Fine properties of minimizers of mechanical Lagrangians with Sobolev potentials

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Abstract

In this paper we study the properties of curves minimizing mechanical Lagrangian where the potential is Sobolev. Since a Sobolev function is only defined almost everywhere, no pointwise results can be obtained in this framework, and our point of view is shifted from single curves to measures in the space of paths. This study is motived by the goal of understanding the properties of variational solutions to the incompressible Euler equations.

1 Introduction

Let $L : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ be a smooth time-dependent Lagrangian which is convex and superlinear in the velocities. In this case the properties of extremal curves have been known for a long time. In particular, they are characterized by being solutions of the Euler-Lagrange equations. Moreover, given an initial datum $u_0 : \mathbb{R}^d \to \mathbb{R}$, the *value function* $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$u(x,t) := \min\Big\{u_0(\gamma(0)) + \int_0^t L\big(\gamma(s),\dot\gamma(s),s\big)\,ds : \gamma\colon [0,t] \to \mathbb{R}^d, \gamma(t) = x\Big\},$$

has been studied by means of the method of characteristics. This function enjoys many interesting properties. For instance it is differentiable in the x variable at a point $(\bar x,\bar t)$ if and only if there exists an unique curve γ which attains the minimum in the definition of $u(\bar x,\bar t)$, and in this case $\nabla_x u(\bar x,\bar t) = \dot\gamma(\bar t)$. In case the minimizing curves at $(\bar x,\bar t)$ are not unique, what remains true is that the superdifferential at $(\bar x,\bar t)$ is the convex hull of $\{\dot\gamma(\bar t) : \gamma \text{ minimizer at } (\bar x,\bar t)\}$. Concerning the second order differentiability of u, the value function enjoys the important property of semi-concavity. Finally, the value function is a viscosity solution of the time-dependent Hamilton-Jacobi equation $\partial_t u + H(x,\nabla_x u,t) = 0$, H being the Legendre transform

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of L. (We refer to [6] for a proof of all these results.) On the other hand, whenever one relaxes the regularity assumptions on L, very little is known.

In this paper we shall consider mechanical Lagrangians for which only Sobolev regularity on the potential is assumed, and we investigate which properties still hold true in this case. As we will see, no pointwise results can be obtained in this more general framework, so the point of view will be shifted from curves to probability measures on the space of curves.

The interest in non-smooth Lagrangians comes from the study of a variational approach to the Euler equations for incompressible fluids: as shown by Arnold [4], the Euler equations in a domain $D \subset \mathbb{R}^d$ can be interpreted as a geodesic equation on the infinite-dimensional manifold $\mathrm{SDiff}(D)$ of the measure-preserving diffeomorphisms of D. Although some well-posedness results can be proven when trying to find minimizing geodesics between two close diffeomorphisms [8], Shnirelman [10, 11] showed that the geodesic problem is ill-posed in the large. To bypass this problem, Brenier has proposed a relaxed approach [5]: he replaces the notion of path in $\mathrm{SDiff}(D)$ with the one of probability measures η on $\Omega_T(D) := C\big([0,T];\overline{D}\big)$ which satisfy an incompressibility constraint. A coherent notion of energy is available also in this wider class of objects, so that it is possible to settle the geodesic problem in this setting and to prove existence of minimizers [5, 1]. By relaxing also the incompressibility constraint, the pressure field arises as a Lagrange multiplier, and as shown in [1] it is possible to look for a generalized solution in $[0,T]$ to the Euler equations by minimizing

$$\min_{\eta} \int_{\Omega_T(D)} \int_0^T \left(\frac{|\dot\gamma(t)|^2}{2} - p(\gamma(t),t) \right) dt\, d\eta(\gamma)$$

among all η with bounded compression satisfying the initial and final constraints (see later for precise definitions). Furthermore, as proven in [2], the best known regularity result for $p(\cdot,t)$ in this situation is BV. This fact motivates the study of non-smooth mechanical Lagrangians. In the present work we shall deal with Sobolev Lagrangians, which, still in their own interest, represent an intermediate step towards the BV case. More precisely, we will define some particular subsets of the spaces of probability measures on $\Omega_T(D)$ which are particularly suitable for studying this kind of problem (see Section 2), and we will investigate the regularity properties of the curves on which minimizing measures are concentrated. Let us observe that, for the model problem we have in mind (i.e. the case when the Lagrangian is given by $\frac{1}{2}|v|^2 - p(t,x)$, p being the pressure associated to a variational solution to the Euler equations), the existence of a minimizer inside the class $\mathcal{P}^{<}_{T,\infty}(D)$ is known [1] (see (2.1) and Definition 2.3 below). Moreover (assuming for simplicity $\mathcal{L}^d(D) = 1$) this minimizer can be chosen to belong to the smaller class $\mathcal{P}^{\approx}_{T,1}(D)$ of incompressible flows. For this reason, in this paper we will never address the question of existence of minimizers, and we will only be interested in studying regularity properties of minimizers.

The paper is structured as follows: in Section 2 we introduce some definition and notation which will be used through the whole paper, and we collect some preliminary technical results. Section 3 is devoted to show that if the potential is $W^{1,p}$, then a.e. extremal curve is $W^{2,p}$ and satisfies the Euler equations (see Theorem 3.3). Finally, in Section 4 we study the properties of the value function. For instance we can show that, if the potential is $W^{2,p}$, then the second spatial derivatives of the value function are measures whose positive part belong to L^p (when $p = \infty$ this correspond to the classical fact that the value function is semiconcave).

2 Notation and preliminary results

Let us introduce the framework for the following sections. Here and in the sequel, D will always denote either a smooth bounded domain of $\mathbb{R}^d$ or the d-dimensional torus $\mathbb{T}^d$. Let $L(x, v, t)$ denote a time-dependent mechanical Lagrangian of the form

$$\begin{aligned} L\colon \overline{D} \times \mathbb{R}^d \times [0, T] &\to \mathbb{R} \\ (x, v, t) &\mapsto \frac{|v|^2}{2} - V(x, t), \end{aligned}$$

where $T > 0$ and $V : \overline{D} \times [0, T] \to \mathbb{R}$ is the potential.

For a continuous curve $\gamma\colon [0, T] \to \overline{D}$, his action is given by

$$\mathcal{A}(\gamma) := \int_0^T \left(\frac{|\dot\gamma(t)|^2}{2} - V\big(\gamma(t), t\big) \right) dt,$$

whenever the integral is well-defined. The set $C\big([0, T]; \overline{D}\big)$ is the most general space of admissible curves for our concerns, and will be denoted by $\Omega_T(D)$. Let us also set $e_t\colon \Omega_T(D) \to \overline{D}$ by $e_t(\gamma) := \gamma(t)$.

Given a probability measure $\eta \in \mathcal{P}(\Omega_T(D))$, his action is given by (with a little abuse of language, we use the same symbol as for the action of a curve)

$$\mathcal{A}(\eta) := \int_{\Omega_T(D)} \int_0^T \left(\frac{|\dot\gamma(t)|^2}{2} - V\big(\gamma(t), t\big) \right) dt\, d\eta(\gamma),$$

whenever this double integral is well-defined.

Let us now introduce some sets of probability measures on $\Omega_T(D)$, whose time marginals are controlled by the Lebesgue measure:

$$\begin{aligned} \mathcal{P}^{\lesssim}_{T,C}(D) &:= \Big\{ \eta \in \mathcal{P}(\Omega_T(D)) : (e_t)_\# \eta \le C\mathcal{L}^d \ \forall\, t \in [0, T] \Big\}, \\ \mathcal{P}^{\lesssim}_{T,\infty}(D) &:= \bigcup_{C>0} \mathcal{P}^{\lesssim}_{T,C}(D), \\ \mathcal{P}^{\approx}_{T,C}(D) &:= \Big\{ \eta \in \mathcal{P}(\Omega_T(D)) : \frac{1}{C}\mathcal{L}^d \le (e_t)_\# \eta \le C\mathcal{L}^d \ \forall\, t \in [0, T] \Big\}, \\ \mathcal{P}^{\approx}_{T,\infty}(D) &:= \bigcup_{C>0} \mathcal{P}^{\approx}_{T,C}(D). \end{aligned} \tag{2.1}$$

Given a measure $\eta \in \mathcal{P}^{<}_{T,\infty}(D)$, the density of $(e_t)_\#\eta$ with respect to $\mathcal{L}^d$ will be denoted by ρ_t. (Although ρ_t depends on η we prefer not to explicit this dependence in order to keep the notation lighter.)

Remark 2.1. If $\eta \in \mathcal{P}^{<}_{T,\infty}(D)$, then for η-a.e. γ

$$\mathcal{L}^1\big(\{t \in [0,T] : \gamma(t) \in \partial D\}\big) = 0.$$

This is a consequence of Fubini Theorem, together with the absolute continuity of $(e_t)_\#\eta$:

$$\begin{aligned}\int_{\Omega_T(D)} \int_0^T \chi_{\partial D}(\gamma(t))\, dt\, d\eta &= \int_0^T \int_{\overline{D}} \chi_{\partial D}(x)\, d(e_t)_\#\eta(x)\, dt \\ &= \int_0^T \int_{\overline{D}} \chi_{\partial D}(x)\rho_t(x)\, dx\, dt = 0.\end{aligned}$$

The same argument also shows that if $f\colon D \times [0,T]$ is a function defined only $\mathcal{L}^{d+1}$-a.e., then for η-a.e. γ the function

$$t \mapsto f(\gamma(t),t), \qquad t \in [0,T],$$

is well-defined $\mathcal{L}^1$-a.e. Indeed if f is modified on a $\mathcal{L}^{d+1}$-negligible set, then the value of $f(\gamma(t),t)$ is modified on a $\eta \otimes dt$-negligible set, which by Fubini Theorem implies that for η-a.e. γ the set of t at which there is a modification is $\mathcal{L}^1$-negligible.

Remark 2.2. If $V \in L^1(D \times [0,T])$ and $\eta \in \mathcal{P}^{<}_{T,\infty}(D)$, then $\mathcal{A}(\eta)$ is well-defined (possibly, it take value $+\infty$), and moreover also $\mathcal{A}(\gamma)$ is well-defined for η-a.e. γ. Indeed this follows easily from

$$\int_{\Omega_T(D)} \int_0^T \big|V(\gamma(t),t)\big|\, dt\, d\eta = \int_0^T \int_D \big|V(x,t)\big|\, d(e_t)_\#\eta(x)\, dt \le C\|V\|_{L^1},$$

which implies that $t \mapsto V(\gamma(t),t) \in L^1(0,T)$ for η-a.e. γ. Moreover, thanks to Remark 2.1 the value of $\mathcal{A}(\gamma)$ is independent of the choice of the representative of V for η-a.e. γ.

Definition 2.3. *Let $V \in L^1(D \times [0,T])$. We say that $\eta \in \mathcal{P}^{<}_{T,\infty}(D)$ is a minimizer for the action at fixed endpoints if $\mathcal{A}(\eta) < +\infty$ and*

$$\mathcal{A}(\eta) = \min\Big\{\mathcal{A}(\nu) : \nu \in \mathcal{P}^{<}_{T,\infty}(D),\ (e_0)_\#\nu = (e_0)_\#\eta,\ (e_T)_\#\nu = (e_T)_\#\eta\Big\}.$$

Convention: in the whole paper, C will denote a positive constant which depends only on the dimension d, the domain D, and the bounds on the density $\rho = (e_t)_\#\eta$ (η will always be a fixed measure in every statement), and may change value from line to line.

We now prove some technical results on the L^p-convergence of the incremental quotients for Sobolev functions and some properties of distributions, which will be used in the next sections.

Proposition 2.4. *Let $u \in L^1(\mathbb{R}^d;\mathbb{R}^m)$, and $z \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ with a bounded gradient. Then, for any measurable set $S \subseteq \mathbb{R}^d$,*

$$\|u(\cdot + \varepsilon z(\cdot)) - u\|_{L^1(S;\mathbb{R}^m)} \to 0 \qquad \text{as } \varepsilon \to 0$$

Proof. Suppose first that $u \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$, and fix $K \subset \mathbb{R}^d$ a compact such that $u(\cdot + \varepsilon z(\cdot))$ is identically zero outside K for ε small. Then

$$\|u(\cdot + \varepsilon z(\cdot)) - u\|_{L^1(S;\mathbb{R}^m)} \leq \|u(\cdot + \varepsilon z(\cdot)) - u\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} \leq \varepsilon \operatorname{Lip} u \|z\|_{L^1(K)} \max_{x\in K} |z(x)|.$$

and the last term converges to 0 as $\varepsilon \to 0$.

If u is arbitrary in $L^1(\mathbb{R}^d;\mathbb{R}^m)$, we can consider a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that

$$u_k \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m), \qquad \|u - u_k\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} \to 0 \quad \text{as } k \to \infty.$$

In this way we get

$$\begin{aligned}\|u(\cdot + \varepsilon z(\cdot)) - u\|_{L^1(S;\mathbb{R}^m)} \leq\ & \|u(\cdot + \varepsilon z(\cdot)) - u_k(\cdot + \varepsilon z(\cdot))\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} \\ &+ \|u_k(\cdot + \varepsilon z(\cdot)) - u_k\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} \\ &+ \|u_k - u\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)}.\end{aligned}$$

Moreover, denoting by $\phi_\varepsilon(x) = x + \varepsilon z(x)$, thanks to the assumptions on z we have

$$\begin{aligned}\|u(\cdot + \varepsilon z(\cdot)) - u_k(\cdot + \varepsilon z(\cdot))\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} &= \int_{\mathbb{R}^d} \big|u(x + \varepsilon z(x)) - u_k(x + \varepsilon z(x))\big|\, dx \\ &= \int_{\mathbb{R}^d} |u(y) - u_k(y)||\det \nabla(\phi_\varepsilon^{-1})(y)|\, dy \\ &\leq C\|u - u_k\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)}.\end{aligned}$$

Hence

$$\|u(\cdot + \varepsilon z(\cdot)) - u\|_{L^1(S;\mathbb{R}^m)} \leq (1 + C)\|u - u_k\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} + \|u_k(\cdot + \varepsilon z(\cdot)) - u_k\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)},$$

so that

$$\limsup_{\varepsilon\to 0} \|u(\cdot + \varepsilon z(\cdot)) - u\|_{L^1(S)} \leq (1 + C)\|u - u_k\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)},$$

which concludes the proof as the right hand side can be made arbitrarily small. $\square$

Lemma 2.5. *Let $V \in L^1([0,T]; W^{1,1}(D))$, $\varphi \in C^0([0,T];\mathbb{R}^d)$. Suppose that $f\colon \overline{D} \times [0,T] \to \mathbb{R}^+$ is bounded and that there exists $\bar\varepsilon > 0$ such that*

$$\operatorname{supp} f \subseteq \{(x,t) \in \overline{D} \times [0,T] : x + \varepsilon\varphi(t) \in \overline{D}\} \qquad \forall\ 0 \leq \varepsilon < \bar\varepsilon.$$

Then

$$\lim_{\varepsilon \searrow 0} \int_{D\times[0,T]} \left| f(x,t)\Big(\frac{V(x + \varepsilon\varphi(t), t) - V(x,t)}{\varepsilon} - \nabla_x V(x,t) \cdot \varphi(t)\Big)\right|\, dt\, dx = 0$$

Proof. We notice that the integrand is well-defined due to the condition on the support of f. Thanks to the boundedness of f we can compute

$$\int_{D\times[0,T]} \left| f(x,t)\Big(\frac{V(x+\varepsilon\varphi(t),t)-V(x,t)}{\varepsilon} - \nabla_x V(x,t)\cdot\varphi(t)\Big)\right| dt\,dx$$
$$\leq \|f\|_\infty\|\varphi\|_\infty \int_0^1 \int_{\operatorname{supp} f} \big|\nabla_x V(x+\varepsilon s\varphi(t),t) - \nabla_x V(x,t)\big|\,dt\,dx\,ds.$$

Up to extending by zero $\nabla_x V$ on the whole of $\mathbb{R}^{d+1}$, we can apply Proposition 2.4 with $u := \nabla_x V$ and $z(x,t) := (s\varphi(t),0)$ to obtain, for s fixed,

$$\int_{\operatorname{supp} f} \big|\nabla_x V(x+\varepsilon s\varphi(t),t) - \nabla_x V(x,t)\big|\,dt\,dx \to 0 \quad \text{as } \varepsilon\to 0.$$

(Observe that, even if z is only C^0, thanks to its particular structure we have $\det\big((x,t)\mapsto(x+\varepsilon s\varphi(t),t\big) = 1$, so the proof still works.) Moreover

$$\int_{\operatorname{supp} f} \big|\nabla_x V(x+\varepsilon s\varphi(t),t) - \nabla_x V(x,t)\big|\,dt\,dx \leq 2\,\big\|\|V(\cdot,t)\|_{W^{1,1}(D)}\big\|_{L^1[0,T]},$$

and we conclude by applying the Dominated Convergence Theorem. □

Lemma 2.6. *Fix $\delta>0$, and let $u\in W^{2,1}(D^\delta)$, with $D^\delta = \{x\in\mathbb{R}^d : d(x,D)<\delta\}$. Then, for any $z: D\to\mathbb{R}^d$ such that $\|z\|_{C^1(D)} < +\infty$ we have*

$$\lim_{\varepsilon\searrow 0}\int_D \left|\frac{u(x+\varepsilon z(x)) + u(x-\varepsilon z(x)) - 2u(x)}{\varepsilon^2} - \big\langle\nabla^2 u(x)z(x), z(x)\big\rangle\right| dx = 0$$

Proof. The assumptions ensure that, for $|\varepsilon|$ small enough, the composition $u(x+\varepsilon z(x))$ is well defined. Let us also point out that, for $|\varepsilon|$ small, the line segment $[x, x+\varepsilon z(x)]$ is entirely contained in D^δ for any $x\in D$.

Suppose first that $u\in C_c^\infty(D)$. Then

$$\begin{aligned}
&\frac{1}{h^2}\Big(u(x+hz(x)) + u(x-hz(x)) - 2u(x)\Big)\\
&\quad= \int_0^h \frac{\nabla u(x+sz(x)) - \nabla u(x-sz(x))}{h^2}\cdot z(x)\,ds\\
&\quad= \int_0^h\int_{-s}^s \frac{\big\langle\nabla^2 u(x+rz(x))\,z(x), z(x)\big\rangle}{h^2}\,dr\,ds\\
&\quad= \int_0^1\int_{-1}^1 \big\langle\nabla^2 u(x+hsrz(x))\,z(x), z(x)\big\rangle\,s\,dr\,ds.
\end{aligned}$$

Thus, the integral in the statement can be written as

$$\begin{aligned}
&\int_D \left|\int_0^1\int_{-1}^1 s\,\big\langle\nabla^2 u(x+hsrz(x))z(x), z(x)\big\rangle - s\,\big\langle\nabla^2 u(x)z(x), z(x)\big\rangle\,dr\,ds\right| dx\\
&\quad\leq C\int_0^1\int_{-1}^1\int_D |z(x)|\,\big|\nabla^2 u(x+hsrz(x)) - \nabla^2 u(x)\big|\,dx\,dr\,ds \qquad (2.2)\\
&\quad\leq C\int_0^1\int_{-1}^1\int_D \big|\nabla^2 u(x+hsrz(x)) - \nabla^2 u(x)\big|\,dx\,dr\,ds.
\end{aligned}$$

We now observe that the function $(r,s) \mapsto \int_D |\nabla^2 u(x+hsrz(x)) - \nabla^2 u(x)|\,dx$ pointwise converges to zero (as $h \to 0$) due to Proposition 2.4. Moreover, if we set $\phi_{h,r,s}(x) := x + hsrz(x)$ we have

$$\begin{aligned}
&\int_D \big|\nabla^2 u(x+hsrz(x)) - \nabla^2 u(x)\big|\,dx \\
&\qquad \le \int_D |\nabla^2 u(x+hsrz(x))| + |\nabla^2 u(x)|\,dx \\
&\qquad \le \int_{D^\delta} |\nabla^2 u(y)|\,|\det(\nabla \phi_{h,r,s}^{-1})(y)|\,dy + \|u\|_{W^{2,1}(D)} \le C\|u\|_{W^{2,1}(D^\delta)},
\end{aligned}$$

Hence, thanks to the Dominated Convergence Theorem applied to the function $(r,s) \mapsto \int_D |\nabla^2 u(x+hsrz(x)) - \nabla^2 u(x)|\,dx$ we obtain the result. In the general case $u \in W^{2,1}(D^\delta)$ it suffices to observe that (2.2) still holds true by approximating u in $W^{2,1}$ by C_c^∞ functions, and one concludes as above. □

The following two propositions are well-known results in functional analysis and measure theory.

Lemma 2.7. *If $u \in \mathcal{D}'(D)$ satisfies $\langle u, \varphi\rangle \ge 0$ for all $\varphi \in C_c^\infty(D), \varphi \ge 0$, then u is a locally finite measure.*

Here and in the sequel we will denote by $\mathcal{M}(X)$ (resp. $\mathcal{M}_{loc}$) the set of finite (resp. locally finite) measures on X, and by $\|\cdot\|_{\mathcal{M}(X)}$ the total variation norm.

Lemma 2.8. *Let $D \subseteq \mathbb{R}^d$ be a bounded open set and let $u \in \mathcal{D}'(D)$ satisfy*

$$\big|\langle u, \varphi\rangle\big| \le \int_D |f\varphi|\,dx, \quad \forall\, \varphi \in C_c^\infty(D), \tag{2.3}$$

for some $f \in L^p(D)$, $1 \le p \le +\infty$. Then $u = h\,dx$ for some $h \in L^p(D)$, with $|h| \le |f|$.

Here we prove a generalization of the Lemma 2.8:

Lemma 2.9. *Let $D \subseteq \mathbb{R}^d$ be a bounded open set and let $u \in \mathcal{D}'(D)$ satisfy*

$$\langle u, \varphi\rangle \le \int_D f\varphi\,dx, \quad \forall\, \varphi \in C_c^\infty(D), \varphi \ge 0,$$

for some nonnegative function $f \in L^p(D)$, $1 \le p \le +\infty$. Then, u is a measure. Moreover, if $u = u^+ - u^-$ denotes the decomposition of v into its positive and negative part, then $u^+ = h\,dx$ for some $h \in L^p(D)$, with $0 \le h \le f$.

Proof. We first prove that $u \in \mathcal{M}(D)$. Indeed, if $U \subset\subset D$ is open, $\varphi \in C_c^\infty(U)$ and $\psi \in C_c^\infty(D, [0,1])$ is a cut-off function such that $\psi \equiv 1$ on U, then the function $(\|\varphi\|_\infty - \varphi)\psi \in C_c^\infty(D)$ is positive. Hence,

$$\langle u, (\|\varphi\|_\infty - \varphi)\psi\rangle \le \int_D f(\|\varphi\|_\infty - \varphi)\psi\,dx \le 2\|f\|_1\|\varphi\|_\infty.$$

This implies $-\langle u,\varphi\rangle \le 3\|f\|_1\|\varphi\|_\infty$. Replacing φ with $-\varphi$, we get

$$\big|\langle u,\varphi\rangle\big| \le 3\|f\|_1\|\varphi\|_\infty, \qquad \forall\,\varphi\in C_c^\infty(D),$$

so that $\|u\|_{\mathcal{M}(D)} \le 3\|f\|_1 < +\infty$. Let us observe that, by approximation, (2.3) now holds for every $\varphi\in C^0(D)$.

Let us then consider the decomposition $u=u^+-u^-$ and a Borel set $B^+\subseteq D$ such that $u^-(B^+)=u^+(D\setminus B^+)=0$. Let us show that $u^+ << \mathcal{L}^d$, which will imply $u^+=h\,d\mathcal{L}^d$ for some $h\in L^1(D)$, $h\equiv 0$ outside B^+. It suffices to show that if $N\subseteq B^+$ and $\mathcal{L}^d(N)=0$, then also $u^+(N)=0$. By Lusin Theorem, for every $\varepsilon>0$ there exists a continuous function φ_ε such that

$$|u|\big(\{x\in D:\varphi_\varepsilon(x)\ne\chi_N(x)\}\big)<\varepsilon, \qquad 0\le\varphi_\varepsilon\le 1.$$

We have

$$\int\varphi_\varepsilon\,du=\int\varphi_\varepsilon\,du^+-\int\varphi_\varepsilon\,du^-\le\int f\varphi_\varepsilon\,dx \qquad \forall\,\varepsilon>0,$$

so that taking the limit as $\varepsilon\to 0$ yields $\int\chi_N\,du^+\le 0$.

It remains to show that if $f\in L^p(D)$ then $h\in L^p(D)$ too. To this aim, let us consider $g\in L^q(B^+)\cap L^\infty(B^+)$, $g\ge 0$, and a Lusin-type approximating sequence $g_\varepsilon\colon D\to\mathbb{R}$ as before. We have

$$\int g_\varepsilon\,du=\int g_\varepsilon h\,dx-\int g_\varepsilon\,du^-\le\int fg_\varepsilon\,dx\le\|f\|_p\|g_\varepsilon\|_q, \qquad \varepsilon>0,$$

and, as $\varepsilon\to 0$, we get

$$\int gh\,dx\le\|f\|_p\|g\|_q \qquad \forall\,g\in L^q(B^+)\cap L^\infty(B^+), g\ge 0.$$

Since h is nonnegative we have $\|h\|_q=\sup\left\{\frac{\int hg\,dx}{\|g\|_q}: g\in L^\infty,\ g\ge 0,\right\}$, and the result is proved. $\square$

3 Euler-Lagrange equations

In this section we generalize to action-minimizing measures the fact that an extremal curve satisfies the Euler-Lagrange equations. We will assume that the potential enjoys a first-order Sobolev regularity in space (not in time).

Definition 3.1. *Given $\gamma\in\Omega_T(D)$ and $\varphi\in W_0^{1,2}([0,T];\mathbb{R}^d)$, we define*

$$\begin{aligned}
\mathrm{Adm}(\gamma) &:= \big\{\varphi\in W_0^{1,2}([0,T];\mathbb{R}^d):\exists\,\varepsilon>0 \ \ s.t. \ \ \gamma+\varepsilon'\varphi\in\Omega_T(D)\ \forall\,\varepsilon'\le\varepsilon\big\};\\
\mathrm{Adm}(\varphi) &:= \big\{\gamma\in\Omega_T(D):\varphi\in\mathrm{Adm}(\gamma)\big\};\\
\mathrm{Adm}_\varepsilon(\varphi) &:= \big\{\gamma\in\Omega_T(D):\gamma+\varepsilon'\varphi\in\Omega_T(D)\ \forall\,\varepsilon'\le\varepsilon\big\};\\
B(\gamma) &:= \big\{t\in[0,T]:\gamma(t)\in\partial D\big\};\\
\overline{\mathrm{Adm}}(\gamma) &:= \overline{\mathrm{Adm}(\gamma)}^{W^{1,2}([0,T];\mathbb{R}^d)}.
\end{aligned}$$

Proposition 3.2. *Let $V \in L^1([0,T];W^{1,p}(D))$, $1 \le p \le \infty$, and let η be a minimizer for the action in $\mathcal{P}^{<}_{T,\infty}(D)$ at fixed endpoints. Fix $\varphi \in W^{1,2}_0([0,T];\mathbb{R}^d)$. Then, for η-a.e. $\gamma \in \mathrm{Adm}(\varphi)$,*

$$\int_0^T \Big(\dot\gamma(t)\cdot\dot\varphi(t) - \nabla_x V(\gamma(t),t)\cdot\varphi(t)\Big)\,dt \ge 0.$$

Proof. First of all, the integral is well-defined for η-a.e. γ. This follows from the integrability assumptions on $\nabla_x V$ arguing as in Remark 2.2, and from the fact $\mathcal{A}(\eta) < +\infty$ (by the definition of minimizer).

Let us then suppose by contradiction that there exists $E \subseteq \mathrm{Adm}(\varphi)$ such that $\eta(E) > 0$ and

$$\int_E\int_0^T \Big(\dot\gamma(t)\cdot\dot\varphi(t) - \nabla_x V(\gamma(t),t)\cdot\varphi(t)\Big)\,dt < 0. \tag{3.1}$$

Since $\eta(E) = \lim_{n\to\infty}\eta(E\cap \mathrm{Adm}_{1/n}(\varphi))$, up to replacing E with $E\cap\mathrm{Adm}_{1/n}(\varphi)$ with n sufficiently big, we can suppose $E \subseteq \mathrm{Adm}_{\bar\varepsilon}(\varphi)$ for a fixed $\bar\varepsilon > 0$.

In such a case, the following function $F_{\varepsilon,E,\varphi} : \Omega_T(D) \to \Omega_T(D)$ is well defined for $\varepsilon < \bar\varepsilon$:

$$F_{\varepsilon,E,\varphi}(\gamma) = \begin{cases} \gamma + \varepsilon\varphi & \text{if } \gamma \in E \\ \gamma & \text{if } \gamma \in \Omega_T(D)\setminus E. \end{cases}$$

We now compute the difference quotients of the action:

$$\begin{aligned}\frac{\mathcal{A}((F_{\varepsilon,E,\varphi})_\#\eta) - \mathcal{A}(\eta)}{\varepsilon} &= \int_E\int_0^T \dot\gamma(t)\cdot\dot\varphi(t)\,dt\,d\eta \\ &\quad + \frac{\varepsilon}{2}\int_E\int_0^T |\dot\varphi(t)|^2\,dt\,d\eta \\ &\quad - \int_E\int_0^T \frac{V(\gamma(t)+\varepsilon\varphi(t),t) - V(\gamma(t),t)}{\varepsilon}\,dt\,d\eta.\end{aligned}$$

We rewrite the last term as

$$\begin{aligned}&\int_E\int_0^T \frac{V(\gamma(t)+\varepsilon\varphi(t),t) - V(\gamma(t),t)}{\varepsilon}\,dt\,d\eta \\ &\qquad\qquad = \int_D\int_0^T \frac{V(x+\varepsilon\varphi(t),t) - V(x,t)}{\varepsilon}\rho_{E,t}(x)\,dt\,dx,\end{aligned}$$

where $\rho_{E,t}$ is the density of $(e_t)_\#(\eta_{\lfloor E})$ with respect to $\mathcal{L}^d$. Let us notice that also $(F_{\varepsilon,E,\varphi})_\#\eta \in \mathcal{P}^{<}_{T,\infty}(D)$, which implies that $\rho_{E,t}$ is bounded. Indeed, if $S \subseteq D$ and C is such that $\eta \in \mathcal{P}^{<}_{T,C}(D)$, then

$$\begin{aligned}(e_t)_\#(F_{\varepsilon,E,\varphi})_\#\eta(S) &= \eta(\{\gamma\in E : \gamma(t)+\varepsilon\varphi(t)\in S\}) + \eta(\{\gamma\in E^c : \gamma(t)\in S\}) \\ &\le (e_t)_\#\eta(S - \varepsilon\varphi(t)) + (e_t)_\#\eta(S) \\ &\le 2C\mathcal{L}^d(S).\end{aligned}$$

Since $E \subseteq \mathrm{Adm}_{\bar{\varepsilon}}(\varphi)$, $W_0^{1,2}([0,T];\mathbb{R}^d) \subset \Omega_T(D)$, and $W^{1,p}(D) \subset W^{1,1}(D)$ (thanks to the boundedness of D), conditions of Lemma 2.5 are satisfied. Thus, we can take the limit as $\varepsilon \to 0$ and obtain

$$
\begin{aligned}
&\lim_{\varepsilon \searrow 0} \frac{\mathcal{A}((F_{\varepsilon,E,\varphi})_{\#}\eta) - \mathcal{A}(\eta)}{\varepsilon} \\
&\qquad = \int_E \int_0^T \dot\gamma(t) \cdot \dot\varphi(t)\, dt\, d\eta - \int_D \int_0^T \nabla_x V(x,t) \cdot \varphi(t) \rho_{E,t}(x)\, dt\, dx \\
&\qquad = \int_E \int_0^T \Big(\dot\gamma(t) \cdot \dot\varphi(t) - \nabla_x V(\gamma(t),t) \cdot \varphi(t) \Big)\, dt\, d\eta.
\end{aligned}
$$

On the other hand we observe that $(F_{\varepsilon,E,\varphi})_{\#}\eta$ and η have the same endpoints (because φ vanishes at the extrema of $[0,T]$), which by the minimality of η implies that the right derivative of the action is nonnegative. This yields a contradiction with (3.1) and proves the result. $\square$

Theorem 3.3. *Let D be either a smooth bounded connected and convex open set in $\mathbb{R}^d$ or $\mathbb{T}^d$, and let $L(x,v,t) = \frac{|v|^2}{2} - V(x,t)$ be a Lagrangian with potential $V \in L^1([0,T]; W^{1,p}(D))$, $1 \le p \le \infty$. If η is a minimizer for the action in $\mathcal{P}^{<}_{T,\infty}(D)$ at fixed endpoints, then, for η-a.e. γ:*

- *(i) $\nabla_x V(\gamma(t),t)$ is well-defined for a.e. t.*
- *(ii) $\gamma \in W^{2,p}((0,T);\overline{D})$. In particular, $\gamma \in C^{1,1/q}$, where q is the dual exponent to p ($\gamma \in C^1$ if $p=1$).*
- *(iii) $\ddot\gamma(t) = -\nabla_x V(\gamma(t),t)$ a.e. in $[0,T]$.*

Proof. The point (i) is a direct consequence of Remark 2.1. We now will show that the distributional derivative (in time) of $\dot\gamma$ is given by $\nabla_x V(\gamma(t),t)$. By

$$
\int_{\Omega_T(D)} \int_0^T |\nabla_x V(\gamma(t),t)|^p\, dt\, d\eta \le C \int_D \int_0^T |\nabla_x V(x,t)|^p\, dt\, dx
$$

and by the estimate

$$
|\dot\gamma(t) - \dot\gamma(s)| \le \int_s^t |\ddot\gamma(\tau)|\, d\tau \le |t-s|^{1/q} \|\ddot\gamma\|_{L^p([0,T])},
$$

this will prove both *(ii)* and *(iii)*.

We split the proof into three cases. In fact, a more careful analysis is needed if the curve γ touches the boundary of D. Of course if $D = \mathbb{T}^d$ then only Step 1 is needed. In what follows, $\{\varphi_n\}_{n\in\mathbb{N}}$ is a countable dense subset of $W_0^{1,2}([0,T];\mathbb{R}^d)$.

• *Case 1. $B(\gamma) = \emptyset$.*
In this case, $\gamma \in \mathrm{Adm}(\varphi)$ for any $\varphi \in W_0^{1,2}([0,T];\mathbb{R}^d)$. In particular, by replacing φ

with $-\varphi$ in Proposition 3.2 we get that, for a fixed φ, η-a.e. γ falling within Case 1 satisfies

$$\int_0^T \Big(\dot\gamma(t)\cdot\dot\varphi(t) - \nabla_x V(\gamma(t),t)\cdot\varphi(t)\Big)\,dt = 0.$$

Now, by taking $\{\varphi_n\}_{n\in\mathbb{N}} \subset W_0^{1,2}([0,T];\mathbb{R}^d)$ a dense countable subset, we deduce that there exists a set of curves γ of full η-measure such that

$$\int_0^T \Big(\dot\gamma(t)\cdot\dot\varphi_n(t) - \nabla_x V(\gamma(t),t)\cdot\varphi_n(t)\Big)\,dt = 0 \qquad \forall\, n\in\mathbb{N},$$

and by density of the φ_n's we actually obtain that, for η-a.e. γ,

$$\int_0^T \Big(\dot\gamma(t)\cdot\dot\varphi(t) - \nabla_x V(\gamma(t),t)\cdot\varphi(t)\Big)\,dt = 0 \qquad \forall\,\varphi\in W_0^{1,2}([0,T];\mathbb{R}^d).$$

Hence, $D_t\dot\gamma(t) = -\nabla_x V(\gamma(t),t)$ in the distributional sense, which implies the result.

• *Case 2. $B(\gamma)\neq\emptyset$ and $D=\{x\in\mathbb{R}^d : x\cdot\hat e_d>0\}$.*
Although now D is an half-space and so is not bounded, this can be seen as a model case where the boundary of D is flat. In this way, in the general case we will be able to locally reducing to this case by flattening the boundary near a point $\gamma(t)$, with $t\in B(\gamma)$ (see Case 3).

By Remark 2.1, η-a.e. γ touches the boundary in a negligible set of times, that is $\mathcal{L}^1\big(B(\gamma)\big)=0$. Moreover $B(\gamma)$ is closed (as γ is continuous), and its complement is a countable union of disjoint open intervals. The same conclusions of the Case 1 are valid in each of such intervals. We now have to treat the set of times where γ touches ∂D. This will be done in two steps, by first proving that $t\mapsto\dot\gamma(t)$ is a function of locally bounded variation, and then by showing that its distributional derivative is absolutely continuous with respect to $\mathcal{L}^1$. Since $\mathcal{L}^1(B(\gamma))=0$, this will allow to conclude that $\ddot\gamma(t) = -\nabla_x V(\gamma(t),t)$ for a.e. $t\in[0,T]$, as desired.

Step a: $\dot\gamma\in BV_{loc}((0,T),\mathbb{R}^d)$. Obviously,

$$\big\{\varphi\in W_0^{1,2}([0,T];\mathbb{R}^d) : \varphi\cdot\hat e_d\geq 0\big\}\subseteq \mathrm{Adm}(\gamma) \quad \forall\,\gamma\in\Omega_T(D).$$

Hence, arguing as in the Case 1, we find that for η-a.e. γ

$$\int_0^T \Big(\dot\gamma(t)\cdot\dot\varphi(t) - \nabla_x V(\gamma(t),t)\cdot\varphi(t)\Big)\,dt \geq 0 \qquad \forall\,\varphi\in W_0^{1,2}([0,T];\mathbb{R}^d) \text{ s.t. } \varphi\cdot\hat e_d\geq 0. \tag{3.2}$$

Let us define $u := -D_t\dot\gamma - \nabla_x V(\gamma(\cdot),\cdot)$. We want to show that u is a locally finite (vector-valued) measure.

We first observe that $u\in\big(W_0^{1,2}([0,T],\mathbb{R}^d)\big)^*$. Let us denote by u_i the components of u, that is $\langle u_i,\psi\rangle := \langle u,\psi\hat e_i\rangle$ for all $\psi\in W_0^{1,2}([0,T];\mathbb{R})$, so that $\langle u,\varphi\rangle = \sum_i\langle u_i,\varphi\cdot\hat e_i\rangle$. Condition (3.2) applied to φ and $-\varphi$, with $\varphi\cdot\hat e_d\equiv 0$, implies that $u_i = 0$ for $1 \leq i \leq d-1$. Hence they are trivially measures. By Lemma 2.7, u_d is a locally finite measure too. Thus u is a locally finite measure. Since $D_t\dot{\gamma} = -u - \nabla_x V(\gamma(\cdot), \cdot) \in \mathcal{M}_{loc}((0,T);\mathbb{R}^d)$, $\dot{\gamma}$ is a function of locally bounded variation, as desired.

We now want to show that $\dot{\gamma}$ is absolutely continuous. We start by proving that $\dot{\gamma}$ has no jumps, and then we will show that it has no Cantor part either.

Step b(i): $\dot{\gamma}$ is continuous. Fix $\bar{t} \in (0,T)$ at which $\gamma(\bar{t}) \in \partial D$. By well-known properties of BV functions in one variable (see for instance [3, Paragraph 3.2]), there exist $\dot{\gamma}(\bar{t}^-) = \lim_{t\to\bar{t}^-} \dot{\gamma}(t)$ and $\dot{\gamma}(\bar{t}^+) = \lim_{t\to\bar{t}^+} \dot{\gamma}(t)$. We want to prove that these limits are actually equal, so that $\dot{\gamma}$ is continuous.

Notice that, since $\gamma(t) \cdot \hat{e}_d \geq 0$ for any t and $\gamma(t) = \gamma(\bar{t}) + \int_{\bar{t}}^t \dot{\gamma}(s)\, ds$, it must hold $\dot{\gamma}(t^+) \cdot \hat{e}_d \geq 0$ and $\dot{\gamma}(t^-) \cdot \hat{e}_d \leq 0$.

Given w a vector in the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ such that $w \cdot \hat{e}_d \geq 0$, we consider the family $\{\varphi_n\}_{n\in\mathbb{N}}$ of test functions defined by

$$\begin{aligned}
&\varphi_n = h_n w, \quad h_n \in C_c^\infty((\bar{t}-\delta_n, \bar{t}+\delta_n);\mathbb{R}), \quad h_n(\bar{t}) = 1,\\
&\dot{h}_n \geq 0 \text{ for } t < \bar{t}, \quad \dot{h}_n \leq 0 \text{ for } t > \bar{t}.
\end{aligned}$$

Here δ_n is chosen in such a way that $\delta_n \leq 1/n$ and

$$\begin{aligned}
|\dot{\gamma}(t) - \dot{\gamma}(t^-)| &\leq \frac{1}{n} \qquad \text{a.e. in } (\bar{t}-\delta_n, \bar{t}),\\
|\dot{\gamma}(t) - \dot{\gamma}(t^+)| &\leq \frac{1}{n} \qquad \text{a.e. in } (\bar{t}+\delta_n, \bar{t}).
\end{aligned}$$

Then condition (3.2) together with Remark 2.2 imply that

$$\begin{aligned}
\int_{\bar{t}-\delta_n}^{\bar{t}+\delta_n} \dot{\gamma}(t) \cdot \dot{\varphi}_n(t)\, dt \geq \int_{\bar{t}-\delta_n}^{\bar{t}+\delta_n} &\nabla_x V(\gamma(t), t) \cdot \varphi_n(t)\, dt\\
&\geq -2\delta_n \int_0^T |\nabla_x V(\gamma(t), t)|\, dt > -\infty.
\end{aligned}$$

On the other hand

$$\Big|\int_{\bar{t}-\delta_n}^{\bar{t}+\delta_n} \dot{\gamma}(t) \cdot \dot{\varphi}_n(t)\, dt - \int_{\bar{t}-\delta_n}^{\bar{t}} \dot{\gamma}(t^-) \cdot \dot{\varphi}_n(t)\, dt - \int_{\bar{t}}^{\bar{t}+\delta_n} \dot{\gamma}(t^+) \cdot \dot{\varphi}_n(t)\, dt\Big| \leq \frac{2}{n},$$

which gives

$$\begin{aligned}
w \cdot (\dot{\gamma}(t^-) - \dot{\gamma}(t^+)) &= \int_{\bar{t}-\delta_n}^{\bar{t}} \dot{\gamma}(t^-) \cdot \dot{\varphi}_n(t)\, dt + \int_{\bar{t}}^{\bar{t}+\delta_n} \dot{\gamma}(t^+) \cdot \dot{\varphi}_n(t)\, dt\\
&\geq \int_{\bar{t}-\delta_n}^{\bar{t}+\delta_n} \dot{\gamma}(t) \cdot \dot{\varphi}_n(t)\, dt - \frac{2}{n} \geq -2\delta_n \int_0^T |\nabla_x V(\gamma(t), t)|\, dt - \frac{2}{n}
\end{aligned}$$

As $n \to \infty$, we get

$$w \cdot (\dot{\gamma}(t^-) - \dot{\gamma}(t^+)) \geq 0 \qquad \forall\, w \in \mathbb{S}^{d-1}, w \cdot \hat{e}_d \geq 0.$$

Combining this with the inequality $\hat{e}_d \cdot (\dot{\gamma}(t^-) - \dot{\gamma}(t^+)) \leq 0$, we get $\hat{e}_d \cdot (\dot{\gamma}(t^-) - \dot{\gamma}(t^+)) = 0$. This implies in particular that $\frac{-(\dot{\gamma}(t^-) - \dot{\gamma}(t^+))}{|(\dot{\gamma}(t^-) + \dot{\gamma}(t^+))|}$ is an admissible choice for w, which yields $\dot{\gamma}(t^+) = \dot{\gamma}(t^-)$. Thus $\dot{\gamma}$ is continuous even at the points where γ touches the boundary.

Step b(ii): $D_t\dot{\gamma}$ has no Cantor part. Let us call $\mu = D_t\dot{\gamma}$ and $w = \frac{d\mu}{d|\mu|}$. It remains to show that μ (or equivalently its total variation $|\mu|$) cannot have a Cantor component either. To this aim, it suffices to prove that, if $\bar{t}$ is a $|\mu|$-Lebesgue point for w, then

$$\liminf_{\varepsilon \searrow 0} \frac{|\mu|(I_\varepsilon)}{|\nabla_x V(\gamma(\cdot), \cdot)|\, d\mathcal{L}^1(I_\varepsilon)} < +\infty. \tag{3.3}$$

Indeed, by well-known results about differentiation of measures (cf. [9, Theorem 2.12]) this would imply that

$$|D_t\dot{\gamma}| = |\mu| << |\nabla_x V(\gamma(\cdot), \cdot)|\, d\mathcal{L}^1 << \mathcal{L}^1,$$

as desired.

For all $\varepsilon > 0$ small set $I_\varepsilon := (\bar{t} - \varepsilon, \bar{t} + \varepsilon)$, and let $\tilde{I}_\varepsilon \supset I_\varepsilon$ be such that

$$|\mu|(\tilde{I}_\varepsilon \setminus I_\varepsilon) + \int_{\tilde{I}_\varepsilon \setminus I_\varepsilon} |\nabla_x V(\gamma(t), t)|\, dt \leq \varepsilon\, |\mu|(I_\varepsilon). \tag{3.4}$$

(Such intervals $\tilde{I}_\varepsilon$ always exist since the measure μ has no atomic part.) Let us then consider a family of test functions $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ such that

$$\varphi_\varepsilon = h_\varepsilon w(\bar{t}), \quad h_\varepsilon \in C_c^\infty(\tilde{I}_\varepsilon; \mathbb{R}), \quad 0 \leq \varphi_\varepsilon \leq 1, \quad h_\varepsilon \equiv 1 \text{ in } I_\varepsilon.$$

Under these conditions, thanks to the fact that $\bar{t}$ is a $|\mu|$-Lebesgue point for w we have

$$\left| \frac{1}{|\mu|(I_\varepsilon)} \left(\int_{\tilde{I}_\varepsilon} h_\varepsilon(t) w(\bar{t})\, d\mu - \int_{\tilde{I}_\varepsilon} h_\varepsilon(t) w(t)\, d\mu \right) \right| = o_\varepsilon(1).$$

We now claim that φ_ε are admissible variations $|\mu|$-a.e. in the sense of condition (3.2), that is $w(\bar{t}) \cdot \hat{e}_d \geq 0$. Indeed, arguing by contradiction, assume that $w(\bar{t}) \cdot \hat{e}_d < 0$. Then, by the Lebesgue point condition we have

$$\begin{aligned} \frac{\dot{\gamma}(\bar{t} + \varepsilon) - \dot{\gamma}(\bar{t} - \varepsilon)}{|\mu|(\bar{t} - \varepsilon, \bar{t} + \varepsilon)} \cdot \hat{e}_d &= \int_{\bar{t} - \varepsilon}^{\bar{t} + \varepsilon} \frac{D_t\dot{\gamma}}{|\mu|(\bar{t} - \varepsilon, \bar{t} + \varepsilon)} \cdot \hat{e}_d \\ &= \int_{\bar{t} - \varepsilon}^{\bar{t} + \varepsilon} \frac{w \cdot \hat{e}_d}{|\mu|(\bar{t} - \varepsilon, \bar{t} + \varepsilon)}\, d|\mu| \quad \stackrel{\varepsilon \to 0}{\longrightarrow} w(\bar{t}) \cdot \hat{e}_d < 0. \end{aligned}$$

This implies that there exists a small number $s_0 > 0$ such that

$$\begin{aligned}\big(\gamma(\bar t + s_0) - \gamma(\bar t)\big) \cdot \hat e_d &= \int_0^{s_0} \dot\gamma(\bar t + \tau) \cdot \hat e_d \, d\tau \\ &< \int_0^{s_0} \dot\gamma(\bar t - \tau) \cdot \hat e_d \, d\tau = \big(\gamma(\bar t) - \gamma(\bar t - s_0)\big) \cdot \hat e_d,\end{aligned}$$

which gives

$$\big(\gamma(\bar t +_0 s) + \gamma(\bar t - s_0) - 2\gamma(\bar t)\big) \cdot \hat e_d < 0.$$

This is impossible since γ takes values in $\overline{D} = \{x \in \mathbb{R}^d : x \cdot \hat e_d \geq 0\}$ and $\gamma(\bar t) \cdot \hat e_d = 0$, and proves the claim.

Now, since φ_ε are admissible variations $|\mu|$-a.e., thanks to the relation $|\mu| = w \cdot \mu$ we get

$$\begin{aligned} 1 &\leq \frac{1}{|\mu|(I_\varepsilon)} \int_{\tilde I_\varepsilon} h_\varepsilon(t) w(t) \cdot \, d\mu = \frac{1}{|\mu|(I_\varepsilon)} \int_{\tilde I_\varepsilon} h_\varepsilon(t) w(\bar t) \cdot \, d\mu + o_\varepsilon(1) \\ &= \frac{1}{|\mu|(I_\varepsilon)} \int_{\tilde I_\varepsilon} \varphi_\varepsilon \, d\mu + o_\varepsilon(1) = -\frac{1}{|\mu|(I_\varepsilon)} \int_{\tilde I_\varepsilon} \dot\gamma \cdot \dot\varphi_\varepsilon \, dt + o_\varepsilon(1) \\ &\leq \frac{1}{|\mu|(I_\varepsilon)} \int_{\tilde I_\varepsilon} -\nabla_x V(\gamma(t), t) \cdot \varphi_\varepsilon(t) \, dt + o_\varepsilon(1) \\ &\leq \frac{1}{|\mu|(I_\varepsilon)} \int_{\tilde I_\varepsilon} |\nabla_x V(\gamma(t), t)| \, dt + o_\varepsilon(1) \leq \frac{1}{|\mu|(I_\varepsilon)} \int_{I_\varepsilon} |\nabla_x V(\gamma(t), t)| \, dt + o_\varepsilon(1)\end{aligned}$$

where at the last step we used (3.4). Inequality (3.3) is now proved, and the proof of Case 2 is completed.

• *Case 3. $B(\gamma) \neq \emptyset$ and $D \subseteq \mathbb{R}^d$ is a smooth convex open set.*
Arguing as in Case 1 we get that, for η-a.e. γ and for any $n \in \mathbb{N}$,

$$\int_0^T \Big(\dot\gamma(t) \cdot \dot\varphi_n(t) - \nabla_x V(\gamma(t), t) \cdot \varphi_n(t)\Big) \, dt \geq 0 \qquad \text{if } \gamma \in \mathrm{Adm}(\varphi_n). \tag{3.5}$$

Here $\{\varphi_n\}_{n\in\mathbb{N}}$ is a dense subset of $W_0^{1,2}([0,T];\mathbb{R}^d)$. From now on, we consider only curves γ for which the above inequality is true. Fix now $\bar t \in B(\gamma)$, and let $\alpha : \mathbb{R}^{d-1} \to \mathbb{R}$ be a smooth convex function such that ∂D coincides with the graph of α in a neighborhood of $\gamma(\bar t)$. We locally reduce to the Case 2 by means of the map

$$\Phi(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1}, x_d - \alpha(x_1, \ldots, x_{d-1})),$$

so that $\Phi(\partial D)$ coincide with the hyperplane $\{x_d = 0\}$ in a neighborhood of $\Phi(\gamma(\bar t))$. Let us show that $D_t\dot\gamma$ is a measure in a neighborhood of $\bar t$, and let $u_\gamma := -D_t\dot\gamma - \nabla_x V(\gamma(\cdot), \cdot) \in \big(W_0^{1,2}([0,T];\mathbb{R}^d)\big)^*$. We have

$$\begin{aligned}\langle u_\gamma, \varphi\rangle &= \big\langle u_\gamma, \nabla(\Phi^{-1} \circ \Phi)(\gamma)\, \varphi\big\rangle = \big\langle u_\gamma, \nabla(\Phi^{-1}) \circ \Phi(\gamma)\, \nabla\Phi(\gamma)\, \varphi\big\rangle \\ &= \langle v_\gamma, \psi(\varphi)\rangle \qquad \forall\, \varphi \in W_0^{1,2}([0,T];\mathbb{R}^d),\end{aligned}$$

where

$$\psi(\varphi) := \nabla\Phi(\gamma(\cdot))\varphi(\cdot) \quad \text{and} \quad v_\gamma := u_\gamma\, \nabla(\Phi^{-1}) \circ \Phi(\gamma).$$

(Observe that $v_\gamma \in \big(W_0^{1,2}([0,T];\mathbb{R}^d)\big)^*$ since $\nabla(\Phi^{-1})(\Phi(\gamma)) \in W_0^{1,2}([0,T];\mathbb{R}^d)$.) Hence, thanks to (3.5) and the fact that the set $\{\psi(\varphi_n)\}_{n\in\mathbb{N}}$ is still dense in $W_0^{1,2}([0,T];\mathbb{R}^d)$, we deduce

$$\langle v_\gamma, \psi\rangle \geq 0 \qquad \forall\, \psi \in \overline{\mathrm{Adm}}(\gamma).$$

Let us now consider $\bar\psi \in W_0^{1,2}([0,T];\mathbb{R}^d)$ such that $\bar\psi \cdot \hat e_d \geq 0$, and fix $\delta > 0$ small. Moreover we take a smooth compactly supported cut-off function $\chi : [0,T] \to [0,1]$ such that $\chi \equiv 1$ in a neighborhood of $\bar t$. Then it is not difficult to check that the function $(\bar\psi + \delta\hat e_d)\chi$ belongs $\mathrm{Adm}(\gamma)$, which implies

$$\big\langle v_\gamma, \big(\bar\psi + \delta\hat e_d\big)\chi\big\rangle \geq 0 \qquad \forall\, \bar\psi \in W_0^{1,2} \text{ s.t. } \bar\psi\cdot\hat e_d \geq 0.$$

By letting $\delta \to 0$ we have recovered the analogous formula to (3.2) in the interior of the interval $\{\chi \equiv 1\}$. By the analysis of the flat case, we deduce that v_γ is a measure in a neighborhood of $\bar t$, which implies that the same holds for $u_\gamma = v_\gamma\,[\nabla(\Phi^{-1})\circ\Phi(\gamma)]^{-1}$.

The rest of the proof follows exactly as in the flat case, where the convexity assumption of D is used to show that the vector $w(\bar t)$ is admissible also in this case (see Step b(ii) of Case 2). □

4 Properties of the value function

In this section, we focus on the properties of the value function. In particular, we will generalize the classical results recalled in the introduction to this more general setting. We start with some preliminary assumptions on V and u_0. We shall work with a precise representative of the potential V and the function u_0, so that in particular both V and u_0 are defined at *every* point.

Definition 4.1. *Let*

$$L(x,v,t) = \frac{|v|^2}{2} - V(x,t), \qquad (x,v,t) \in \overline{D} \times \mathbb{R}^d \times [0,T]$$

where V is bounded from above. Let $u_0\colon \overline{D} \to \mathbb{R}$ be a function bounded from below. The value function $u : \overline{D}\times[0,T] \to \mathbb{R}$ associated to L and u_0 is defined as

$$u(x,t) := \inf\left\{ u_0(\gamma(0)) + \int_0^t L(\gamma(\tau),\dot\gamma(\tau),\tau)\,d\tau : \gamma \in \Omega_t(D),\ \gamma(t) = x \right\}. \tag{4.1}$$

The assumptions on V and u_0 ensure that the infimum in the above definition of u is not $-\infty$. Moreover, the condition for V and u_0 to be defined everywhere ensures that u is well-defined. Indeed a priori, by changing for instance u_0 on a negligible set, u may change in a non-negligible set.

Definition 4.2. *Given V, u_0 and u as in the previous definition, we say that $\gamma \in \Omega_t(D)$ is a* minimizer for the evolutive problem (at time t) *if*

$$u(\gamma(t), t) = u_0(\gamma(0)) + \mathcal{A}(\gamma).$$

In order to deduce some regularity properties on $u(\cdot, T)$, in all the following results we will assume the existence of a measure $\eta \in \mathcal{P}^{\approx}_{T,\infty}(D)$ concentrated on minimizers for the evolutive problem at time T. Let us observe that a variational solution to the Euler equation will not generally satisfy this assumption, as the curves on which it is concentrated only minimizes the action at fixed endpoints (see [1, Theorem 6.8]). However, it is likely that a variant of the minimization problem related to the Euler equations [5, 1], where one removes the endpoint constraint but keep the incompressibility, may allow to find such measures. As we said in the introduction, in this paper we will make no attempts to prove existence of minimizers, but we will only study their regularity properties.

Theorem 4.3. *Let D be as in Theorem 3.3, V, u_0 and u as in Definition 4.1. Assume that there exists $\eta \in \mathcal{P}^{\approx}_{T,\infty}(D)$ concentrated on minimizers for the evolutive problem at time T. Moreover, suppose that $V \in L^p([0,T]; W^{1,p}(D))$, $1 \leq p \leq \infty$. Then for every $v \in \mathbb{R}^d, |v| = 1$, if $\partial_v u(\cdot, T)$ denotes the distributional derivative of $u(\cdot, T)$ in the direction v, it holds:*

(i) $\partial_v u(\cdot, T) \in L^p(D)$. *More precisely, there exists $h_v \in L^p(D)$ such that*

$$\langle \partial_v u(\cdot, T), \varphi \rangle = \int_D h_v \varphi \, dx \qquad \forall \varphi \in C_c^\infty(D).$$

In particular, by Sobolev's embeddings, if $p > d$ then $u(\cdot, T)$ is continuous.

(ii) The distributional gradient $\nabla_x u_T = (h_{\hat{e}_1}, \ldots, h_{\hat{e}_d})$ is given by

$$\nabla_x u_T(x) = \int_{\{\gamma : \gamma(T) = x\}} \dot{\gamma}(T) \, d\eta_{T,x}, \qquad \text{for } \mathcal{L}^d\text{-a.e. } x \in D,$$

where $\eta_{T,x}$ are the probability measures on $\{\gamma \in \Omega_T(D) : \gamma(T) = x\}$ obtained by disintegrating η through the map e_T, i.e.

$$\eta = \int \eta_{T,x} \, d(e_T)_\# \eta(x)$$

(we refer to [7, Chapter III] for the notion of disintegration of a measure).

Proof. Fix $v \in \mathbb{S}^{d-1}$. To simplify the notation, we shall denote by u_T the function $u(\cdot, T)$ all along the proof. Remark 2.1 ensures that η-a.e. γ touches the boundary in a set of times of zero measure. Let us point out that, under our assumptions, if η is concentrated on minimizers for the evolutive problem, then it is easily seen that it also minimizes the action at fixed endpoints (see for instance the proof of

[1, Theorem 6.12]). This implies that all the conclusions of Theorem 3.3 hold. In particular

$$\ddot\gamma(t) = -\nabla_x V(\gamma(t),t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T].$$

We now claim that there exists a universal constant C, depending only on D and the dimension, such that

$$\|\dot\gamma\|_{L^\infty([0,T])} \le C + \int_0^T |\nabla_x V(\gamma(t),t)|\,dt \tag{4.2}$$

Indeed, we apply the intermediate value theorem to any component $\gamma_i(t) = \gamma(t)\cdot\hat e_i$, $i = 1,\dots,d$, to find a time $t_{\gamma_i} \in [0,T]$ such that

$$|\dot\gamma_i(t_{\gamma_i})| \le C|\gamma_i(T)-\gamma_i(0)| \le C\,\mathrm{diam}(D).$$

Hence, for every $\tau \in [0,T]$,

$$|\dot\gamma_i(t)| \le |\dot\gamma_i(t_{\gamma_i})| + \int_{t_{\gamma_i}}^{\tau} |\nabla_x V(\gamma(t),t)|\,dt \le C + \int_0^T |\nabla_x V(\gamma(t),t)|\,dt,$$

and (4.2) follows easily.

Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(D;[0,1])$ be an increasing sequence of smooth cut-off functions such that, if D_n denotes the interior of the set $\{\varphi_n = 1\}$, then $\cup_n D_n = D$. We define

$$\phi_{\gamma,n}(t) := \frac{t}{T}\varphi_n(\gamma(t)).$$

Let us point out that, for any $\gamma \in \Omega_T(D)$ and for any $n \in \mathbb{N}$, since $\phi_{\gamma,n}(t) = 0$ whenever $\gamma(t) \notin \operatorname{supp}\varphi_n$, the curve $t \mapsto \gamma(t) + \phi_{\gamma,n}(t)hv$ belongs to $\Omega_T(D)$ for $|h|$ smaller than $\mathrm{dist}(\operatorname{supp}\varphi_n, \partial D)$.

Step 1. Let γ be a minimizer for the evolution problem at time T, fix $n \in \mathbb{N}$, and assume that $|h| \le \mathrm{dist}(\operatorname{supp}\varphi_n, \partial D)$. Then

$$\frac{u_T(\gamma(T)+\phi_{\gamma,n}(T)hv) - u_T(\gamma(T))}{h} \le \int_0^T v\cdot\dot\phi_n(t)\dot\gamma(t)\,dt$$
$$+\, h\int_0^T \frac{\dot\phi_n(t)^2}{2}\,dt - \int_0^T \frac{V(\gamma(t)+\phi_{\gamma,n}(t)hv,t) - V(\gamma(t),t)}{h}\,dt.$$

Let us first remark that $|\dot\phi_{\gamma,n}(t)| \le \frac{1}{T} + \|\nabla\varphi_n\|_\infty|\dot\gamma(t)|$, which implies that $\phi_{\gamma,n} \in W^{1,2}([0,T];\mathbb{R}^d)$. Moreover, thanks to our assumptions the curve $t \mapsto \gamma(t)+\phi_{\gamma,n}(t)hv$ belongs to $\Omega_T(D)$, and starts from $\gamma(0)$ at $t=0$. In particular it is admissible in the definition of u_T, which implies

$$u_T(\gamma(T)+\phi_{\gamma,n}(T)hv) \le u_0(\gamma(0)) + \int_0^T \left[\frac{1}{2}\left|\dot\gamma(t)+\dot\phi_n(t)hv\right|^2 - V\Big(\gamma(t)+\phi_{\gamma,n}(t)hv, t\Big)\right] dt.$$

On the other hand, thanks to the minimality of γ we have

$$u_T(\gamma(T)) = u_0(\gamma(0)) + \int_0^T \frac{|\dot\gamma(t)|^2}{2} - V(\gamma(t),t)\,dt.$$

The conclusion of Step 1 follows easily.

Step 2. There exists a function $f \in L^p(D)$ such that $\big|\langle \partial_v u_T, \varphi\rangle\big| \le \int_D |f\varphi|\,dy$ for all $\varphi \in C_c^\infty(D)$.

Fix $\varphi \in C_c^\infty(D), \varphi \ge 0$. Then there exists $n \in \mathbb{N}$ such that $\operatorname{supp}\varphi \subset D_n$. Since D_n is open this implies the following: whenever $\varphi(y) \ne 0$, then $y, y+hv \in D_n$ for $|h| < \operatorname{dist}(\operatorname{supp}\varphi_n, \partial D)$. Hence $\phi_{\gamma,n}(T) = 1$ for all curves γ such that $\gamma(T) \in \operatorname{supp}\varphi$, and thanks to Step 1 we obtain

$$\begin{aligned}
\langle \partial_v u_T, \varphi\rangle &= -\langle u_T, \partial_v\varphi\rangle = \lim_{h\to 0}\int_D u_T(y)\frac{\varphi(y-hv)-\varphi(y)}{h}\,dy \\
&= \lim_{h\to 0}\int_D \varphi(y)\frac{u_T(y+hv)-u_T(y)}{h}\,dy \\
&= \lim_{h\to 0}\int_{\Omega_T(D)} \left(\rho_T^{-1}(\gamma(T))\varphi\big(\gamma(T)\big)\frac{u_T\big(\gamma(T)+hv\big)-u_T\big(\gamma(T)\big)}{h}\right)d\eta \\
&\le \lim_{h\to 0}\int_{\Omega_T(D)} \Bigg(\rho_T^{-1}(\gamma(T))\varphi(\gamma(T)) \\
&\quad \Big(\int_0^T v\cdot\dot\phi_n(t)\dot\gamma(t)\,dt - \int_0^T \frac{V(\gamma(t)+\phi_{\gamma,n}(t)hv,t)-V(\gamma(t),t)}{h}\,dt\Big)\Bigg)\,d\eta \\
&= \int_{\Omega_T(D)} \Bigg(\rho_T^{-1}(\gamma(T))\varphi(\gamma(T)) \\
&\qquad\qquad \Big(v\cdot\int_0^T\Big(\dot\phi_n(t)\dot\gamma(t) - \phi_{\gamma,n}(t)\nabla_x V(\gamma(t),t)\Big)\,dt\Big)\Bigg)\,d\eta \\
&= \int_{\Omega_T(D)} \rho_T^{-1}(\gamma(T))\,\varphi(\gamma(T))\,v\cdot\dot\gamma(T)\,d\eta.
\end{aligned} \tag{4.3}$$

(Observe that, by Theorem 3.3, η-a.e. γ is of class C^1, so that in particular $\dot\gamma(T)$ is well-defined η-a.e.) Here the last but one equality is a consequence of Lemma 2.5 together with the identities

$$\begin{aligned}
\mathit{wehave} \int_{\Omega_T(D)}\int_0^T &\frac{V(\gamma(t)+\phi_{\gamma,n}(t)hv,t)-V(\gamma(t),t)}{h}\,dt\,d\eta \\
&= \int_D\int_0^T \frac{V(x+\frac{t}{T}\varphi_n(x)hv,t)-V(x,t)}{h}\rho_t(x)\,dt\,dx,
\end{aligned}$$

$$\int_{\Omega_T(D)}\int_0^T \phi_{\gamma,n}(t)\nabla_x V(\gamma(t),t)\,dt\,d\eta = \int_D\int_0^T \frac{t}{T}\varphi_n(x)\nabla_x V(x,t)\rho_t(x)\,dt\,dx,$$

while the last one follows from an integration by parts together with Theorem 3.3. By exchanging v with $-v$ in (4.3) and exploiting the linearity with respect to φ, we actually obtain the equality

$$\langle \partial_v u_T, \varphi\rangle = \int_{\Omega_T(D)} \rho_T^{-1}(\gamma(T))\, \varphi(\gamma(T))\, v\cdot \dot\gamma(T)\, d\eta \qquad \forall\, \varphi \in C_c^\infty(D). \tag{4.4}$$

From the above equation together with (4.2), we get

$$\begin{aligned}
&\left|\int_{\Omega_T(D)} \rho_T^{-1}(\gamma(T))\varphi(\gamma(T))\, v\cdot\dot\gamma(T)\, d\eta\right| \\
&\qquad \le C\int_{\Omega_T(D)} \rho_T^{-1}(\gamma(T))\varphi(\gamma(T))\, d\eta \\
&\qquad\qquad + \int_{\Omega_T(D)} \rho_T^{-1}(\gamma(T))\varphi(\gamma(T)) \int_0^T |\nabla_x V(\gamma(t),t)|\, dt\, d\eta \\
&\qquad = C\|\varphi\|_1 + \int_D \varphi(y) \int_{\Omega_T(D)} \int_0^T \big|\nabla_x V(\gamma(t),t)\big|\, dt\, d\eta_{T,y}\, dy \\
&\qquad = C\|\varphi\|_1 + \int_D \tilde f\, \varphi\, dy,
\end{aligned}$$

where $\tilde f\colon D\to\mathbb{R}$ is given by

$$\tilde f(y) = \int_{\Omega_T(D)} \int_0^T |\nabla_x V(\gamma(t),t)|\, dt\, d\eta_{T,y}. \tag{4.5}$$

The function $\tilde f$ belongs to $L^p(D)$: indeed, thanks to Jensen inequality and the fact that ρ_T is bounded from below, we get

$$\begin{aligned}
\|\tilde f\|_p^p &\le C\int_D \int_{\Omega_T(D)} \int_0^T \big|\nabla_x V(\gamma(t),t)\big|^p\, dt\, d\eta_{T,y}\, dy \\
&\le C\int_{\Omega_T(D)} \int_0^T \rho_T^{-1}(y) \big|\nabla_x V(\gamma(t),t)\big|^p\, dt\, d\eta \\
&\le C\|V\|^p_{L^p([0,T];W^{1,p}(D))}.
\end{aligned}$$

Thus Step 2 is achieved with $f = C+\tilde f$.

Step 3. Proof of (*i*) *and* (*ii*).

Statement (i) follows from Step 2 together with the Lemma 2.8. To get (ii), we disintegrate the right-hand side of (4.4) through the map e_T to obtain

$$\langle \partial_v u_T, \varphi\rangle = \int_D \varphi(y) \left(\int_{\Omega_T(D)} \dot\gamma(T)\, d\eta_{T,y}\right) dy.$$

□

Theorem 4.4. *Let D be as in Theorem 3.3, V, u_0 and u as in Definition 4.1. Assume that there exists $\eta \in \mathcal{P}^{\approx}_{T,\infty}(D)$ concentrated on minimizers for the evolutive problem at time T. Moreover, suppose that $V \in L^p([0,T]; W^{2,p}(D))$, $1 \le p \le \infty$. Then for every $v \in \mathbb{R}^d$, $|v| = 1$, if $\partial_{vv}u(\cdot,T)$ denotes the second distributional derivative of $u(\cdot,T)$ in the direction v, it holds:*

(i) $\partial_{vv}u(\cdot,T) \in \mathcal{M}_{loc}(D)$;

(ii) there exists $h \in L^p_{loc}(D)$ *such that* $\partial_{vv}u(\cdot,T))^+ = h\,\mathcal{L}^d$.

Proof. To simplify the notation, we denote by u_T the function $u(\cdot,T)$. Remark 2.1 ensures that η-a.e. γ touches the boundary in a set of times of zero measure. Moreover, as we already observed at the beginning of the proof of Theorem 4.3, if η is concentrated on minimizers for the evolutive problem then it is also a minimizer for the action at fixed endpoints. This implies that all the conclusions of Theorem 3.3 hold, and in particular η-a.e. γ is of class C^1.

As in the proof of Theorem 4.3, we consider an increasing sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^\infty_c(D;[0,1])$ such that if D_n denotes the interior of $\{\varphi_n = 1\}$ then $\cup_n D_n = D$, and we define

$$\phi_{\gamma,n}(t) := \frac{t}{T}\varphi_n(\gamma(t)).$$

We observe that for any $\gamma \in \Omega_T(D)$ and for any $n \in \mathbb{N}$ the curve $t \mapsto \gamma(t) + \phi_{\gamma,n}(t)hv$ belongs to $\Omega_T(D)$ for $|h| < \operatorname{dist}(\operatorname{supp}\varphi_n, \partial D)$.

Step 1. If γ is a minimizer at time T and $\gamma(T) \in D$, then for any $n \in \mathbb{N}$ and for any $h > 0$ small enough,

$$\frac{u_T\big(\gamma(T) + \phi_{\gamma,n}(T)hv\big) + u_T\big(\gamma(T) - \phi_{\gamma,n}(T)hv\big) - 2u_T(\gamma(T))}{h^2}$$
$$\le \int_0^T \left(|\dot\phi_{\gamma,n}(t)|^2 - \frac{V\big(\gamma(t) + \phi_{\gamma,n}(t)hv, t\big) + V\big(\gamma(t) - \phi_{\gamma,n}(t)hv, t\big) - 2V(\gamma(t),t)}{h^2} \right) dt.$$

For $|h|$ small, the curve $t \mapsto \gamma(t) + \phi_{\gamma,n}(t)hv$ is admissible in the definition of u_T, hence

$$u_T\big(\gamma(T) + \phi_{\gamma,n}(T)hv\big) \le u_0(\gamma(0)) + \int_0^T \frac{1}{2}\Big|\dot\gamma(t) + \dot\phi_{\gamma,n}(t)hv\Big|^2 - V\big(\gamma(t) + \phi_{\gamma,n}(t)hv, t\big)\, dt.$$

Analogously,

$$u_T\big(\gamma(T) - \phi_{\gamma,n}(T)hv\big) \le u_0(\gamma(0)) + \int_0^T \frac{1}{2}\Big|\dot\gamma(t) - \dot\phi_{\gamma,n}(t)hv\Big|^2 - V\big(\gamma(t) - \phi_{\gamma,n}(t)hv, t\big)\, dt.$$

By using the minimality of γ, the conclusion of Step 1 follows easily.

Step 2. There exists a function $f \in L^p(D)$ such that the following holds: for any $n \in \mathbb{N}$ there exists a constant $\bar C_n$ such that $\langle \partial_{vv}u_T, \varphi\rangle \le \bar C_n \int f\varphi\, dy$ for all $\varphi \in C^\infty_c(D_n)$, $\varphi \ge 0$.

Fix $\varphi \in C_c^\infty(D_n), \varphi \geq 0$, and observe that since $\operatorname{supp}\varphi \subseteq D_n$ then $y + hv \in D_n$ for all $y \in \operatorname{supp}\varphi$, $|h| < \operatorname{dist}(\operatorname{supp}\varphi_n, \partial D)$. This gives that $\phi_{\gamma,n}(T) = 1$ for all curves γ such that $\gamma(T) \in \operatorname{supp}\varphi$, and by Step 1 we get

$$
\begin{aligned}
&\langle \partial_{vv} u_T, \varphi\rangle = \langle u_T, \partial_{vv}\varphi\rangle \\
&\quad = \lim_{h\to 0} \int_{\Omega_T(D)} \Bigg(\varphi(\gamma(T)) \frac{u_T(\gamma(T)+hv) + u_T(\gamma(T)-hv) - 2u_T(\gamma(T))}{h^2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rho_T^{-1}(\gamma(T)) \Bigg)\, d\eta \\
&\quad \leq \int_{\Omega_T(D)} \int_0^T \varphi(\gamma(T))\rho_T^{-1}(\gamma(T)) |\dot\phi_{\gamma,n}(t)|^2 \, dt\, d\eta \\
&\qquad - \lim_{h\to 0} \int_{\Omega_T(D)} \int_0^T \Bigg(\varphi(\gamma(T))\rho_T^{-1}(\gamma(T)) \\
&\qquad\qquad \Big(\frac{V\big(\gamma(t)+\phi_{\gamma,n}(t)hv, t\big) + V\big(\gamma(t)-\phi_{\gamma,n}(t)hv, t\big) - 2V\big(\gamma(t), t\big)}{h^2} \Big) \Bigg)\, dt\, d\eta \\
&\quad = \int_{\Omega_T(D)} \int_0^T \varphi(\gamma(T))\rho_T^{-1}(\gamma(T)) |\dot\phi_{\gamma,n}(t)|^2 \, dt\, d\eta \\
&\qquad - \int_{\Omega_T(D)} \int_0^T \varphi(\gamma(T))\rho_T^{-1}(\gamma(T)) \partial_{vv} V(\gamma(t), t)\, \phi_{\gamma,n}(t)^2 \, dt\, d\eta
\end{aligned} \tag{4.6}
$$

Here the inequality follows from Step 1 together with positivity of φ, while the last equality is a consequence of Lemma 2.6 together with the assumption on V. Now, since $|\phi_{\gamma,n}| \leq 1$, the last term is easily estimated as in the proof of Step 2 in Theorem 4.3:

$$
\int_{\Omega_T(D)} \int_0^T \varphi(\gamma(T))\rho_T^{-1}(\gamma(T)) |\partial_{vv} V(\gamma(t), t)|\, \phi_{\gamma,n}(t)^2 \, dt\, d\eta \leq C \int_D f_1\, \varphi\, dx
$$

where

$$
f_1(y) = \int_{\Omega_T(D)} \int_0^T |\partial_{vv} V(\gamma(t), t)| \, dt\, d\eta_{T,y}, \qquad y \in D.
$$

(the measures $\eta_{T,y}$ are defined as in Theorem 4.3). Concerning the other term, we observe that $|\dot\phi_{\gamma,n}(t)|^2 \leq \frac{1}{T^2} + \frac{2}{T}\|\nabla\varphi_n\|_\infty |\dot\gamma(t)| + \|\nabla\varphi_n\|_\infty^2 |\dot\gamma(t)|^2$. Moreover, thanks to Theorem 3.3 and (4.2) we have

$$
\begin{aligned}
\int_0^T |\dot\gamma(t)|^2 \, dt &= \gamma(T)\cdot\dot\gamma(T) - \gamma(0)\cdot\dot\gamma(0) - \int_0^T \gamma(t)\cdot\ddot\gamma(t)\, dt \\
&= \gamma(T)\cdot\dot\gamma(T) - \gamma(0)\cdot\dot\gamma(0) + \int_0^T \gamma(t)\cdot\nabla_x V(\gamma(t), t)\, dt \\
&\leq C\Big(1 + \int_0^T |\nabla_x V(\gamma(t), t)|\, dt\Big),
\end{aligned}
$$

where we used that the L^∞-norm of γ is uniformly bounded since $\gamma(t) \in D$ for all $t \in [0,T]$. Thanks to these facts we easily obtain

$$\langle \partial_{vv} u_T, \varphi \rangle \le C_n \|\varphi\|_1 + C \int_D f_1 \, \varphi \, dy + C_n \int_D f_2 \, \varphi \, dy, \qquad \forall \, \varphi \in C_c^\infty(D), \, \varphi \ge 0$$

where $f_2 = \tilde f$ (with $\tilde f$ defined in (4.5)) and C_n is a constant depending on $\|\nabla \varphi_n\|_\infty$. Arguing as in the proof of Step 2 in Theorem 4.3, one can easily prove that $f_1 \in L^p(D)$, and Step 2 is proved with $f = 1 + f_1 + f_2$.

Step 3. Conclusion of the theorem.
It is a direct consequence of Lemma 2.9. □

Now we focus on the generalization of the property of the value function to be a viscosity solution of Hamilton-Jacobi. Since the notion of viscosity solution requires at least continuity of the value function, we assume that the potential is $W^{1,p}$ with $p > d$. We start with a preliminary result:

Lemma 4.5. *Let D be as in Theorem 3.3, V, u_0 and u as in Definition 4.1, and suppose that u_0 is continuous and $V \in L^\infty([0,T]; W^{1,p}(D))$, $p > d$. Then u is continuous on $\overline{D} \times [0,T]$.*

Proof. First of all we remark that under the above assumptions the existence of a minimizing curve for any (x,t) follows by standard methods in the calculus of variations. (Although all the proof could be done by only considering a sequence of minimizing curves.)

Thanks to the assumption $V \in L^\infty([0,T]; W^{1,p}(D)) \subset L^\infty([0,T]; C^{0,\alpha}(D))$, $\alpha = 1 - d/p$, the continuity of u in space follows easily arguing as in the proof of Step 1 of Theorem 4.3.

To prove the continuity in time, fix a point x and two times $0 < s < t \le T$. If $\gamma_s : [0,s] \to D$ is such that $\gamma(s) = x$, then the concatenation of γ with the curve constantly equal to x on $[s,t]$ gives

$$u(t,x)) \le u_0(\gamma(0)) + \int_0^s \frac{1}{2}\Big[|\dot\gamma(\tau)|^2 - V(\gamma(\tau),\tau) \Big] \, d\tau - \int_s^t V(x,\tau) \, d\tau,$$

and by the arbitrariness of γ we get

$$u(x,t) \le u(x,s) + (t-s) \|V\|_{L^\infty([0,T]\times D)}.$$

On the other hand, given a minimizing curve $\gamma_t : [0,t] \to D$ for the evolutive problem such that $\gamma(t) = x$, we construct a competitor for u at (x,s) by considering $\gamma_{t,s}(\tau) := \gamma\left(\frac{t}{s}\tau\right)$. Thanks to the estimate

$$\begin{aligned}
\int_0^s \big| V(\gamma_t(\tau),\tau) - V(\gamma_{t,s}(\tau),\tau) \big| \, d\tau &\le C \int_0^s \big| \gamma_t(\tau) - \gamma_{t,s}(\tau) \big|^\alpha \, d\tau \\
&\le C \, \frac{s}{t^\alpha} \left(\int_0^t \big| \gamma_t(\tau) - \gamma_t\left(\tfrac{s}{t}\tau\right) \big| \, d\tau \right)^\alpha \le C \, \frac{s}{t^\alpha} \left(\int_0^t \int_{\frac{s}{t}\tau}^\tau |\dot\gamma_t(u)| \, du \, d\tau \right)^\alpha \\
&= C \, \frac{s}{t^\alpha} \, (t-s)^\alpha \left(\int_0^t u |\dot\gamma_t(u)| \, du \right)^\alpha \le C \, s \, (t-s)^\alpha \left(\int_0^t |\dot\gamma_t(u)| \, du \right)^\alpha
\end{aligned}$$

and the bound

$$\int_0^t |\dot\gamma_t(t)|^2\, d\tau \le \mathcal{A}(\gamma_t) + \|V\|_{L^\infty([0,T]\times D)}\, T,$$

since the action of $\mathcal{A}(\gamma_t)$ is easily seen to be bounded by a universal constant (thanks to the minimality) we easily get the inequality

$$u(x,s) \le u(x,t) + C\,(t-s)^\alpha.$$

This concludes the proof. □

In the next theorem we show that u is a viscosity solution to the evolutive Hamilton-Jacobi equation at almost every point. Although both u and V are continuous, it is not clear to us whether one may expect u to be a viscosity solution at *every* point, since the continuity of V does not ensure that a minimizing curve is C^1. However, thanks to the Sobolev regularity of V, we can apply Theorem 3.3 to say that almost every curve satisfies the Euler-Lagrange equation and is C^1.

Theorem 4.6. *Let $D \subseteq \mathbb{R}^d$ be as in Theorem 3.3, V, u_0 and u as in Definition 4.1, and suppose that u_0 is continuous and $V \in C^0([0,T]; W^{1,p}(D))$, $p > d$. Furthermore, assume that there exists $\eta \in \mathcal{P}^{\approx}_{T,\infty}(D)$ concentrated on minimizers for the evolutive problem at time T. Set $H(x,p,t) := \frac{1}{2}|p|^2 + V(x,t)$. Then, for every fixed $t \in (0,T)$, u is a viscosity solution of the evolutive Hamilton-Jacobi equation*

$$\begin{cases} \partial_t u(x,t) + H(x, \nabla_x u(x,t), t) = 0, \\ u(\cdot, 0) = u_0, \end{cases} \tag{4.7}$$

at $\mathcal{L}^d$-a.e. point x.

Proof. As we already observed in the proof of Theorem 4.3, under the above hypotheses all the conclusions of Theorem 3.3 hold, and in particular η-a.e. γ is of class C^1. Moreover, let us remark that if γ is a minimizer for the evolutive problem at time T, it is the case also for every $t < T$.

Fix $t \in (0,T)$, and consider a point x such that there exists a minimizer $\gamma \in C^1([0,t], \mathbb{R}^d)$. (This holds true for $\mathcal{L}^d$-a.e. x, thanks to the fact that $\mathcal{L}^d \le C(e_t)_\# \eta$.) We will prove that u is a viscosity solution at (x,t).

Supersolution: If φ is a C^1 function touching from below u at $(x,t) = (\gamma(t), t)$, then

$$\varphi(\gamma(t),t) - \varphi(\gamma(t'),t') \ge u(\gamma(t),t) - u(\gamma(t'),t') = \int_{t'}^t L\big(\gamma(s), \dot\gamma(s), s\big)\, ds.$$

Recalling that V is continuous and $\gamma \in C^1$, dividing by $t - t' > 0$ and letting $t' \nearrow t$ we obtain

$$\nabla_x \varphi(x,t) \cdot \dot\gamma(t) + \frac{\partial \varphi}{\partial t}(x,t) \ge L(x, \dot\gamma(t), t).$$

Since $L(y,v,t) - p\cdot v \ge -H(x,p,t)$ for all y,v,p,t (by the definition of H), the above equation implies that u is a viscosity supersolution at the point $(\gamma(t),t)$.

Subsolution: For this part, we do not need the existence of a C^1 minimizing curve. Given a vector $v \in \mathbb{R}^d$ and $h > 0$ small, if φ is a C^1 function touching from above u at (x, t) we have

$$\frac{\varphi(x,t) - \varphi(x - hv, t - h)}{h} \leq \frac{u(x,t) - u(x - hv, t - h)}{h}.$$

Now, if $\gamma : [0, t - h] \to \overline{D}$ is a minimizer for the evolutive problem, by considering the concatenation of γ with the curve $\sigma : [t - h, t] \to \overline{D}$, $\sigma(\tau) = x - (t - \tau)v$, we get

$$\frac{u(x,t) - u(x - hv, t - h)}{h} \leq \frac{1}{h} \int_{t-h}^{t} L\big(x - (t - \tau)v, v, \tau\big)\, d\tau.$$

Letting $h \to 0$ we obtain

$$\frac{\partial \varphi}{\partial t}(x,t) \leq L(x, v, t) - \nabla_x \varphi(x,t) \cdot v,$$

and the result follows by taking the supremum among all vectors $v \in \mathbb{R}^d$. $\square$

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