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ON A 1-CAPACITARY TYPE PROBLEM IN THE PLANE

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Abstract. We study a 1-capacitary type problem in $\mathbb{R}^2$: given a set E, we minimize the perimeter (in the sense of De Giorgi) among all the sets containing E (modulo $\mathcal{H}^1$) and satisfying an indecomposability constraint (according to the definition by [1]). By suitably choosing the representant of the relevant set E, we show that a convexification process characterizes the minimizers.

As a consequence of our result we determine the 1-capacity of (a suitable representant of) sets with finite perimeter in the plane.

1. **Introduction.** Capacities provide a powerful analytical tool to deal with fine properties of spaces of weakly differentiable functions.

In particular, 1-capacity, which is defined for any set $E \subset \mathbb{R}^N$ as

$$\mathrm{C}_1(E) := \inf\{\mathrm{Per}(D) : D\ \mathcal{L}^N\text{-measurable}, \mathcal{L}^N(D) < +\infty, \mathcal{H}^{N-1}(E \setminus D^+) = 0\},$$

arises in the theory of the spaces of functions of bounded variation : Poincaré type inequalities and imbeddings theorems being prominent examples ([9, Section 5.12 and 5.13]). We defer the precise definitions of the quantities appearing in the formula above to Section 2, here we only mention that $\mathrm{Per}(\cdot)$ denotes the perimeter in the sense of De Giorgi.

Recently capacitary Brunn-Minkowski and dual Brunn-Minkowski inequalities have been investigated, and find application in convex geometry [8].

As it turns out from the very definition, 1-capacity is related to minimal surface problems with obstacle conditions; the theory developed by [4] (see also [3]) shows that it is well suited to deal also with "thin" obstacles, that is to consider sets E with negligible Lebesgue measure, though of positive $(N-1)$-dimensional Hausdorff measure.

In this paper we study a variant of the 1-capacitary problem in the plane by imposing the additional constraint of *indecomposability* to the class of competitors.

Roughly speaking a set of finite perimeter D is indecomposable if it cannot be partitioned into non trivial components whose perimeters also partition that of D.

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This notion is the analogue of topological connectedness in this measure theoretical framework (see subsection 2.4 for further comments and details).

Without the attempt to be precise the problem we are going to investigate can be described as follows: given a set $E \subset \mathbb{R}^2$, find the set with minimal perimeter (if any exists) among the "connected" sets in the plane "containing" E. Depending on the regularity of the set E different interpretations can be given to the words "connected" and "containing". In the classical framework, i.e. when E is a sufficiently regular bounded open set, it is well known that the the convex hull of E is the bounded, topologically connected open set of minimal perimeter containing E.

This framework is however not satisfying, indeed it does not allow to consider objects with non regular geometries such as cusps and spikes. These issues can be solved working in the more general functional setting of sets with finite perimeter paying the price of giving a measure theoretic interpretation to the notion of connectedness (indecomposability) and of set inclusion.

An attempt in this direction has been done in [5], under the additional constraint of boundedness for competitors, where the minimum of the problem

$$\mu(E) := \inf\{\mathrm{Per}(D) : D \text{ indecomposable and bounded}, \quad \mathcal{L}^2(E \setminus D) = 0\}$$

is characterized. However, the set function $\mu(E)$ is not completely satisfactory, too. Indeed, since the set inclusion in the definition of $\mu(\cdot)$ is taken up to $\mathcal{L}^2$-negligible sets, if the set E has a spike (modeled for example as a regular curve) a competitor D is not affected by its presence.

In this note we propose a more accurate model, where the set inclusion is considered up to $\mathcal{H}^1$-measure zero. More precisely, we study the variational problem

$$\gamma(E) := \inf\{\mathrm{Per}(D) \,:\, D \,\text{indecomposable}, \, \mathcal{L}^2(D) < +\infty, \; \mathcal{H}^1(E \setminus D^+) = 0\}.$$

Given these limitations for the competitors, it turns out that, in the two dimensional setting, the convexification is the right geometric tool to solve the problem: $\gamma(E)$ is computed by calculating the perimeter of the convex hull of a suitable set related to E. Indeed, being $\gamma(\cdot)$ equal for sets which coincide up to $\mathcal{H}^1$ negligible variations, some care is needed in selecting the set to be convexified. More precisely, in Theorem 3.2, our main result, we show that upon taking the set of points with positive one-dimensional upper density for E, denoted by $E^{1,+}$ in what follows, the equality $\gamma(E) = \mathrm{Per}(\overline{\mathrm{co}}(E^{1,+}))$ holds true, with $\overline{\mathrm{co}}(\cdot)$ denoting the closure of the convex hull. Actually, a more detailed statement which characterizes also all the minimizers of the problem and further properties of the set function $\gamma(\cdot)$ is established.

Building upon these results we are able to determine the 1-capacity of sets with finite perimeter in the plane satisfying a mild regularity assumption which essentially guarantees the existence of a minimizer for the related variational problem (see Corollaries 4 and 5). Again, the problem is reduced to the computation of the perimeters of the convex hulls of suitable representants of the indecomposable components of the given set. Let us remark that the regularity condition referred to above is not restrictive, since on one hand sets with finite perimeter are specified $\mathcal{L}^2$ a.e., and on the other $\mathrm{C}_1(\cdot)$ is affected by $\mathcal{H}^1$ non-negligible changes of the set under examination. Then the choice of a suitable representant for the relevant set is required.

Finally, we remark that our results not only recover those by [5] (in particular see Corollary 3), but also enlarge the stage to the case of thin sets, i.e. sets E such that $\mathcal{L}^2(E) = 0$, though satisfying $\mathcal{H}^1(E) > 0$.

The plan of the paper is as follows. In Section 2 we settle down the notations and list the required prerequisites needed for our analysis. In particular, subsection 2.2 is devoted to a detailed description of the representants of $\mathcal{L}^2$ equivalence classes of sets in order to justify the subsequent choices. Section 3 contains the main result of the paper, its relation with other variational problems is discussed in Section 4.

2. Notations and preliminaries.

2.1. Basic notations and definitions.

In the ensuing sections the (topological) closure, boundary and interior of a set $E \subseteq \mathbb{R}^n$ will be denoted by $\overline{E}$, ∂E and $\mathring{E}$, respectively. The ball centered in x and with radius $r > 0$ with respect to the euclidean distance is denoted by $B_r(x)$. The symbol $\triangle$ denotes the symmetric difference between sets.

In the sequel for every set $E \subseteq \mathbb{R}^2$ we denote by $\overline{\mathrm{co}}(E)$ the closure of the *convex envelope* of E, or equivalently

$$\overline{\mathrm{co}}(E) = \{\lambda x + (1-\lambda)y \,|\, x, y \in \overline{E}, \lambda \in [0,1]\}.$$

Standard notations are used for Lebesgue and Hausdorff measures. In particular, given two sets E, $F \subseteq \mathbb{R}^N$ with $\mathcal{L}^N(E \setminus F) = 0$ we will also write $E \subseteq F \mod \mathcal{L}^N$. Analogously, $E = F \mod \mathcal{L}^N$ stands for $\mathcal{L}^N(E \triangle F) = 0$.

2.2. Representants of equivalence classes of sets.

In order to state our main result we set some preliminary definitions.

For any Borel set $E \subseteq \mathbb{R}^2$ we introduce the sets of points with 2-dimensional and 1-dimensional positive upper density, defined respectively by

$$E^+ := \left\{ x \in \mathbb{R}^2 : \limsup_{r \to 0^+} \frac{\mathcal{L}^2(E \cap B_r(x))}{r^2} > 0 \right\} \tag{1}$$

and

$$E^{1,+} := \left\{ x \in \mathbb{R}^2 : \limsup_{r \to 0^+} \frac{\mathcal{H}^1(E \cap B_r(x))}{r} > 0 \right\}. \tag{2}$$

It is well known that the set E^+ represents the measure theoretic closure of E in a suitable sense (see [9]). In the last section of the paper we will also deal with the measure theoretic interior of the set E, defined as

$$E^1 := \left\{ x \in \mathbb{R}^2 \ : \ \lim_{r \to 0^+} \frac{\mathcal{L}^2(E \cap B_r(x))}{\pi r^2} = 1 \right\}.$$

All the sets E^1, E^+ and $E^{1,+}$ are Borel sets, and $E^1 \subseteq E^+ \subseteq E^{1,+}$ with possibly strict inequality holding (see figure 1).

In the following we collect some further results on the sets E^1, E^+ and $E^{1,+}$ that will be useful in the rest of the paper.

By Lebesgue differentiation theorem it follows that $\mathcal{L}^2(E \triangle E^+) = 0$, and the direct implication in the following equivalence

$$E^+ \subseteq F^+ \iff \mathcal{L}^2(E \setminus F) = 0. \tag{3}$$

Instead, the opposite implication follows directly from the very definition of E^+ in (1). It is evident from (3) that E^+ depends on the set E only through the $\mathcal{L}^2$-equivalence class. In particular, we infer that $(E^+)^+ = E^+$.

Clearly the same arguments used above imply $\mathcal{L}^2(E \triangle E^1) = 0$ and (3) replacing E^+, F^+ with E^1, F^1, respectively.

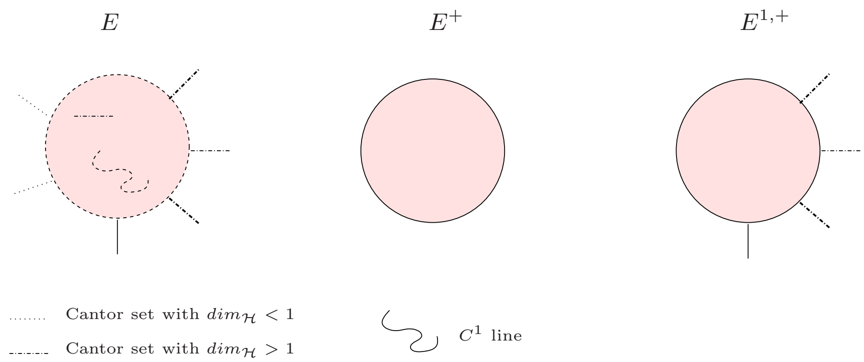

FIGURE 1. Computation of E^+ and $E^{1,+}$ for a generic Borel set E

Similarly, the very definition of $E^{1,+}$ in (2) yields that

$$\mathcal{H}^1(E \setminus F) = 0 \Longrightarrow E^{1,+} \subseteq F^{1,+}, \tag{4}$$

while simple examples show that the opposite implication is false. In particular the set $E^{1,+}$ is affected by $\mathcal{H}^1$-positive variations of the Borel set E.

In addition, in Lemma 3.3 we will prove that $\mathcal{H}^1(E \setminus E^{1,+}) = 0$ for any Borel set E. The previous property is a consequence of the standard estimates for densities of positive Radon measures (see for instance [2, Theorem 2.56] and the subsequent formulas (2.42) and (2.43) therein), which actually settle the case $\mathcal{H}^1 \llcorner E$ σ-finite. The statement below is restricted to the case of interest in this paper.

Lemma 2.1. *For any Borel set $E \subset \mathbb{R}^2$ with $\mathcal{H}^1(E) < +\infty$ the estimates*

$$\frac{1}{2} \leq \limsup_{r \to 0} \frac{\mathcal{H}^1(E \cap B_r(x))}{2r} \leq 1$$

hold for $\mathcal{H}^1$ a.e. $x \in E$.

Instead, simple examples show that in general $\mathcal{H}^1(E^{1,+} \setminus E) > 0$ (see also figure 1 above).

Despite of all these facts the sets E^1, E^+ and $(E^+)^{1,+}$ have all the same (topological) closure. More precisely, the following result holds.

Lemma 2.2. *For any Borel set E it holds $\overline{E^1} = \overline{E^+} = \overline{(E^+)^{1,+}}$.*
In particular, $\overline{\mathrm{co}}(E^1) = \overline{\mathrm{co}}(E^+) = \overline{\mathrm{co}}((E^+)^{1,+})$.

Proof. To establish the result it is enough to prove that $E^1 \cap B_r(x_0) \neq \emptyset$ for any point $x_0 \in (E^+)^{1,+}$ and for any radius $r > 0$ since $E^1 \subseteq E^+ \subseteq (E^+)^{1,+}$. For, by the very definition of $(E^+)^{1,+}$ we have $\mathcal{H}^1(E^+ \cap B_r(x_0)) > 0$ for any $r > 0$, and thus $E^+ \cap B_r(x_0) \neq \emptyset$. To conclude, observe that $E^1 \cap B_r(x) \neq \emptyset$ for any $x \in E^+$ and $r > 0$ since by Lebesgue differentiation theorem $\mathcal{L}^2(E^+ \setminus E^1) = 0$. □

2.3. **Functional capacity of degree 1.** We briefly recall here the notion and some properties of 1-*capacity* also called *functional capacity of degree 1*. This notion dates back to Federer and Ziemer [7] and different minimization problems characterize it (see [7] and [3] for a deeper insight). Here we prefer to express it simply in terms of the perimeter of the sets "containing" E in a suitable sense, namely

$$\mathrm{C}_1(E) := \inf\{\mathrm{Per}(D) : D \text{ is } \mathcal{L}^2\text{-measurable}, \mathcal{L}^2(D) < +\infty, \mathcal{H}^1(E \setminus D^+) = 0\}. \quad (5)$$

In formula (5) above $\mathrm{Per}(\cdot)$ denotes the perimeter of the relevant set. Sets with finite perimeter in the sense of De Giorgi will be object of our analysis. For the definition and their theory we refer to the book [2] (see also [9]). In particular, detailed references will be indicated for all the properties employed in the sequel. We limit ourselves to recall that the set $\partial^* D$ denotes the essential (or measure theoretical) boundary of D defined by $\mathbb{R}^2 \setminus (D^1 \cup (\mathbb{R}^2 \setminus D)^1)$. In addition, by De Giorgi's and Federer's theorems (see [2, Theorem 3.59 and 3.61]), we have $\mathrm{Per}(D) = \mathcal{H}^1(\partial^* D)$.

As shown in [7] the set function $\mathrm{C}_1(\cdot)$ turns out to be a strongly subadditive outer measure and a Choquet capacity. In the next proposition we recall only those properties needed in the ensuing sections.

Proposition 1. *$\mathrm{C}_1(\cdot)$ is a monotone increasing set function. In addition, the following properties hold*

(i) *$\mathrm{C}_1(\cdot)$ is* positively 1-homogeneous*, i.e. $\mathrm{C}_1(rE) = r\mathrm{C}_1(E)$ for any set $E \subseteq \mathbb{R}^2$ and $r > 0$;*
(ii) *for any Borel set $E \subseteq \mathbb{R}^2$*

$$\mathrm{C}_1(E) = 0 \Longleftrightarrow \mathcal{H}^1(E) = 0. \quad (6)$$

(iii) *$\mathrm{C}_1(\cdot)$ is continuous on increasing sequences of sets, i.e.*

$$E_n \nearrow E \Longrightarrow \lim_{n\to\infty} \mathrm{C}_1(E_n) = \mathrm{C}_1(E).$$

The existence of extremals for the capacitary problem above fails for many sets E with $\mathrm{C}_1(E) < +\infty$, e.g. if E is a line segment. This phenomenon can occur only if $\mathcal{H}^1(E \setminus E^+) > 0$ as stated in the next proposition which enlights the role of E^+ in this setting (for the proof see [6, Proposition 2.6]).

Proposition 2. *For every $\mathcal{L}^2$-measurable set $E \subset \mathbb{R}^2$ with $\mathrm{C}_1(E) < +\infty$ it holds*

(i) *$\mathrm{C}_1(E^+) \le \mathrm{C}_1(E)$;*
(ii) *problem (5) for E^+ has always solution and*

$$\mathrm{C}_1(E^+) = \min\{\mathrm{Per}(D) : D \text{ is } \mathcal{L}^2\text{-measurable}, \mathcal{L}^2(D) < +\infty, \mathcal{L}^2(E \setminus D) = 0\}. \quad (7)$$

Moreover, if $\mathcal{H}^1(E \setminus E^+) = 0$ then $\mathrm{C}_1(E^+) = \mathrm{C}_1(E)$ and problem (5) for E has solution.

Take note that the relaxed form of the 1-capacitary problem has been determined in [4], in connection with minimal surfaces problems with obstacle (see also [3] for further extensions to non-parametric problems).

Remark 1. For any bounded set E it is easy to prove that $\mathrm{C}_1(E) < +\infty$. Moreover, if E is contained in the interior of a bounded convex set C, one can restrict the class of competing sets in the capacitary problem for E to those contained in C (see [6, Remark 2.8]).

We give the proof for the sake of the readers' convenience. By using the formulation (5), given a test set D, note that the set $D \cap C$ has finite perimeter and, being $E \subset \overset{\circ}{C}$ and $C^+ = \overline{C}$, it holds $\mathcal{H}^1(E \setminus (D \cap C)^+) = \mathcal{H}^1(E \setminus D^+) = 0$. If Π_C denotes the projection on the convex set C, then $\mathcal{H}^1(\Pi_C(D\cap(\mathbb{R}^2\setminus C)))\leq \mathrm{Per}(D\cap(\mathbb{R}^2\setminus C))$. Hence, we have

$$\begin{aligned}\mathrm{Per}(D\cap C) &\leq \mathcal{H}^1(\Pi_C(D\setminus \mathring{C}))+\mathcal{H}^1(\partial^* D\cap \mathring{C})\\ &\leq \mathrm{Per}(D\setminus \mathring{C})+\mathcal{H}^1(\partial^* D\cap \mathring{C})\\ &\leq \mathrm{Per}(D).\end{aligned}$$

2.4. **Indecomposable sets in $\mathbb{R}^2$.** Let us recall the notion of *indecomposable sets* according to [1].

A set of finite perimeter $E\subset\mathbb{R}^N$ is said to be *decomposable* if there exists a partition of E in two $\mathcal{L}^N$-measurable sets A, B with strictly positive measure such that

$$\mathrm{Per}(E)=\mathrm{Per}(A)+\mathrm{Per}(B).$$

Accordingly, a set of finite perimeter is said to be *indecomposable* if the previous property does not hold for any pair A,B. Notice that the properties of being decomposable or indecomposable depend only on the $\mathcal{L}^N$-equivalence class of the relevant set.

These notions have been introduced in [1] to treat two dimensional problems in Image Processing in order to extend the topological notion of connectedness (see [1, Remark 4] and the references therein for further extensions and comparisons in the setting of normal integer currents).

Before stating the main result of this subsection we recall that Γ is called (the image of) a Jordan curve if it is image of a simple closed curve, i.e. if $\Gamma=\varphi([a,b])$ for some $a,b\in\mathbb{R}$ and some continuous function φ , one-to-one on $[a,b)$ such that $\varphi(a)=\varphi(b)$. According to the Jordan curve theorem, Γ splits $\mathbb{R}^2\setminus\Gamma$ in two open connected components, a bounded and an unbounded one, having common boundary Γ. These components will be denoted respectively $\mathrm{int}(\Gamma)$ and $\mathrm{ext}(\Gamma)$. According to [1, Lemma 4] if Γ is rectifiable it holds that

$$\mathcal{H}^1(\Gamma)=\mathrm{Per}(\mathrm{int}(\Gamma))=\mathrm{Per}(\mathrm{ext}(\Gamma)) \tag{8}$$

Next we summarize some results proved in [1, Theorem 1, Corollary 1] which provide a decomposition property for sets of finite perimeter into "indecomposable components". Moreover, a bounded indecomposable set in the plane essentially coincides with the interior of a rectifiable Jordan curve minus a finite or countable number of holes.

Theorem 2.3. *Let E be a set of finite perimeter in $\mathbb{R}^N$, then we have:*

(i) *there exists a unique finite or countable family of pairwise disjoint indecomposable sets E_i, $i\in I\subseteq\mathbb{N}$, with $\mathcal{L}^N(E_i)>0$ such that*

$$E=\bigcup_{i\in I}E_i \ (\mathrm{mod}\ \mathcal{L}^{\mathrm{N}}) \quad \textit{and} \quad \mathrm{Per}(E)=\sum_{i\in I}\mathrm{Per}(E_i).$$

Moreover the sets E_i are maximal indecomposable components in the sense that for any indecomposable set $F\subseteq E$ (mod $\mathcal{L}^N$) we have that F is contained (mod $\mathcal{L}^N$) in some E_i;

(ii) *if $N=2$ and E is a bounded indecomposable set, then there exists a unique decomposition* (mod $\mathcal{H}^1$) *of its reduced boundary into a finite or countable number of rectifiable Jordan curves C_0, C_i, $i\in I\subseteq\mathbb{N}$, such that* $\mathrm{int}(C_i)\subset$

$\mathrm{int}(C_0)$, *the* $\mathrm{int}(C_i)$ *are pairwise disjoint,*

$$E = \mathrm{int}(C_0) \setminus \bigcup_{i\in I} \mathrm{int}(C_i) \ (\mathrm{mod}\ \mathcal{L}^2) \quad \textit{and} \quad \mathrm{Per}(E) = \mathcal{H}^1(C_0) + \sum_{i\in I} \mathcal{H}^1(C_i).$$

Remark 2. Let us point out that the statement of [1, Theorem 1] requires the stronger assumption $F \subseteq E$ to show that the sets E_i are maximal indecomposable. Nevertheless, a careful inspection of the (last part of the) proof shows that the pointwise inclusion is not needed and the weaker requirement $\mathcal{L}^N(F\setminus E) = 0$ suffices since only estimates on N-dimensional densities are involved in that argument.

3. **Main result.** In this section we introduce a constrained minimum problem and provide a complete characterization of its minimizers. Let E be a Borel set in $\mathbb{R}^2$, then define

$$\gamma(E) := \inf\{\mathrm{Per}(D) \,:\, D\,\text{indecomposable},\ \mathcal{L}^2(D) < +\infty,\ \mathcal{H}^1(E \setminus D^+) = 0\}, \quad (9)$$

where we adopt the standard convention that the infimum of the empty set is $+\infty$.

Clearly for any set E we have $\mathrm{C}_1(E) \leq \gamma(E)$ with possibly strict inequality holding. Consider for instance $E = B_1(x_1) \cup B_1(x_2)$ with $|x_1 - x_2| > \pi$, then $\mathrm{C}_1(E) \leq 4\pi < \gamma(E) = 2\pi + 2|x_1 - x_2|$. The latter equality follows from Theorem 3.2 below, while the first inequality by considering E itself as a competitor for the capacitary problem. Actually, as a consequence of Corollary 5 in the last section it holds $\mathrm{C}_1(E) = 4\pi$.

Remark 3. A sufficient condition for $\gamma(E)$ and $\mathrm{C}_1(E)$ to be equal is the convexity of the set E. More precisely, under this hypothesis, the minimizer or the minimizing sequence for the capacitary problem is attained in the class of indecomposable competitors.

To prove this fact consider first the case in which E is a singleton or a segment (possibly infinite). Then $\mathrm{C}_1(E)$ equals zero or $2\mathcal{H}^1(E)$, respectively, being an optimizing sequence obtained in both cases by tubular neighbourhoods of E with infinitesimal radii (see [4, Chapter IV, Theorem 4.10], [3, Proposition 4.6]).

Assume now that $\overset{\circ}{E} \neq \emptyset$. If E is unbounded then it contains at least an half line thus, by a simple monotonicity argument, we easily get $\gamma(E) = \mathrm{C}_1(E) = +\infty$. Otherwise, since $E^+ = \overline{E}$, by (3) any competitor D for $\mathrm{C}_1(E)$ satisfies also $D^+ \supseteq \overline{E}$. In particular $\partial^* D \subseteq \mathbb{R}^2 \setminus \overset{\circ}{E}$. Arguing as in Remark 1, a standard projection argument implies that $\mathrm{Per}(D) \geq \mathcal{H}^1(\Pi_E(\partial^* D)) = \mathrm{Per}(E)$, being Π_E the projection on $\overline{E}$. As the set E is admissible for $\mathrm{C}_1(E)$ we get $\mathrm{C}_1(E) = \mathrm{Per}(E)$, i.e. E is an indecomposable minimizer and $\gamma(E) = \mathrm{C}_1(E)$.

Remark 4. Note that the set function $\gamma(\cdot)$ inherits many properties of the capacity $\mathrm{C}_1(\cdot)$ such as monotonicity, sub-additivity and positive 1-homogeneity. In particular, as a consequence of the next Theorem 3.2, the analogous of (6) holds true, i.e. $\gamma(E) = 0 \Longleftrightarrow \mathcal{H}^1(E) = 0$ for any Borel set $E \subseteq \mathbb{R}^2$.

As one may expect, thanks to the indecomposability constraint and the two dimensional setting, the value of $\gamma(E)$ can be obtained through a convexification argument of a suitable representant (in a measure theoretical sense) of the given set E.

Before stating the main result of the paper we need the following definition.

Definition 3.1. Given a Borel set E in $\mathbb{R}^2$ we say that E is $\mathcal{H}^1$-*essentially bounded* if there exists a bounded Borel set $F \subseteq E$ such that $\mathcal{H}^1(E \setminus F) = 0$.

Clearly, this is equivalent to the condition $\mathcal{H}^1(E \setminus B_r(x)) = 0$ for some $r > 0$ and $x \in \mathbb{R}^2$.

Theorem 3.2. *For any Borel set $E \subset \mathbb{R}^2$ we have*

$$\gamma(E) = \gamma(E^{1,+}) = \gamma(\overline{\text{co}}(E^{1,+})) = \mathrm{C}_1(\overline{\text{co}}(E^{1,+})). \tag{10}$$

Moreover the following properties hold:

(i) $\gamma(E) = 0 \Longleftrightarrow \mathcal{H}^1(E) = 0$;
(ii) $\gamma(E) = +\infty \Longleftrightarrow E$ *is not* $\mathcal{H}^1$*-essentially bounded;*
(iii) *If $\gamma(E) < +\infty$, a minimizer D of (9) (if any exists) satisfies $D^+ = \overline{\text{co}}(E^{1,+})$;*
(iv) *If $0 < \gamma(E) < +\infty$, the minimum problem defining $\gamma(E)$ has a solution if and only if $\overline{\text{co}}(E^{1,+})$ has positive $\mathcal{L}^2$ measure, being $\overline{\text{co}}(E^{1,+})$ itself a minimizer.*

The proof of Theorem 3.2 will be a consequence of several intermediate results.

As a first step we show that $\mathcal{H}^1$ almost every point of a Borel set has positive 1-dimensional upper density. Before giving the proof let us point out that the conclusion of Lemma 3.3 below holds more generally in arbitrary dimension N and for an arbitrary k-dimensional density, $0 \le k < N$. We confined ourselves to the case of interest in this paper only for the sake of simplicity.

Lemma 3.3. *Let E be a Borel set in $\mathbb{R}^2$, then $\mathcal{H}^1(E \setminus E^{1,+}) = 0$.*

Proof. First note that by (2) we have that

$$E \setminus E^{1,+} = \left\{ x \in E : \lim_{r \to 0} \frac{\mathcal{H}^1(E \cap B_r(x))}{r} = 0 \right\}.$$

Then, by monotonicity, it turns out that any point $x \in E \setminus E^{1,+}$ has zero 1-dimensional density in $E \setminus E^{1,+}$ itself, i.e. for every $x \in E \setminus E^{1,+}$

$$\lim_{r \to 0} \frac{\mathcal{H}^1\big((E \setminus E^{1,+}) \cap B_r(x)\big)}{r} = 0. \tag{11}$$

In particular, for every $x \in E \setminus E^{1,+}$ there exists $r(x) > 0$ such that

$$\mathcal{H}^1\big((E \setminus E^{1,+}) \cap B_r(x)\big) \le r \;\; , \;\; \forall \;\; 0 < r \le r(x).$$

Consider the family of sets $\mathcal{F} := \{(E \setminus E^{1,+}) \cap B_{r(x)}(x)\}_{x \in E \setminus E^{1,+}}$, and observe that $\mathcal{F}$ is an open cover of $E \setminus E^{1,+}$ in the induced topology. Since the latter has a countable base, $E \setminus E^{1,+}$ turns out to be a Lindelöf space, i.e. every open cover has a countable subcover. Then,

$$E \setminus E^{1,+} = \bigcup_{k \in \mathbb{N}} \big((E \setminus E^{1,+}) \cap B_{r_k}(x_k)\big)\,, \tag{12}$$

for some sequence $(x_k) \subseteq E \setminus E^{1,+}$. Furthermore, by construction

$$\mathcal{H}^1\big((E \setminus E^{1,+}) \cap B_{r_k}(x_k)\big) \le r_k. \tag{13}$$

In turn, the latter two properties imply that the measure $\mathcal{H}^1 \llcorner (E \setminus E^{1,+})$ is σ-finite.

We are now in a position to apply the usual estimates for 1-dimensional densities of positive Radon measures (see Lemma 2.1) and deduce from (11)

$$\mathcal{H}^1\left((E \setminus E^{1,+}) \cap B_{r_k}(x_k)\right) = 0$$

for all $k \in \mathbb{N}$. Equality (12) then gives the conclusion. $\square$

In the next lemmata we show that any competitor for $\gamma(E)$ can be modified into a competitor for $\gamma(\overline{\mathrm{co}}(E^{1,+}))$. Furthemore, this can be done by decreasing the perimeter provided indecomposability is assumed.

Lemma 3.4. *Let D be a $\mathcal{L}^2$-measurable set in $\mathbb{R}^2$ such that $\mathcal{H}^1(E \setminus D^+) = 0$, then*

$$\overline{\mathrm{co}}(D^+) \supseteq \overline{\mathrm{co}}(E^{1,+}).$$

Proof. Let H be a closed half-plane in $\mathbb{R}^2$ containing D^+. Since, by Hahn-Banach Theorem, $\overline{\mathrm{co}}(D^+) = \bigcap_{H\in\mathcal{S}} H$ where $\mathcal{S} := \{H :\ H \text{ closed half-plane with } H \supseteq D^+\}$, it is enough to prove that $H \supseteq E^{1,+}$. Arguing by contradiction assume there exists $x_0 \in E^{1,+} \setminus H$. As H is closed we can find a radius $\bar{r} > 0$ such that $H \cap B_r(x_0) = \emptyset$ for any $r \leq \bar{r}$. A straightforward computation shows that for any $r \leq \bar{r}$

$$\begin{aligned}\mathcal{H}^1(E \cap B_r(x_0)) &= \mathcal{H}^1\big((E \cap B_r(x_0)) \setminus H\big)\\ &\leq \mathcal{H}^1\big((E \cap B_r(x_0)) \setminus D^+\big)\\ &\leq \mathcal{H}^1(E \setminus D^+) = 0.\end{aligned}$$

This yields immediately that $x_0 \notin E^{1,+}$, contradicting the assert. □

Lemma 3.5. *Let D be a bounded indecomposable set in $\mathbb{R}^2$, then*

$$\mathrm{Per}(\overline{\mathrm{co}}(D^+)) \leq \mathrm{Per}(D).$$

In addition, equality $\mathrm{Per}(\overline{\mathrm{co}}(D^+)) = \mathrm{Per}(D)$ holds if and only if $D^+ = \overline{\mathrm{co}}(D^+)$.

Proof. By the indecomposability hypothesis and by using Theorem 2.3 we can find an at most countable family of rectifiable Jordan curves C_0, C_i, $i \in I \subseteq \mathbb{N}$, such that $\mathrm{int}(C_i) \subset \mathrm{int}(C_0)$, the $\mathrm{int}(C_i)$ are pairwise disjoint,

$$D^+ = \left(\mathrm{int}(C_0) \setminus \bigcup_{i\in I} \mathrm{int}(C_i)\right)^+ \tag{14}$$

and

$$\mathrm{Per}(D) = \mathcal{H}^1(C_0) + \sum_{i\in I} \mathcal{H}^1(C_i). \tag{15}$$

Let us define $\tilde{D} := (\mathrm{int}(C_0))^+$, then, by locality of the perimeter functional and using (8), we infer that $\mathrm{Per}(\tilde{D}) = \mathcal{H}^1(C_0)$; hence, by (15) we have

$$\mathrm{Per}(\tilde{D}) \leq \mathrm{Per}(D). \tag{16}$$

Moreover, from (3) and (14), we infer that

$$\overline{\mathrm{co}}(D^+) \subseteq \overline{\mathrm{co}}(\tilde{D})$$

and, by monotonicity of the perimeter with respect to the inclusion on convex sets, we deduce that

$$\mathrm{Per}(\overline{\mathrm{co}}(D^+)) \leq \mathrm{Per}(\overline{\mathrm{co}}(\tilde{D})). \tag{17}$$

Next we observe that, being $\partial\tilde{D} = C_0$ a rectifiable Jordan curve we have (see Proposition 3 in the Appendix)

$$\mathrm{Per}(\overline{\mathrm{co}}(\tilde{D})) \leq \mathrm{Per}(\tilde{D}), \tag{18}$$

which, together with (16) and (17) implies the first part of the result.

If equality $\mathrm{Per}(\overline{\mathrm{co}}(D^+)) = \mathrm{Per}(D)$ holds true, inequalities (16)-(18) yield that $\mathrm{Per}(\overline{\mathrm{co}}(D^+)) = \mathrm{Per}(\overline{\mathrm{co}}(\tilde{D})) = \mathrm{Per}(\tilde{D}) = \mathrm{Per}(D^+)$. The last equality, together with (15) and Theorem 2.3, implies $\tilde{D} = D^+$. Finally, since the perimeter functional is strictly decreasing under convexification of sets whose boundary is a rectifiable Jordan curve (by Proposition 3), we infer that $\tilde{D} = \overline{\mathrm{co}}(\tilde{D}) = \overline{\mathrm{co}}(D^+)$, and so the claim follows. □

We are now ready to prove our main result.

proof of Theorem 3.2. Equality $\gamma(\overline{\mathrm{co}}(E^{1,+})) = \mathrm{C}_1(\overline{\mathrm{co}}(E^{1,+}))$ follows from Remark 3, thus we limit ourselves to prove that

$$\gamma(E) = \gamma(E^{1,+}) = \gamma(\overline{\mathrm{co}}(E^{1,+})). \tag{19}$$

Let D be any competitor for $\gamma(E^{1,+})$. Using the sub-additivity of the one dimensional Hausdorff measure $\mathcal{H}^1$ and Lemma 3.3 we have

$$\mathcal{H}^1(E \setminus D^+) \leq \mathcal{H}^1(E \setminus E^{1,+}) + \mathcal{H}^1(E^{1,+} \setminus D^+) = 0.$$

Hence D is a competitor also for $\gamma(E)$, and thus $\gamma(E) \leq \gamma(E^{1,+})$. The latter estimate together with the monotonicity of γ as a set function gives the following chain of inequalities

$$\gamma(E) \leq \gamma(E^{1,+}) \leq \gamma(\overline{\mathrm{co}}(E^{1,+})).$$

To conclude the proof of (19) it remains to show that $\gamma(\overline{\mathrm{co}}(E^{1,+})) \leq \gamma(E)$.

Note that if $\mathcal{H}^1(E) = 0$ the equality is trivially proved by considering the emptyset as a minimizer for $\gamma(E)$ and taking into account that $E^{1,+} = \emptyset$.

Thus we assume in what follows that $\mathcal{H}^1(E) > 0$. We will prove the assert first in the case that E is $\mathcal{H}^1$-essentially bounded. Take note that if F is a bounded Borel set contained in E and such that $\mathcal{H}^1(E \setminus F) = 0$ clearly $\gamma(F) = \gamma(E)$. Thus, in the sequel we assume E to be bounded for the sake of simplicity. Furthermore, under this hypothesis on E, by taking into account Remark 1, we can restrict the class of competitor to the bounded ones.

Let D be a bounded indecomposable set such that $\mathcal{H}^1(E \setminus D^+) = 0$; then by Lemma 3.4 we have $\overline{\mathrm{co}}(E^{1,+}) \subseteq \overline{\mathrm{co}}(D^+)$, and thus in particular

$$\mathcal{H}^1\big(\overline{\mathrm{co}}(E^{1,+}) \setminus \overline{\mathrm{co}}(D^+)\big) = 0.$$

Moreover, Lemma 3.5 implies that $\mathrm{Per}(\overline{\mathrm{co}}(D^+)) \leq \mathrm{Per}(D)$. Thus, for any competitor D for $\gamma(E)$ we can find a competitor $\overline{\mathrm{co}}(D^+)$ for $\gamma(\overline{\mathrm{co}}(E^{1,+}))$ decreasing the perimeter value. Passing to the infimum we get the desired inequality. This concludes the proof of (19) in case the set E is $\mathcal{H}^1$-essentially bounded.

It remains to deal with the case in which E does not satisfy this extra condition.

As $\mathcal{H}^1(E) > 0$, by Lemma 3.3 we can find a point $x_0 \in E \cap E^{1,+}$. Rephrasing the fact the E is not $\mathcal{H}^1$-essentially bounded, there exists an increasing sequence of radii r_n diverging to $+\infty$ such that

$$\mathcal{H}^1\big(E \cap (B_{r_{n+1}}(x_0) \setminus B_{r_n}(x_0))\big) > 0.$$

Thus, by taking into account Lemma 3.3, for every $n \in \mathbb{N}$ we can find a point $x_n \in \big(E \cap (B_{r_{n+1}}(x_0) \setminus B_{r_n}(x_0))\big)^{1,+}$.

Using the monotonicity of γ and the fact that we have established (10) for bounded sets we infer the estimate

$$\gamma(E) \geq \gamma\big(E \cap B_{r_{n+1}}(x_0)\big) = \mathrm{C}_1\big(\overline{\mathrm{co}}((E \cap B_{r_{n+1}}(x_0))^{1,+}\big).$$

Since for any $n \in \mathbb{N}$ the points x_0 and x_n belong to the set $\big(E \cap B_{r_{n+1}}(x_0)\big)^{1,+}$, the set $\overline{\mathrm{co}}\big(E \cap B_{r_{n+1}}(x_0)\big)^{1,+}$ contains the segment joining x_0 and x_n. Denoted the latter by $[x_0, x_n]$, the monotonicity of $\mathrm{C}_1(\cdot)$ and Remark 3 yield

$$\gamma(E) \geq \mathrm{C}_1\big([x_0, x_n]\big) \geq 2r_n.$$

In conclusion, by letting $n \to +\infty$ we obtain

$$\gamma(E) = \gamma(\overline{\mathrm{co}}(E^{1,+})) = +\infty.$$

The previous analysis also characterizes the minimizers of the problem if $\gamma(E) < +\infty$ and then proves (iii). Indeed, suppose that D is an indecomposable set with $\mathcal{L}^2(D) < +\infty$ satisfying $\gamma(E) = \mathrm{Per}(D) < +\infty$. If $\gamma(E) > 0$ then $\mathcal{L}^2(D) > 0$, and by the argument employed above we deduce that the set D is $\mathcal{H}^1$-essentially bounded. In addition, the previous analysis also entails that the set $\overline{\mathrm{co}}(D^+)$ is a minimizer for $\gamma(E)$ and $\gamma(\overline{\mathrm{co}}(E^{1,+}))$ with $\mathrm{Per}(\overline{\mathrm{co}}(D^+)) = \mathrm{Per}(D)$. Then the last part of the statement of Lemma 3.5 yields that $D^+ = \overline{\mathrm{co}}(D^+)$, and a standard projection argument between convex sets implies $\mathrm{Per}(\overline{\mathrm{co}}(D^+)) \geq \mathrm{Per}(\overline{\mathrm{co}}(E^{1,+}))$ with strict inequality if $\overline{\mathrm{co}}(D^+)$ strictly contains $\overline{\mathrm{co}}(E^{1,+})$.

On the other hand, if D is such that $\gamma(E) = \mathrm{Per}(D) = 0$ and $\mathcal{L}^2(D) < +\infty$, the isoperimetric inequality [2, Theorem 3.46] entails $\mathcal{L}^2(D) = 0$, and then $D^+ = \overline{\mathrm{co}}(D^+) = \emptyset$.

It remains to show the validity of (i), (ii) and (iv). As respect to (i), let E be such that $\gamma(E) = 0$. By applying (10) we deduce that $\mathrm{C}_1(\overline{\mathrm{co}}(E^{1,+})) = 0$. Taking into account (6) we then infer $\mathcal{H}^1(\overline{\mathrm{co}}(E^{1,+})) = 0$, that in turn implies $\mathcal{H}^1(E^{1,+}) = 0$. By Lemma 3.3 the same holds for the set E. The converse inequality is trivial.

To prove (ii) notice that if E is $\mathcal{H}^1$-essentially bounded there exists a bounded Borel set F such that $\gamma(E) = \gamma(F)$. Since F is contained in a suitable ball $B_r(x)$ then

$$\gamma(E) = \gamma(F) \leq \gamma(B_r(x)) = \mathrm{C}_1(\overline{B}_r(x)) = 2\pi r.$$

This proves that if $\gamma(E) = +\infty$ then E is not $\mathcal{H}^1$-essentially bounded; the converse implication has been already proved by the arguments developed above to get the validity of (10).

We are left with proving (iv). Let E be a Borel set with $0 < \gamma(E) < +\infty$. If $\mathcal{L}^2(\overline{\mathrm{co}}(E^{1,+})) > 0$, a minimizer for $\gamma(E)$ is given by $\overline{\mathrm{co}}(E^{1,+})$ itself as shown above. Otherwise, $\overline{\mathrm{co}}(E^{1,+})$ is contained in a line segment. Assume by contradiction that there exists a minimizer D for $\gamma(E)$. Then $\mathcal{L}^2(D) > 0$, and $\overline{\mathrm{co}}(D^+)$ is a minimizer for $\gamma(\overline{\mathrm{co}}(E^{1,+}))$ with $\mathcal{L}^2(\overline{\mathrm{co}}(D^+)) > 0$. By a projection argument on the line containing $E^{1,+}$ we get at once the contradiction $\mathrm{Per}(\overline{\mathrm{co}}(D^+)) > 2\mathcal{H}^1(\overline{\mathrm{co}}(E^{1,+})) = \gamma(\overline{\mathrm{co}}(E^{1,+}))$ (for the last equality see Remark 3). □

To conclude the section we notice that as a consequence of the properties stated in Theorem 3.2 we can establish the following result.

Corollary 1. *E^+ is bounded for any indecomposable set $E \subseteq \mathbb{R}^2$ of finite measure.*

Proof. By taking the set E itself as a competitor we get $\gamma(E) \leq \mathrm{Per}(E) < +\infty$. On the other hand, arguing by contradiction, if E^+ was unbounded then E would not be $\mathcal{H}^1$-essentially bounded, and thus $\gamma(E) = +\infty$ as stated in point (ii) of Theorem 3.2. □

4. **Further results.** This section is devoted to prove some consequences that can be deduced from our main result and to show its links with some other variational problems.

The first straightforward consequence of Theorem 3.2 is the following.

Corollary 2. *For any $\mathcal{L}^2$-measurable set $E \subset \mathbb{R}^2$ we have*

$$\gamma(E^+) = \gamma(\overline{\mathrm{co}}(E^+)). \tag{20}$$

Proof. Taking into account that, by Lemma 2.2, $\overline{\mathrm{co}}((E^+)^{1,+}) = \overline{\mathrm{co}}(E^+)$, the result follows by applying Theorem 3.2 to the Borel set E^+. □

Using the previous corollary and our main result we recover [5, Theorem 1].

Corollary 3. *Let $E \subset \mathbb{R}^2$ be a bounded $\mathcal{L}^2$-measurable set with $\mathcal{L}^2(E) > 0$. The problem*

$$\mu(E) := \inf\{\mathrm{Per}(D) : D \,\text{indecomposable}, \ \mathcal{L}^2(D) < +\infty\,, \mathcal{L}^2(E \setminus D^+) = 0\}$$

has the unique solution $\overline{\mathrm{co}}(E^1)$ (mod $\mathcal{L}^2$).

Proof. Take first note that $\mathcal{L}^2(E \triangle F) = 0$ implies $\mu(E) = \mu(F)$, in particular $\mu(E) = \mu(E^+)$. Thus we may assume E to be a Borel set without loss of generality.

We claim that $\mu(E) = \gamma(E^+)$. Given this for granted, the conclusion follows as $\gamma(E^+) = \gamma\,(\overline{\mathrm{co}}(E^+))$ by Corollary 2 and $\overline{\mathrm{co}}(E^+) = \overline{\mathrm{co}}(E^1)$ by Lemma 2.2. Moreover, by Theorem 3.2 the minimizer of the minimum problem defining $\gamma(E^+)$ is determined up to $\mathcal{L}^2$-negligible sets by $\overline{\mathrm{co}}(E^+)$.

We are left with proving $\mu(E) = \gamma(E^+)$. To this aim let D be a competitor for $\mu(E)$, then $E^+ \subseteq (D^+)^+ = D^+$ by (3). Hence, D is admissible for $\gamma(E^+)$, and thus $\gamma(E^+) \leq \mu(E)$.

On the other hand, $\mathcal{L}^2(E^+ \setminus D^+) = 0$ for any competitor D for $\gamma(E^+)$ since $\mathcal{H}^1(E^+ \setminus D^+) = 0$. Therefore, D is admissible for $\mu(E)$ and thus $\mu(E) \leq \gamma(E^+)$. The claim then follows. □

As a further byproduct of Theorem 3.2 and the analysis in [1] we characterize the minimizers of capacitary problems for indecomposable sets in the plane $\mathcal{H}^1$ almost all contained in their measure theoretic closure E^+. Recall that otherwise minimizers for the capacitary problem might not exist.

Corollary 4. *Let $E \subset \mathbb{R}^2$ be an indecomposable set such that $\mathcal{L}^2(E) > 0$ and $\mathcal{H}^1(E \setminus E^+) = 0$. Then*

$$\mathrm{C}_1(E) = \mathrm{Per}(\overline{\mathrm{co}}(E^+)).$$

In particular, the unique minimizer (mod $\mathcal{L}^2$) is $\overline{\mathrm{co}}(E^+)$.

Proof. Take note that $\mu(E) = \mu(E^+) = \mathrm{Per}(\overline{\mathrm{co}}(E^+))$ by Corollary 3, and in addition that $\mathrm{C}_1(E) = \mathrm{C}_1(E^+)$ because of the assumption $\mathcal{H}^1(E \setminus E^+) = 0$ (see Proposition 2). Thus, to conclude it suffices to show equality $\mathrm{C}_1(E^+) = \mu(E^+)$.

Since $\mathrm{C}_1(E^+) \leq \mu(E^+)$ by the very definitions, we are left with proving $\mu(E^+) \leq \mathrm{C}_1(E^+)$.

Let D be a competing set for $\mathrm{C}_1(E^+)$, then by Theorem 2.3 there exists an indecomposable set $\tilde{D}$ with $\mathrm{Per}(\tilde{D}) \leq \mathrm{Per}(D)$ such that $\mathcal{L}^2(E^+ \setminus \tilde{D}) = 0$. Hence, for any admissible set D for $\mathrm{C}_1(E^+)$ we have $\mu(E^+) \leq \mathrm{Per}(D)$, by passing to the infimum on such sets we conclude.

Finally, uniqueness mod $\mathcal{L}^2$ is a consequence of Corollary 3. □

The previous result can be further extended. It reduces the calculation of the capacity of a set of finite perimeter E satisfying $\mathcal{H}^1(E \setminus E^+) = 0$, to the minimal perimeter configuration among the convex hulls of all the subsets of the family of its indecomposable components.

Corollary 5. *Let $E \subset \mathbb{R}^2$ be a set with finite perimeter with $\mathcal{L}^2(E) > 0$ and $\mathcal{H}^1(E \setminus E^+) = 0$. Denote by $\{E_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, the family of its indecomposable components. Then, if $\#I = k \in \mathbb{N}$ setting*

$$\beta_k(E) := \min\left\{\sum_{r=1}^{s} \mathrm{Per}\left(\overline{\mathrm{co}}\left(\cup_{j\in I_r} E_j^+\right)\right) \,:\, \{1,\dots,k\} = \sqcup_{r=1}^{s} I_r\right\}$$

we have

$$\mathrm{C}_1(E) = \beta_k(E). \tag{21}$$

Otherwise, if $I = \mathbb{N}$ we have

$$\mathrm{C}_1(E) = \lim_{k\to+\infty} \beta_k(\cup_{i\le k} E_i). \tag{22}$$

Proof. Let us first prove identity (21). As noticed in Corollary 4 we have $\mathrm{C}_1(E) = \mathrm{C}_1(E^+)$. In addition, it is easy to check from the very definitions equality $\beta_k(E) = \beta_k(E^+)$, and also that

$$\mathrm{C}_1(E^+) \le \beta_k(E^+).$$

Hence, we may suppose without loss of generality $\mathrm{C}_1(E^+) < +\infty$.

We argue by induction on k, the first step $k = 1$ being established by Corollary 4. Then assume $k \ge 2$, fix $\varepsilon > 0$ and let D be a competitor for the capacitary problem related to E^+ such that $\mathrm{Per}(D) \le \mathrm{C}_1(E^+) + \varepsilon$. Either D is indecomposable or not. In the former case, Corollaries 2 and 4 yield $\beta_k(E^+) \le \mu(E^+) = \gamma(E^+) \le \mathrm{Per}(D)$ and thus $\beta_k(E^+) \le \mathrm{C}_1(E^+) + \varepsilon$. In the latter case, there exist at least two indecomposable components of E, E_{i_1} and E_{i_2} contained (mod $\mathcal{L}^2$) into two different indecomposable components of D, D_{j_1} and D_{j_2} (see item (i) in Theorem 2.3). Furthermore, since $\mathrm{Per}(D) = \mathrm{Per}(D_{j_1}) + \mathrm{Per}(\cup_{j\ne j_1} D_j)$ and $E^+ = \cup_{i=1}^{k} E_i^+$, Corollary 4 and the inductive assumption yield

$$\begin{aligned}
\beta_k(E^+) &\le \mu(E_{i_1}^+) + \beta_{k-1}(\cup_{i\ne i_1} E_i)\\
&= \mu(E_{i_1}^+) + \mathrm{C}_1(\cup_{i\ne i_1} E_i)\\
&\le \mathrm{Per}(D) \le \mathrm{C}_1(E^+) + \varepsilon.
\end{aligned}$$

The conclusion then follows at once as $\varepsilon \downarrow 0^+$.

Finally, we establish (22). Being $E^+ = \cup_{i\in\mathbb{N}} E_i^+$ and by taking into account that $\mathrm{C}_1(\cdot)$ is a continuous set function on increasing sequences of sets (sse item (iii) in Proposition 1) we have

$$\mathrm{C}_1(E^+) = \lim_{k\to+\infty} \mathrm{C}_1(\cup_{i\le k} E_i^+) = \lim_{k\to+\infty} \beta_k(\cup_{i\le k} E_i).$$

□

5. **Appendix.** Here we recall a well known property of the perimeter functional. The proof of the statement is presented for the reader's convenience since we have not found any explicit reference in literature. In the following, given a simple closed curve $\Gamma \subset \mathbb{R}^2$, we will denote by $\mathscr{L}(\Gamma)$ its length and we will refer to a *polygonal inscribed* in Γ as a closed polygonal with vertices lying on Γ. Moreover, we will tacitly consider all the simple curves oriented in the counterclockwise sense and use the notation $\prec$ for the order.

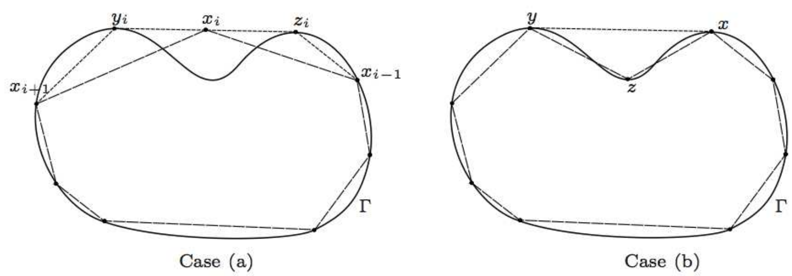

FIGURE 2. Monotonicity of perimeter under convexification.

Proposition 3. *Let* Γ *be a rectifiable Jordan curve, then*

$$\mathrm{Per}\big(\overline{\mathrm{co}}(\mathrm{int}(\Gamma))\big) \leq \mathrm{Per}\big(\mathrm{int}(\Gamma)\big). \tag{23}$$

Moreover the inequality is strict whenever $\mathrm{int}(\Gamma)$ *is not convex.*

Proof. We first observe that, by (8) and by the regularity of convex sets, $\mathrm{Per}(\mathrm{int}(\Gamma))$ and $\mathrm{Per}(\overline{\mathrm{co}}(\mathrm{int}(\Gamma)))$ are nothing but the length of Γ and of $\partial\big(\overline{\mathrm{co}}(\mathrm{int}(\Gamma))\big)$ respectively. Then the inequality (23) will follow once we have proved that for any polygonal, Θ, inscribed in $\partial(\overline{\mathrm{co}}(\mathrm{int}(\Gamma)))$, one can find a polygonal, Θ', inscribed in Γ with

$$\mathscr{L}(\Theta) \leq \mathscr{L}(\Theta'). \tag{24}$$

With this aim in mind, let $\{x_i\}_{i=1}^k$ be the vertices of Θ ordered in a counterclockwise sense. If for some $i \in \{1, \ldots, k\}$, $x_i \notin \Gamma$, then x_i can be written as a convex combination of extreme points of $\overline{\mathrm{co}}(\mathrm{int}(\Gamma))$, then lying in Γ (see figure 2.a), i.e. there exist $\lambda \in (0,1)$, $y_i, z_i \in \Gamma$ such that

$$x_i = \lambda y_i + (1-\lambda) z_i.$$

The polygonal Θ_i with vertices $\{x_1, \ldots, x_{i-1}, y_i, z_i, x_{i+1}, \ldots, x_k\}$, since it contains the point x_i, by triangular inequality, clearly satisfies

$$\mathscr{L}(\Theta_i) > \mathscr{L}(\Theta).$$

We can proceed inductively. If Θ_i is not inscribed in Γ, using the previous argument, by adding at most $k-1$ vertices we end up with a polygonal $\widetilde{\Theta}$ inscribed in Γ satisfying (24). Thus, (23) is established.

Suppose now that $\mathrm{int}(\Gamma)$ is not convex, then $\partial\big(\overline{\mathrm{co}}(\mathrm{int}(\Gamma))\big)$ contains at least one segment with end points x, y, say $[x,y]$,with $[x,y] \cap \Gamma = \{x,y\}$ (see figure 2.b). Hence we may restrict the class of competitor in the definition of the length of $\partial\big(\overline{\mathrm{co}}(\mathrm{int}(\Gamma))\big)$ to the class P_{xy} of the polygonal inscribed in $\partial\big(\overline{\mathrm{co}}(\mathrm{int}(\Gamma))\big)$ containing $[x,y]$, i.e.

$$\mathscr{L}\left(\partial\big(\overline{\mathrm{co}}(\mathrm{int}(\Gamma))\big)\right) = \sup_{\Theta \in P_{xy}} \mathscr{L}(\Theta).$$

Let now $z \in \Gamma$ with $x \prec z \prec y$, since we have

$$0 < h = |x-z| + |y-z| - |x-y|,$$

adding the vertex z we can modify any polygonal Θ in P_{xy} in a new polygonal, Θ_z, with

$$\mathscr{L}(\Theta_z) \geq \mathscr{L}(\Theta) + h$$

Applying again the previous inductive construction we can therefore construct a polygonal $\widetilde{\Theta}$ inscribed in Γ such that

$$\mathscr{L}(\widetilde{\Theta}) \geq \mathscr{L}(\Theta) + h,$$

that gives the claim by the arbitrariness of $\Theta \in P_{xy}$. $\square$

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