsilon$. By the previous computation we have

$$\begin{aligned} \frac{1}{Vol(\mathbb{S}^1)} \int_{\mathbb{S}^2 \times \mathbb{S}^1} \left(-|Ric_{\tilde{g}}|^2_{\tilde{g}} + \left(\frac{3}{8} + \varepsilon\right) R^2_{\tilde{g}}\right) dV_{\tilde{g}} \;&=\; \int_{\mathbb{S}^2} \left(-|Ric_g|^2_g + \alpha R^2_g\right) \, dV_g + \\ &+ \int_{\mathbb{S}^2} \Big[(9\varepsilon - \frac{1}{8})(\Delta_g f)^2 + 4\varepsilon |\nabla_g f|^4_g \\ &\quad -8\varepsilon |\nabla_g f|^2_g - (12\varepsilon + \frac{1}{2})\Delta_g f |\nabla_g f|^2_g\Big] \, dV_g \\ &\simeq\; 2\pi \Big[(8\varepsilon - 1) + (9\varepsilon - \frac{1}{8}(1.2121) + 4\varepsilon(0.0359) \\ &\quad -8\varepsilon(0.171) - (12\varepsilon + \frac{1}{2})(0.0058)\Big] \,. \end{aligned}$$

This turns out to be positive if

$$\varepsilon > \frac{1}{15} \,.$$

□

1.6 The case of non–negative scalar curvature

For the proof of the main theorem we used substantially the fact that the initial metric g has positive scalar curvature. However, we can deform a metric g_0 with non–negative (but not zero) scalar curvature to a metric g with positive scalar curvature. Here is the result

Theorem 1.6.1 *Let (M, g_0) be a closed three–dimensional Riemannian manifold with non–negative scalar curvature. If*

$$\int_M |Ric_{g_0}|^2_{g_0}\, dV_{g_0} \leq \frac{3}{8}\int_M R^2_{g_0}\, dV_{g_0}\,,$$

then M is diffeomorphic either to flat or a spherical space form.

Proof. If the metric g_0 has identically zero scalar curvature, then the integral pinching condition implies that the manifold must be Ricci flat. Since we are in dimension three, this implies that the manifold must be flat.
Now suppose that the metric g_0 has, non–flat, non–negative scalar curvature. We will flow this metric using the Yamabe flow (for general result see Hamilton [33]) with initial condition $g(0) = g_0$, namely

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(t) = -R_{g(t)} g_{ij}(t) \\ g(0) = g_0 \end{cases}$$

This flow is the negative L^2–gradient flow of the (normalized) total scalar curvature, restricted to a given conformal class. It was introduced in order to deform a Riemannian metric to a conformal metric of constant scalar curvature. For our purposes we can use this flow to deform our metric g_0 to a metric with positive scalar curvature. In fact we know that there exists a unique smooth solution $g(t)$ in $[0, T)$ at least for small time T (for the proof see [33]). Moreover, the evolution of the scalar curvature takes the form

$$\frac{\partial}{\partial t} R_{g(t)} = 2\Delta_{g(t)} R_{g(t)} + 2R^2_{g(t)}\,.$$

By the strong maximum principle (minimum principle in this case), since the scalar curvature at time 0 is non–negative, we know that the scalar curvature at time $t > 0$ must be strictly positive, unless it is zero for all t (which cannot be the case, since we are assuming that at time zero the metric is not scalar flat). Since we want to apply Corollary 1.1.3, we have to show that, for sufficient small time t, we have

$$\int_M \sigma_2(g^{-1}(t)A_{g(t)})\,dV_{g(t)} + C'Y(M,[g(t)])^2 > 0\,, \tag{1.10}$$

where $C' = C'(\mathrm{diam}(M,g), \|\nabla^2_{g(t)} Rm\|)$ is the constant needed in order to conclude. Since the Yamabe flow moves the metric $g(t)$ in the same conformal class, for all $t \in [0,T)$, clearly we have

$$Y(M,[g(t)]) = Y(M,[g(0)])\,.$$

Moreover, the integral above and quantity $C' = C'(\mathrm{diam}(M,g), \|\nabla^2_{g(t)} Rm\|)$ depend continuously on time, by the smoothness of the flow. Since the metric $g(0)$ satisfies condition (1.10), for t sufficiently small, also the metric $g(t)$ must satisfy the integral pinching condition (1.10). Then, we conclude that the manifold must be diffeomorphic to a spherical space form, since we constructed a metric $g(t)$ satisfying all the hypothesis of Corollary 1.1.3.

□

Chapter 2

Four manifolds

2.1 Hamilton and Margerin's results in dimension four

In this chapter we will concentrate on smooth, closed Riemannian manifolds of dimension four. The main question the following: under which conditions on the curvature can we conclude that a manifold is diffeomorphic (or homeomorphic) to the sphere? A result of this type is usually referred to as a *sphere theorem*, and we have lots of examples in this direction (for instance, see Petersen's book [42]).

In dimension four, at the homeomorphism level, we have of course the work of Freedman (see Freedman [17]), which asserts that any curvature conditions which implies the vanishing of the de Rham cohomology groups $H^1(M^4, \mathbb{R})$ and $H^2(M^4, \mathbb{R})$ will imply that the universal cover M^4 must be homeomorphic to the sphere.

At the same time, there are also some results which characterize the smooth sphere. The two examples of particular importance are the results of Hamilton [32] and Margerin [38].

Some years after the proof of his famous result on three–manifolds with positive Ricci curvature, Hamilton applied Ricci flow techniques to four manifolds. In this work he was able to deform a metric with positive curvature operator to a metric of constant positive sectional curvature.

Recall that, a Riemannian manifold (M, g) has positive curvature operator if, for all non–zero 2–forms $\omega \in \Lambda^2(TM)$, there holds

$$R_{ijkl}\,\omega_{ij}\omega_{kl} > 0\,.$$

Here the result is:

Theorem 2.1.1 (Hamilton, 1986) *Let (M, g) be a closed four–dimensional Riemannian manifold with positive curvature operator. The M is diffeomorphic to a spherical space form, i.e. M admits a metric with constant positive sectional curvature. Moreover, we get that the manifold M is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{R}P^4$.*

More generally, the same conclusion holds for compact four-manifolds with 2–positive curvature operators (see [13]). Recall that a curvature operator is called 2–positive, if the sum of its two smallest eigenvalues is positive. Recently, Böhm and Wilking in [3] generalized this result to manifolds of every dimension.

In the same year, more or less, Margerin proved a pinching theorem on four manifolds, which was an improvement of the results of Huisken [34] and Nishikawa [40]. To explain this we will need to establish some notations. Given a Riemannian manifold (M, g) of dimension four, the orthogonal decomposition of the Riemann curvature can be written

$$Rm_g = W_g + \frac{1}{2}E_g \owedge g + \frac{1}{24}R_g\, g \owedge g\,,$$

where Rm_g is the curvature operator, Ric_g is the Ricci curvature, R_g the scalar curvature, $E_g = Ric_g - \frac{1}{4}R_g g$ is the trace-free Ricci tensor and $\owedge$ denotes the Kulkarni-Nomizu product which is defined as follow: let A, B two symmetric $(0, 2)$-tensors, then

$$(A \owedge B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} - A_{jk}B_{il} + A_{jl}B_{ik}.$$

If we let $Z_g = W + \frac{1}{2}R_g\, g \owedge g$ (this $(0, 4)$–tensor is called the concircular curvature), then

$$Rm_g = Z_g + \frac{1}{24}R_g\, g \owedge g\,.$$

Note that (M, g) has constant curvature if, and only if $Z_g \equiv 0$. We now define the scale–invariant "weak pinching" quantity

$$WP_g = \frac{|Z_g|_g^2}{R_g^2} = \frac{|W_g|_g^2 + 2|E_g|_g^2}{R_g^2}\,,$$

where $|\cdot|_g^2$ denotes the usual norm of a tensor with respect to the metric g.

Theorem 2.1.2 (Margerin, 1986) *Let (M, g) be a closed four–dimensional Riemannian manifold with positive scalar curvature and the weak pinching quantity satisfies*

$$WP_g < \frac{1}{6}\,.$$

Then M is diffeomorphic to a spherical space form, i.e. M admits a metric with constant positive sectional curvature. Moreover, we get that the manifold M is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{R}P^4$.

Moreover, this condition is sharp (see [39]): the spaces $(\mathbb{C}P^2, g_{fs})$ and $(\mathbb{S}^3 \times \mathbb{S}^1, g_{pd})$ both have positive scalar curvature and satisfy $WP \equiv \frac{1}{6}$. Using a holonomy reduction argument, Margerin also showed the converse, in the sense that these manifolds (and any quotients) are characterized by the property that $WP \equiv \frac{1}{6}$.

2.2 A conformally invariant sphere theorem

As we mentioned already in the introduction, these results require one to verify a pointwise condition on the components of the curvature tensor, in contrast to the case of surfaces (dimension two), where one can get a complete classification, just having a look on the integral of the Gauss curvature.

In 2003, Chang, Gursky and Yang [11, 12] generalized this situation, by showing that the smooth sphere in dimension four can be characterized by an integral curvature condition. Moreover, they showed that the condition is sharp, as in the case of Margerin.

We will denote by $Y(M,[g])$ the Yamabe invariant associated to (M,g), where $[g]$ denotes the conformal class of the metric g (that is $[g] = \{\tilde{g} = e^{-2u}g \text{ for } u \in C^\infty(M)\}$). Recall that, in dimension four, the Yamabe invariant is defined by

$$Y(M,[g]) = \inf_{\tilde{g}\in[g]} \frac{\int_M R_{\tilde{g}} dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{\frac{1}{2}}}\,.$$

Positivity of the Yamabe invariant implies that g is conformal to a metric of (strictly) positive scalar curvature. Conversely, if g has positive scalar curvature, then $Y(M,[g])$ is positive.

The first way of stating the result is the following

Theorem 2.2.1 (Chang–Gursky–Yang, 2003) *Let (M,g) be a closed four–dimensional Riemannian manifold with positive Yamabe invariant. If the Weyl curvature satisfies*

$$\int_M |W_g|_g^2\, dV_g < 16\pi^2\chi(M)\,,$$

then M is diffeomorphic to either $\mathbb{S}^4$ or $\mathbb{R}P^4$.

By the Chern–Gauss–Bonnet formula, we know that

$$\int_M \left(\frac{1}{4}|W_g|_g^2 - \frac{1}{2}|E_g|_g^2 + \frac{1}{24}R_g^2\right)\, dV_g\,.$$

So it is possible to replace the hypothesis on the Weyl curvature with an equivalent condition which does not involve the Euler characteristic. So the previous theorem is equivalent to the following

Theorem 2.2.2 *Let (M,g) be a closed four–dimensional Riemannian manifold with positive Yamabe invariant. If the curvatures satisfy*

$$\int_M \left(-\frac{1}{4}|W_g|_g^2 - \frac{1}{2}|E_g|_g^2 + \frac{1}{24}R_g^2\right)\, dV_g > 0\,, \tag{2.1}$$

then M is diffeomorphic to either $\mathbb{S}^4$ or $\mathbb{R}P^4$.

In this formulation, it is very easy to see the connection with the result of Margerin. Indeed, any metric g for which the integrand of (2.1) is positive must satisfy the pinching condition of Margerin, i.e.

$$|W_g|_g^2 + 2|E_g|_g^2 < \frac{1}{6}R_g^2 \,.$$

In other words, this theorem says that we can pass from positivity in an integral sense to pointwise positivity. This is exactly what they proved: by a conformal change of the metric, they were able to find a solution of a certain fully nonlinear equation.

Note that the integral pinching hypothesis can be written as

$$\int_M \sigma_2(g^{-1}A_g^1)\, dV_g - \frac{1}{16}\int_M |W_g|_g^2\, dV_g > 0 \,,$$

where $\sigma_2(g^{-1}A_g)$ denotes the second elementary symmetric function of the eigenvalues of $g^{-1}A_g$ (see the Appendix for general definitions).

2.3 A simple proof

We present a different proof of Theorem 2.2.1, which can be obtained by an easy modification of the work of Gursky–Viaclovsky for manifolds of dimension four (see [30]).

Preliminaries

Let (M,g) be a closed four–dimensional Riemannian manifold. As in Chapter 1, we define the tensor

$$A_g^t = \frac{1}{2}\left(Ric_g - \frac{t}{6}R_g g\right) \,,$$

where Ric_g and R_g denote the Ricci and the scalar curvature of g respectively. We want to prove the following:

Theorem 2.3.1 *Let (M, g) be a closed four–dimensional Riemannian manifold with positive scalar curvature. Fix $0 \leq \alpha \leq 1$. If*

$$\int_M \sigma_2(g^{-1}A_g^1)\, dV_g - \frac{\alpha}{16}\int_M |W_g|_g^2\, dV_g + \frac{1}{24}(1-t_0)(2-t_0)Y(M,[g])^2 > 0\,,$$

for some $t_0 \leq 1$, then there exists a conformal metric $\tilde{g} = e^{-2u}g$ whose curvature satisfies

$$\sigma_2(\tilde{g}^{-1}A_{\tilde{g}}^{t_0}) - \frac{\alpha}{16}|W_{\tilde{g}}|_{\tilde{g}}^2 > 0\,.$$

As an application for $t_0 = 1$, we obtain

Corollary 2.3.2 *Let (M, g) be a closed four–dimensional Riemannian manifold with positive scalar curvature. Fix $0 \leq \alpha \leq 1$. If*

$$\int_M \sigma_2(g^{-1}A_g^1)\, dV_g - \frac{\alpha}{16}\int_M |W_g|_g^2\, dV_g > 0\,,$$

then there is a conformal metric $\tilde{g} = e^{-2u}g$ whose curvature satisfies

$$\sigma_2(\tilde{g}^{-1}A_{\tilde{g}}) - \frac{\alpha}{16}|W_{\tilde{g}}|_{\tilde{g}}^2 > 0\,.$$

If we choose $\alpha = 1$, using Margerin result (see Theorem 2.1.2), we obtain:

Theorem 2.3.3 *Let (M, g) be a closed four–dimensional Riemannian manifold with positive scalar curvature. If*

$$\int_M \sigma_2(g^{-1}A_g^1)\, dV_g - \frac{1}{16}\int_M |W_g|_g^2\, dV_g > 0\,,$$

then M is diffeomorphic to either S^4 or $\mathbb{R}P^4$.

For the proof of Theorem 2.3.1, we will be concerned with the following equation for a conformal metric $\tilde{g} = e^{-2u}g$:

$$\sigma_2^{1/2}(g^{-1}A_{\tilde{g}}^t) = \frac{\sqrt{\alpha}}{4}|W_g|_g + fe^{2u}\,, \tag{2.2}$$

where $0 \leq \alpha \leq 1$ and f is a positive smooth function. Let $\sigma_1(g^{-1}A^1_g)$ be the trace of A^1_g with respect to the metric g. We have the following formula for the transformation of A^t under this conformal change of metric:

$$A^t_{\tilde{g}} = A^t_g + \nabla^2_g u + \frac{1-t}{2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g\,. \tag{2.3}$$

Since

$$A^t_g = A^1_g + \frac{1-t}{2}\sigma_1(g^{-1}A^1_g)g\,,$$

this formula follows easily from the standard formula for the transformation of the Schouten tensor (see [45]):

$$A^1_{\tilde{g}} = A^1_g + \nabla^2_g u + du \otimes du - \frac{1}{2}|\nabla_g u|^2_g g\,. \tag{2.4}$$

Using these formulas we may write (2.2) with respect to the background metric g

$$\sigma_2^{1/2}\left(g^{-1}\left(A^t_g + \nabla^2_g u + \frac{1-t}{2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g\right)\right) = \frac{\sqrt{\alpha}}{4}|W_g|_g + f(x)e^{2u}\,.$$

Definition 2.3.4 *For a conformal metric $\tilde{g} = e^{-2u}g$, we define the set*

$$\Lambda^+_{\tilde{g}} = \left\{ t \in [\delta, t_0] \mid A^t_{\tilde{g}} \in \Gamma^+_2 \quad \text{and} \quad \sigma_2^{1/2}(g^{-1}A^t_{\tilde{g}}) - \frac{\sqrt{\alpha}}{4}|W_g|_g > 0 \right\}.$$

In particular if $t \in \Lambda^+_{\tilde{g}}$ then $A^t_{\tilde{g}} \in \Gamma^+_2$.

Upper bound and higher order estimates

Throughout the sequel, (M, g) will be a closed four–dimensional Riemannian manifold with positive scalar curvature.

Since $R_g > 0$, there exists $\delta > -\infty$ such that A^δ_g is positive definite (i.e. $Ric - \frac{\delta}{6}R > 0$ on M). Moreover we can choose δ so small such that $\delta \in \Lambda^+_g$, in particular

$$\sigma_2^{1/2}(g^{-1}A^\delta_g) - \frac{\sqrt{\alpha}}{4}|W_g|_g > 0\,.$$

Note that δ depends only on $\|Rm\|$. For $t \in [\delta, t_0]$, consider the path of equations (in the sequel we use the notation $A^t_{u_t} = A^t_{g_t}$ for g_t given by $g_t = e^{-2u_t} g$)

$$\sigma_2^{1/2}(g^{-1}A^t_{u_t}) = \frac{\sqrt{\alpha}}{4}|W_g|_g + f e^{2u_t}\,, \tag{2.5}$$

where $0 \leq \alpha \leq 1$ and $f(x) = \sigma_2^{1/2}(g^{-1}A^\delta_g) - \frac{\sqrt{\alpha}}{4}|W_g|_g > 0$. Note that $u \equiv 0$ is a solution of (2.5) for $t = \delta$.

Proposition 2.3.5 (Upper bound) *Let $u_t \in C^2(M)$ be a solution of (2.5) for some $t \in [\delta, t_0]$, with $t \in \Lambda^+_{u_t}$. Then $u_t \leq \overline{\delta}$, where $\overline{\delta}$ depends only on $\|Rm\|$.*

Proof. From Newton's inequality $\frac{4}{\sqrt{6}}\sigma_2^{1/2} \leq \sigma_1$ (see Lemma A.2.3–(iv)), so for all $x \in M$

$$\frac{4}{\sqrt{6}}\frac{\sqrt{\alpha}}{4}|W_g|_g + \frac{4}{\sqrt{6}} f e^{2u_t} \leq \sigma_1(g^{-1}A^t_{u_t})\,.$$

Let $p \in M$ be the maximum of u_t, then using (2.3), since the gradient terms vanish at p and $(\Delta u_t)(p) \leq 0$

$$\begin{aligned} \frac{4}{\sqrt{6}}\frac{\sqrt{\alpha}}{4}(|W_g|_g)(p) + \frac{4}{\sqrt{6}} f(p) e^{2u_t(p)} &\leq \sigma_1(g^{-1}A^t_{u_t})(p) \\ &= \sigma_1(g^{-1}A^t_g)(p) + (3-2t)(\Delta u_t)(p) \\ &\leq \sigma_1(g^{-1}A^t_g)(p) \\ &\leq \sigma_1(g^{-1}A^\delta_g)(p)\,. \end{aligned}$$

This implies

$$\frac{4}{\sqrt{6}} f(p) e^{2u_t(p)} \leq \sigma_1(g^{-1}A^\delta_g)(p) - \frac{4}{\sqrt{6}}\frac{\sqrt{\alpha}}{4}(|W_g|_g)(p)\,,$$

where the last term has positive sign. Since M is compact, this implies $u_t \leq \overline{\delta}$, for some $\overline{\delta}$ depending only on $\|Rm\|$.

□

As we have seen in the previous chapter, an upper bound for the solutions of equation (2.5) gives us C^1 and C^2 estimates. For the proof see the Appendix, Theorem A.4.1. Thus we have

Proposition 2.3.6 (C^1 and C^2 estimates) *Let $u_t \in C^4(M)$ be a solution of (2.5) for some $t \in [\delta, t_0]$, with $t \in \Lambda^+_{u_t}$. Then*

$$\sup_M \left(|\nabla_g u_t|^2_g + |\nabla^2_g u_t|_g \right) \le C_1 \,,$$

where C_1 depends only on diam(M, g) and $\|\nabla^2 Rm\|$.

Lower bound

Proposition 2.3.7 (Lower bound) *Assume that for some $t \in [\delta, t_0]$ the following estimate holds*

$$\int_M \sigma_2(g^{-1} A^1_g)\, dV_g - \frac{\alpha}{16} \int_M |W_g|^2_g \, dV_g + \frac{1}{24}(1-t)(2-t) Y(M, [g])^2 = \mu_t > 0 \,. \tag{2.6}$$

Then there exist $\underline{\delta}$ depending only on diam(M, g) and $\|\nabla^2 Rm\|$ such that if $u_t \in C^2(M)$ is a solution of (2.5) and if $t \in \Lambda^+_{u_t}$ then $u_t \ge \underline{\delta}$.

Proof. Since $A^t_g = A^1_g + \frac{1-t}{2} \sigma_1(A^1_g) g$, we easily have

$$\sigma_2(A^t_g) = \sigma_2(A^1_g) + \frac{3}{2}(1-t)(2-t)\sigma_1(A^1_g)^2 \,.$$

Letting $\tilde{g} = e^{-2u_t} g$, since u_t is a solution of equation (2.5), we have

$$f^2 e^{4u_t} + \frac{\sqrt{\alpha}}{2} f |W_g|_g e^{2u_t} = \sigma_2(g^{-1} A^t_{u_t}) - \frac{\alpha}{16} |W_g|^2_g \,.$$

The left–hand side can be estimated by

$$f^2 e^{4u_t} + \frac{\sqrt{\alpha}}{2} f |W_g|_g e^{2u_t} \le C' e^{2u_t} \,,$$

where the positive constant C' depends only on $\|Rm\|$. So we get

$$\begin{aligned} C' e^{2u_t} &\ge \sigma_2(g^{-1} A^t_{u_t}) - \frac{\alpha}{16} |W_g|^2_g \\ &= e^{-4u_t} \left(\sigma_2(\tilde{g}^{-1} A^1_{u_t}) + \frac{1}{24}(1-t)(2-t) R^2_{\tilde{g}} \right) - \frac{\alpha}{16} |W_g|^2_g \,. \end{aligned}$$

Integrating this with respect to dV_g, we obtain

$$\begin{aligned} C' \int_M e^{2u_t}\, dV_g &\geq \int_M \sigma_2(\tilde{g}^{-1}A^1_{u_t})\, dV_{\tilde{g}} - \frac{\alpha}{16}\int_M |W_g|^2_g\, dV_g + \frac{1}{24}(1-t)(2-t)\int_M R^2_{\tilde{g}}\, dV_{\tilde{g}} \\ &= \int_M \sigma_2(g^{-1}A^1_g)\, dV_g - \frac{\alpha}{16}\int_M |W_g|^2_g\, dV_g + \frac{1}{24}(1-t)(2-t)\int_M R^2_{\tilde{g}}\, dV_{\tilde{g}} \\ &\geq \int_M \sigma_2(g^{-1}A^1_g)\, dV_g - \frac{\alpha}{16}\int_M |W_g|^2_g\, dV_g + \frac{1}{24}(1-t)(2-t)Y(M,[g])^2 = \mu_t > 0\,, \end{aligned}$$

where we used the conformal invariance of the integral of σ_2, and the fact that, for any conformal metric $g' \in [g]$ one has

$$\int_M R^2_{g'}\, dV_{g'} \geq Y(M,[g])^2\,.$$

This gives

$$\max_M u_t \geq \log\mu_t - C(\mathrm{diam}(M,g), \|Rm\|)\,.$$

Since $\max_M |\nabla_g u_t|_g \leq C_1$ by Proposition 2.3.6, this implies the Harnack inequality

$$\max_M u_t \leq \min_M u_t + C(\mathrm{diam}(M,g), \|\nabla^2 Rm\|)\,,$$

by simply integrating along a geodesic connecting points at witch u_t attains its maximum and minimum. Combining these two inequalities, we obtain

$$u_t \geq \min_M u_t \geq \log\mu_t - C =: \underline{\delta}\,,$$

where C depends only on $\mathrm{diam}(M,g)$ and $\|\nabla^2 Rm\|$.

□

Proof of the main result

Again, we will use continuity method. Our 1-parameter family of equations, for $t \in [\delta, t_0]$, is

$$\sigma_2^{1/2}(g^{-1}A^t_{u_t}) = \frac{\sqrt{\alpha}}{4}|W_g|_g + f(x)e^{2u_t}\,, \tag{2.7}$$

for $0 \geq \alpha \geq 1$, $f(x) = \sigma_2^{1/2}(g^{-1}A_g^\delta) - \frac{\sqrt{\alpha}}{4}|W_g|_g > 0$, and δ was chosen so that $\delta \in \Lambda_g^+$. Define

$$\mathcal{S} = \left\{ t \in [\delta, t_0] \mid \exists \, \text{a solution} \, u_t \in C^{2,\alpha}(M) \, \text{of} \, (2.7) \, \text{with} \, t \in \Lambda_{u_t}^+ \right\} .$$

Clearly with our choice of f, $u \equiv 0$ is a solution for $t = \delta$. Since $\delta \in \Lambda_g^+$, $\delta \in \mathcal{S}$, and $\mathcal{S} \neq \emptyset$. Let $t \in \mathcal{S}$, and u_t be a solution. By Proposition A.3.1, the linearized operator at u_t, $\mathcal{L}^t : C^{2,\alpha}(M) \to C^\alpha(M)$, is invertible. The implicit function theorem implies that $\mathcal{S}$ is open. By Proposition 2.3.5 we get a uniform upper bound on the solutions u_t, independent of t. We may then apply Proposition 2.3.6 to obtain uniform gradient and Hessian bounds on u_t, and by Proposition 2.3.7, we get a uniform lower bound. The compactness for a sequence of solutions in $\mathcal{S}$ follows from Theorem A.5.1. Finally from Ascoli–Arzelà's theorem, we get that $\mathcal{S}$ must be closed, therefore $\mathcal{S} = [\delta, t_0]$. The metric $\tilde{g} = e^{-2u_{t_0}} g$ then satisfies $t_0 \in \Lambda_{\tilde{g}}^+$, in particular

$$\sigma_2(g^{-1}A_{\tilde{g}}^{t_0}) - \frac{\alpha}{16}|W_g|_g^2 > 0 \, ,$$

i.e.

$$e^{-4u_{t_0}} \sigma_2(\tilde{g}^{-1}A_{\tilde{g}}^{t_0}) - e^{-4u_{t_0}} \frac{\alpha}{16}|W_{\tilde{g}}|_{\tilde{g}}^2 > 0 \, ,$$

which conclude the proof of Theorem 2.3.1. Now, the classification result follows from the fact that the metric $\tilde{g}$ satisfies the pointwise condition

$$|W|^2 + 2|E|^2 < \frac{1}{6}R^2 \, ,$$

where $E = Ric - \frac{1}{4}R\, g$, denote the trace-free Ricci tensor. Hence we obtain exactly the pointwise pinching condition in order to apply Margerin's result (see Theorem 2.1.2) and get that M is diffeomorphic to either $\mathbb{S}^4$ or $\mathbb{R}P^4$.

Chapter 3

Locally conformally flat manifolds

3.1 A classification of locally conformally flat manifolds

In this chapter we will concentrate on locally conformally flat manifolds of dimensions larger or equal to four. We recall that a n–dimensional Riemannian manifold (M^n, g) is said to be locally conformally flat (LCF) if it admits a coordinate covering $\{U_\alpha, \phi_\alpha\}$ such that the maps $\phi_\alpha : (U_\alpha, g_\alpha) \longrightarrow (\mathbb{S}^n, g_{\mathbb{S}^n})$ is a conformal map, where $g_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}^n$. This turns out to be equivalent to the vanishing of the Weyl tensor of g (if $n \geq 4$). In particular the full curvature tensor of g can be recovered from the Ricci tensor. Hence, on locally conformally flat metrics, conditions on the Ricci curvature impose strong restrictions on their metrics. Indeed, it is possible to classify locally conformally flat metrics with non–negative Ricci curvature. For more references see the papers by Schoen and Yau [44] for the compact case and Zhu [46] for the complete one.

Theorem 3.1.1 *If (M^n, g) is a complete locally conformally flat Riemannian manifold with non–negative Ricci curvature, then the universal cover $\widetilde{M}$ of M with the pull–back metric is either conformally equivalent to $\mathbb{S}^n$, $\mathbb{R}^n$ or is isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$. If M itself is compact, then $\widetilde{M}$ is either conformally equivalent to $\mathbb{S}^n$ or isometric to $\mathbb{R}^n$, $\mathbb{R} \times \mathbb{S}^{n-1}$.*

As we said in the Introduction, now we ask the following question: Is it possible to classify locally conformally flat manifolds satisfying an integral pinching assumption?

As usual, we define $\sigma_k(g^{-1}A)$ the k–th elementary function of the eigenvalues of $g^{-1}A$ (see the Appendix for general results), and choose the tensor (here t is a real number)

$$A_g^t = \frac{1}{n-2}\left(Ric_g - \frac{t}{2(n-1)}R_g g\right) .$$

A Gauss–Bonnet type formula was proved by Viaclovsky in [45] for locally conformally flat manifolds, which relates the classical Gauss–Bonnet–Chern to the integral of the $\frac{n}{2}$–th elementary function of the eigenvalues of the Schouten tensor. More precisely we have

$$\frac{1}{(2\pi)^{(n-1)}\left((n-2)!!\right)^2}\int_M \sigma_{\frac{n}{2}}(g^{-1}A_g^1)\, dV_g = \chi(M)\,,$$

where $\chi(M)$ denotes the Euler–Poincaré characteristic of M.

Hence positivity of the Euler–Poincaré characteristic can be read as an integral pinching condition on the curvature of the metric g. Under this assumption, Gursky [26] was able to classify four and six dimensional LCF manifolds with positive Euler–Poincaré characteristic. Namely

Theorem 3.1.2 (Gursky, 1994) *Let (M, g) be a closed, locally conformally flat, n–dimensional Riemannian manifold, $n = 4$ or 6, with non–negative scalar curvature. Then $\chi(M) \leq 2$. Furthermore, $\chi(M) = 2$ if and only if (M, g) is conformally diffeomorphic to the standard sphere, and $\chi(M) = 1$ if and only if (M, g) is conformally diffeomorphic to the standard real projective space.*

As pointed out by Gursky, this result is not true for higher dimensions. Take for example the product of $\mathbb{S}^4$ equipped with the canonical metric and a four dimensional hyperbolic space form. This manifold has zero scalar curvature and positive Euler–Poincaré characteristic. So, in view of generalizing the classification result to higher dimensions, one has to add some additional condition on the geometry of the manifold.

3.2 A sphere theorem on locally conformally flat even–dimensional manifolds

Our main result is the following (joint work with Djadli and Ndiaye, see [9]):

Theorem 3.2.1 *Let (M, g) be a closed, locally conformally flat, n–dimensional Riemannian manifold, $n \geq 8$ even, with positive scalar curvature and with positive Euler–Poincaré characteristic.*
There exists a constant $t_0 = t_0(n, diam(M, g), \|\nabla^2 Rm\|) < 1$ such that, if

$$A_g^t \in \Gamma^+_{\frac{n}{2}}\,,$$

for some $t \in (t_0, 1]$, then there exists a metric $\tilde{g}$ conformal to g such that $A^1_{\tilde{g}} \in \Gamma^+_{\frac{n}{2}}$. In particular $(M, \tilde{g})$ has non–negative Ricci curvature ($Ric_{\tilde{g}} \geq 0$).

Using the classification in Theorem 3.1.1 for compact, LCF manifolds with non–negative Ricci curvature and a vanishing type result, we can prove the following result

Theorem 3.2.2 *Let (M, g) be a closed, locally conformally flat, n–dimensional Riemannian manifold, $n \geq 8$ even, with positive scalar curvature and with positive Euler–Poincaré characteristic.*
There exists a constant $t_0 = t_0(n, diam(M, g), \|\nabla^2 Rm\|) < 1$ such that if

$$A_g^t \in \Gamma^+_{\frac{n}{2}}\,,$$

for some $t \in (t_0, 1]$, then M is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{RP}^n$.

For the proof of Theorem 3.2.1 and Theorem 3.2.2, we will be concerned with the following equation for a conformal metric $\tilde{g} = e^{-2u} g$:

$$\left(\sigma_{\frac{n}{2}}(g^{-1} A^t_{\tilde{g}})\right)^{2/n} = f e^{2u}\,, \tag{3.1}$$

where f is a positive function on M. As usual $\sigma_1(g^{-1} A^1_g)$ will be the trace of A^1_g with respect to the metric g. We have the following formula for the transformation of A^t_g under this conformal change of metric:

$$A^t_{\tilde{g}} = A^t_g + \nabla^2_g u + \frac{1-t}{n-2}(\Delta_g u) g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g \,. \tag{3.2}$$

Using this formula we may write (3.1) with respect to the background metric g

$$\sigma_{\frac{n}{2}}\left(g^{-1}\left(A^t_g + \nabla^2_g u + \frac{1-t}{n-2}(\Delta_g u) g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g\right)\right)^{2/n} = f(x) e^{2u} \,.$$

Remark 3.2.3 *The assumption that there exists a constant* $t_0 = t_0(n, diam(M,g), \|\nabla^2 Rm\|) < 1$ *such that*

$$A^t_g \in \Gamma^+_{\frac{n}{2}} \,,$$

for some $t \in (t_0, 1]$*, by the Guan–Viaclovsky–Wang inequality (see Theorem A.6.1 in the Appendix), does not imply that the metric* g *has non–negative Ricci curvature. This would be true, if one could get* $t_0 = 1$.

We need to point out that there is another result in the same direction due to Guan–Lin–Wang [22], where they showed, as a corollary of a more general result, that if (M, g) is a locally conformally flat manifold of even dimension, with positive Euler–Poincaré characteristic and with $A^1_g \in \Gamma_{\frac{n}{2}-1}$, then M is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{RP}^n$. Also in this case, there is not a relation between their and our assumption.

3.3 Upper bound and higher order estimates

Throughout the sequel, (M, g) will be a closed n–dimensional Riemannian manifold (n even) with positive scalar curvature and locally conformally flat. Since $R_g > 0$, there exists $\delta > -\infty$ such that $A^\delta_g \in \Gamma^+_{\frac{n}{2}}$. (for example we can take δ such that A^δ_g is positive definite, i.e. $Ric_g - \frac{\delta}{2(n-1)} R_g g > 0$ on M). Note that δ only depends on $\|Rm\|$. For

$t \in [\delta, 1]$, consider the path of equations (in the sequel we use the notation $A^t_{u_t} = A^t_{g_t}$ for g_t given by $g_t = e^{-2u_t} g$)

$$\sigma^{2/n}_{\frac{n}{2}}(g^{-1} A^t_{u_t}) = f e^{2u_t}\,, \tag{3.3}$$

where $f = \sigma^{2/n}_{\frac{n}{2}}(g^{-1} A^\delta_g) > 0$. Note that $u \equiv 0$ is a solution of (3.3) for $t = \delta$.

Proposition 3.3.1 (Upper bound) *Let $u_t \in C^2(M)$ be a solution of* (3.3) *for some $t \in [\delta, 1]$,*
with $A^t_{u_t} \in \Gamma^+_{\frac{n}{2}}$. Then $u_t \leq \overline{\delta}$, where $\overline{\delta}$ depends only on $\|Rm\|$.

Proof. From Newton's inequality (see Lemma A.2.3–(iv)) we have

$$\sigma^{\frac{2}{n}}_{\frac{n}{2}} \leq C_n \sigma_1\,,$$

for some $C_n > 0$. So for all $x \in M$

$$f e^{2u_t} \leq C_n \sigma_1(g^{-1} A^t_{u_t})\,.$$

Let $p \in M$ be a maximum of u_t, then using (3.2), since the gradient terms vanish at p and $(\Delta u_t)(p) \leq 0$,

$$\begin{aligned} f(p) e^{2u_t(p)} &\leq C_n \sigma_1(g^{-1} A^t_{u_t})(p) \\ &= C_n \sigma_1(g^{-1} A^t_g)(p) + C_n \frac{2n - 2 - nt}{n - 2} (\Delta u_t)(p) \\ &\leq C_n \sigma_1(g^{-1} A^t_g)(p) \\ &\leq C_n \sigma_1(g^{-1} A^\delta_g)(p)\,. \end{aligned}$$

Since M is compact, then $u_t \leq \overline{\delta}$, for some $\overline{\delta}$ depending only on $\|Rm\|$.

□

Once we have an upper bound for the solutions of equation (3.3), we immediately get C^1 and C^2 estimates. For the proof see in the Appendix, Theorem A.4.1. We have

Proposition 3.3.2 (C^1 and C^2 estimates) *Let $u_t \in C^4(M)$ be a solution of (3.3) for some $t \in [\delta, 1]$, with $A^t_{u_t} \in \Gamma_2^+$. Then*

$$\sup_M \left(|\nabla_g u_t|_g^2 + |\nabla_g^2 u_t|_g \right) \leq C_1 \,,$$

where C_1 depends only on n, diam(M, g) and $\|\nabla^2 Rm\|$.

Now, by the Yamabe equation for the conformal deformation of the scalar curvature, we have that

$$R_{g_t} = e^{2u_t} \left(R_g + 2(n-1)\Delta_g u_t - (n-1)(n-2)|\nabla_g u_t|_g^2 \right) \,.$$

So we obtained an uniform estimates for the scalar curvature of g_t, i.e.

Proposition 3.3.3 *Let $u_t \in C^4(M)$ be a solution of (3.3) for some $t \in [\delta, 1]$, with $A^t_{u_t} \in \Gamma_2^+$. Then*

$$0 < R_{g_t} \leq \Lambda \,,$$

where Λ is a positive constant depending only on n, diam(M, g) and $\|\nabla^2 Rm\|$.

3.4 Lower bound

Proposition 3.4.1 (Lower Bound) *Assume that for some $t \in [\delta, 1]$ the following estimate holds*

$$\int_M \sigma_{\frac{n}{2}}(g^{-1}A_g^1)\, dV_g + C_1(1-t)^{\frac{n}{2}} (Y(M,[g]))^{\frac{n}{2}} - C_2(1-t)^{\frac{n}{2}} \int_M R_{g_t}^{\frac{n}{2}}\, dV_{g_t} = \lambda_t > 0 \,,$$

for some positive constants C_1 and C_2 depending only on n. Then there exists $\underline{\delta}$ depending only on diam(M, g) and $\|\nabla^2 Rm\|$ such that if $u_t \in C^2(M)$ is a solution of (3.3) and if $A^t_{u_t} \in \Gamma^+_{\frac{n}{2}}$, then $u_t \geq \underline{\delta}$.

Proof. It is easy to see that the following formula holds

$$\sigma_{\frac{n}{2}}(g^{-1}A_g^t) = \sigma_{\frac{n}{2}}(g^{-1}A_g) + C_1(1-t)^{\frac{n}{2}} \sigma_1(g^{-1}A_g)^{\frac{n}{2}} + \sum_{i=1}^{\frac{n}{2}-1} c_{n,i} \left(\frac{1-t}{n-2} \right)^{\frac{n}{2}-i} \sigma_i(g^{-1}A_g)(\sigma_1(g^{-1}A_g))^{\frac{n}{2}-i} \,,$$

for some positive constants C_1 and $c_{n,i}$ depending only on n and i.

Since by assumption $A^t_{u_t} \in \Gamma^+_{\frac{n}{2}}$, we have $\sigma_i(g^{-1}_{u_t} A^t_{u_t}) > 0$ for all $1 \leq i \leq n/2$. So, iterating the previous formula, we can easily check that

$$\sigma_i(g^{-1}_{u_t} A^1_{u_t}) > -C_i(1-t)^i \left(\sigma_1(g^{-1}_{u_t} A^1_{u_t})\right)^i \,,$$

for some positive constants C_i depending only on n. Hence, by the previous formula, we have

$$\sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t}) \geq \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^1_{u_t}) + C_1(1-t)^{\frac{n}{2}} \sigma_1(g^{-1}_{u_t} A^1_{u_t})^{\frac{n}{2}} - C_2(1-t)^{\frac{n}{2}} \sigma_1(g^{-1}_{u_t} A^1_{u_t})^{\frac{n}{2}} \,.$$

On the other hand, since u_t is a solution of equation (3.3), we have

$$\sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t}) = e^{nu_t} \sigma_{\frac{n}{2}}(g^{-1} A^t_{u_t}) = e^{2nu_t} f^{\frac{n}{2}} \,,$$

or equivalently

$$e^{-nu_t} \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t}) = e^{nu_t} f^{\frac{n}{2}} \,.$$

Integrating on M this with respect to dV_g, we obtain

$$\begin{aligned} C \int_M e^{nu_t}\, dV_g &\geq \int_M e^{nu_t} f^{\frac{n}{2}}\, dV_g \\ &= \int_M e^{-nu_t} \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t})\, dV_g \\ &= \int_M \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^t_{u_t})\, dV_{g_{u_t}} \\ &\geq \int_M \sigma_{\frac{n}{2}}(g^{-1}_{u_t} A^1_{u_t})\, dV_{g_{u_t}} + C_1(1-t)^{\frac{n}{2}} \int_M R^{\frac{n}{2}}_{g_{u_t}}\, dV_{g_{u_t}} - C_2(1-t)^{\frac{n}{2}} \int_M R^{\frac{n}{2}}_{g_{u_t}}\, dV_{g_{u_t}} \,, \end{aligned}$$

where $C > 0$ is chosen so that $f^{\frac{n}{2}} \leq C$ (recall that, since $f = \sigma^{2/n}_{\frac{n}{2}}(g^{-1} A^\delta_g)$, C depends only on $\|Rm\|$).

Using Hölder inequality and the definition of the Yamabe invariant (which is positive), we get

$$\int_M R^{\frac{n}{2}}_{g_{u_t}}\, dV_{g_{u_t}} \geq (Y(M,[g]))^{\frac{n}{2}} \,.$$

Moreover, by the result of Viaclovsky in [45], we have the conformal invariance

$$\int_M \sigma_{\frac{n}{2}}(g_{u_t}^{-1}A_{u_t}^1)\,dV_{g_{u_t}} = \int_M \sigma_{\frac{n}{2}}(g^{-1}A_g^1)\,dV_g\,.$$

Thus we get

$$C\int_M e^{nu_t}\,dV_g \geq \int_M \sigma_{\frac{n}{2}}(g^{-1}A_g^1)\,dV_g + C_1(1-t)^{\frac{n}{2}}(Y(M,[g]))^{\frac{n}{2}} - C_2(1-t)^{\frac{n}{2}}\int_M R_{g_{u_t}}^{\frac{n}{2}}\,dV_{g_{u_t}} = \lambda_t > 0\,.$$

This gives

$$\max_M u_t \geq \frac{1}{n}\log\lambda_t - C(\mathrm{diam}(M,g),\|Rm\|)\,.$$

By Proposition 3.3.2, $\max_M |\nabla_g u_t|_g \leq C_1$. This implies the Harnack inequality

$$\max_M u_t \leq \min_M u_t + C(\mathrm{diam}(M,g),\|\nabla^2 Rm\|)\,,$$

by simply integrating along a geodesic connecting points at which u_t attains its maximum and minimum. Combining these two inequalities, we obtain

$$u_t \geq \min_M u_t \geq \frac{1}{n}\log\lambda_t - C =: \underline{\delta}\,,$$

where C only depends on $\mathrm{diam}(M,g)$ and $\|\nabla^2 Rm\|$. This ends the proof of the Lemma.

□

3.5 Proof of the main result

By hypothesis, there exist $t_0 = t_0(n,\mathrm{diam}(M,g),\|\nabla^2 Rm\|) < 1$ such that

$$A_g^{t_0} \in \Gamma_{\frac{n}{2}}^+\,.$$

This parameter t_0 will play the role of δ, i.e. will be the starting point in order to use the continuity method. Our one–parameter family of equations, for $t \in [t_0, 1]$, is

$$\sigma_{\frac{n}{2}}^{2/n}(g^{-1}A_{u_t}^t) = f(x)e^{2u_t}\,, \tag{3.4}$$

with $f(x) = \sigma_{\frac{n}{2}}^{2/n}(g^{-1}A_g^{t_0}) > 0$,. Define the set

$$\mathcal{S} = \left\{ t \in [t_0, 1] \mid \exists \, \text{a solution} \, u_t \in C^{2,\alpha}(M) \, \text{of} \; (3.4) \, \text{with} \, A^t_{u_t} \in \Gamma^+_{\frac{n}{2}} \right\} .$$

Clearly, with our choice of f, $u \equiv 0$ is a solution for $t = t_0$. By assumption, $t_0 \in \mathcal{S}$, and $\mathcal{S} \neq \emptyset$. Let $t \in \mathcal{S}$, and u_t be a solution. By Proposition A.3.1, the linearized operator at u_t, $\mathcal{L}^t : C^{2,\alpha}(M) \to C^{\alpha}(M)$, is invertible. The implicit function theorem tells us that $\mathcal{S}$ is open. By Proposition 3.3.1 we get a uniform upper bound on the solutions u_t, independent of t. We may then apply Proposition 3.3.2 to obtain uniform gradient and Hessian bounds on u_t. Now suppose that we prove a uniform lower bound as in Proposition 3.4.1. By Proposition A.5.1 and the classical Ascoli–Arzelà's theorem, we will get that $\mathcal{S}$ must be closed, therefore $\mathcal{S} = [t_0, 1]$. The metric $\tilde{g} = e^{-2u_1}g$ then satisfies $\sigma_k(A^1_{\tilde{g}}) > 0$ for all $1 \leq k \leq \frac{n}{2}$. The inequality on the Ricci curvature in Theorem 3.2.1 follows from Theorem A.6.1.

So, in order to conclude we need to prove that the hypothesis of Proposition 3.4.1 holds. But this can be proved easily, since by Proposition 3.3.3, we have an uniform estimate for the scalar curvature of g_t. Moreover, we get that the quantity

$$\int_M R_{g_t}^{\frac{n}{2}} \, dV_{g_t} = \int_M R_{g_t}^{\frac{n}{2}} e^{-nu_t} \, dV_g$$

is uniformly bounded. Hence for t_0 sufficiently close to 1, since the Euler–Poincaré characteristic of M is positive (i.e. larger or equal to 1), we can always assume that

$$\int_M \sigma_{\frac{n}{2}}(g^{-1}A^1_g) \, dV_g + C_1(1-t)^{\frac{n}{2}}(Y(M,[g]))^{\frac{n}{2}} - C_2(1-t)^{\frac{n}{2}} \int_M R_{g_t}^{\frac{n}{2}} \, dV_{g_t} = \lambda_t > 0 \, ,$$

for every $t \in [t_0, 1]$.

In Theorem 3.2.1 we have proved that if (M, g) is an even–dimensional, closed, locally conformally flat manifolds with positive scalar curvature, positive Euler-Poincaré characteristic, and "close to" be $\frac{n}{2}$–admissible, then M admits a metric $\tilde{g}$ which is $\frac{n}{2}$–admissible, i.e. with $A_{\tilde{g}} \in \Gamma^+_{\frac{n}{2}}$. In particular it turns out that the Ricci curvature ($Ric_{\tilde{g}}$) must be non–negative. By the classification we have that M must be conformally equivalent to either a space form or a finite quotient of a Riemannian $\mathbb{S}^{n-1}(c) \times \mathbb{S}^1$, for some $c > 0$. Moreover, for locally conformally flat manifolds which admit k–admissible metrics (i.e. metrics g such that $A_g \in \Gamma_k^+$) we have topological restrictions (see, for example, [20, 23]). In [20] the authors proved the following vanishing theorem

Proposition 3.5.1 *Let (M, g) be a closed, locally conformally flat manifold, with $A_g \in \Gamma_k^+$, $k < n/2$. Then the q-th Betti number $b_q = 0$, for*

$$\frac{n-2k}{2} + 1 \leq q \leq \frac{n+2k}{2} - 1\,.$$

Since $A_{\tilde{g}} \in \Gamma^+_{\frac{n}{2}} \subset \Gamma^+_{\frac{n}{2}-1}$, we can apply this proposition to the case $k = \frac{n}{2} - 1$ to get that the q-th Betti number

$$b_q = 0, \text{ for } 2 \leq q \leq n-2\,.$$

Since the Euler characteristic can be defined in terms of the Betti numbers, by Poincaré duality, we get that

$$\chi(M) = 2 - 2b_1\,.$$

Then $0 < \chi(M) \leq 2$, which forces the manifold M to be diffeomorphic to either $\mathbb{R}P^n$ (if $\chi(M) = 1$) or $\mathbb{S}^n$ (if $\chi(M) = 2$).

Appendix A

Fully nonlinear elliptic equations

A.1 Preliminaries

Let S_n be the space of $n \times n$ real symmetric matrices, and $F(A, x)$ be a function on $S_n \times M$ which is of class C^2.

Definition A.1.1 *We say that F is uniformly elliptic if there are two positive constants λ, Λ (called the ellipticity constants) such that for any $A \in S_n$ and $x \in M$*

$$\lambda\|B\| \leq F(A+B, x) - F(A, x) \leq \Lambda\|B\| \qquad \forall B \geq 0\,.$$

We say that $B \geq 0$ whenever B is a non–negative definite symmetric matrix and

$$\|B\| = \sup_{X \in TM,\ |X|=1} |B_{ij}X^j|_g\,.$$

Remark A.1.2 *One can easily check that a linear operator $L(u) = a_{ij}(x)\nabla^2_{ij}u$, where $a_{ij}(x)$ is a symmetric matrix with eigenvalues in $[\lambda, \Lambda]$, is uniformly elliptic with ellipticity constants λ, $n\Lambda$.*

We can extend the function F to the whole space of $n \times n$ real matrices by $F(A, x) = F(\frac{1}{2}(A + A^T), x)$. Then F is a function of $n \times n$ variables a_{ij} and x. If we define

$$F^{ij}(A, x) = \frac{\partial F}{\partial a_{ij}}(A, x)\,.$$

It is clear that if A and B are symmetric, then the differential $dF(A,x)B = F^{ij}(A,x)B_{ij}$ does not depend on the extension of F. Now if F is uniformly elliptic with ellipticity constants λ and Λ then

$$\lambda|X|_g^2 \leq F^{ij}(A,x)X_iX_j \leq \Lambda|X|_g^2 \quad \forall A \in S_n \;\; \forall x \in M \;\; \forall X \in TM\,. \tag{A.1}$$

On the other hand, (A.1) implies that F is uniformly elliptic with ellipticity constants λ, $n\Lambda$.

A.2 The σ_k–curvature

Let (M,g), a compact, smooth, n–dimensional Riemannian manifold without boundary. Given a section A of the bundle of symmetric 2–tensors, we can use the metric to raise an index and view A as a tensor of type $(1,1)$, or equivalently as a section of $End(TM)$. This allows us to define $\sigma_k(g^{-1}A)$ the k–elementary function of the eigenvalues of $g^{-1}A$. More precisely we define

Definition A.2.1 *Let $(\lambda_1,\cdots,\lambda_n) \in \mathbb{R}^n$. We view the k–elementary symmetric function as a function on $\mathbb{R}^n$:*

$$\sigma_k(\lambda_1,\cdots,\lambda_n) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}\,,$$

and we define

$$\Gamma_k^+ = \bigcap_{1\leq j\leq k} \{\sigma_j(\lambda_1,\cdots,\lambda_n) > 0\} \subset \mathbb{R}^n\,,$$

For a symmetric linear transformation $A : V \to V$, where V is an n–dimensional inner product space, the notation $A \in \Gamma_k^+$ will mean that the eigenvalues of A lie in the corresponding set. We note that this notation also makes sense for a symmetric 2–tensor on a Riemannian manifold. If $A \in \Gamma_k^+$, let $\sigma_k^{1/k}(A) = \{\sigma_k(A)\}^{1/k}$.

Definition A.2.2 *Let $A : V \to V$, where V is an n–dimensional inner product space. The $(k-1)$–Newton transformation associated to A is*

$$T_{(k-1)}(A) = \sum_{j=0}^{k-1} (-1)^{k-1-j} \sigma_j(A) A^{k-1-j} \,.$$

Also, for $t \in \mathbb{R}$ we define the linear transformation

$$L^t(A) = T_{(k-1)}(A) + \frac{1-t}{n-2} \sigma_1(T_{(k-1)}(A)) \cdot I \,.$$

We have the following list of properties (the proofs can be found in [5])

Lemma A.2.3 (i) *Γ_k^+ is an open convex cone with vertex at the origin, and we have the following sequence of inclusions*

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+ \,.$$

(ii) *If $A \in \Gamma_k^+$, then $T_{k-1}(A)$ is positive definite. Hence for all $t \leq 1$, $L^t(A)$ is positive definite.*

(iii) *We have the identities*

$$T_{k-1}(A)^{ij} A_{ij} = k \, \sigma_k(A) \,,$$

$$T_{k-1}(A)^{ll} = (n-k+1) \sigma_{k-1}(A) \,.$$

(iv) *If $A \in \Gamma_k^+$, then*

$$\sigma_{k-1}(A) \geq \frac{k}{n-k+1} \binom{n}{k}^{\frac{1}{k}} \sigma_k(A)^{\frac{(k-1)}{k}} \,.$$

(v) *If A and B are symmetric linear transformations, $A, B \in \Gamma_k^+$, then $\forall \rho \in [0,1]$, $\rho A + (1-\rho) B \in \Gamma_k^+$, and*

$$\sigma_k^{\frac{1}{k}}(\rho A + (1-\rho)B) \geq \rho \sigma_k^{\frac{1}{k}}(A) + (1-\rho) \sigma_k^{\frac{1}{k}}(B) \,.$$

In particular this gives the concavity of the function $\sigma_k^{\frac{1}{k}}$ in the cone Γ_k^+.

Lemma A.2.4 *If $A : \mathbb{R} \to Hom(V, V)$, then*

$$\frac{d}{ds}\sigma_k(A)(s) = \sum_{i,j} T_{(k-1)}(A)_{ij}(s)\frac{d}{ds}(A)_{ij}(s)\,,$$

i.e, the $(k-1)$–Newton transformation is what arises when we differentiate σ_k.

We choose the tensor (here t is a real number)

$$A^t_g = \frac{1}{n-2}\left(Ric_g - \frac{t}{2(n-1)}R_g g\right)\,,$$

where Ric_g and R_g denote the Ricci and the scalar curvature of g respectively. Note that for $t = 1$, A^1_g is the classical Schouten tensor (see [1]). Hence, with our notations, $\sigma_k(g^{-1}A^t_g)$ denotes the k–elementary symmetric function of the eigenvalues of $g^{-1}A^t_g$. We will be concerned with the following equation for a conformal metric $\tilde{g} = e^{-2u}g$:

$$\text{(A.2)} \qquad \left(\sigma_k(g^{-1}A^t_{\tilde{g}})\right)^{1/k} = fe^{2u}\,,$$

where f is a positive function on M. Let $\sigma_1(g^{-1}A^1_g)$ be the trace of A^1_g with respect to the metric g. We have the following formula for the transformation of A^t_g under this conformal change of metric:

$$\text{(A.3)} \qquad A^t_{\tilde{g}} = A^t_g + \nabla^2_g u + \frac{1-t}{n-2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g\,.$$

Since

$$A^t_g = A^1_g + \frac{1-t}{n-2}\sigma_1(g^{-1}A^1_g)g\,,$$

this formula follows easily from the standard formula for the transformation of the Schouten tensor (see [45]):

$$\text{(A.4)} \qquad A^1_{\tilde{g}} = A^1_g + \nabla^2_g u + du \otimes du - \frac{1}{2}|\nabla_g u|^2_g g\,.$$

Using this formula we may write (A.2) with respect to the background metric g

$$\sigma_k\left(g^{-1}\left(A^t_g + \nabla^2_g u + \frac{1-t}{n-2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2_g g\right)\right)^{1/k} = f(x)e^{2u}\,.$$

If we define the 2–tensor (see the previous section)

$$F^{ij} = \frac{\partial}{\partial a_{ij}} \left(\sigma_k (g^{-1} A^t_{\tilde{g}})^{1/k} \right) ,$$

from Lemma A.2.3–(iii) we get

$$F^{ij} = \frac{1}{k} \sigma_k (g^{-1} A^t_{\tilde{g}})^{\frac{1}{k}-1} T_{k-1} (g^{-1} A^t_{\tilde{g}})^{ij} . \tag{A.5}$$

By this, we immediately get

Lemma A.2.5 *Let* $u \in C^2(M)$ *be a solution of equation* (A.2) *for some* $t \leq 1$ *and let* $\tilde{g} = e^{-2u} g$*. Assume that* $A^t_{\tilde{g}} \in \Gamma^+_k$*. Then the 2-tensor* F^{ij} *is positive definite. Moreover, we have the identity*

$$F^{ij} W_{ij} = f e^{2u},$$

where $W_{ij} = (A^t_{\tilde{g}})_{ij}$*. Under this notation, we also have the following*

$$F^{ij} W_{ij,k} = (F^{ij} W_{ij})_k ,$$

$$F^{ij} W_{ij,kk} \geq (F^{ij} W_{ij})_{kk} .$$

Proof. The first identity follows immediately by formula (A.5) and Lemma A.2.3–(iii). Moreover, the positivity of F^{ij} comes from the positivity of the $(k-1)$–Newton transformation and $\sigma_k(g^{-1} A^t_{\tilde{g}})$. If we take the k–covariant derivative of the equation (A.2), we get

$$\begin{aligned} \nabla_k \left(\left(\sigma_k (g^{-1} A^t_{\tilde{g}}) \right)^{1/k} \right) &= \nabla_k \left(f e^{2u} \right) \\ &= \nabla_k (F^{ij} W_{ij}) , \end{aligned}$$

where in the last equality we used the first identity. Now differentiating the left–hand side and using the definition of F^{ij}, we obtain

$$\frac{\partial}{\partial a_{ij}} \left(\sigma_k (g^{-1} A^t_{\tilde{g}})^{1/k} \right) W_{ij,k} = F^{ij} W_{ij,k} = (F^{ij} W_{ij})_k .$$

To get the second estimate, we differentiate again the previous equation to get

$$\begin{aligned}
(F^{ij}W_{ij})_{kk} &= \nabla_k \left(\frac{\partial}{\partial a_{ij}} \left(\sigma_k (g^{-1}A^t_{\tilde{g}})^{1/k} \right) W_{ij,k} \right) \\
&= \nabla_k \left(\frac{\partial}{\partial a_{ij}} \left(\sigma_k (g^{-1}A^t_{\tilde{g}})^{1/k} \right) \right) W_{ij,k} + \frac{\partial}{\partial a_{ij}} \left(\sigma_k (g^{-1}A^t_{\tilde{g}})^{1/k} \right) W_{ij,kk} \\
&= \frac{\partial^2}{\partial a_{ij} a_{lm}} \left(\sigma_k (g^{-1}A^t_{\tilde{g}})^{1/k} \right) W_{lm,k} W_{ij,k} + \frac{\partial}{\partial a_{ij}} \left(\sigma_k (g^{-1}A^t_{\tilde{g}})^{1/k} \right) W_{ij,kk} \\
&\leq \frac{\partial}{\partial a_{ij}} \left(\sigma_k (g^{-1}A^t_{\tilde{g}})^{1/k} \right) W_{ij,kk} \\
&= F^{ij} W_{ij,kk} \,,
\end{aligned}$$

where we used the concavity of $\sigma_k^{\frac{1}{k}}$ (see Lemma A.2.3–(v)), i.e.

$$\frac{\partial^2}{\partial a_{ij} a_{lm}} \left(\sigma_k (g^{-1}A^t_{\tilde{g}})^{1/k} \right) \leq 0 \,.$$

□

Since we consider differential equations on a compact Riemannian manifold (M, g), all derivatives are the covariant derivatives with respect to the metric g. So, if u is a smooth function on M, when we consider derivatives of order higher than two we should get some curvature terms if we change the order of differentiation. If we assume $g_{ij} = \delta_{ij}$ at the point we are evaluating, without loss of generality, we have the following formulae (see for example [1])

Lemma A.2.6

$$\begin{aligned}
u_{ij} &= u_{ji} \\
u_{kij} &= u_{ijk} + R_{mikj} u_m \\
u_{ijkl} &= u_{ijlk} + R_{mjkl} u_{mi} + R_{mikl} u_{mj} \\
u_{kkij} &= u_{ijkk} + 2R_{mikj} u_{mk} - R_{mj} u_{mi} - R_{mi} u_{mj} - R_{mi,j} u_m + R_{mikj,k} u_m \,.
\end{aligned}$$

Hence, if M is compact, we get

$$\begin{aligned}
u_{kij} &= u_{ijk} + O(|\nabla_g u|_g) \\
u_{kkij} &= u_{ijkk} + O(|\nabla^2_g u|_g + |\nabla_g u|_g) \,.
\end{aligned}$$

A.3 Ellipticity

We will discuss the ellipticity properties of equation (A.2).

Proposition A.3.1 (Ellipticity property) *Let $u \in C^2(M)$ be a solution of equation* (A.2) *for some $t \leq 1$ and let $\tilde{g} = e^{-2u}g$. Assume that $A^t_{\tilde{g}} \in \Gamma^+_k$. Then the linearized operator at u, $\mathcal{L}^t : C^{2,\alpha}(M) \to C^\alpha(M)$, is elliptic and invertible* $(0 < \alpha < 1)$.

Proof. Define the operator

$$F_t[u, \nabla_g u, \nabla^2_g u] = \sigma_k(g^{-1}A^t_{\tilde{g}}) - f(x)^k e^{2ku}\,,$$

so that solutions of the equation (A.2) are exactly the zeroes of F_t. Define the function $u_s = u + s\varphi$, then the linearization at u of the operator F_t is defined by

$$\begin{aligned}\mathcal{L}^t(\varphi) &= \frac{d}{ds}F_t[u_s, \nabla_g u_s, \nabla^2_g u_s]\Big|_{s=0} \\ &= \frac{d}{ds}\left(\sigma_k(g^{-1}A^t_{\tilde{g}})\right)\Big|_{s=0} - \frac{d}{ds}\left(f(x)^k e^{2ku_s}\right)\Big|_{s=0}\,.\end{aligned}$$

From Lemma A.2.4 we have

$$\frac{d}{ds}\left(\sigma_k(g^{-1}A^t_{\tilde{g}})\right)\Big|_{s=0} = T_{k-1}(g^{-1}A^t_{\tilde{g}})_{ij}\frac{d}{ds}\left((A^t_{\tilde{g}})_{ij}\right)\Big|_{s=0}\,.$$

We compute

$$\frac{d}{ds}\left((A^t_{\tilde{g}})_{ij}\right)\Big|_{s=0} = (\nabla^2_g\varphi)_{ij} + \frac{1-t}{n-2}(\Delta_g\varphi)g_{ij} - (2-t)\nabla_g u\cdot\nabla_g\varphi\, g_{ij} + 2du\otimes d\varphi\,.$$

Easily we have also

$$\frac{d}{ds}\left(f(x)^k e^{2ku_s}\right)\Big|_{s=0} = 2kf(x)^k e^{2ku}\,\varphi\,.$$

Putting all together, we conclude

$$\mathcal{L}^t(\varphi) = T_{k-1}(g^{-1}A^t_{\tilde{g}})_{ij}\left((\nabla^2_g\varphi)_{ij} + \frac{1-t}{n-2}(\Delta_g\varphi)g_{ij}\right) - 2kf(x)^k e^{2ku}\,\varphi + \cdots$$

where the last terms denote additional ones witch are linear in $\nabla_g\varphi$. The first term of the linearization is exactly the one defined in A.2.2, i.e.

$$L^t(A^t_{\tilde{g}})_{ij} = T_{k-1}(A^t_{\tilde{g}})_{ij} + \frac{1-t}{n-2}T_{k-1}(A^t_{\tilde{g}})_{pp}\,\delta_{ij}\,.$$

So finally, we have

$$\mathcal{L}^t(\varphi) = L^t(A^t_{\tilde{g}})_{ij}(\nabla^2_g \varphi)_{ij} - 2kf(x)^k e^{2ku}\,\varphi + \cdots$$

Since $A^t_{\tilde{g}} \in \Gamma^+_k$, by Lemma A.2.3, we have that the tensor $L^t(A^t_{\tilde{g}})$ is positive definite. So, the linearized operator at any solution u must be elliptic. Note also that, by the previous formula, the operator is of the form

$$\mathcal{L}^t(\varphi) = E(\varphi) - c(x)\varphi\,,$$

where $E(\varphi)$ is a second order linear elliptic operator and $c(x)$ is a strictly positive function on M, since $c(x) = 2kf(x)^k e^{2ku}$ and $f(x) > 0$. This allows us to invert this operator between the Hölder spaces $C^{2,\alpha}(M)$ and $C^\alpha(M)$ (see for instance [19]).

□

A.4 C^1 and C^2 estimates

We want to prove a priori gradient and Hessian estimates for solutions of the equation (A.2). These problems have been investigate by many authors, like Guan–Wang [25] and Gursky–Viaclovsky [30, 28]. Here we present a recent proof done by Chen [14], which allows us to incorporate two statements in one.

Theorem A.4.1 *Let $u_t \in C^4(M)$ be a solution of* (A.2)*, in a geodesic ball B_r, for some $t \le 1$ and let $\tilde{g} = e^{-2u_t}g$. Assume that $A^t_{\tilde{g}} \in \Gamma^+_k$. Then*

$$\sup_{B_{\frac{r}{2}}} \left(|\nabla_g u_t|^2_g + |\nabla^2_g u_t|_g\right) \le C\left(1 + \sup_{B_r} e^{2u_t}\right)\,,$$

where $C = C\left(n, k, r, \|\nabla^2 Rm\|, \|f\|_{C^2}\right) > 0$ but is independent of t and $\inf f$.

Proof. Assume $g_{ij} = \delta_{ij}$ at the point we are evaluating. Let $u = u_t$ and , by (A.3),

$$W = A^t_{\tilde{g}} = \nabla^2_g u + \frac{1-t}{n-2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|^2 g + A^t_g\,.$$

Step 1: $\Delta_g u$ has a lower bound.

By the condition $W \in \Gamma_k^+ \subset \Gamma_1^+$, we have

$$0 < \sigma_1(g^{-1}W) = a\Delta_g u - b|\nabla_g u|_g^2 + \sigma_1(g^{-1}A_g^t)\,,$$

where

$$a = \frac{2n-2-nt}{n-2} \quad \text{and} \quad b = \frac{4-3t}{2}$$

are positive constants for all $t \leq 1$. Note that this condition is equivalent to the positivity of the scalar curvature of $\tilde{g}$. Since M is compact we have that

$$\sigma_1(g^{-1}A_g^t) = \frac{4-3t}{4} R_g$$

must be bounded. This gives us a lower bound for $\Delta_g u$ and

$$|\nabla_g u|_g^2 < C(\Delta_g u + 1)\,, \tag{A.6}$$

where $C = C(n, \|Rm\|) > 0$. Let

$$H = \eta(\Delta_g u + |\nabla_g u|_g^2) = \eta L\,,$$

where η in a cutoff function, $0 \leq \eta \leq 1$, such that $\eta = 1$ in $B_{\frac{r}{2}}$ and $\eta = 0$ outside B_r, and also we have $|\nabla_g \eta|_g < C\frac{\sqrt{\eta}}{r}$ and $|\nabla_g^2 \eta| < \frac{C}{r^2}$. Without loss of generality, we assume $r = 1$. Since we have a lower bound for $\Delta_g u$, we immediately get a lower bound for L. Hence we only need to get an upper bound of L.

Step 2: $\Delta_g u$ is bounded.

Suppose $x_0 \in M$ is the maximal point of H. Then at x_0 we have

$$H_i = \nabla_i H = \eta_i L + \eta L_i = \eta_i(\Delta_g u + |\nabla_g u|_g^2) + \eta(u_{kki} + 2u_k u_{ki}) = 0\,, \tag{A.7}$$

and

$$H_{ij} = \eta_{ij} L + \eta_i L_j + \eta_j L_i + \eta L_{ij} = \left(\eta_{ij} - 2\frac{\eta_i \eta_j}{\eta}\right) L + \eta L_{ij}$$

is negative semi–definite, where in the second equality we have used (A.7). Moreover,

$$L_{ij} = u_{kkij} + 2u_{ki}u_{kj} + 2u_k u_{kij}\,. \tag{A.8}$$

Now we define the tensor

$$\overline{F}^{ij} = F^{ij} + \frac{1-t}{n-2} F^{ll}\delta^{ij}\,,$$

which is positive definite by the positivity of F^{ij} . If we sum, by the conditions on η, we get

$$0 \geq \overline{F}^{ij} H_{ij} = \overline{F}^{ij}\left(\left(\eta_{ij} - 2\frac{\eta_i\eta_j}{\eta}\right)L + \eta L_{ij}\right) \geq -C\sum_i \overline{F}^{ii} L + \eta\overline{F}^{ij} L_{ij}\,. \tag{A.9}$$

By (A.8) we have

$$\overline{F}^{ij} L_{ij} = \overline{F}^{ij}\left(u_{kkij} + 2u_{ki}u_{kj} + 2u_k u_{kij}\right)\,.$$

Changing the order of the covariant derivatives and Lemma A.2.6, using the definition of $\overline{F}^{ij}$ and (A.6) give

$$\begin{aligned}
\overline{F}^{ij} L_{ij} &\geq F^{ij}u_{ijkk} + \frac{1-t}{n-2}F^{ii}u_{llkk} + \overline{F}^{ij}(2u_{ki}u_{kj} + 2u_k u_{ijk}) - C\sum_i F^{ii}\left(1 + |\nabla_g^2 u|_g^{\frac{3}{2}}\right) \\
&= A + B - C\sum_i F^{ii}\left(1 + |\nabla_g^2 u|_g^{\frac{3}{2}}\right)\,.
\end{aligned}$$

In order to compute A, notice that

$$\begin{aligned}
W_{ij,kk} = \nabla_k\nabla_k W_{ij} &= u_{ijkk} + \frac{1-t}{n-2}u_{llkk}\delta_{ij} + u_{ikk}u_j + 2u_{ik}u_{jk} + u_i u_{jkk} \\
&\quad -(2-t)u_l u_{lkk}\delta_{ij} - (2-t)|\nabla_g^2 u|_g^2\delta_{ij} + (A_g^t)_{ij,kk}\,.
\end{aligned}$$

Then

$$\begin{aligned}
A &= F^{ij}u_{ijkk} + \frac{1-t}{n-2}F^{ii}u_{llkk} \\
&= F^{ij}\big(W_{ij,kk} - u_{ikk}u_j - 2u_{ik}u_{jk} - u_{jkk}u_i + (2-t)u_{lkk}u_l\delta_{ij} \\
&\quad +(2-t)|\nabla_g^2 u|_g^2\delta_{ij} + (A_g^t)_{ij,kk}\big) \\
&\geq F^{ij}W_{ij,kk} - F^{ij}\left(2u_{ikk}u_j + 2u_{ik}u_{jk} - (2-t)u_{lkk}u_l\delta_{ij}\right) \\
&\quad +(2-t)\sum_i F^{ii}|\nabla_g^2 u|_g^2 - C\sum_i F^{ii}\left(1 + |\nabla_g^2 u|_g^{\frac{3}{2}}\right)\,.
\end{aligned}$$

Changing the order of covariant derivatives again yields

$$\begin{aligned} A &\geq F^{ij}W_{ij,kk} - F^{ij}\left(2u_{kki}u_j + 2u_{ik}u_{jk} - (2-t)u_{kkl}u_l\delta_{ij}\right) \\ &\quad +(2-t)\sum_i F^{ii}|\nabla_g^2 u|_g^2 - C\sum_i F^{ii}\left(1+|\nabla_g^2 u|_g^{\frac{3}{2}}\right). \end{aligned}$$

Now we replace the terms u_{kki} and u_{kkl} using (A.7) to get

$$\begin{aligned} A &\geq F^{ij}W_{ij,kk} + F^{ij}\left(2\frac{\eta_i}{\eta}u_j L + 4u_k u_j u_{ki} - 2u_{ik}u_{jk}\right) \\ &\quad -(2-t)F^{ij}\left(\frac{\eta_l}{\eta}u_l L\delta_{ij} + 2u_k u_l u_{kl}\delta_{ij}\right) \\ &\quad +(2-t)\sum_i F^{ii}|\nabla_g^2 u|_g^2 - C\sum_i F^{ii}\left(1+|\nabla_g^2 u|_g^{\frac{3}{2}}\right). \end{aligned}$$

Using (A.6) again and the conditions on η, finally, we have

$$\begin{aligned} A &\geq F^{ij}W_{ij,kk} + F^{ij}\left(4u_k u_j u_{ki} - 2u_{ik}u_{jk} - 2(2-t)u_k u_l u_{kl}\delta_{ij}\right) \\ &\quad +(2-t)\sum_i F^{ii}|\nabla_g^2 u|_g^2 - C\eta^{-\frac{1}{2}}\sum_i F^{ii}|\nabla_g u|_g L - C\sum_i F^{ii}\left(1+|\nabla_g^2 u|_g^{\frac{3}{2}}\right). \end{aligned}$$

To estimate B, we use the formula

$$W_{ij,k} = u_{ijk} + \frac{1-t}{n-2}u_{llk}\delta_{ij} + u_{ik}u_{jk} + u_{jk}u_i - (2-t)u_{lk}u_l\delta_{ij} + (A_g^t)_{ij,k}.$$

Hence we obtain

$$\begin{aligned} B &= \overline{F}^{ij}(2u_{ki}u_{kj} + 2u_k u_{ijk}) \\ &= F^{ij}(2u_{ki}u_{kj} + 2u_k u_{ijk}) + \frac{1-t}{n-2}F^{ii}(2u_{ki}u_{ki} + 2u_k u_{llk}) \\ &\geq 2u_k F^{ij}W_{ij,k} + 2F^{ij}u_{ki}u_{kj} - 4F^{ij}u_k u_{ik}u_j + 2(2-t)F^{ij}u_k u_{lk}u_l\delta_{ij} \\ &\quad +\frac{2(1-t)}{n-2}\sum_i F^{ii}|\nabla_g^2 u|_g^2 - C\sum_i F^{ii}\left(1+|\nabla_g^2 u|_g^{\frac{3}{2}}\right). \end{aligned}$$

Combining A and B together, we find that

$$\begin{aligned} \overline{F}^{ij}L_{ij} &\geq A + B - C\sum_i F^{ii}\left(1+|\nabla_g^2 u|_g^{\frac{3}{2}}\right) \\ &\geq F^{ij}W_{ij,kk} + 2u_k F^{ij}W_{ij,k} + F^{ij}\left(4u_k u_j u_{ki} - 2u_{ik}u_{jk} - 2(2-t)u_k u_l u_{kl}\delta_{ij}\right) \\ &\quad +2F^{ij}u_{ki}u_{kj} - 4F^{ij}u_k u_{ik}u_j + 2(2-t)F^{ij}u_k u_{lk}u_l\delta_{ij} \\ &\quad +(2-t)\sum_i F^{ii}|\nabla_g^2 u|_g^2 - C\eta^{-\frac{1}{2}}\sum_i F^{ii}|\nabla_g u|_g L - C\sum_i F^{ii}\left(1+|\nabla_g^2 u|_g^{\frac{3}{2}}\right) \\ &\quad +\frac{2(1-t)}{n-2}\sum_i F^{ii}|\nabla_g^2 u|_g^2. \end{aligned}$$

After the cancellations, finally we get

$$\begin{aligned}\overline{F}^{ij}L_{ij} \;\; &\geq \;\; F^{ij}W_{ij,kk} + 2u_kF^{ij}W_{ij,k} + \Lambda \sum_i F^{ii}|\nabla_g^2 u|_g^2 \\ &\quad -C\eta^{-\frac{1}{2}}\sum_i F^{ii}|\nabla_g u|_g L - C\sum_i F^{ii}\left(1+|\nabla_g^2 u|_g^{\frac{3}{2}}\right) ,\end{aligned}$$

where

$$\Lambda = (2-t) + \frac{2(1-t)}{n-2} > 0$$

is positive, for all $t \leq 1$.

Now, returning to inequality (A.9), and applying η on both sides produces

$$\begin{aligned}0 \;\; &\geq \;\; \eta\overline{F}^{ij}H_{ij} \geq -C\eta\sum_i \overline{F}^{ii}L + \eta^2\overline{F}^{ij}L_{ij} \\ &\geq \;\; \eta^2F^{ij}W_{ij,kk} + 2\eta^2u_kF^{ij}W_{ij,k} + \Lambda\eta^2\sum_i F^{ii}|\nabla_g^2u|_g^2 \\ &\quad -C\eta\sum_i F^{ii}L - C\eta^{\frac{3}{2}}\sum_i F^{ii}|\nabla_g u|_g L - C\eta^2\sum_i F^{ii}\left(1+|\nabla_g^2u|_g^{\frac{3}{2}}\right) .\end{aligned}$$

Using inequality (A.6) again and the fact that $L \leq C|\nabla_g^2 u|_g$, we arrive at

$$\begin{aligned}0 \;\; &\geq \;\; \eta^2F^{ij}W_{ij,kk} + 2\eta^2u_kF^{ij}W_{ij,k} + \Lambda\eta^2\sum_i F^{ii}|\nabla_g^2u|_g^2 \\ &\quad -C\sum_i F^{ii}\left(1+\eta|\nabla_g^2u|_g + (\eta|\nabla_g^2u|_g)^{\frac{3}{2}}\right) .\end{aligned}$$

Here is the key point of the estimates, where we use the equation. By Lemma A.2.5 we have $F^{ij}W_{ij,k} = (F^{ij}W_{ij})_k = (fe^{2u})_k$ and $F^{ij}W_{ij,kk} \geq (F^{ij}W_{ij})_{kk} = (fe^{2u})_{kk}$. So we obtain

$$\begin{aligned}0 \;\; &\geq \;\; \eta^2(fe^{2u})_{kk} + 2\eta^2u_k(fe^{2u})_k + \Lambda\eta^2\sum_i F^{ii}|\nabla_g^2u|_g^2 \\ &\quad -C\sum_i F^{ii}\left(1+\eta|\nabla_g^2u|_g + (\eta|\nabla_g^2u|_g)^{\frac{3}{2}}\right) \\ &= \;\; \eta^2(f_{kk} + 2fu_{kk} + 4f_ku_k + 4f|\nabla_g u|_g^2)e^{2u} + \eta^2(2f_ku_k + 4f|\nabla_g u|_g^2)e^{2u} \\ &\quad +\Lambda\eta^2\sum_i F^{ii}|\nabla_g^2u|_g^2 - C\sum_i F^{ii}\left(1+\eta|\nabla_g^2u|_g + (\eta|\nabla_g^2u|_g)^{\frac{3}{2}}\right) .\end{aligned}$$

Using again inequality (A.6), finally we obtain

$$0 \geq \sum_i F^{ii}\left(\Lambda(\eta|\nabla_g^2u|_g)^2 - C(\eta|\nabla_g^2u|_g) + (\eta|\nabla_g^2u|_g)^{\frac{3}{2}}\right) .$$

Now, from formula (A.5) and Lemma A.2.3–(iii), we have

$$\begin{aligned}\sum_i F^{ii} = F^{ii} &= \frac{1}{k}\sigma_k(g^{-1}A^t_{\tilde{g}})^{\frac{1}{k}-1}T_{k-1}(g^{-1}A^t_{\tilde{g}})^{ii}\\ &= \frac{n-k+1}{k}\sigma_k(g^{-1}A^t_{\tilde{g}})^{\frac{1}{k}-1}\sigma_{k-1}(g^{-1}A^t_{\tilde{g}})\,.\end{aligned}$$

The right-hand side can be estimate from below using Lemma A.2.3–(iv), since $A^t_{\tilde{g}} \in \Gamma^+_k$, to arrive

$$\begin{aligned}F^{ii} &\geq \frac{n-k+1}{k}\sigma_k(g^{-1}A^t_{\tilde{g}})^{\frac{1}{k}-1}\frac{k}{n-k+1}\binom{n}{k}^{\frac{1}{k}}\sigma_k(g^{-1}A^t_{\tilde{g}})^{\frac{(k-1)}{k}}\\ &= \binom{n}{k}^{\frac{1}{k}}.\end{aligned}$$

This gives

$$0 \geq \Lambda(\eta|\nabla^2_g u|_g)^2 - C(\eta|\nabla^2_g u|_g) + (\eta|\nabla^2_g u|_g)^{\frac{3}{2}}\,.$$

Since $\Lambda > 0$, the first term dominates and we obtain $(\eta|\nabla^2_g u|_g)(x_0) \leq C$ and hence $H = \eta(\Delta_g u + |\nabla_g u|^2_g) = L$ is bounded in $B_{\frac{1}{2}}$. Now, by inequality (A.6), since L is bounded, we have $\Delta_g u \leq C$ and $|\nabla_g u|^2 \leq C$.

Step 3: $\nabla^2_g u$ is bounded.

To get the Hessian bounds, simply consider the maximum of the function

$$H(e_p) = \eta(\nabla^2_g u + \Lambda|\nabla_g u|^2_g g)(e_p, e_p)$$

over the set $S(TM)$, where $S(TM)$ denote the unit tangent bundle on M. Since M is compact, $S(TM)$ is compact too. With an appropriate choice of Λ, following the previous step, with almost the same computation, we get the result.

□

A.5 Uniform ellipticity and $C^{2,\alpha}$ estimates

In the previous section we showed that for a solution u of the equation (A.2), with $A_u^t \in \Gamma_k^+$, we have a priori C^1 and C^2 estimates just depending on the upper bound of the function u. Now suppose that we already know that the solution $u \in C^2(M)$ has a lower bound too, clearly from Theorem A.4.1 we have

$$\|u\|_{L^\infty(M)} + \|\nabla_g u\|_{L^\infty(M)} + \|\nabla_g^2 u\|_{L^\infty(M)} \leq C\,. \tag{A.10}$$

We want to prove the following

Theorem A.5.1 *Under assumption* (A.10) *equation* (A.2) *is uniformly elliptic. Moreover, we have that $u \in C^{2,\alpha}(M)$, with*

$$\|u\|_{C^{2,\alpha}(M)} \leq C\|u\|_{C^2(M)}\,,$$

for some positive constant C.

Proof. For simplicity we let

$$F(\lambda) = \sigma_k(g^{-1}A_u^t)^{1/k}\,,$$

since the equation depends only on the eigenvalues of the tensor A_u^t. In order to prove that F is uniformly elliptic, we need to verify the conditions (A.1), or equivalently that there exist a positive constant γ such that

$$\gamma^{-1} \geq \frac{\partial F}{\partial \lambda_i}(\lambda) \geq \gamma\,. \tag{A.11}$$

First note that if $u \in C^2(M)$ is a solution of equation (A.2) satisfying the bound (A.10), then the eigenvalues of A_u^t will satisfy this two conditions

$$F(\lambda) \geq c_1 > 0\,,$$

$$|\lambda| \leq c_2\,.$$

Let λ be the vector of the eigenvalues of A_u^t. By assumption $\lambda \in \Gamma_k^+$. If we define the set $\Gamma^1 \subset \Gamma_k^+$ of the vector of length one which lives in the k–cone, clearly we have $\lambda^1 = \lambda/|\lambda| \in \Gamma^1$. We claim that there is a constant $\delta > 0$ such that

$$dist(\lambda^1, \partial\Gamma^1) \geq \delta\,. \tag{A.12}$$

Assume this for the moment. Since F is homogeneous of degree one, $\partial F/\partial\lambda_i$ is homogenous of degree zero. Hence

$$0 < \frac{\partial F}{\partial \lambda_i}(\lambda) = \frac{\partial F}{\partial \lambda_i}(\lambda^1)\,.$$

But inequality (A.12) says that λ^1 is at fixed distance from the boundary of Γ^1. Therefore inequality (A.11) follows from the continuity of $\partial F/\partial\lambda_i$. So, we need to prove (A.12). Suppose there is a sequence $\{\lambda_s\} \in \Gamma_k^+$ with $|\lambda_s| \leq c_2$ and $F(\lambda_s) \geq c_1 > 0$, but with

$$dist(\lambda_s^1, \partial\Gamma^1) \to 0$$

as $s \to \infty$, where $\lambda_s^1 = \lambda_s/|\lambda_s|$. Now choose a subsequence (still denoted $\{\lambda_s\}$) with $\lambda_s^1 \to \lambda_0 \in \partial\Gamma^1$. By continuity, $F(\lambda_s^1) \to 0$. On the other hand, by homogeneity,

$$F(\lambda_s) = F(|\lambda_s|\lambda_s^1) = |\lambda_s| F(\lambda_s^1) \to 0\,,$$

since $|\lambda_s| \leq c_2$. However, this contradicts the assumption $F(\lambda_s) \geq c_1 > 0$. This proves the claim (A.12) and the uniform ellipticity of F.

By the works of Krylov [36] and Evans [16] we obtain $C^{2,\alpha}$ estimates. For a complete overview on the subject see the book of Cabré–Caffarelli [4].

□

Remark A.5.2 *By a bootstrap argument, we immediately get $C^\infty(M)$ estimates from $C^{2,\alpha}(M)$ estimates. Indeed, using the notation of Lemma A.2.5, if we differentiate the equation in the k direction, we get*

$$\text{(A.13)} \qquad (F^{ij} W_{ij})_k = F^{ij} W_{ij,k} = \nabla_k (f\, e^{2u}) \qquad \forall k = 1 \dots n\,.$$

Since

$$W_{ij} = (A_u^t)_{ij} = \nabla_{ij}^2 u + \frac{1-t}{n-2}(\Delta_g u) g_{ij} + \nabla_i u \nabla_j u - \frac{2-t}{2}|\nabla_g u|_g^2 g_{ij} + (A_g^t)_{ij}\,,$$

using equation (A.13) *and changing the order of differentiation, we have that the functions $\nabla_k u$ satisfy an elliptic equation where the coefficients in front of the second derivatives are* *in* C^α*, since* $u \in C^{2,\alpha}(M)$*. Now we can apply directly Schauder's estimates (see, for instance, [19]) to the function* $\nabla_k u$*, which give us* $C^{3,\alpha}(M)$ *estimates for the solution* u*. Iterating this argument we get higher orders estimates.*

A.6 σ_k–curvature and Ricci curvature

In this section we recall some algebraic properties of the σ_k curvatures and how it is related to the classical curvatures. This results can be found in the work by Guan, Viaclovsky and Wang (see [24]).

Theorem A.6.1 (Guan–Viaclovsky–Wang) *Let* (M, g) *be a Riemannian manifold with* $A_g \in \Gamma_k^+$*, for some* $k \geq n/2$*. Then its Ricci curvature is positive. Moreover, if* $k > 1$*, then*

$$Ric_g > \frac{2k-n}{2n(k-1)} R_g \, g \, .$$

In particular for three and four–dimensional manifolds with positive scalar curvature we find that if the Schouten tensor of a metric g is in the cone Γ_2^+, then the Ricci curvature of g must be positive definite. More in general, for the one–parameter Schouten tensor

$$A_g^t = \frac{1}{n-2}\left(Ric_g - \frac{t}{2(n-1)} R_g g\right) ,$$

we have the following

Proposition A.6.2 *If for some metric* g_1 *on* M *we have* $A_{g_1}^t \in \Gamma_2^+$*, then*

$$\begin{aligned} -A_{g_1}^t + \sigma_1(g_1^{-1} A_{g_1}^t) g_1 &> 0, \\ A_{g_1}^t + \frac{n-2}{n} \sigma_1(g_1^{-1} A_{g_1 1}^t) g_1 &> 0 \, . \end{aligned}$$

The easy proof of this fact can be found in Gursky and Viaclovsky [30, Lemma 5.1].

The σ_k curvatures give some informations on the sign of the sectional curvatures. Namely we have

Proposition A.6.3 *If for some metric g_1 on M^n the Weyl curvature is zero and we have $A_{g_1} \in \Gamma^+_{n-1}$, then g_1 has positive sectional curvatures. Moreover, in dimension three, if $\sigma_2(g_1^{-1} A_{g_1}) > 0$ then the sectional curvatures have a sign.*

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