From rate-dependent to quasi-static brittle crack propagation

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Abstract. On the base of many experimental results, e.g. [18], [19], [21], [12], the object of our analysis is a rate-dependent model for the propagation of a crack in brittle materials. Our goal is a mathematical study of the evolution equation in the geometries of the 'Single Edge Notch Tension' (SENT) and of the 'Compact Tension' (ASTM-CT). Besides existence and uniqueness, emphasis is placed on the regularity of the evolution making reference to the 'velocity gap'. The transition to the quasi-static regime of Griffith's model is obtained by time rescaling, proving convergence of the rescaled evolutions and of their energies. Further, the discontinuities of the quasi-static propagation are characterized in terms of unstable branches of evolution in real time frame. The results are illustrated by a couple of numerical examples in the above mentioned geometries.

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1 Introduction

For almost one century Griffith's criterion [11] has played a prominent role in the mechanics of brittle materials. It provides a convenient approximation of the physical behaviour in real life applications and a good framework for analytical studies and computer simulations. On the other hand, models based solely on Griffith's criterion should cover only the quasi-static evolution of a brittle crack (along a straight path). Outside this relatively small field many other aspects should be taken into account to improve the predictability of the model. Our effort, pointing in the direction of dynamic fracture [10], consists in the use of a rate-dependent dissipation, as suggested by many experimental investigations (e.g. [18], [19], [21] and [12]), which represents on the macro-scale the effect of micro-cracking.

Specifically, our intention is to give a mathematical study of the experimental results cited above, considering in particular the geometries of the 'Single Edge Notch Tension' (SENT) and of the 'Compact Tension' (ASTM-CT) as presented in [12]. In a broad perspective, the results of this work should serve as a base toward the analysis of a fully dynamical model. At this point, it is important to make a remark on the mathematical tools: the ones employed in this work are based essentially on functional analysis (e.g. Sobolev and BV spaces [8], [1]), ordinary differential equations (ODE) and partially on differential inclusions [3]. Some readers more familiar with other mathematical tools in fracture mechanics [20] may wonder if our technical approach is really worth. To our credit let us mention that existence, uniqueness, regularity of the evolutions are among the targets of this investigation, even if the task may not be trivial, whereas the use of Sobolev functions allows to set the problem in a finite domains and thus to consider boundary effects. Further, the application of these tools has been unquestionably fruitful for quasi-static evolutions, especially in the last decade; the interested reader can find in [4] and [15] a comprehensive overview with a complete list of references. On the contrary, rate-dependent models are still at an early stage of developments; to our knowledge the only example is [14], a work that deserves some attention since it provides an argument in favor of the phenomenological rate-dependent dissipation by means of a relaxation result, where a microscopic pattern of micro-crack appears.

Let us present the content of this work and its main features, skipping as much as possible on the technical details. After the geometrical setting and the physical assumptions, contained in Section 2, the evolution law for the crack tip, written in terms of an ODE, is the object of Section 3: existence and uniqueness of a C^1 evolution are proved, accompanied with the energy balance, an equivalent incremental variational approach and some considerations about stable and unstable points. Section 4 considers a larger class of dissipation potential, closer to those employed in the physical literature. In this setting the evolution law is to be written in terms of a differential inclusions (rather than a differential equation) and the evolution in general is not of class C^1; notably, this loss of regularity results in a discontinuity (the velocity gap) in the speed of the tip which may occur at incipient crack propagation. In Section 5 the rate-independent quasi-static model of Griffith is derived from the rate-dependent model by means of a natural time rescaling. It turns out that a strong loss of regularity occurs: the quasi-static evolutions are only BV functions and the evolution law itself takes a completely different form, in terms of Kuhn-Tucker conditions (in the sense of measures). In particular, the quasi-static propagation may have discontinuity points which physically represent instantaneous abrupt propagations of the crack while mathematically they are characterized in terms of unstable [10] or catastrophic evolutions [5]. Further, rescaling the energy balance it turns out that the amount of energy dissipated in the discontinuity points corresponds exactly to the energy dissipated in the truly dynamic part of dissipation (i.e. by micro-cracking). A brief remark about the variational approach concludes the section. Finally, Section 6 contains a couple of numerical simulations in the SENT and ASTM-CT geometries while Appendix A contains a useful result on the regularity of the energy release rate.

2 Preliminaries and basic assumptions

Let us start to describe the experimental tests reported in [12] (cf. Figure 2) which are the prototype examples for our setting. Since it is not restrictive a two dimensional framework will be employed from the beginning. For a gallery of other standard experimental procedures in fracture mechanics the reader is referred to [2], while for the experimental results about rate dependence in brittle fracture let us mention also [18], [19] and [21].

In the Single Edge Notch Tension (SENT) the specimen is a rectangle (usually with sizes $h \ll L$) having an initial side crack of length l_0 along the horizontal middle line. On the upper and lower edges it is imposed a uniform vertical displacement. As the boundary displacement increases (in time) the material deforms and then the crack starts running along a straight path. Being the boundary condition uniform the crack tip runs at almost uniform speed until the specimen is splitted.

For the Compact Tension (ASTM-CT) the specimen is similar but the loading is quite different. A uniform time dependent displacement is imposed on a couple of symmetric holes, placed in the vicinity of the initial crack. In practice, the effect is a localization of the load around the tip of the initial crack. Indeed, increasing slowly the load, the crack starts to propagate along the middle line but it arrests before reaching the other side of the specimen.

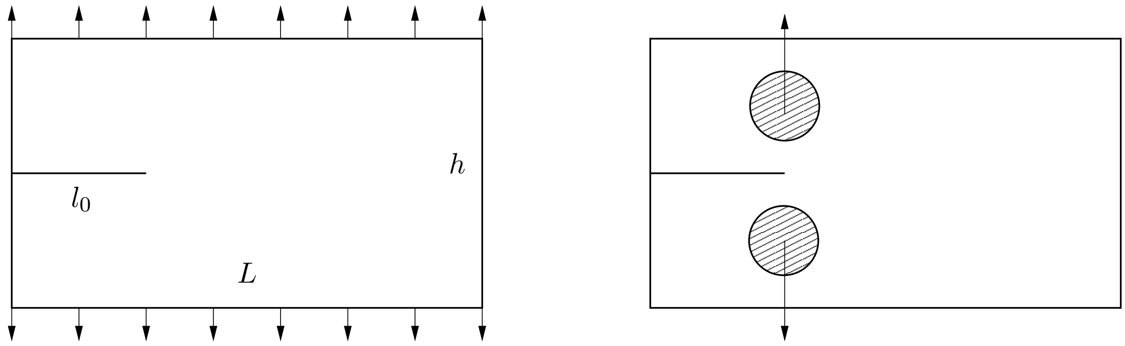

Figure 1: Standard specimen geometries used in [12]: SENT and ASTM-CT.

Keeping in mind the previous examples and other standard experimental tests in fracture mechanics [2], let us state our general assumptions. The reference uncracked configuration is represented by a Lipschitz domain Ω in $\mathbf{R}^2$. The crack path (known a priori) is a straight line segment in Ω identified, by length parametrization, with the interval $[0, L]$. Assume that the path disconnects the domain and that the admissible cracks are simply the line segments K_l of length l, identified with the interval $[0, l]$. Note that in this way an extremum of K_l is always on the boundary while the other (the crack tip) is the interior. The initial crack is denoted by K_{l_0}.

Consider a time frame $[0, T]$ and a subset $\partial_D\Omega$ of the boundary $\partial\Omega$. On $\partial_D\Omega$ it is imposed a Dirichlet boundary condition for the displacement u having the form $u(t, \cdot) = \alpha(t)\hat{g}(\cdot)$, where $\hat{g} \in H^{1/2}(\partial_D\Omega, \mathbf{R}^2)$ represents the 'geometry' of the boundary condition while $\alpha \in W^{1,\infty}(0, T)$ is a scalar (dimensionless) quantity that represents a control parameter. On the rest of boundary, denoted by $\partial_N\Omega$, and on the lips of the fracture it is imposed the homogeneous Neumann condition $\boldsymbol{\sigma}(t, \cdot)\hat{n} = 0$.

Remark 2.1 *As the reader could check, it is not hard to consider a more general setting where the crack path (known a priori) is a C^1 simple curve; the case of multiple cracks should need some more care because of possible interactions between the branches.*

2.1 Elastic energy and energy release rate

Employing the classical context of linear elastic fracture mechanics, let us consider the static equilibrium problem with boundary conditions $u = \hat{g}$ on $\partial_D\Omega$. The elastic energy, as a function of crack length l and displacement u, is given by

$$\widehat{E}(l, u) = \int_{\Omega\setminus K_l} W^e(\boldsymbol{\varepsilon})\, dx\,,$$

where the energy density is simply $W^e(\boldsymbol{\varepsilon}) = \mu|\boldsymbol{\varepsilon}| + (\lambda/2)\mathrm{tr}(\boldsymbol{\varepsilon})$, being $\boldsymbol{\varepsilon}$ the symmetrized gradient, μ and λ the Lamè coefficients. For l fixed, the equilibrium configuration is given by the (unique) minimizer $u_l \in H^1(\Omega \setminus K_l, \mathbf{R}^2)$; the corresponding elastic energy is denoted by

$$\widehat{\mathcal{E}}(l) = \min\left\{\widehat{E}(l, u) \text{ for } u = \hat{g} \text{ on } \partial_D\Omega\right\} = E(l, u_l)\,.$$

As usual in fracture mechanics, the key role in the evolution will be played by the variations of $\widehat{\mathcal{E}}$, more precisely by the energy release rate

$$\widehat{G}(l) = -\,\widehat{\mathcal{E}}'(l)\,.$$

For sake of simplicity the energy release rate is not defined in L, which corresponds to the case in which the domain is splitted. Let us summarize the properties of $\widehat{\mathcal{E}}$ and $\widehat{G}$ in the next Proposition, a proof is given is the Appendix.

Proposition 2.2 *The energy $\widehat{\mathcal{E}}$ is non-increasing and of class C^1 in $(0, L)$. Consequently $\widehat{G}$ is non-negative and continuous in $(0, L)$.*

By linearity of the elasticity problem it is immediate to check that $\mathcal{E}(t, l) = \alpha^2(t)\,\widehat{\mathcal{E}}(l)$ and $G(t, l) = \alpha^2(t)\,\widehat{G}(l)$.

Remark 2.3 *Notice that, given l and u the elastic energy E can be computed explicitly. On the contrary, given l the energy $\widehat{\mathcal{E}}$ and is no longer explicit, since its computation requires a minimization operator. Obviously, the energy release rate $\widehat{G}$ is not explicit as well. In some cases approximate formulas are available but they seem not suitable for our general framework. Therefore, in the analysis of the evolution the only piece of information on G to be used are the abstract properties contained in Proposition 2.2 (in particular the continuity); in the explicit examples of section §6 it is employed a numerical computations of G.*

2.2 Dissipated energy and dissipation potential

Let us start again from the experimental results of [12]. Consider the SENT geometry. Depending on the length of the initial notch the fracture starts to propagates at different times and then runs with different velocities. (This behaviour is due to the fact that the energy release rate increases rapidly for short initial cracks, see for instance the numerical computations of Section 6). Consequently, knowing the amount of elastic energy released in the propagation it is possible to measure, by a simple balance, the amount of energy dissipated by the crack.

According to Griffith's theory the energy dissipated by the fracture is proportional to the length of the fracture. In other terms, the dissipated energy per unit of crack length should be constant. Actually, the careful experimental results mentioned above reveled a strong dependence on the speed of the crack. This is clearly visible in Figure 2 where the spots represent dissipated energy per unit of crack length versus crack speed while the horizontal line corresponds to the constant value predicted by Griffith. Different spots correspond to different propagation velocities, i.e. to experiments with different initial notches.

Of course, it may be argued that this rate dependence is due to dynamical effects occurring at high speeds; this objection has been extensively discussed and refused in the physical literature since the values predicted by the dynamic energy release rate [10] are not consistent with the experimental data. According to the evidence of the experimental results this phenomenon is essentially due to the microstructure of the fracture set. Some nice pictures of the microscopic pattern of the crack (see e.g. [12]) show clearly that surface roughness and micro branching increases significantly at high crack velocities. Hence, besides the energy dissipated in the macroscopic crack a considerable amount of energy is dissipated at a subscale level.

A multiscale approach should be the right theoretical solution, however it seems not feasible since the model should be able to resolve very complex patterns. A phenomenological description in the macroscopic setting seems to be an easier and effective way to describe this phenomenon.

From the mathematical point of view the dissipated energy per unit of crack length is well represented by a monotone graph ϕ. Let $G_c > 0$ be the material toughness. Assume that $\phi(0) = (-\infty, G_c]$ and that $\phi : (0, +\infty) \to [G_c, +\infty]$ is a convex non-decreasing function such that $\lim_{\dot\ell \searrow 0} \phi(\dot\ell) = G_c$ and $\lim_{\dot\ell \to +\infty} \phi(\dot\ell) = +\infty$. A couple of typical choices are contained in the next Example.

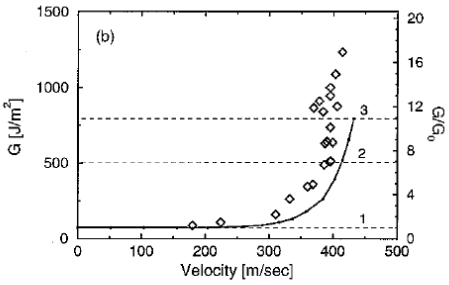

Figure 2: Rate dependence in Homalite-100: energy released by the crack per unit of 'length' (J/m) vs (averaged) crack speed (courtesy of [12]).

Example 2.4 *Figure 2 and [12] suggest a function of the form*

$$\phi(\dot\ell) = \begin{cases} (-\infty, G_c] & \dot\ell = 0 \\ G_c & \dot\ell \in [0, \dot\ell_c] \\ G_c + \big((\dot\ell - \dot\ell_c)/(\dot\ell_m - \dot\ell)\big)^p & \dot\ell \in [\dot\ell_c, \dot\ell_m] \\ +\infty & \dot\ell > \dot\ell_m\,, \end{cases}$$

where $0 < \dot\ell_c < \dot\ell_m$ and $p > 1$. This is the case of a brittle material in which micro-cracks, and possibly other velocity dependent phenomena, occurs when the propagation velocities are higher than $\dot\ell_c$; these sub-scale dissipation mechanisms limit the maximum speed of the tip which cannot exceed $\dot\ell_m$. Note that in general $\dot\ell_m$ does not coincide with the Rayleigh speed as in the dynamic theory of fracture [10].

A much simpler choice, without thresholds, is given by

$$\phi(\dot\ell) = \begin{cases} (-\infty, G_c] & \dot\ell = 0 \\ G_c + \dot\ell^p & \dot\ell > 0\,. \end{cases}$$

As it will be explained in Section 3 and 4 the analytical properties of the evolution obtained in the first and second case are different. In particular the first allows for the velocity gap.

We remark that Griffith model, which applies to the quasi-static case, corresponds to a non-decreasing function of the form

$$\phi^{qs}(\dot\ell) = \begin{cases} (-\infty, G_c] & \dot\ell = 0 \\ G_c & \dot\ell > 0\,. \end{cases}$$

Note in particular that it does not satisfy the condition $\lim_{\dot\ell \to +\infty} \phi^{qs}(\dot\ell) = +\infty$. The resulting evolution is qualitatively very different from those obtained with a diverging ϕ; its behaviour has been studied [16] and in [15], here it is recovered as a limiting case in the quasi-static evolution of Section 5. Note also, in view of Section 5, that both the functions ϕ of Example 2.4 can be written in the form $\phi = \phi^{qs} + \phi^{dyn}$ where ϕ^{qs} (defined above) accounts for the macroscopic dissipation while $\phi^{dyn} > 0$ for the microscopic rate dependent dissipation.

Let us define the dissipation potential (dimensionally the energy dissipated per unit of time). It is given formally by $\mathcal{D}(\dot\ell) = \phi(\dot\ell)\dot\ell$. More precisely

$$\mathcal{D}(\dot\ell) = \begin{cases} +\infty & \text{for } \dot\ell < 0 \\ \phi(\dot\ell)\dot\ell & \text{for } \dot\ell \geq 0. \end{cases} \qquad (1)$$

Note that $\mathcal{D} : \mathbf{R} \to [0, +\infty]$ is convex and that $\mathcal{D}(0) = 0$.

In our picture the energy dissipated by the fracture in time interval $(0, t)$ depends on the whole trajectory ℓ and is given by

$$\mathcal{K}(t, \ell) = \int_0^t \mathcal{D}(\dot{\ell}(s))\, ds\,. \tag{2}$$

Note in $\mathcal{K}$ the dependence on evolution ℓ and not on value $\ell(t)$.

Finally, note that Griffith's dissipation corresponds to the one-homogeneous potential

$$\mathcal{D}^{qs}(\dot{\ell}) = \begin{cases} +\infty & \dot{\ell} < 0 \\ G_c \dot{\ell} & \dot{\ell} > 0 \end{cases}$$

and thus to $\mathcal{K}^{qs}(t, \ell) = G_c(\ell(t) - l_0)$.

As observed for ϕ, it is convenient to represent $\mathcal{D}$ also as $\mathcal{D}(\dot{\ell}) = \mathcal{D}^{qs}(\dot{\ell}) + \mathcal{D}^{dyn}(\dot{\ell})$ where $\mathcal{D}^{qs}(\dot{\ell}) = G_c \dot{\ell}$ and hence $\mathcal{D}^{dyn}(\dot{\ell}) > 0$. Accordingly we may write $\mathcal{K}(t, \ell) = \mathcal{K}^{qs}(t, \ell) + \mathcal{K}^{dyn}(t, \ell)$. We remark once more that $\mathcal{D}^{qs}$ will account for the macroscopic crack while $\mathcal{D}^{dyn}$ will account (phenomenologically) for the microscopic effects.

In conclusion, the free energy of the system at time t is given by

$$\mathcal{F}(t, \ell) = \mathcal{E}(t, \ell(t)) + \mathcal{K}(t, \ell)\,. \tag{3}$$

Clearly, for the energy to be finite it is necessary that $\mathcal{K}(t, \ell)$ is finite, i.e. that $\dot{\ell} \geq 0$ for a.e. time (which renders the irreversibility of the crack).

2.3 Crack propagation law

A mathematically convenient and concise way of writing the evolution law is by means of the (doubly non-linear) differential inclusion

$$\begin{cases} \phi(\dot{\ell}(t)) \ni G(t, \ell(t)) \\ \ell(0) = l_0\,. \end{cases} \tag{4}$$

Let us spend few words to understand the meaning of (4). First of all, note that the differential inclusion is dimensionally like a balance of forces, since ϕ and G here are measured in $[N]$; in particular G is the driving force of the crack and ϕ is a sort of viscosity. To strengthen this interpretation it is convenient to re-write (4) taking into account the fact that $\phi(0) = (-\infty, G_c]$:

$$\begin{cases} G(t, \ell(t)) \leq G_c & \dot{\ell}(t) = 0 \\ G(t, \ell(t)) = \phi(\dot{\ell}(t)) & \dot{\ell}(t) > 0 \\ \ell(0) = l_0\,, \end{cases}$$

that is formally like a Coulomb dry friction for the external force $G(t, \ell(t))$. More simply, (4) combines the activation condition and the flow rule:

$$\dot{\ell}(t) = 0 \;\Rightarrow\; G(t, \ell(t)) \leq G_c\,, \qquad \dot{\ell}(t) > 0 \;\Rightarrow\; G(t, \ell(t)) = \phi(\dot{\ell}(t))\,.$$

In the next two sections are contained the existence, uniqueness and regularity results of the evolution defined by (4). For sake clarity the case in which ϕ is strictly increasing (in its proper domain) is considered in Section 3 while the case in which ϕ is only non-decreasing is considered in Section 4; the corresponding solutions are indeed qualitatively different.

3 Evolution

This section deals with the case in which ϕ is strictly increasing (in its proper domain). Let $\phi^{-1} : \mathbf{R} \to \mathbf{R}$ be its inverse, then ϕ^{-1} is continuous, non-decreasing and non-negative, in particular $\phi^{-1} = 0$ in $(-\infty, G_c]$ while ϕ^{-1} is positive, strictly increasing and concave in $(G_c, +\infty)$. Note that by definition the range of ϕ^{-1} is the proper domain of ϕ.

Example 3.1 *Consider ϕ from Example 2.4, then*

$$\phi^{-1}(G) = \begin{cases} 0 & \text{if } G \in (-\infty, G_c] \\ (G - G_c)^{1/p} & \text{otherwise.} \end{cases}$$

See Figure 3.

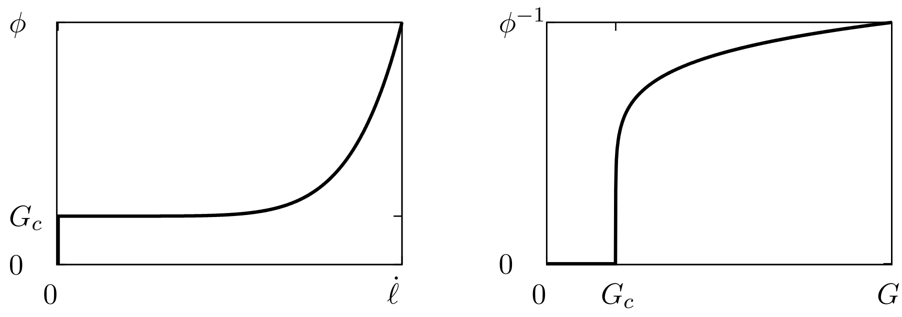

Figure 3: The dissipation potential ϕ of Example 3.1 and its inverse ϕ^{-1}.

In this case the differential inclusion (4) takes a simpler form, it is indeed equivalent to the ODE

$$\begin{cases} \dot{\ell}(t) = \phi^{-1}\big(G(t, \ell(t))\big) \\ \ell(0) = l_0 \,. \end{cases} \tag{5}$$

Note that in general the right hand side of (5) is not uniformly Lipschitz continuous with respect to ℓ; this is indeed the case of Example 3.1.

Let us denote by t_0 the initiation time and by T the failure time, i.e.

$$t_0 = \sup\{t : \ell(t) = l_0\} \,, \qquad T = \sup\{t : \ell(t) < L\} \,. \tag{6}$$

By definition, the Cauchy problem (5) makes sense in $[0, T)$, as $\widehat{G}(L)$ is not defined.

Theorem 3.2 *Let ϕ be strictly increasing, then there exists an evolution ℓ of class C^1 in $[0, T)$. If α is strictly increasing then the solution ℓ is unique.*

Proof. By the continuity of G and ϕ^{-1} there exists a solution in a small right neightborhood of 0; moreover ϕ^{-1} is non-negative, therefore ℓ is non decreasing. It follows that there exists a solution in the maximal interval $(0, T)$. The C^1 regularity is obvious.

Consider a strictly increasing control α. Assume by contradiction that there exist solutions $\ell_1 \neq \ell_2$. By continuity we can assume that there exist t_c such that $\ell_1(t_c) = \ell_2(t_c) = \ell_c$ and $\ell_1 < \ell_2$ in $(t_c, t_c + \delta)$.

First, let us see that $G(t_c, \ell_c) = G_c$. We argue by contradiction. If $G(t_c, \ell_c) < G_c$ then by continuity $G(t, l) < G_c$ in a neighborhood U_c of (t_c, ℓ_c). Hence, the solutions of the ODE satisfy $\dot\ell_i = 0$ in a neighborhood of t_c. This is a contradiction, since $\ell_1 \neq \ell_2$. Similarly, if $G(t_c, \ell_c) > G_c$ then $G(t, l) \geq \bar G > G_c$ in a neighborhood U_c. As ϕ^{-1} is locally Lipschitz in $(\bar G, +\infty)$ (being ϕ^{-1} concave) by a standard result on ODE we get a unique solution, which is again a contradiction.

Second, let us show that ℓ_i $(i = 1, 2)$ are invertible in $(t_c, t_c + \delta)$. The solutions ℓ_i are non-decreasing. If ℓ_i is constant (say $\ell_i(t) = \bar\ell$) in an interval $(t_a, t_b) \subset (t_c, t_c + \delta)$ then $\dot\ell_i = 0$, hence $G(t, \ell_i(t)) \leq G_c$. As α is strictly increasing we get $G(t, \bar\ell) < G(t_b, \bar\ell) \leq G_c$ for $t < t_b$. Hence $\bar\ell(t) \equiv \bar\ell$ solves the ODE in $(0, t_b)$. As $G(t, \bar\ell) < G_c$ in $(0, t_b)$ and ϕ^{-1} is locally Lipschitz in $(-\infty, G_c)$ then the solution is unique, which contradicts our assumptions. Note that $\{t : \dot\ell_i(t) = 0\}$ is closed with empty interior (being ℓ_i continuous and strictly increasing). It follows (by contradiction) that $G(t, \ell_i(t)) \geq G_c$ in $[t_c, t_c + \delta]$.

Third, let us define $\tau(s) = \ell_1^{-1} \circ \ell_2(s)$. Hence $\ell_1(\tau(s)) = \ell_2(s)$. As ℓ_i are indcreasing, τ itself is increasing. Moreover, by $\ell_1(t_c) = \ell_2(t_c)$ we get $\tau(t_c) = t_c$ and by $\ell_1 < \ell_2$ we get $\tau(s) > s$.

Moreover, being α strictly increasing and $\tau(s) > s$ we have

$$G_c \leq G(s, \ell_2(s)) = \alpha^2(s)\, \widehat G(\ell_2(s)) < \alpha^2(\tau(s))\, \widehat G(\ell_1(\tau(s))) = G(\tau(s), \ell_1(\tau(s)))\,.$$

Hence $G(\tau, \ell_1(\tau)) > G_c$, and $\dot\ell_1(\tau) > 0$. As a consequence τ is of class C^1.

By the monotonicity of ϕ^{-1} and the previous inequality (for $\tau = \tau(s)$) we can write

$$\dot\ell_1(\tau) = \phi^{-1}\big(G(\tau, \ell_1(\tau))\big) > \phi^{-1}\big(G(s, \ell_2(s))\big) = \dot\ell_2(s)\,.$$

Hence

$$\dot\ell_2(s) = \dot\ell_1(\tau(s))\tau'(s) < \dot\ell_1(\tau(s))\,.$$

As $\dot\ell_1 > 0$ it follows that $\tau' < 1$. This inequality is a contradiction with $\tau(s) > s$. ■

Proposition 3.3 *Let ℓ be a solution of* (5). *The following energy balance, in differential form, holds true:*

$$d\mathcal{E}(t, \ell(t)) = \mathcal{P}^{ext}(t, \ell(t))\, dt - \mathcal{D}(\dot\ell(t))\, dt \tag{7}$$

where $\mathcal{P}^{ext}$ is the power of external forces, namely

$$\mathcal{P}^{ext}(t, \ell(t)) = \int_{\partial_D\Omega} \dot g(t, x) \cdot \boldsymbol{\sigma}_{\ell(t)}(t, x)\hat n\, dx$$

and $\boldsymbol{\sigma}_{\ell(t)}$ is the stress tensor for the equilibrium configuration $u_{\ell(t)}$.

Proof. By the chain rule and by (5)

$$\begin{aligned} d\mathcal{E}(t, \ell) &= \big[\mathcal{P}^{ext}(t, \ell(t)) - G(t, \ell(t))\dot\ell(t)\big] dt \\ &= \big[\mathcal{P}^{ext}(t, \ell(t)) - \phi(\dot\ell(t))\dot\ell(t)\big]\, dt = \mathcal{P}^{ext}(t, \ell(t))\, dt - \mathcal{D}(\dot\ell(t))\, dt\,. \end{aligned}$$

■

The above energy balance in time interval (t_1, t_2) reads

$$\mathcal{E}(t_2, \ell(t_2)) = \mathcal{E}(t_1, \ell(t_1)) + \int_{t_1}^{t_2} \mathcal{P}^e(s, \ell(s))\, ds - \int_{t_1}^{t_2} \mathcal{D}(\dot\ell(s))\, ds\,. \tag{8}$$

As already mentioned, the two integrals on the right hand side are the work of external forces and the dissipated energy (respectively).

3.1 Stable and unstable points

Let us define the sets of stable, unstable and critical points. A length $l < L$ is stable at time t if there exists $\eta > 0$ such that $G(t, z) < G_c$ for $l < z < l + \eta$, otherwise it is called unstable. A length l is critical at time t if $G(t, l) = G_c$. Clearly, by continuity of the energy release rate if $G(t, l) < G_c$ then (t, l) is stable, if $G(t, l) > G_c$ then (t, l) is unstable. Moreover, again by continuity, $G(t, l) \leq G_c$ in the stable points while $G(t, l) \geq G_c$ in the unstable ones. Critical points are either stable or unstable. The sets of stable, critical and unstable points at time t are denoted respectively by $\mathcal{S}(t)$, $\mathcal{C}(t)$ and $\mathcal{U}(t)$. Note that if ℓ is a solution of (5) then by the properties of ϕ^{-1} we deduce easily that

$$\ell(t) \in \mathcal{S}(t) \;\Rightarrow\; \dot{\ell}(t) = 0\,. \tag{9}$$

Equivalently if $\dot{\ell}(t) > 0$ then $\ell(t) \in \mathcal{U}(t)$, while the reverse of (9) is false, since it may happen that $\dot{\ell}(t) = 0$ if $\ell \in \mathcal{U}(t) \cap \mathcal{C}(t)$.

3.2 Examples

For the sake of completeness, we give also a couple of counter example to uniqueness.

Example 3.4 *Let* $\widehat{G}(l) = l$, $G_c = l_0$ *and let*

$$\phi(\dot{\ell}) = \begin{cases} (-\infty, G_c] & \text{if } \dot{\ell} = 0 \\ G_c + \dot{\ell}^2 & \text{if } \dot{\ell} > 0\,, \end{cases} \qquad \phi^{-1}(G) = \begin{cases} 0 & \text{if } G \in (-\infty, G_c] \\ |G - G_c|^{1/2} & \text{otherwise.} \end{cases}$$

Assume that α *is non decreasing and of class* C^1 *in* $(0, +\infty)$, *with* $\alpha(0) = 0$, $0 < \alpha(t) < 1$ *for* $t \in (0, t_0)$, $\alpha(t) = 1$ *for* $t \geq t_0$. *Then it is easy to check that in* $(0, t_0)$ *the unique solution is* $\ell(t) = l_0$. *Therefore* (5) *reduces to*

$$\begin{cases} \dot{\ell}(t) = |G(t, \ell(t)) - G_c|^{1/2} \\ \ell(t_0) = l_0\,. \end{cases}$$

Besides $\ell(t) \equiv l_0$, *there are infinitely many solutions of the form*

$$\ell(t) = l_0 + |t - t'|_+^2/4$$

where $|\cdot|_+$ *denotes the positive part and* $t' \geq t_0$.

Example 3.5 *Let us give also an example of non-uniqueness with crack arrest in finite time. Let* $l_0 < l^* < L$ *and*

$$\widehat{G}(l) = \begin{cases} l & \text{if } l \leq l^* \\ 2l^* - l & \text{if } l^* < l \leq 2l^* \\ 0 & \text{if } l \geq 2l^*\,. \end{cases}$$

Let G_c, ϕ *and* α *as in the previous example. Let* t^* *be defined by* $l_0 + (t^* - t_0)^2/4 = l^*$ *and let* $t_a = 2t^* - t_0$. *Then, it is easy to check that besides* $\ell(t) \equiv l_0$ *the function*

$$\ell(t) = \begin{cases} l_0 & \text{if } t \leq t_0 \\ l_0 + (t - t_0)^2/4 & \text{if } t_0 < t \leq t^* \\ (2l^* - l_0) - (t - t_a)^2/4 & \text{if } t^* < t \leq t_a \\ (2l^* - l_0) & \text{if } t > t_a \end{cases}$$

is a C^1 *solution of* (5) *in* $[0, +\infty)$. *Clearly, the crack arrests at time* t_a.

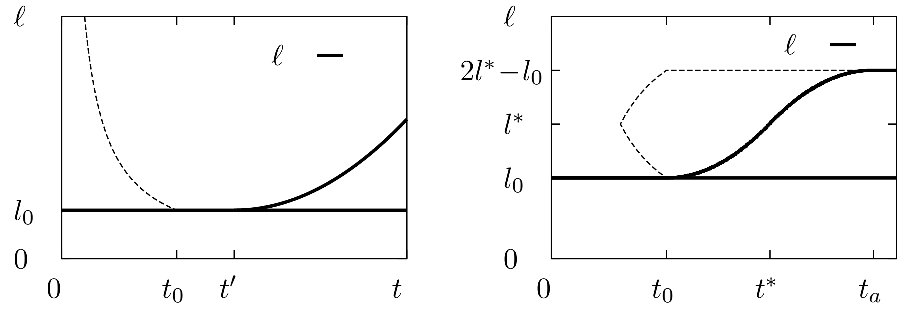

Figure 4: Evolutions ℓ of Example 3.4 and 3.5 (bold), loci of stationary points (dashed).

3.3 Variational approach

In this section we show that the evolution ℓ which solves (5) can be derived from a variational approach by means of incremental problems.

Given ϕ let $\psi : \mathbf{R} \to [0, +\infty]$ be a dissipation potential such that $\psi(\dot\ell) = +\infty$ if $\dot\ell < 0$, $\psi(0) = 0$ and $\partial\psi(\dot\ell) = \phi(\dot\ell)$ if $\dot\ell \geq 0$ (in particular $\partial\psi(0) = (-\infty, G_c]$). Obviously, ψ is convex, being ϕ non-decreasing. We remark that ψ is dimensionally a power but it is not the dissipation $\mathcal{D}$. Let Δt^k be a positive infinitesimal sequence of time increment. Given $k, n \in \mathbf{N}$ let $t^k_n = n\Delta t^k$ be a (uniform) discretization of the time interval. Let $\ell^k_0 = l_0$ and define by induction

$$\ell^k_{n+1} \in \operatorname{argmin} \left\{ \Delta t^k \psi\big((l - \ell^k_n)/\Delta t^k\big) + \mathcal{E}(t^k_{n+1}, l) : l \in [l_0, L] \right\} . \tag{10}$$

Note that, being $\psi(\dot\ell) = +\infty$ for $\dot\ell < 0$, the above minimization problem would be equivalent with $l \in [\ell^k_n, L]$. Let ℓ^k be the piecewise affine interpolation of the values ℓ^k_n defined above. For every k the function $\ell^k : [0, +\infty) \to [l_0, L]$ are continuous and non-decreasing. Therefore, by Helly's Theorem there exists a subsequence (not relabelled) converging pointwise to a (limit) non-decreasing function ℓ. Let us see that ℓ is continuous and that it is a solution of the (5). For simplicity choose T' such that $\ell(T') < L$ and consider the time interval $[0, T']$. It is not restrictive to assume that $\ell^k(T') < L$ (this is clearly true for $k \gg 1$).

Before proceeding, let us see the variational properties of ℓ^k. Note that the derivative, with respect to l of the incremental energy (10) is (by definition of ψ and G)

$$\phi\big((l - \ell^k_n)/\Delta t^k\big) - G(t^k_{n+1}, l) \,.$$

If $\ell^k_{n+1} = \ell^k_n$ minimality gives

$$(-\infty, G_c] - G(t^k_{n+1}, \ell^k_{n+1}) \ni 0 \quad \Leftrightarrow \quad G(t^k_{n+1}, \ell^k_{n+1}) \leq G_c \,.$$

It follows that for $t \in (t^k_n, t^k_{n+1})$ we have $\dot\ell^k(t) = 0$ and

$$\dot\ell^k(t) = \phi^{-1}\big(G(t^k_{n+1}, \ell^k_{n+1})\big) \,. \tag{11}$$

Similarly, if $\ell^k_{n+1} > \ell^k_n$ minimality gives

$$\phi\big((\ell^k_{n+1} - \ell^k_n)/\Delta t^k\big) - G(t^k_{n+1}, \ell^k_{n+1}) = 0 \,,$$

hence (11) holds true in general. From (11) follows the uniform Lipschitz continuity of the sequence (of functions) ℓ^k. Indeed, the control α is bounded in $[0, T']$ and the energy release rate $\widehat{G}$ is bounded in $[l_0, \ell(T')]$ (as $\ell(T') < L$), hence $G(t^k_{n+1}, \ell^k_{n+1}) = \alpha^2(t^k_{n+1})\,\widehat{G}(\ell^k_{n+1})$ is bounded independently of k and n. It follows that the velocities $\dot\ell^k$ are bounded, independently of k. As a consequence, by Ascoli-Arzelà Theorem ℓ^k converges uniformly (up to subsequences) to a continuous function ℓ.

It remains to see that ℓ satysfies (5). Given $s \in [0, T']$ let $n_s = [s/\Delta t^k] + 1$ (where $[\cdot]$ denotes the integer part) so that $t^k_{n_s - 1} \le s < t^k_{n_s}$. By (11) we can write

$$\ell^k(t) = l_0 + \int_0^t \phi^{-1}\big(G\big(t^k_{n_s}, \ell^k_{n_s}\big)\big)\, ds\,.$$

Let us take the limit as $\Delta t^k \searrow 0$. Obviously $\ell^k(t) \to \ell(t)$. Let us check that

$$\int_0^t \phi^{-1}\big(G\big(t^k_{n_s}, \ell^k_{n_s}\big)\big)\, ds \quad\longrightarrow\quad \int_0^t \phi^{-1}\big(G(s, \ell(s))\big)\, ds\,.$$

Note that the integrand is uniformly bounded, therefore by dominated convergence and continuity of $\phi^{-1} \circ G$ it is sufficient to see that

$$\big(t^k_{n_s}, \ell^k_{n_s}\big) \quad\longrightarrow\quad (s, \ell(s)) \quad \text{for every } s \in [0, T'].$$

As $|t^k_{n_s} - s| \le \Delta t^k$ (by definition of n_s) it is clear that $t^k_{n_s} \to s$. Moreover, $\ell^k_{n_s} = \ell^k(t^k_{n_s})$, thus

$$|\ell^k(t^k_{n_s}) - \ell(s)| \le |\ell^k(t^k_{n_s}) - \ell(t^k_{n_s})| + |\ell(t^k_{n_s}) - \ell(s)|\,.$$

This concludes the proof by the uniform convergence of ℓ^k to ℓ and by the continuity of ℓ.

4 Velocity gap

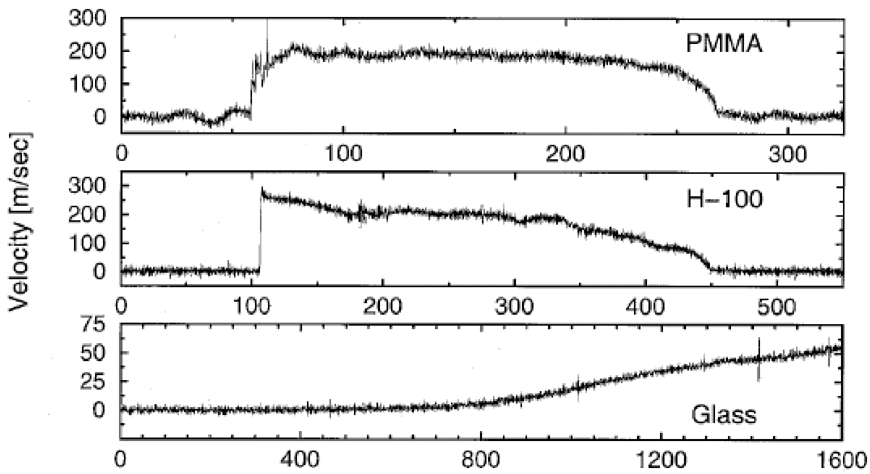

Figure 5: A velocity gap for Homalite-100 (courtesy of [12]).

In this section we consider the case of a non-decreasing dissipation potential ϕ such that $\phi(\dot\ell) = G_c$ in $[0, \dot\ell_c]$, with $0 < \dot\ell_c < +\infty$, and $\phi(\dot\ell) > G_c$ in $(\dot\ell_c, +\infty)$.

Example 4.1 *For $0 < \dot\ell_c < \dot\ell_m$ and $p > 1$ consider the function*

$$\phi(\dot\ell) = \begin{cases} (-\infty, G_c] & \text{if } \dot\ell = 0 \\ G_c & \text{if } 0 < \dot\ell < \dot\ell_c \\ G_c + (\dot\ell - \dot\ell_c)^p/(\dot\ell_m - \dot\ell)^p & \text{if } \dot\ell_c < \dot\ell_m. \end{cases}$$

In this case the inverse of ϕ is given by the maximal monotone graph

$$\phi^{-1}(G) = \begin{cases} 0 & \text{if } G \in (-\infty, G_c] \\ [0, \dot\ell_c] & G = G_c \\ \big(\dot\ell_c + \dot\ell_m (G - G_c)^{1/p}\big)/\big((G - G_c)^{1/p} + 1\big) & \text{otherwise.} \end{cases}$$

Alternatively, a simpler model can be used, without the upper bound $\dot\ell_m$; this is for instance the case of a dissipation potential of the form

$$\phi(\dot\ell) = \begin{cases} (-\infty, G_c] & \text{if } \dot\ell = 0 \\ G_c & \text{if } 0 < \dot\ell < \dot\ell_c \\ G_c + (\dot\ell - \dot\ell_c)^p & \text{if } \dot\ell_c < \dot\ell, \end{cases}$$

with inverse

$$\phi^{-1}(G) = \begin{cases} 0 & \text{if } G \in (-\infty, G_c] \\ [0, \dot\ell_c] & G = G_c \\ \dot\ell_c + (G - G_c)^{1/p} & \text{otherwise.} \end{cases}$$

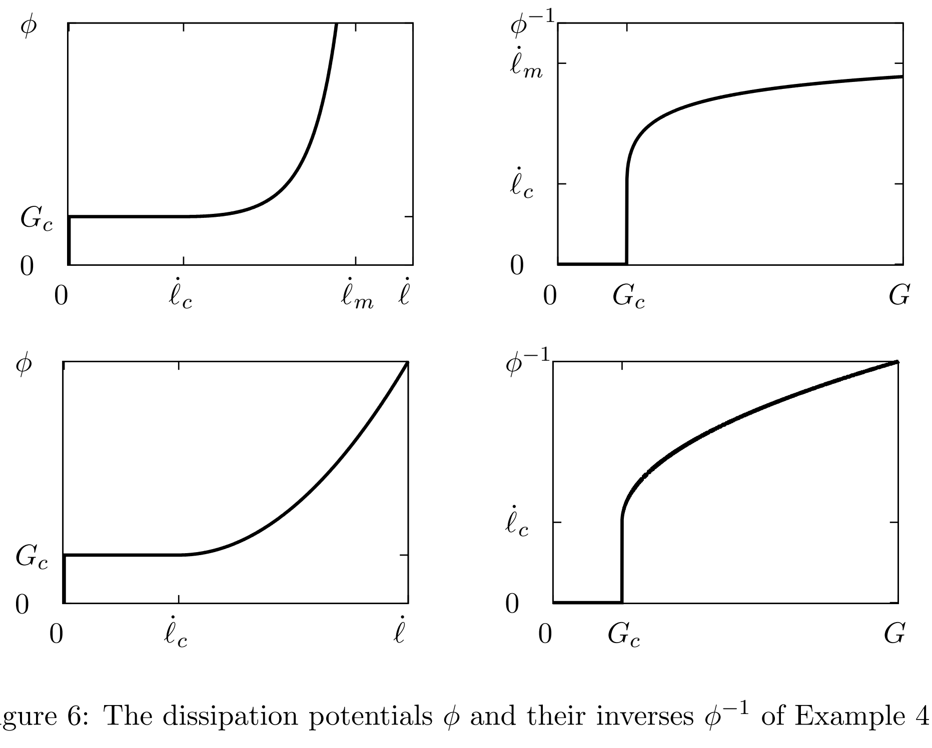

Figure 6: The dissipation potentials ϕ and their inverses ϕ^{-1} of Example 4.1.

Unlike what stated in the previous section, when the dissipation potential ϕ is just non-decreasing the evolution is continuous but in general it is not of class C^1 (an explicit example is given hereafter). This loss of regularity is a remarkable feature in connection with the 'velocity gap'. Let us briefly explain this behaviour: experimental measurements on brittle materials seem to indicate that in many cases the velocity of the crack at the initiation time is strictly positive; to be more precise, the velocity $\dot\ell$ has a jump (or gap) at t_0 since $\dot\ell^-(t_0) = 0$ while $\dot\ell^+(t_0) > 0$ (e.g. H-100 in Figure 5). From the physical point of view it is questionable wether this behaviour should be accepted or not. For instance, Hauch & Marder [12] provided also a set of very fine measures of $\dot\ell$ around t_0 to show that the velocity changes continuously in time; rightfully they concluded that the velocity gap is an apparent feature due to fact that the velocity changes very rapidly and thus its graph looks like a jump if it is seen at a large time scale. Our result may provide a further element in the understanding of the 'velocity gap' or (equivalently) a selection criterion for ϕ. In fact, considering the velocity gap as an artificial effect leads to rule out the non-decreasing cases (such as those of Example 4.1) since the evolution could exhibit a velocity gap, i.e. a discontinuity in the velocity at the initiation time.

Example 4.2 *Let $l_0 = 1$, $G_c = 1$ and*

$$\widehat{G}(l) = \frac{4(\sqrt{l}-1)^2+1}{\sqrt{l}} .$$

Let $\alpha(t) = \sqrt{t}$ and

$$\phi(\dot\ell) = \begin{cases} (-\infty, 1] & \text{if } \dot\ell = 0 \\ 1 & \text{if } 0 < \dot\ell < 2 \\ 1 + (\dot\ell - 2)^2 & \text{if } 2 \le \dot\ell . \end{cases}$$

Under these assumptions it is not difficult to check that the function $\ell(t) = 1 + |t^2 - 1|_+$ satisfies the differential inequality (4) for every $t \neq t_0 = 1$. Note that $\dot\ell(t) = 0$ for $t \in (0, t_0)$ and $\dot\ell(t) = 2t$ for $t > t_0$ while ℓ is not differentiable in the initiation time. In particular, the crack starts the propagation with a finite speed $\dot\ell^+(t_0) > 0$ and hence the evolution exhibits a velocity gap.

Note that if the evolution is not of class C^1 it is necessary to give a suitable meaning to (5), as ℓ may be non-differentiable in every point of $[0, T)$. Following the standard theory of differential inclusions (e.g. [3]) we say that ℓ is a solution if it is absolutely continuous (i.e. in $W^{1,1}$) and if

$$\begin{cases} \dot\ell(t) \in \phi^{-1}\big(G(t, \ell(t))\big) & \text{for a.e. } t \in [0, T) \\ \ell(0) = l_0 . \end{cases} \tag{12}$$

We recall (e.g. [3, Lemma 1 on page 99]) that ℓ solves (12) if and only if there exists a measurable selection of $\phi^{-1}\big(G(t, \ell(t))\big)$ in such a way that for every $t_1 < t_2$

$$\ell(t_2) \in \ell(t_1) + \int_{t_1}^{t_2} \phi^{-1}\big(G(s, \ell(s))\big)\, ds .$$

Comparing with (9), if ℓ is a solution of (12) then

$$\begin{cases} \dot\ell(t) = 0 & \text{if } \ell(t) \in \mathcal{S}(t) \\ \dot\ell(t) \in [0, \dot\ell_c] & \text{if } \ell(t) \in \mathcal{C}(t) \\ \dot\ell(t) > \dot\ell_c & \text{if } \ell(t) \in \mathcal{U}(t). \end{cases} \tag{13}$$

Now, let us state the existence result.

Theorem 4.3 *Let ϕ be non-decreasing then there exists an (absolutely continuous) evolution ℓ in $[0, T)$.*

Proof. Under the above assumptions, our problem (12) fits into the framework of [3, Theorem 3 on page 98] which ensures existence. ■

We conclude the section with an Example to show that in some cases there exists a C^1 evolution and hence there is no velocity gap.

Example 4.4 *Assume that $\widehat{G}(\ell) = 2l_0 - \ell$ for $\ell \geq l_0$. Let $G_c = l_0$, α strictly increasing with $\alpha(t) = (t - t_0)^2 + 1$ for $t > t_0$. Let ϕ be as in Example 4.1. Let*

$$\ell(t) = \begin{cases} l_0 & \text{if } t \leq t_0 \\ 2l_0 - l_0/\big((t-t_0)^2+1\big)^2 & \text{if } t > t_0 . \end{cases}$$

Clearly ℓ is of class C^1. Let us see that ℓ is a solution of (12) (at least for $t \approx t_0$). For $t \leq t_0$ we have $\alpha(t) < 1$ and $G(t, l_0) < l_0 = G_c$, hence $\ell(t) = l_0$ is the unique solution. Moreover, being $G_c = l_0$, it is easy to check that $G(t, \ell(t)) = G_c$ for $t > t_0$. Moreover $\dot{\ell}(t) < \dot{\ell}_c$ for $t \approx t_0$, hence (12) is satisfied.

5 Quasi-static evolution by time rescaling

The goal of this section is to obtain a quasi-static evolution by means of a suitable time rescaling. For simplicity, let us restrict ourselves to the case of ϕ strictly increasing.

Given a control α let us consider for $0 < \varepsilon \ll 1$ the slow control $\alpha_\varepsilon(t) = \alpha(\varepsilon t)$. Intuitively, as the Dirichlet boundary conditions $u = \alpha_\varepsilon(t)\hat{g}$ change slowly in time the fracture is expected to evolve slowly and thus the rate dependence of the dissipation should became negligible. Let us be more precise. Denoting by $G_\varepsilon(t, l) = \alpha_\varepsilon^2(t)\, \widehat{G}(l)$ our 'slow evolution' will be the solution of the Cauchy problem

$$\begin{cases} \dot{\ell}_\varepsilon(t) = \phi^{-1}\big(G_\varepsilon(t, \ell_\varepsilon(t))\big) \\ \ell_\varepsilon(0) = \ell_0 . \end{cases} \tag{14}$$

Notice that the initiation times of the evolution ℓ_ε are of the form t_0/ε; hence the limit of the slow evolutions as $\varepsilon \searrow 0$ is identically equal to l_0. In order to recover some information about the evolutions ℓ_ε for small values of ε it is therefore necessary to rescale the time variable. Let us employ the change of variables $\tau = \varepsilon t$. Note that τ is not the real time variable, which is instead denoted by t. Denote by $l_\varepsilon(\tau) = \ell_\varepsilon(\tau/\varepsilon)$ the rescaled slow evolutions. By the chain rule $\dot{l}_\varepsilon(\tau) = \dot{\ell}_\varepsilon(\tau/\varepsilon)/\varepsilon$. Moreover with the change of variable $t = \tau/\varepsilon$ the energy release rate G_ε can be written as

$$G_\varepsilon(\tau/\varepsilon, l) = \alpha_\varepsilon^2(\tau/\varepsilon)\, \widehat{G}(l) = \alpha^2(\tau)\, \widehat{G}(l) = G(\tau, l) .$$

Thus, from (14) it follows that l_ε solves

$$\begin{cases} \varepsilon \dot{l}_\varepsilon(\tau) = \phi^{-1}\big(G(\tau, l_\varepsilon(\tau))\big) \\ l_\varepsilon(0) = l_0 . \end{cases} \tag{15}$$

Clearly by Theorem 3.2 there exists a solution l_ε of class C^1 in a maximal interval $(0, T_\varepsilon)$ and l_ε is unique if α is strictly increasing. In the sequel we will study the properties of the rescaled slow evolutions l_ε and of their quasi-static limit. Some facts about the evolution ℓ_ε in the real time variable will be recovered later, in Section 5.5.

Example 5.1 *Note that that* (15) *can be written equivalently as*

$$\begin{cases} \dot{l}_\varepsilon(\tau) = \phi_\varepsilon^{-1}\big(G(\tau, l_\varepsilon(\tau))\big) \\ l_\varepsilon(0) = l_0 \end{cases}$$

where $\phi_\varepsilon^{-1} = \phi^{-1}/\varepsilon$*. In this form* (15) *corresponds to a rescaled rate of dissipated energy* $\phi_\varepsilon(\dot{\ell}) = \phi(\varepsilon\dot{\ell})$*. Since* $\phi_\varepsilon \to \phi_{qs}$ (15) *provides also a vanishing viscosity approach to the quasi-static evolution (note that* ϕ_{qs} *doesn't fit in the framework of differential inclusions [3]). For instance, even if this is not a good interpolation of the experimental data, consider*

$$\phi(\dot{\ell}) = \begin{cases} (-\infty, G_c] & \text{for } \dot{\ell} = 0 \\ \dot{\ell} & \text{otherwise.} \end{cases}$$

Then

$$\phi^{-1}(G) = |G - G_c|^+ = \begin{cases} 0 & \text{if } G \le G_c \\ G - G_c & \text{otherwise,} \end{cases}$$

where $|\cdot|^+$ *stands for the positive part. Then* (15) *reads*

$$\begin{cases} \varepsilon\dot{l}_\varepsilon(\tau) = |G(\tau, l_\varepsilon(\tau)) - G_c|^+ \\ l_\varepsilon(0) = l_0\,, \end{cases}$$

as in the viscosity approach proposed in [22].

5.1 Convergence of the evolutions

5.1.1 Heuristic limit

Before studying in detail the convergence of the rescaled slow evolution ℓ_ε let us try to understand what happens in a simplified situation. Let $\tau \in (0, +\infty)$ such that $l_\varepsilon(\tau) \to l(\tau)$ and $\dot{l}_\varepsilon(\tau) \to \dot{l}(\tau)$ then

$$G(\tau, l_\varepsilon(\tau)) \to G(\tau, l(\tau)) \qquad \text{and} \qquad \phi(\varepsilon\dot{l}_\varepsilon(\tau)) \to \phi(0) = (-\infty, G_c]\,.$$

Hence the differential inclusion $\phi(\varepsilon\dot{l}_\varepsilon(\tau)) \ni G(\tau, l_\varepsilon(\tau))$, which defines the evolution, in the limit takes the form

$$(-\infty, G_c] \ni G(\tau, l(\tau)) \qquad \Leftrightarrow \qquad G(\tau, l(\tau)) \le G_c\,,$$

that is Griffith's equilibrium for the quasi-static evolution. In the sequel the reader will see that in general the above assumption about the convergence of the velocities $\dot{l}_\varepsilon$ is not fullfilled for every $\tau \in (0, +\infty)$ and a more general framework is needed. Moreover, the equilibrium inequality $G(\tau, l(\tau)) \le G_c$ gives scan information about the limit evolution l. A rigorous analysis of the convergence of the evolutions l_ε requires some more effort and is the subject of the next subsection.

5.1.2 Precise limit

Consider the function l_ε to be defined in the whole $(0, +\infty)$ setting $l_\varepsilon(\tau) = L$ for $\tau \ge T_\varepsilon$, where T_ε is the failure time for l_ε. Then $\{l_\varepsilon\}$ is a family of bounded monotone functions; thus by Helly's theorem there exists a subsequence of l_ε converging pointwise to a limit non-decreasing function l. In general such a limit depends on the subsequence and is not unique. Moreover it may happen that l is not continuous, even if the l_ε are of class C^1; this is indeed the case in many situations, e.g. if the crack is close to the boundary $\partial\Omega$ (an numerical example with this feature is presented in Section 6). In general l is non-decreasing and bounded, hence it belongs to $BV_{loc}(0,+\infty)$, and (in general) its set of jumps $S(l)$ is not empty (for the properties of function with bounded variation the reader is referred to [1]). From a mechanical point of view it may seem absurd that an evolution is discontinuous in time, in our context it means that a crack increment appears suddenly. As we will see, the discontinuities in the quasi-static evolution represent (in the quasi-static time scale) a fast regime of propagation (in the real time frame) governed by dynamics, i.e. by rate-dependent effects. A rigorous explanation is given in the next sections, more details can be found in [16] and [15]. In other terms, discontinuities represent what is called in the mechanical vocabulary an unstable or catastrophic propagation (the reader interested in fracture can find a clear exposition in [10] and a couple of interesting examples in [5] while a general mathematical results is provided in [23]).

For later convenience we introduce the left and right-continuous representatives of l, defined respectively by

$$l^-(t) = \lim_{s\to t^-} l(s)\,, \qquad l^+(t) = \lim_{s\to t^+} l(s)\,.$$

Some usefull properties of $l^\pm$ are listed in the next Lemma; its elementary proof, based just on the monotonicity of l, is omitted.

Lemma 5.2 *Let $l^\pm$ be the right and left-continuous representatives of l. Then*

1. *$l^\pm$ are non-decreasing,*
2. *l^- is left-continuous and l^+ is right-continuous,*
3. *$l^- \le \ell \le l^+$ and $l^\pm = l$ a.e. in $(0,\infty)$,*
4. *$S(l^-) = S(l^+) = S(l)$,*
5. *$l^+(t) = \lim_{s\to t^+} l^-(s)$ and $l^-(t) = \lim_{s\to t^-} l^+(s)$.*

Definition 5.3 *Let l be a limit of the rescaled slow evolutions l_ε, $l^\pm$ its right and left-continuous representatives. A quasi-static evolution l_{qs} is any function such that for every $\tau \ge 0$ either $l_{qs}(\tau) = l^+_{qs}(\tau)$ or $l_{qs}(\tau) = l^-_{qs}(\tau)$.*

Note that, by Lemma 5.2 all the quasi-static evolutions l_{qs} coincide a.e. in $(0,+\infty)$. In particular $l^\pm$ are quasi-static evolutions, according to Definition 5.3. Let us first supply some properties of the functions l_{qs}.

Lemma 5.4 *Let l_{qs} be defined as above, then*

1. *l_{qs} belongs to $BV_{loc}(0,+\infty)$*
2. *l_{qs} is monotonically non-decreasing,*
3. *$S(l^-) = S(l^+) = S(l_{qs})$,*
4. *$l^- \le l_{qs} \le l^+$ and $l_{qs} = l^\pm$ everywhere in $(0,\infty)$,*

Now, let us see the properties of l_{qs} which make it a good notation of quasi-static evolution. By definition l_{qs} is defined in $(0,+\infty)$; let t_0 and T be its initiation and failure time.

Theorem 5.5 *Let l_{qs} be a quasi-static evolution in the sense of Definition 5.3. Then*

$$G(\tau, l_{qs}(\tau)) \le G_c \text{ in } [0,T)\,, \tag{16}$$

$$G(\tau, l_{qs}(\tau)) \text{ is continuous in } [0,T)\,, \tag{17}$$

$$\big(G(\tau, l_G^-(\tau)) - G_c\big)\, dl_{qs}(\tau) = 0 \text{ in the sense of measures in } [0,T). \tag{18}$$

For $\tau \in S(l_{qs})$

$$G(\tau, l_{qs}^-(\tau)) = G(\tau, l_{qs}^+(\tau)) = G_c\,, \tag{19}$$

$$\big(G(\tau, l) - G_c\big) \ge 0 \qquad \text{for every } l \in [l_{qs}^-(\tau), l_{qs}^+(\tau)]. \tag{20}$$

Proof. Clearly, by pointwise convergence $l_{qs}(0) = l(0) = l_0$.

Let us prove (16). By (15) we can write

$$\varepsilon l_\varepsilon(\tau) = \varepsilon l_0 + \int_0^\tau \phi^{-1}\big(G(s, l_\varepsilon(s))\big)\, ds\,.$$

Passing to the limit as $\varepsilon \searrow 0$ yields

$$0 = \int_0^\tau \phi^{-1}\big(G(s, l(s))\big)\, ds\,,$$

thus $\phi^{-1}(G(s,l(s))) = 0$ for a.e. $s \in (0,\tau)$ and thus $G(s,l(s)) \le G_c$ for a.e. $s \in [0,T)$. By the left-continuity of l^- and the continuity of G we get $G(s, l^-(s)) \le G_c$ for every $s \in (0,T)$. Similarly $G(s, l^+(s)) \le G_c$, hence $G(s, l_{qs}(s)) \le G_c$ for every $s \in (0,T)$.

Let us prove (19). Let $\tau \in S(l_{qs})$ and assume by contradiction that $G(\tau, l_G^+(\tau)) < G_c$ (the other case is similar). Then by the continuity of G there exists a neighborhood $\mathcal{U}$ of $(\tau, l_G^+(\tau))$ (in the (s,l) space) where $G(s,l) < G_c$. By pointwise convergence, $(s, l_\varepsilon(s))$ is then contained in $\mathcal{U}$ for $\varepsilon \ll 1$ and $s \sim \tau$. Thus $G(s, l_\varepsilon(s)) < G_c$ and $\dot{l}_\varepsilon = 0$. It follows that the functions l_ε are constant and thus l_{qs} is constant in $\mathcal{U}$; this contradicts $\tau \in S(l_{qs})$.

Let us prove (17). Consider, by monotonicity of l_{qs} and continuity of G, the limits

$$\lim_{s\to\tau^+} G(s, l_{qs}(s)) = G(\tau, l_{qs}^+(\tau))\,, \qquad \lim_{s\to\tau^-} G(s, l_{qs}(s)) = G(\tau, l_{qs}^-(\tau))\,.$$

If $\tau \notin S(l_{qs})$ then $l_{qs}^- = l_{qs}^+ = l_{qs}$, hence $G(s, l_{qs}(s))$ is continuous in τ. If $\tau \in S(l_{qs})$ then $l_{qs}^-(\tau) \ne l_{qs}^+(\tau)$ and by (19) $G(\tau, l_{qs}^\pm(\tau)) = G_c$. By Lemma 5.4, $l_{qs} = l^\pm$ everywhere in $(0,\infty)$, hence $G(\tau, l_{qs}(\tau)) = G_c$, that proves the continuity in τ.

Let us prove (18). It is sufficient to consider the case $G(\tau, l_G^-(\tau)) < G_c$. By the continuity of G there exists a neighborhood $\mathcal{U}$ of $(\tau, l_{qs}^-(\tau))$ where $G(s,l) < G_c$. Hence for $\varepsilon \ll 1$ and $s \sim \tau$ the evolutions l_ε are constant in $\mathcal{U}$ it follows that l_{qs} is constant, hence $dl_{qs}(\tau) = 0$.

It remains to prove (20). Let $\tau_* \in S(l_{qs})$ and assume by contra that $G(\tau_*, l_*) < G_c$ for some $l_* \in (l_{qs}^-, l_{qs}^+)$ (clearly the case $l_* = l_{qs}^\pm$ is ruled out by (19)). Again by the continuity of G, there exists a neighborhood $\mathcal{U} = (\tau_* - \delta, \tau_* + \delta) \times (l_* - \eta, l_* + \eta)$, with $l_{qs}^- < l_* - \eta < l_* + \eta < l_{qs}^+$, where $G(s,l) < G_c$. Then for (ε small enough) $l_\varepsilon(s') < l_* - \eta$ for $\tau_* - \delta < s' < \tau_*$ while $l_\varepsilon(s'') > l_* + \eta$ for some $\tau_* < s'' < \tau_* + \delta$. This is impossible since l_ε, being continuous, should cross the neighborhood $\mathcal{U}$ where $G < G_c$ and hence $\dot{l}_\varepsilon = 0$ by (15). ■

Remark 5.6 *As* (18) *holds true in the sense of measures and since dl_{qs} is the sum of the mutually singular measures dl_{qs}^a, dl_{qs}^c, dl_{qs}^j it follows that* (18) *holds true separately for each of these measures. Hence*

$$\big(G(\tau, l_G^-(\tau)) - G_c\big)\, dl_{qs}^j(\tau) = 0\,, \qquad \text{in the sense of measures in } [0,T),$$

and

$$\big(G(\tau, l_{qs}(\tau)) - G_c\big)\, dl^{ac}_{qs}(\tau) = 0\,, \qquad \text{in the sense of measures in } [0,T),$$

since by continuity $l_{qs}(\tau) = l^-_{qs}(\tau)$. *In terms of the sets of stable, critical and unstable points defined in Section 3.1, in this case* $(\tau, l_{qs}(\tau)) \in \mathcal{S} \cup \mathcal{C}$ *for every time* τ*; moreover from* (20) *it follows that in the jump points for every* $l \in (l^-_{qs}(s), l^+_{qs}(s))$ *we have* $(s,l) \in \mathcal{C} \cup \mathcal{U}$.

The evolution defined here coincides with the one defined in [16], [22], [13]; for more detail we refer the interested reader to [15]. Moreover, invoking Corollary 7.3 in [15], we deduce easily the following uniqueness result.

Corollary 5.7 *If* α *is a strictly increasing control then* l_{qs} *is a.e. unique. In particular it does not depend on the subsequence extracted by Helly's theorem.*

Remark 5.8 *Let us make some remarks about the regularity of the evolutions. When* ϕ *is increasing the evolution* ℓ *is of class* C^1 *(Theorem 3.2) and it is defined for every time* $t \in (0,T)$ *by the ordinary differential equation* (5). *When* ϕ *is non-decreasing* ℓ *is of class* C^0 *(Theorem 4.3) and it is defined for a.e. time by the differential inclusion* (12). *Finally, when* ϕ *is constant (i.e. in the quasi-static case)* l_{qs} *is only of class* BV *and the evolution is defined by a set of Kuhn-Tucker conditions in the sense of measures (Theorem 5.5). In synthesis, the regularity of the evolution depends strongly on the rate of dissipated energy* ϕ *and even the 'equation of motion' changes accordingly.*

5.2 Stable and unstable points

It is very interesting to study the properties of the quasi-static evolution in terms of critical, stable and unstable points. Indeed by (20) it easy to see that

$$l^-_G(\tau) \in \mathcal{S}(\tau) \;\Rightarrow\; \tau \notin S(l_G)\,. \tag{21}$$

In other terms, stable points are continuity points. In the same way, if $\tau \in S(l_G)$ then $l^-_G(\tau) \in \mathcal{U}(\tau)$. Hence discontinuity points are unstable points. As in the previous cases the characterization is not complete since it depends on the control α whether all the unstable points are discontinuity points.

5.3 Energy balance

Let us write the energy balance for the quasi-static evolution l_{qs}. By the chain rule in BV it follows that the total derivative of $\mathcal{E}(\tau, l_{qs}(\tau))$ is given by

$$d\mathcal{E}(\tau, l_{qs}(t)) = \mathcal{P}^{ext}(\tau, l_{qs}(\tau))\, d\tau - G(\tau, l_{qs}(\tau))\, dl^{ac}_{qs}(\tau) + \sum_{\tau \in S(l_{qs})} [\![\mathcal{E}(\tau, l_{qs}(\tau))]\!]\, \delta_\tau\,.$$

By Remark 5.6 we have

$$-G(\tau, l_{qs}(\tau))\, dl^{ac}_{qs}(\tau) = -G_c\, dl^{ac}_{qs}(\tau)\,.$$

Moreover the jump of $\mathcal{E}$ in the discontinuity points of l_{qs} is

$$
\begin{aligned}
[\![\mathcal{E}(\tau, l_{qs}(\tau))]\!] &= -\int_{l^-_{qs}(\tau)}^{l^+_{qs}(\tau)} G(\tau,z)\,dz \\
&= -G_c\big(l^+_{qs}(\tau) - l^-_{qs}(\tau)\big) - \int_{l^-_{qs}(\tau)}^{l^+_{qs}(\tau)} (G(\tau,z) - G_c)\,dz \\
&= -G_c dl^j_{qs}(\tau) - \int_{l^-_{qs}(\tau)}^{l^+_{qs}(\tau)} (G(\tau,z) - G_c)\,dz\,.
\end{aligned}
$$

With the notation

$$
\mathcal{D}^{qs}(dl_{qs}) = G_c dl_{qs}\,, \qquad \mathcal{D}^j(\tau, l_{qs}(\tau)) = \int_{l^-_{qs}(\tau)}^{l^+_{qs}(\tau)} (G(\tau,z) - G_c)\,dz
$$

the total derivative becomes

$$
d\mathcal{E}(\tau, l_{qs}(\tau)) = \mathcal{P}^{ext}(\tau, l_{qs}(\tau))\,d\tau - \mathcal{D}^{qs}(dl_{qs}) - \sum_{\tau\in S(l_{qs})} \mathcal{D}^j(\tau, l_{qs}(\tau))\,\delta_\tau\,. \tag{22}
$$

Note that both $\mathcal{D}^{qs}$ and $\mathcal{D}^j$ are non-negative measures, the first by the monotonicity of l_{qs} and the second by (18). In a time interval (τ_1, τ_2), with $\tau_i \notin S(l_{qs})$, the energy balance reads

$$
\begin{aligned}
\mathcal{E}(\tau_2, l_{qs}(\tau_2)) &= \mathcal{E}(\tau_1, l_{qs}(\tau_1)) + \int_{\tau_1}^{\tau_2} \mathcal{P}^e(\tau, l_{qs}(\tau))d\tau \\
&\quad - G_c\big(l_{qs}(\tau_2) - l_{qs}(\tau_1)\big) - \sum_{\tau\in(\tau_1,\tau_2)\cap S(l_{qs})} \mathcal{D}^j(\tau, l_{qs}(\tau))\,.
\end{aligned} \tag{23}
$$

It is interesting to remark that considering only the energy terms of Griffith's theory the energy balance would be given by the inequality

$$
d\mathcal{E}(\tau, l_{qs}(\tau)) \le \mathcal{P}^{ext}(\tau, l_{qs}(\tau))\,d\tau - \mathcal{D}^{qs}(dl_{qs})\,,
$$

since $\mathcal{D}^j$ is positive. In some sense, when the evolution l_{qs} is continuous the amount energy supplied by the external forces is partly stored in the bulk and partly dissipated in the crack. On the contrary, when the evolution has a discontinuity an extra source of dissipation appears, as if converting elastic energy into fracture was energy consuming. It is therefore interesting to investigate the physical origin of $\mathcal{D}^j$. An answer is given in the next section in terms of dynamic dissipation.

5.4 Convergence of the energy balance

Let us write the energy balance for the rescaled evolutions l_ε. By the chain rule and by (14) we get

$$
\begin{aligned}
d\mathcal{E}(\tau, l_\varepsilon(\tau)) &= \mathcal{P}^{ext}(\tau, l_\varepsilon(\tau))\,d\tau - G(\tau, l_\varepsilon(\tau))\,\dot{l}_\varepsilon(\tau)\,d\tau \\
&= \mathcal{P}^{ext}(\tau, l_\varepsilon(\tau))\,d\tau - \phi(\varepsilon\dot{l}_\varepsilon(\tau))\,\dot{l}_\varepsilon(\tau)\,d\tau \\
&= \mathcal{P}^{ext}(\tau, l_\varepsilon(\tau))\,d\tau - \mathcal{D}_\varepsilon(\dot{l}_\varepsilon(\tau))\,d\tau\,,
\end{aligned}
$$

where $\mathcal{D}_\varepsilon(\dot{l}) = \mathcal{D}(\varepsilon\dot{l})/\varepsilon$. Hence, for every interval (τ_1, τ_2) we have

$$\mathcal{E}(\tau_2, l_\varepsilon(\tau_2)) = \mathcal{E}(\tau_1, l_\varepsilon(\tau_1)) + \int_{\tau_1}^{\tau_2} \mathcal{P}^{ext}(\tau, l_\varepsilon(\tau))\, d\tau - \int_{\tau_1}^{\tau_2} \mathcal{D}_\varepsilon(\dot{l}_\varepsilon(\tau))\, d\tau\,.$$

For simplicity, assume that $\tau_i \notin S(l_{qs})$. Then by pointwise convergence

$$\mathcal{E}(\tau_i, l_\varepsilon(\tau_i)) \to \mathcal{E}(\tau_i, l_{qs}(\tau_i))\,. \tag{24}$$

Remembering that $u_l(\tau, \cdot) = \alpha(\tau)\hat{u}_l(\cdot)$ and using the divergence theorem the power of external forces can be written (for a generic l) as

$$\mathcal{P}^{ext}(\tau, l) = \int_{\partial_D\Omega} \dot{g}(\tau, x) \cdot \boldsymbol{\sigma}_l(\tau, x)\hat{n}\, dx = \alpha(\tau)\dot{\alpha}(\tau)\, \widehat{\mathcal{E}}(\hat{u}_l)\,.$$

Therefore, by Theorem A.1,

$$\mathcal{P}^{ext}(\tau, l_\varepsilon(\tau)) \to \mathcal{P}^{ext}(\tau, l_{qs}(\tau)) \qquad \text{for a.e. } \tau \in (\tau_1, \tau_2).$$

By dominated convergence it follows that

$$\int_{\tau_1}^{\tau_2} \mathcal{P}^{ext}(\tau, l_\varepsilon(\tau))\, d\tau \to \int_{\tau_1}^{\tau_2} \mathcal{P}^{ext}(\tau, l_{qs}(\tau))\, d\tau\,. \tag{25}$$

Let us consider the dissipation. Note that $\mathcal{D}_\varepsilon \to \mathcal{D}_{qs}$ pointwise. However in general

$$\int_{\tau_1}^{\tau_2} \mathcal{D}_\varepsilon(\dot{l}_\varepsilon(\tau))\, d\tau \not\to \int_{\tau_1}^{\tau_2} \mathcal{D}_{qs}(dl_{qs}(\tau)) = G_c\big(l_{qs}(\tau_2) - l_{qs}(\tau_1)\big)\,.$$

Let us be more precise, writing

$$\mathcal{D}_\varepsilon(\dot{l}_\varepsilon) = \mathcal{D}^{qs}(\varepsilon\dot{l}_\varepsilon)/\varepsilon + \mathcal{D}^{dyn}(\varepsilon\dot{l}_\varepsilon)/\varepsilon\,.$$

By 1-homogeneity $\mathcal{D}^{qs}(\varepsilon\dot{l}_\varepsilon)/\varepsilon = G_c\dot{l}_\varepsilon$. Hence

$$\int_{\tau_1}^{\tau_2} \mathcal{D}^{qs}(\varepsilon\dot{l}(\tau))/\varepsilon\, d\tau = G_c\big(l_\varepsilon(\tau_2) - l_\varepsilon(\tau_1)\big) \to G_c\big(l_{qs}(\tau_2) - l_{qs}(\tau_1)\big)\,. \tag{26}$$

It remains to consider the dynamic term of the dissipation which (in general) does not vanish in the quasi-static limit. Indeed, invoking (23) together with (24)-(26) we get

$$\int_{\tau_1}^{\tau_2} \mathcal{D}^{dyn}(\varepsilon\dot{l}(\tau))/\varepsilon\, d\tau \;\to \sum_{\tau\in(\tau_1,\tau_2)\cap S(l_{qs})} \mathcal{D}^j(\tau, l_{qs}(\tau))\,. \tag{27}$$

Thus, the energy $\mathcal{D}^j$ dissipated in the jump accounts (in the quasi-static time frame) for the amount of energy dissipated by $\mathcal{D}^{dyn}$ (in the real time frame).

Remark 5.9 *In a more mathematical fashion, the above convergence result for the energies can be restated in terms of measures, saying that $\mathcal{D}^{dyn}(\varepsilon\dot{l}(\tau))/\varepsilon\, d\tau$ converges weakly* to $\sum \mathcal{D}^j(\tau, l_{qs}(\tau))\, \delta_\tau$. Alternatively, this convergence can be seen also in the sense of graphs [7] or by means of Cartesian currents [15]. A similar result has been obtained also in [17] in a different context.*

5.5 Back to the real time variable

As observed at the beginning of Section 5 in the real time frame it is not usefull to consider the pointwise limit of the slow evolutions ℓ_ε since $\ell_\varepsilon(t) \to l_0$ for every $t \in [0, +\infty)$. A better way is to consider rescaled times. Given τ let $t_\varepsilon = \tau/\varepsilon$. Then $\ell_\varepsilon(t_\varepsilon) = \ell_\varepsilon(\tau/\varepsilon) = l_\varepsilon(\tau) \to l_{qs}(\tau)$ for a.e. τ. Let us try to draw some conclusions about velocities in the rescaled times $t_\varepsilon = \tau/\varepsilon$. By (14)

$$\dot{\ell}_\varepsilon(t_\varepsilon) = \varepsilon \dot{l}_\varepsilon(\tau) = \phi^{-1}\big(G(\tau, l_\varepsilon(\tau))\big)\,.$$

At this point, it is better to distinguish between continuity and discontinuity points.

Let $\tau_c \notin S(l_{qs})$ then $l_\varepsilon(\tau_c) \to l_{qs}(\tau_c)$ and $G(\tau_c, l_{qs}(\tau_c)) \le G_c$ by (16). It follows that the evolution ℓ_ε become slower and slower (as $\varepsilon \searrow 0$) since

$$\dot{\ell}_\varepsilon(t_\varepsilon) \to \phi^{-1}\big(G(\tau_c, l_{qs}(\tau_c))\big) = 0\,.$$

On the contrary, if $\tau_s \in S(l_{qs})$ the evolution ℓ_ε behaves (for ε small) as if the system was autonomous. Let us try to explain this statement. Let $\tau_\varepsilon^- = \sup\{\tau : l_\varepsilon(\tau) = l_{qs}^-(\tau_s)\}$ and $\tau_\varepsilon^+ = \inf\{\tau : l_\varepsilon(\tau) = l_{qs}^+(\tau_s)\}$. Clearly $\tau_\varepsilon^\pm \to \tau$. Let $t_\varepsilon^\pm = \tau_\varepsilon^\pm/\varepsilon$. Then by (14)

$$\ell_\varepsilon(t_\varepsilon^+) = \ell_\varepsilon(t_\varepsilon^-) + \int_{t_\varepsilon^-}^{t_\varepsilon^+} \phi^{-1}\big(\alpha(\varepsilon t)\, \widehat{G}(\ell_\varepsilon(t))\big)\, dt\,.$$

Using the change of variable $s = t - t_\varepsilon^-$ let $\varepsilon t = \varepsilon(t_\varepsilon^- + s) = \tau_\varepsilon^- + \varepsilon s$ and let $\delta_\varepsilon = t_\varepsilon^+ - t_\varepsilon^- = (\tau_\varepsilon^+ - \tau_\varepsilon^-)/\varepsilon$. Introducing the notation $\lambda_\varepsilon(s) = \ell_\varepsilon(t_\varepsilon^- + s)$ the previous equation becomes

$$\lambda_\varepsilon(\delta_\varepsilon) = \lambda_\varepsilon(0) + \int_0^{\delta_\varepsilon} \phi^{-1}\big(\alpha(\tau_\varepsilon^- + \varepsilon s)\, \widehat{G}(\lambda_\varepsilon(s))\big)\, ds\,.$$

Clearly $\lambda_\varepsilon(\delta_\varepsilon) = \ell_\varepsilon(t_\varepsilon^+) = l_{qs}^+(\tau_s)$ and $\lambda_\varepsilon(0) = \ell_\varepsilon(t_\varepsilon^-) = l_{qs}^-(\tau_s)$ (by definition of $t_\varepsilon^\pm$). Up to subsequences $\delta_\varepsilon \to \delta \in (0, +\infty]$ (note that in general it may happen that $\delta = +\infty$, corresponding to an infinite transition time). By Helly's theorem $\lambda_\varepsilon \to \lambda_{qs}$ (up to subsequences); the evolution λ_{qs} represents the evolution l_{qs} in the real time frame. Then by dominated convergence we get

$$l_{qs}^+(\tau_s) = l_{qs}^-(\tau_s) + \int_0^{\delta} \phi^{-1}\big(\alpha(\tau)\, \widehat{G}(\lambda_{qs}(s))\big)\, ds\,,$$

namely the solution of the system

$$\begin{cases} \dot{\lambda}_{qs}(s) = \phi^{-1}\big(G(\tau, \lambda_G(s))\big) \text{ for } s \in [0, \delta) \\ \lambda_{qs}(0) = l_{qs}^-(\tau_s)\,. \end{cases}$$

Note that $\lambda_{qs}(\delta) = l_{qs}^+(\tau_s)$. In conclusion, the discontinuity points of the quasi-static evolution l_{qs} correspond to an evolution λ_{qs} that solves an autonomous equation of motion in the real time frame. Loosely speaking, the functions λ_ε (or equivalently ℓ_ε) are much faster than α and in the limit for $\varepsilon \searrow 0$ they converge to λ_{qs} that evolves with α constant. Equivalently, the fracture propagates much faster than the boundary condition and in the limit it moves without any change in the boundary condition.

5.6 Variational approach

In the light of Section 3.3 it is reasonable to expect that the variational approach, applied in the quasi-static framework, gives an equivalent construction of the evolution l_{qs}. As a matter of fact, in general this is not true. Let us be more precise. Let ψ^{qs} be the quasi-static dissipation potential

$$\psi^{qs}(\dot{l}) = \begin{cases} +\infty & \text{if } \dot{l} < 0 \\ G_c \dot{l} & \text{otherwise.} \end{cases}$$

Note that ψ^{qs} is again convex but not strictly convex. Consider a positive, infinitesimal sequence of time increment $\Delta\tau^{(k)}$. Then, let $l_0^k = l_0$ and define by induction

$$l_{n+1}^k \in \operatorname{argmin} \left\{ \Delta\tau^k \psi^{qs}\big((l - l_n^k)/\Delta t^k\big) + \mathcal{E}(\tau_{n+1}^k, l) : l \in [l_n^k, L] \right\} .$$

By the above definition this is equivalent to

$$l_{n+1}^k \in \operatorname{argmin} \left\{ G_c(l - l_n^k) + \mathcal{E}(\tau_{n+1}^k, l) : l \in [l_n^k, L] \right\} . \tag{28}$$

Denote by l^k the piecewise affine (or piecewise constant) interpolation of the values l_n^k. Being l^k non-decreasing, Helly's Theorem yields, up to a subsequence, a non-decreasing pointwise limit, denoted by l^{var}. This is the approach suggested in [9]. As already shown in [16] in general $l^{var} \neq l_{qs}$. This is essentially due to the fact that jumps in l^{var} are related to multiple wells of the incremental energy while jumps in l_{qs} are related to unstable branches of the evolution. The interested reader is referred to [4] and to the references therein for the literature on the variational approach and to [15] for a detailed comparison of the evolutions.

6 A couple of explicit numerical examples

6.1 Single Edge Notch Tension

Assume that the virgin domain is a rectangle $\Omega = (-1, 1) \times (0, 3)$ of glass with Lamè coefficient $\mu = 0.5$ and $\lambda = 0.5$. A Dirichlet boundary condition of the form $u(t, x) = \alpha(t)\hat{g}(x)$ is imposed on the top and on the bottom face, i.e. on $\partial_D\Omega = \{-1, 1\} \times [0, 3]$. Let $\hat{g}(x_1, 1) = (0, 0.1)$ (on the upper face) and $\hat{g}(x_1, -1) = (0, -0.1)$ (on the lower face). The remaining part of the boundary $\partial_N\Omega$ is traction free. A crack is expected to run horizontally along the x-axis from left to right.

First of all, the elastic energy $\widehat{\mathcal{E}}$ and the energy release rate $\widehat{G}$ have been computed numerically by a finite element code on a fine structured triangulation. The results are shown in Figure 7. As expected the elastic energy decays almost linearly if the crack is far from the boundary; accordingly the release rate has a significant variation if the crack is either short ($l \approx 0$) or very long ($l \approx L$) and it is roughly constant in the middle.

Note that the values of $\widehat{\mathcal{E}}$ and $\widehat{G}$ are defined only on the discrete set of lenghts, corresponding to the finite element discretization of the domain. Then, the values of the energy release rate $\widehat{G}$ have been interpolated by continuous piecewise polynomial functions $\widehat{G}_i$ (see Figure 8) with a relative error approximately of 0.02, apart from a small left neighborhood of L.

Fracture toughness is set to $G_c = 0.06$ and the dissipation function ... is given by

$$\phi(\dot{\ell}) = \begin{cases} (-\infty, G_c] & \dot{\ell} = 0 \\ G_c + \dot{\ell}^2 & \dot{\ell} > 0 \,. \end{cases}$$

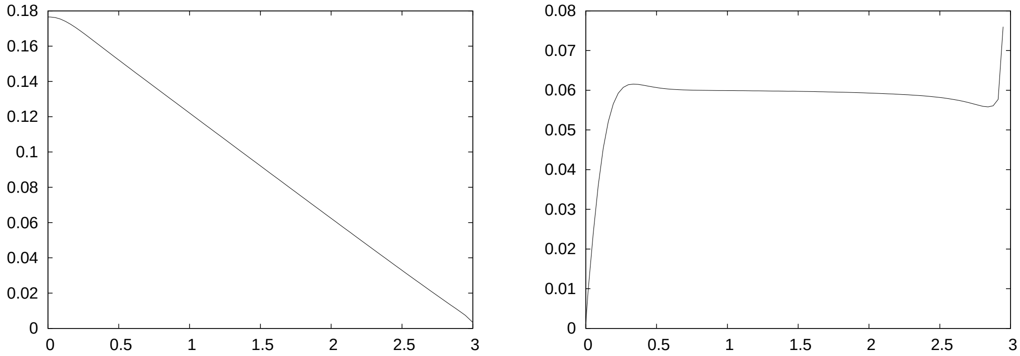

Figure 7: Numerical values of elastic energy $\widehat{\mathcal{E}}$ (left) and energy release rate $\widehat{G}$ (right) versus crack length l.

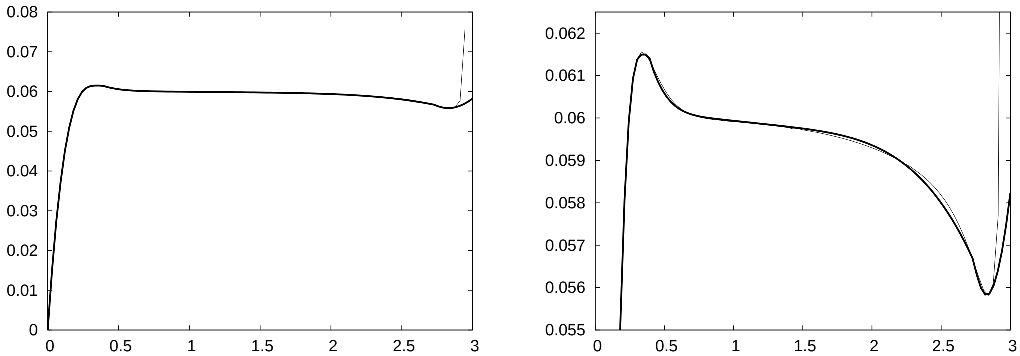

Figure 8: Comparison between $\widehat{G}$ and $\widehat{G}_i$ (bold) in the range $[0, 0.8]$. A zoom for values in the range $[0.055, 0.0625]$ (right).

We consider an initial side crack of length 0.3 and the control $\alpha(t) = \sqrt{t}$. The evolution is described by the ODE

$$\begin{cases} \dot{\ell}(t) = \phi^{-1}\big(t\, \widehat{G}_i(\ell(t))\big) \\ \ell(0) = l_0 \,. \end{cases} \tag{29}$$

The rescaled slow evolution are the solutions of

$$\begin{cases} \varepsilon \dot{l}_\varepsilon(\tau) = \phi^{-1}\big(\tau\, \widehat{G}_i(l_\varepsilon(\tau))\big) \\ l_\varepsilon(0) = l_0 \,. \end{cases} \tag{30}$$

The numerical solutions have been obtained using the solver `ode15s` of MATLAB for stiff differential equations, since the right hand side in (30) is not Lipschitz continuous with respect to l. Figure 9 shows l_ε for $\varepsilon = 1.0, 0.6, 0.2$. It is intuitively clear that l_ε converge to l_G when $\varepsilon \searrow 0$. From Figure 9 (left) it may seem that l_{qs} has a discontinuity in the initiation time t_0 and that $l_{qs}^+(t_0) = L$. Actually, a closer look (right) in the time interval $[0.96, 1.08]$ shows that a l_{qs} has a small jump in t_0, it follows the equilibrium curve up to $\tau \approx 1.075$, then it jumps again disconnecting the domain.

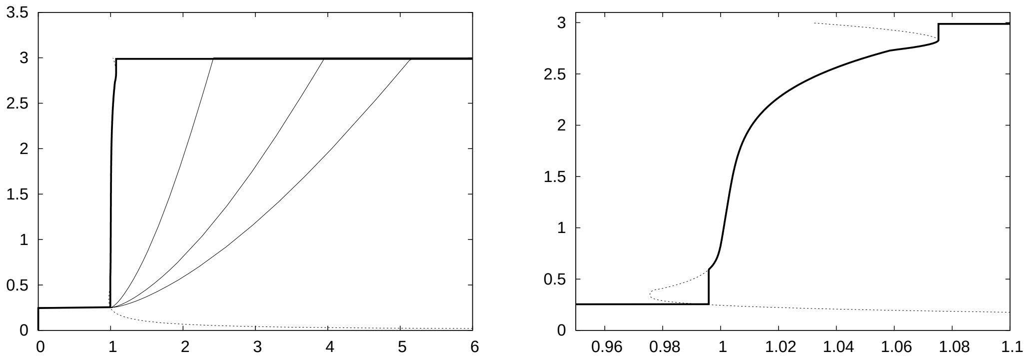

Figure 9: Plot of the rescaled slow evolutions l_ε for $\varepsilon = 1.0, 0.6, 0.2$ (from right to left). As ε becames smaller the plots move to the left (see the arrow). In the limit they converge to the quasi-static evolution l_{qs} (bold). A detail of l_{qs} in the time frame $[0.96, 1.08]$.

6.2 Compact Tension

In this section we present a numerical example obtained in the ASTM-CT geomerty (see Figure 2). As expected, freezing the boundary condition just after the intiation time t_0 the crack arrests in finite time both in the rate dependent and in the quasi-static case.

In detail, the uncracked domain is the square $\Omega = (0, 2) \times (0, 2)$ of glass with the holes $B^\pm$, where B^+ is the ball with centre $(0.5, 1.5)$ and radius 0.25 and B^- the symmetric one. A uniform Dirichlet boundary condition of the form $u(t, x) = \alpha(t)\hat{g}$ is imposed in the holes. Specifically $\hat{g} = (0, \pm 0.1)$ in $B^\pm$ and $\alpha(t) = \min\{t, 1.2\}$. Thank to the symmetry the initial side crack of length 0.25 propagates along the middle line from left to tight.

As in the previous example, the energy release rate has been first computed numerically and then interpolated by a continuous piecewise polynomial function. In the ASTM-CT geometry the deformation is in some localized (since the boundary condition is imposed on the holes), thus the energy release rate has a peak around 0.5, corresponding to the center of the holes.

The rescaled slow propagations, solutions of (30), have been computed again with the solver `ode15s` of MATLAB. The results, for $\varepsilon = 1.0, 0.6, 0.2$ are plotted in Figure 11. It is evident that l_ε stops at different finite times when the crack length is approximately 0.76. The solutions l_ε are of class C^1 and converge to the discontinuous quasi-static propagation l_{qs}. In particular l_{qs} has a jump at the intiation time t_0, it runs along the equilibrium curve and finally, when the control is frozen, it arrests.

A The energy release rate

This appendix deals with the regularity of the energy release rate $\widehat{G}$ as a function l. The proof of the main result (Theorem A.1) is quite simple and follows closely the path of reasoning employed in [22]; it is reported in full length for sake of self consistency and since a standard reference is seemingly missing in the literature.

Theorem A.1 *The elastic energy $\widehat{\mathcal{E}}$ is of class C^1 in $(0, L)$ (hence $\widehat{G}$ is of class C^0).*

Proof. For convenience the regularity is proved in (l', L) where $0 < l' < L' < L$; the complete statement follows by arbitrarity of l' and L'. Denote $K = K_{l'}$.

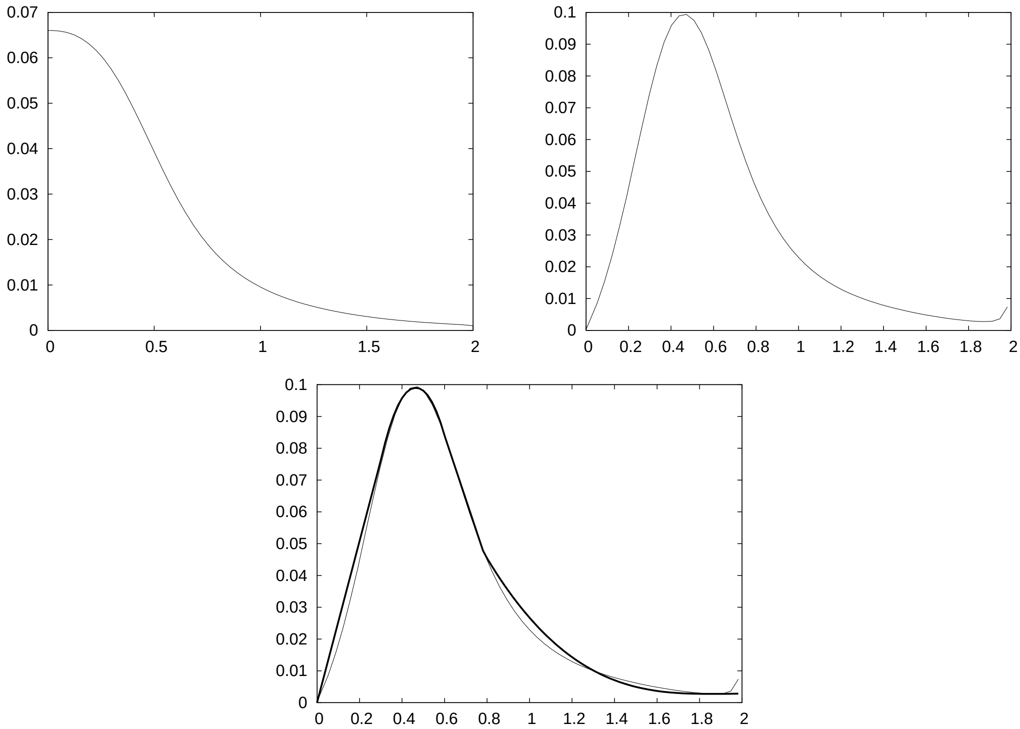

Figure 10: Plot of elastic energy $\widehat{\mathcal{E}}$ (left) and energy release rate (right) versus crack length l. Graphs of $\widehat{G}$ and $\widehat{G}_i$ (below).

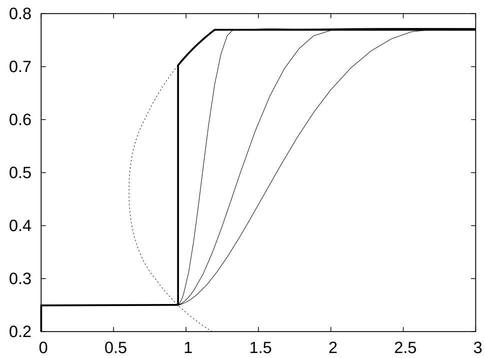

Figure 11: Plot of the rescaled slow evolutions l_ε for $\varepsilon = 1.0, 0.6, 0.2$ (from right to left) and the quasi-static limit l_{qs}.

1. Let us write the elastic energy in a more convenient way. Let $\eta \in C_0^\infty(0, L)$ such that $0 \le \eta \le 1$, $\eta = 1$ in a (sufficiently small) neighborhood of l' and such that $d\eta/dx > -1/(L'-l')$ in $(0, L)$. Let $\mu \in C_0^\infty(-1, 1)$ with $0 \le \mu \le 1$, $\mu = 1$ in a (small) neighborhood of 0. Define $\rho(x, y) = \eta(x)\mu(y)$; clearly ρ belongs to $C_0^\infty(\Omega)$ and $\rho = 1$ in a (small) neighborhood of $(0, l')$.

Given $l' < l < L'$ let us define the map $\psi_l : \Omega \setminus K \to \Omega \setminus K_l$ as

$$\psi_l(x, y) = \big(x + (l - l')\,\rho(x, y), y\big) = (x, y) + (l - l')\,\rho(x, y)\,\hat{e}_1\,.$$

By the properties of η and μ it is easy to check that ψ_l is a smooth invertible map of $\Omega \setminus K$ onto $\Omega \setminus K_l$ such that $\psi_l(x, y) = (x, y)$ in a (small) neighborhood of $\partial\Omega$ and such that $\psi_l(x, y) = (l - l')\hat{e}_1$ in a (small) neighborhood of $(0, l')$. Note also that ρ does not depend on l. Clearly,

$$D\psi_l(x, y) = I + (l - l')\, \hat{e}_1 \otimes \nabla\rho(x, y)\,.$$

For $l' < l < L'$ and $u \in H^1(\Omega \setminus K_l, \mathbf{R}^2)$ it is convenient to write the elastic energy

$$\widehat{E}(l, u) = \int_{\Omega\setminus K_l} W^e(\boldsymbol{\varepsilon}) = \int_{\Omega\setminus K_l} \boldsymbol{W}^e(Du)\,,$$

where the energy density $\boldsymbol{W}^e$ is given (in terms of the full displacement gradient Du) by

$$\boldsymbol{W}^e(Du) = Du\,\mathbf{C}\,Du^T\,,$$

where $\mathbf{C}$ is a fourth order symmetric tensor which depends only on the Lamé coefficients.

Consider $\mu \in C^\infty(\Omega)$ with $\mu = 1$ on $\partial\Omega$ and $\operatorname{supp}(\rho) \cap \operatorname{supp}(\mu) = \emptyset$. Let $\tilde{g} = \hat{g}\mu$. Given $u \in H^1(\Omega \setminus K_l, \mathbf{R}^2)$ with $u = \hat{g}$ on $\partial_D\Omega$ consider $w = u \circ \psi_l$. Obviously $w \in H^1(\Omega \setminus K, \mathbf{R}^2)$ with $w = \hat{g} = \tilde{g}$ on $\partial_D\Omega$. In terms of w the elastic energy turns out to be

$$\widehat{E}(l, u) = \int_{\Omega\setminus K} Dw\,\mathbf{C}_l\,Dw^T\,dx\,,$$

where

$$\mathbf{C}_l = D\psi_l^{-1}\,\mathbf{C} D\psi_l^{-T} |\mathrm{det} D\psi_l|$$

is a fourth order symmetric tensor smooth with respect to l. Note that by the properties of ψ_l $\mathbf{C} = \mathbf{I}$ (the identity tensor) in a neighborhood of $\partial\Omega$ and in particular in $\operatorname{supp}(\tilde{g})$.

Now, let V denote the space $\{v \in H^1(\Omega \setminus K) \mathrm{with} v = 0 \mathrm{on} \partial_D\Omega\}$ and write $w = v + \tilde{g}$ for $v \in V$. Then, for $v = u \circ \psi_l^{-1} - \tilde{g}$,

$$\begin{aligned}\widehat{E}(l, u) &= \int_{\Omega\setminus K} Dv\,\mathbf{C}_l\,Dv^T\ dx + 2\int_{\Omega\setminus K} D\tilde{g}\,\mathbf{C}_l\,Dv^T\ dx + \int_{\Omega\setminus K} D\tilde{g}\,\mathbf{C}_l\,D\tilde{g}^T\,dx \\ &= \int_{\Omega\setminus K} Dv\,\mathbf{C}_l\,Dv^T\ dx + 2\int_{\Omega\setminus K} D\tilde{g}\cdot Dv^T\ dx + \int_{\Omega\setminus K} D\tilde{g}\cdot D\tilde{g}^T\,dx\,,\end{aligned}$$

where the last two terms follows from the fact that $\mathbf{C} = \mathbf{I}$ in $\operatorname{supp}(\tilde{g})$. For convenience let us define the energy

$$\widetilde{E}(l, v) = \int_{\Omega\setminus K} Dv\,\mathbf{C}_l\,Dv^T\ dx + 2\int_{\Omega\setminus K} D\tilde{g} Dv^T\,dx \qquad \text{for } v \in V$$

and the semi-norms

$$|v|_1 = \left(\int_{\Omega\setminus K} Dv Dv^T\,dx\right)^{1/2}, \qquad |v|_{1,l} = \left(\int_{\Omega\setminus K} Dv\,\mathbf{C}_l\,Dv^T\,dx\right)^{1/2}.$$

As $0 < l' \le l \le L' < L$ it easy to check that theyare equivalent, in particular $|v_1| \le C|v|_{1,l}$ for a suitable choice of C independent of l. Then, by Korn's inequality, e.g. [6, Theorem 6.3-4],

$$\|v\|_1 \le C|v|_{1,l}\,,$$

where $\|\cdot\|_1$ denote the norm in $H^1(\Omega\setminus K, R^2)$ while C is independent of l.

Let v_l denote the unique minimizer of $\widetilde{E}(l,v)$ in V. Obviously, v_l corresponds to the minimizer u_l of $\widehat{E}(l,u)$ by the change of variable $u_l = v_l\circ\psi_l^{-1} + \tilde g\circ\psi_l^{-1}$ and is characterized by the variational formulation, i.e. by

$$\int_{\Omega\setminus K} Dv\,\mathbf{C}_l\,D\phi^T\,dx = \int_{\Omega\setminus K} D\tilde g D\phi^T\,dx \qquad \text{for every } \phi\in V. \tag{31}$$

2. Let us prove that the map $l\mapsto v_l$ form $(0,L')$ to V is continuous (with respect to $\|\cdot\|_1$). Note that $\mathbf{C}_{l+h}\to\mathbf{C}_l$ uniformly, in particular for $\phi\in V$ $\mathbf{C}_{l+h}D\phi\to\mathbf{C}_lD\phi$ strongly in $L^2(\Omega\setminus K,\mathbf{R}^{2\times2})$. For $h\ll 1$, $\|v_{l+h}\|_1$ is uniformly bounded, hence (up to subsequences) $v_{l+h}\rightharpoonup v_*$ in $H^1(\Omega\setminus K,\mathbf{R}^2)$. Passing to the limit with respect to h in the variational formulation gives $v_*=v_l$, indeed

$$\int_{\Omega\setminus K} Dv_{l+h}\,\mathbf{C}_{l+h}\,D\phi^T\,dx = \int_{\Omega\setminus K} D\tilde g D\phi^T\,dx$$

converges to

$$\int_{\Omega\setminus K} Dv_*\,\mathbf{C}_l\,D\phi^T\,dx = \int_{\Omega\setminus K} D\tilde g D\phi^T\,dx\,.$$

Let us check that $|v_{l+h}|_{l+h}\to|v_l|_l$. Using v_{l+h} as test function in (31) gives

$$|v_{l+h}|^2_{l+h} = \int_{\Omega\setminus K} D\tilde g Dv^T_{l+h} = \int_{\Omega\setminus K} Dv_l\,\mathbf{C}_l\,Dv^T_{l+h}\,.$$

Passing to the limit we get

$$\int_{\Omega\setminus K} D\tilde g Dv_l^T = \int_{\Omega\setminus K} Dv_l\,\mathbf{C}_l Dv_l^T = |v_l|_l^2\,.$$

Note that for every $v\in V$

$$|v|^2_{l+h} - |v|^2_l = \int_{\Omega\setminus K} Dv\big(\mathbf{C}_l - \mathbf{C}_{l+h}\big)Dv\,dx \le |\,\mathbf{C}_{l+h} - \mathbf{C}_l|_\infty |v|_1^2\,.$$

Hence, writing $|v_{l+h}|_l - |v_l|_l = (|v_{l+h}|_l - |v_{l+h}|_{l+h}) + (|v_{l+h}|_{l+h} - |v_l|_l)$ it follows that $|v_{l+h}|_l$ converges to $|v_l|_l$.

3. Finally, let us see that $l\mapsto v_l$ is of class C^1. First we check the (weak) differentiability, i.e. that $(v_{l+h}-v_l)/h\rightharpoonup v_l'$ in $H^1(\Omega\setminus K,\mathbf{R}^2)$. Note that

$$\int_{\Omega\setminus K} Dv_{l+h}\,\mathbf{C}_{l+h}D\phi^T\,dx = \int_{\Omega\setminus K} Dv_l\,\mathbf{C}_lD\phi^T\,dx\,.$$

Hence
$$\frac1h\int_{\Omega\setminus K}(Dv_{l+h}-Dv_l)\,\mathbf{C}_lD\phi^T\,dx = \frac1h\int_{\Omega\setminus K} Dv_{l+h}(\,\mathbf{C}_{l+h}-\mathbf{C}_l)D\phi^T\,dx$$
and then
$$\frac1h\int_{\Omega\setminus K}(Dv_{l+h}-Dv_l)\,\mathbf{C}_lD\phi^T\,dx \quad\longrightarrow\quad \int_{\Omega\setminus K} Dv_l\,\mathbf{C}_l'D\phi^T\,dx\,.$$
By Riesz representation Theorem, there exists $v_*\in H^1(\Omega\setminus K,\mathbf{R}^2)$ such that

$$\int_{\Omega\setminus K} Dv_*\,\mathbf{C}_lD\phi^T\,dx = \int_{\Omega\setminus K} Dv_*\,\mathbf{C}_l'D\phi^T\,dx\,.$$

By definition $v_* = v_l'$. In particular v_l' solves

$$\int_{\Omega\setminus K} Dv_l' \, \mathbf{C}_l D\phi^T \, dx = \int_{\Omega\setminus K} \big(Dv_l \, \mathbf{C}_l'\big) D\phi^T \, dx \, .$$

Arguing as in the previous step and using the continuous (linear) dependence with respect to $Dv_l \, \mathbf{C}_l'$ follows the continuous dependence of the derivative v_l'. ■

Remark A.2 *Note that the proof of Theorem A.1 does not require any knowledge of the singularity of the displacement field around the crack tip.*

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