NON CONVEX HOMOGENIZATION PROBLEMS FOR SINGULAR STRUCTURES

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Abstract. We prove a homogenization theorem for non-convex functionals depending on vector-valued functions, defined on Sobolev spaces with respect to oscillating measures. The proof combines the use of the localization methods of Γ-convergence with a 'discretization' argument, which allows to link the oscillating energies to functionals defined on a single Lebesgue space, and to state the hypothesis of p-connectedness of the underlying periodic measure in a handy way.

Introduction. In this paper we consider homogenization problems on singular structures, with in mind the model case of an energy defined on smooth functions over a periodic piecewise-smooth k-dimensional manifold E. Starting from such a geometry, after the usual homogenization scaling we are led to dealing with functionals of the form

$$\varepsilon^{n-k}\int_{\Omega\cap\varepsilon E} f\Big(\frac{x}{\varepsilon}, Du\Big)\, d\mathcal{H}^k(x), \tag{1}$$

where f is a Borel function, one-periodic in the first variable. Here $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure, and the factor ε^{n-k} follows from the scaling properties of $\mathcal{H}^k$. In order that the functional above be well defined one can consider it as defined only on smooth functions. Note that if we denote by μ_ε the measure $\varepsilon^{n-k}\mathcal{H}^k$ restricted to εE, then this integral can be equivalently written as

$$\int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\, d\mu_\varepsilon. \tag{2}$$

Following the choice made by several authors (see e.g. Bouchitté and Fragalà [8], Zhikov, [14, 15], Pastukhova [16, 17], etc.) the study of these types of problems can be set in a more general framework by fixing a general 1-periodic measure μ and defining

$$\mu_\varepsilon(B) = \varepsilon^n \mu\Big(\frac{1}{\varepsilon}B\Big).$$

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Moreover to each such measure, one can associate the 'relaxed' Sobolev spaces $W^{1,p}(\Omega,\mu_\varepsilon;\mathbb{R}^m)$ of functions whose 'tangential derivatives with respect to the measure μ_ε' are p-integrable. The definition of tangential derivative D_λ with respect to a measure λ coincides with the usual one if $\lambda=\mathcal{H}^k \llcorner E$ is the restriction of the k-dimensional Hausdorff measure to a smooth k-dimensional manifold as above, and is defined by relaxation for an arbitrary measure, bringing along additional joint conditions if for example $\lambda=\mathcal{H}^k \llcorner E$ is as above but E is only piecewise smooth (see Bouchitté, Buttazzo, and Seppecher [4], Zhikov [14, 15]).

The homogenization problem can be then stated as the characterization of the asymptotic behaviour of integrals of the form

$$\int_\Omega f\Big(\frac{x}{\varepsilon}, D_{\mu_\varepsilon}u\Big)\,d\mu_\varepsilon, \tag{3}$$

defined on $W^{1,p}(\Omega,\mu_\varepsilon;\mathbb{R}^m)$ if f is a 1-periodic Borel function with p-growth. Since the functionals we take into account are defined on varying spaces, a notion of convergence of functions

$$u_\varepsilon \xrightarrow{\mu_\varepsilon} u \qquad \Longleftrightarrow \qquad u_\varepsilon\mu_\varepsilon \rightharpoonup \mu(Y)u\mathcal{L}^n$$

as $\varepsilon\to 0$, must be introduced to rephrase the problem in terms of the computation of a Γ-limit. In this framework, a general result for convex integrands, for which the second formulation is derived from the first by relaxation, is obtained by Bouchitté and Fragalà [8] by means of two-scale convergence techniques, and in a series of works by Zhikov and Pastukhova [14, 15, 16, 17].

In this paper we consider the general case where u is vector valued and no convexity hypothesis is required on the function f with respect to the second variable. In this framework we cannot resorts to techniques used in previous papers such as two- scale convergence, and we deal with the problem by using the more complex localization methods of Γ-convergence (see Dal Maso [10], Braides [5, 6], Braides and Defranceschi [7]). Note that in the non convex case no general result ensuring that the relaxed energy of functionals of the form (2) is still of the form (3) is available; hence, they two homogenization processes may give different limits. Our procedure can be anyhow applied to both cases, obtaining different homogenization formulas. We perform in detail the proof for the functionals (3) only.

A difference to be remarked between the present paper and the previous literature is a new 'discrete' way to state the condition of 'p-connectedness' on (the power p and) the measure μ (introduced by Zhikov in [13]), that ensures that the Γ-limit is still a coercive local functional, in terms of properties of the averages

$$u^i=\int_{i+[0,1)^n} u(y)\,d\mu(y) \qquad i\in\mathbb{Z}^n;$$

namely, that $C,\delta\geq 0$ exist such that for $u\in W^{1,p}_{\rm loc}(\mathbb{R}^n,\mu)$

$$\begin{aligned} |u^i-u^j|^p &\leq C\int_{((i+[0,1)^n)\cup(j+[0,1)^n))+(-\delta,\delta)^n} |D_\mu u|^p d\mu \text{ if } |i-j|=1;\\ \int_{i+[0,1)^n}|u-u^i|^p d\mu &\leq C\int_{(i+[0,1)^n)+(-\delta,\delta)^n}|D_\mu u|^p d\mu. \end{aligned}$$

The two joint properties above are implied by the strong p-connectedness of μ (and indeed are slightly more general), and the first one, when scaled by ε, allows to consider in place of functions defined on varying Sobolev spaces with respect to measures, simply subspaces of piecewise-constant functions in $L^p(\Omega;\mathbb{R}^m)$ (with respect to the Lebesgue measure). A suitable compactness result ensures that the two notions of convergence are equivalent. A little technical issue must be mentioned at this point: some extra conditions must be added in order to obtain such compactness results since those may fail if the boundary of Ω disconnects the support of the measures μ_ε. We have chosen to add a (continuous) term to the functional of the form

$$\int_\Omega |u|^p d\mu_\varepsilon$$

as done by Zhikov in [14]. An alternative way could have been to add some boundary conditions as done by Bouchitté and Fragalà in [8]. This option actually brings along better compactness properties, and is briefly hinted at in the paper.

1. **Statement of the problem.** Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, let $Y=(0,1)^n$, and let $\mathcal{B}(\mathbb{R}^n)$ denote the σ-algebra of Borel sets of $\mathbb{R}^n$. By $\mathbb{M}^{m\times n}$ we denote the set of $m\times n$ matrix with real entries, and by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$, so that we may use dx or $d\mathcal{L}^n$ indifferently. If λ is a Radon measure on Ω then the *Sobolev space with respect to the measure* λ $W^{1,p}(\Omega,\lambda)$ is defined by relaxation as the domain of the lower-semicontinuous envelope of the functional

$$\int_\Omega |Du|^p\, d\mu$$

on $L^p(\Omega,\lambda)$. For such functions a *tangential gradient* $D_\lambda u\in(L^p(\Omega,\lambda))^n$ of u exists. In the paper we will consider vector-valued functions $u\in(W^{1,p}(\Omega,\lambda))^m$, but we will drop the apex m for simplicity in the notation. Note that in this case the tangential gradient is a $\mathbb{M}^{m\times n}$-valued function. For the definition and properties of the spaces $W^{1,p}(\Omega,\lambda)$, $W^{1,p}_{\rm per}(Y,\lambda)$, and of the tangential gradient $D_\mu u$, we refer to [4], [8] (with a slightly different notation). When $\mu=\mathcal{L}^n$ we shall use the standard notation for the corresponding spaces $W^{1,p}(\Omega)$, and $W^{1,p}_{\rm per}(Y)$.

We consider a positive, Y-periodic Radon measure μ on $\mathbb{R}^n$. Up to a translation of the periodicity cell Y, it is not restrictive to assume that $\mu(\partial Y)=0$. For every $\varepsilon>0$ we denote by μ_ε the measure defined by

$$\mu_\varepsilon(B)=\varepsilon^n\mu(\varepsilon^{-1}B)\qquad\text{for all } B\in\mathcal{B}(\mathbb{R}^n).$$

Note that if $Y^i_\varepsilon=\varepsilon(i+Y)$, $i\in\mathbb{Z}^n$, then

$$\mu_\varepsilon(Y^i_\varepsilon)=\varepsilon^n\mu(i+Y)=\varepsilon^n\mu(Y)\qquad\text{for all }\varepsilon>0,$$

from which it immediately follows that

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu(Y)\mathcal{L}^n$$

weakly in the sense of measures; i.e.,

$$\lim_{\varepsilon\to0}\int_\Omega\varphi(x)d\mu_\varepsilon(x)=\int_\Omega\varphi(x)dx\qquad\text{for all }\varphi\in\mathcal{C}_c(\Omega),$$

where $\mathcal{C}_c(\Omega)$ denotes the space of continuous functions with compact support in Ω.

Let $f=f(y,\xi):\mathbb{R}^n\times\mathbb{M}^{m\times n}\to\mathbb{R}$ be a Borel function such that $f(\cdot,\xi)$ is Y-periodic for every $\xi\in\mathbb{R}^n$ and

$$c_1|\xi|^p\le f(y,\xi)\le c_2(1+|\xi|^p),\qquad\text{for }\mu\text{-a.e. } y\in\mathbb{R}^n\text{ and for all }\xi\in\mathbb{M}^{m\times n},\tag{4}$$

with $0 < c_1 \le c_2$, $p > 1$. We want to study the asymptotic behaviour as $\varepsilon \to 0$ of the functionals $F_\varepsilon + G_\varepsilon$ where

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, D_{\mu_\varepsilon} u\Big) d\mu_\varepsilon, \quad G_\varepsilon(u) = \int_\Omega |u|^p d\mu_\varepsilon, \tag{5}$$

$u : \Omega \to \mathbb{R}^m$, and $u \in W^{1,p}(\Omega, \mu_\varepsilon)$.

2. **Main assumptions and preliminary results.** In this section we state some assumptions on the measure μ and their consequences.

Condition (H_1)**.** *Coerciveness.* There exist two constants $c_0 > 0$ and $\delta \ge 0$ such that, for every $i, j \in \mathbb{Z}^n$, with $|i - j| = 1$,

$$|u^i - u^j|^p \le c_0 \int_{(Y^i \cup Y^j) + (-\delta,\delta)^n} |D_\mu u|^p d\mu$$

for every $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n, \mu)$, where

$$u^i = \frac{1}{\mu(Y)} \int_{i+Y} u d\mu.$$

Condition (H_2)**.** *Poincaré-Wirtinger's inequality.* There exist two constants $c = c(n,p) > 0$, $\delta \ge 0$ such that

$$\int_Y |u - \overline{u}|^p d\mu \le c \int_{(-\delta, 1+\delta)^n} |D_\mu u|^p d\mu,$$

for every $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n, \mu)$, where

$$\overline{u} = \frac{1}{\mu(Y)} \int_Y u d\mu.$$

Note that these conditions involve only p and the measure μ, and can be stated for u scalar. They then are also valid for u vector valued by arguing component-wise.

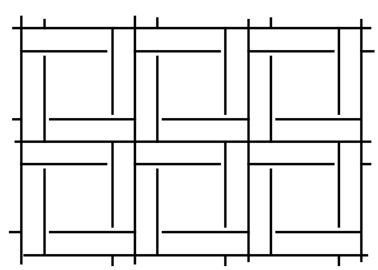

FIGURE 1. The support of a measure satisfying assumptions (H_1) and (H_2)

Remark 1. We note that if μ is *strongly p-connected* (see [8], Section 4); i.e., there exists $C > 0$ such that for all $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n, \mu)$ and for all $k \in \mathbb{N}$

$$\int_{kY} |u|^p d\mu \le Ck^p \int_{kY} |D_\mu u|^p d\mu \quad \text{whenever } \int_{kY} u d\mu = 0, \tag{6}$$

then (H_2) holds true with $\delta = 0$. Moreover it is easy to see that (H_1) holds true with $\delta = 1$.

If (H_2) is satisfied, then μ is *strongly p-connected on the torus* (see [8], Section 4); i.e., there exists $C > 0$ such that for all $u \in W^{1,p}_{\rm per}(\mathbb{R}^n, \mu)$

$$\int_Y |u|^p d\mu \le C \int_Y |D_\mu u|^p d\mu \quad \text{whenever} \quad \int_Y u d\mu = 0, \tag{7}$$

Note that the joint conditions (H_1) and (H_2) are weaker than the strong p-connectedness condition as stated above (see Example 2 below), even though it is likely that some slight modification of (6) is indeed equivalent to (H_1) and (H_2). Fig. 1 pictures an example of a two-dimensional set E for which strong p-connectedness is not satisfied for the restriction of the measure $\mathcal{H}^1$ to E but satisfying our assumptions (the necessity of introducing a δ as above is also illustrated by the examples in [1]).

Condition (H_1*)**.** We remark that (H_1) can be iterated and implies that for every $i, l \in \mathbb{Z}^n$,

$$|u^i - u^l|^p \le c_0 2^p 2n \int_{S_{il}+(-\delta,\delta)^n} |D_{\mu_\varepsilon} u|^p d\mu \tag{8}$$

for every $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n, \mu)$, where

$$u^i = \frac{1}{\mu(Y)} \int_{i+Y} u d\mu,$$

and S_{il} is a chain of neighbour cubes joining Y^i to Y^l, such that each two consecutive cubes have one face in common; i.e. $S_{il} = \bigcup_{k=0}^M (x_k + Y)$, where $\|x_k - x_{k-1}\| = 1$, $x_0 = i$ and $x_M = l$.

Example 1. We give a simple example in which we illustrate how $(H1)$ and $(H2)$ can be easily derived. We define

$$E = \{x = (x_1 \dots, x_n) : \exists i \text{ such that } x_i \in \mathbb{Z}\}, \qquad \mu = \mathcal{H}^{n-1} \llcorner (E + (1/2, \dots, 1/2))$$

(the translation is necessary in order to have $\mu(\partial(0,1)^n) = 0$)) For simplicity we prove the validity of $(H1)$ and $(H2)$ in the two-dimensional case, the proof being easily extended to the general setting. In this case, in both conditions we may take $\delta = 0$.

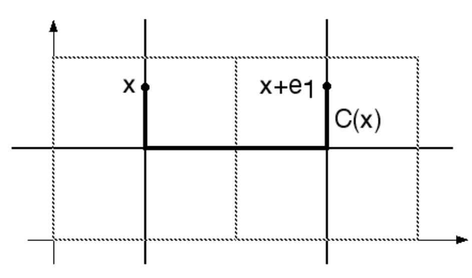

FIGURE 2. The set E in Example 1

It is enough to check $(H1)$ for $i = (0,0)$ and $j = e_1 = (1,0)$, and $Y^i = Y$, $Y^j = Y + e_1 =: Y^1$. Let $u \in W^{1,p}_{\rm loc}(\mathbb{R}^2, \mu)$. For all $x \in Y \cap E$ let $C(x)$ be the minimal path in E joining x and $x+e_1$. Note that $C(x)$ lies entirely in $Y \cup Y^1$ and the length of $C(x)$ is at most 2. We can estimate

$$\begin{aligned}
|\mu(Y)|^p|u^i-u^j|^p &= \Big|\int_Y (u(x+e_1)-u(x))d\mu\Big|^p \\
&= \Big|\int_Y \int_{C(x)} D_\mu u(\tau)d\mu(\tau)d\mu(x)\Big|^p \\
&\leq \Big|\int_Y (\mathcal{H}^1(C(x)))^{1-1/p}\Big(\int_{C(x)}\Big|D_\mu u(\tau)\Big|^p d\mu(\tau)\Big)^{1/p} d\mu(x)\Big|^p \\
&\leq 2^{p-1}\Big|\int_Y \Big(\int_{Y\cup Y^1}\Big|D_\mu u(\tau)\Big|^p d\mu(\tau)\Big)^{1/p} d\mu(x)\Big|^p \\
&= 2^{p-1}(\mu(Y))^p \int_{Y^i\cup Y^j}\Big|D_\mu u(\tau)\Big|^p d\mu(\tau),
\end{aligned}$$

which proves $(H1)$.

The proof of $(H2)$ is easily obtained as for the usual Poincaré-Wirtinger inequality, for example by arguing by contradiction. Note in fact that if we take a sequence u_j such that $\int_Y |D_\mu u_j|^p d\mu \to 0$, then we easily deduce that u_j converge (up to translations) to a constant in each segment of $Y \cap E$, and that by connectedness indeed it converges to a unique constant c on $Y \cap E$. If $(H2)$ does not hold then we may find such u_j satisfying $\int_Y u_j d\mu = 0$ and $\int_Y |u_j|^p d\mu = 1$, from which a contradiction follows.

Example 2. A simple variation of the geometrical structure in the previous example exhibits a situation where δ must be taken strictly positive. Such a structure E is pictured in Fig. 3. The proof of the validity of $(H1)$ and $(H2)$ is obtained in the same way, but choosing $C(x)$ as the minimal path between x and $x+e_1$ in $(Y\cup Y^1)+(-\delta,\delta)^2$ and $\delta>0$ large enough such that such a path exists for all $x \in Y \cap E$.

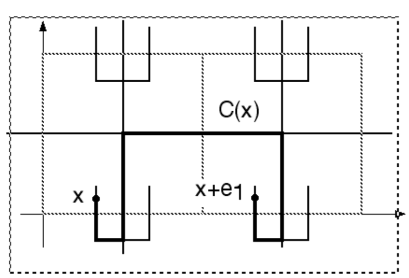

FIGURE 3. The set E in Example 2

Rescaled inequalities. By the change of variable $y=\frac{x}{\varepsilon}$ assumption (H_1) implies that for every $i,j \in \mathbb{Z}^n$, with $|i-j|=1$,

$$|u^i_\varepsilon - u^j_\varepsilon|^p \leq c_0\varepsilon^{p-n}\int_{Y^i_\varepsilon\cup Y^j_\varepsilon+(-\varepsilon\delta,\varepsilon\delta)^n} |D_{\mu_\varepsilon} u|^p d\mu_\varepsilon$$

for every $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mu_\varepsilon)$, where

$$u^i_\varepsilon = \frac{1}{\mu_\varepsilon(Y^i_\varepsilon)} \int_{Y^i_\varepsilon} u d\mu_\varepsilon. \tag{9}$$

Analogously, from (H_2) it follows that

$$\int_{Y^i_\varepsilon} |u - u^i_\varepsilon|^p d\mu_\varepsilon \le c\varepsilon^p \int_{\varepsilon i + \varepsilon(-\delta, 1+\delta)^n} |D_{\mu_\varepsilon} u|^p d\mu_\varepsilon,$$

for every $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mu_\varepsilon)$, $i \in \mathbb{Z}^n$, and u^i_ε defined by (9).

Finally, condition (H_1*) corresponds to

$$|u^i_\varepsilon - u^l_\varepsilon|^p \le c_0 2^p 2n \varepsilon^{p-n} \int_{\varepsilon S_{il} + (-\varepsilon\delta, \varepsilon\delta)^n} |D_{\mu_\varepsilon} u|^p d\mu_\varepsilon \tag{10}$$

for every $i, l \in \mathbb{Z}$ and every $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mu_\varepsilon)$.

The following lemma is the main preliminary result, allowing to link the convergence of functions in $W^{1,p}(\Omega, \mu_\varepsilon)$ to the convergence of the corresponding piecewise-constant interpolations in usual L^p-spaces, and easily proving coerciveness properties.

Lemma 2.1. *Let $u_\varepsilon \in W^{1,p}(\Omega, \mu_\varepsilon)$ and let $\overline{u}_\varepsilon$ be defined by*

$$\overline{u}_\varepsilon = \sum_{i \in \mathbb{Z}^n} u^i_\varepsilon \chi^i_\varepsilon,$$

where $\chi^i_\varepsilon = \chi_{Y^i_\varepsilon}$ is the characteristic function of Y^i_ε and

$$u^i_\varepsilon = \begin{cases} \dfrac{1}{\mu_\varepsilon(Y^i_\varepsilon)} \displaystyle\int_{Y^i_\varepsilon} u_\varepsilon d\mu_\varepsilon & \text{if } Y^i_\varepsilon \subset \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following statements hold true.

(a) *If*

$$\int_\Omega |u_\varepsilon|^p d\mu_\varepsilon \le c \qquad \text{for all } \varepsilon > 0, \tag{11}$$

then there exists $\overline{u} \in L^p(\Omega)$ such that, up to a subsequence,

$$\overline{u}_\varepsilon \rightharpoonup \overline{u} \qquad \text{weakly in } L^p(\Omega) \tag{12}$$

$$u_\varepsilon \mu_\varepsilon \overset{*}{\rightharpoonup} \mu(Y) \overline{u} d\mathcal{L}^n \qquad \text{as } \varepsilon \to 0 \tag{13}$$

(b) *If, in addition, μ satisfies condition (H_1), and*

$$\int_\Omega \Big(|u_\varepsilon|^p + |D_{\mu_\varepsilon} u_\varepsilon|^p \Big) d\mu_\varepsilon \le c \qquad \text{for all } \varepsilon > 0, \tag{14}$$

then

$$\overline{u} \in W^{1,p}(\Omega) \tag{15}$$

and

$$\int_\Omega |D\overline{u}|^p \, dx \le k_0 \liminf_{\varepsilon \to 0} \int_\Omega |D_{\mu_\varepsilon} u_\varepsilon|^p d\mu_\varepsilon, \tag{16}$$

with $k_0 > 0$.

(c) *If* (H_1), (H_2), (14) *hold true, then*

$$\overline{u}_\varepsilon \to \overline{u} \quad \text{strongly in } L^p_{\rm loc}(\Omega). \tag{17}$$

$$\int_{\Omega'} |u_\varepsilon|^p d\mu_\varepsilon \to \mu(Y) \int_{\Omega'} |\overline{u}|^p d\mathcal{L}^n, \quad \textit{for all } \Omega' \subset\subset \Omega. \tag{18}$$

Proof. Proof of (a). In order to prove (12), it is enough to show that $\overline{u}_\varepsilon$ is bounded in $L^p(\Omega)$. So we compute

$$\begin{aligned} \|\overline{u}_\varepsilon\|^p_{L^p(\Omega)} &= \sum_i \int_{Y^i_\varepsilon} |u^i_\varepsilon|^p dx = \varepsilon^n \sum_i \Big| \frac{1}{\mu_\varepsilon(Y^i_\varepsilon)} \int_{Y^i_\varepsilon} u_\varepsilon d\mu_\varepsilon \Big|^p \\ &\le \varepsilon^n \sum_i \frac{1}{\mu_\varepsilon(Y^i_\varepsilon)} \int_{Y^i_\varepsilon} |u_\varepsilon|^p d\mu_\varepsilon = \frac{1}{\mu(Y)} \sum_i \int_{Y^i_\varepsilon} |u_\varepsilon|^p d\mu_\varepsilon \le c, \end{aligned}$$

where we have used Hölder's inequality and assumption (11). Hence, up to a subsequence we have that

$$\overline{u}_\varepsilon \rightharpoonup \overline{u} \qquad \text{weakly in } L^p(\Omega). \tag{19}$$

In order to show that (13) holds true we introduce the measure $\nu_\varepsilon = u_\varepsilon \mu_\varepsilon$; i.e.,

$$\nu_\varepsilon(E) = \int_E u_\varepsilon d\mu_\varepsilon \qquad \text{for all } E \in \mathcal{B}(\Omega)$$

By (11) ν_ε has uniformly bounded total variation, i.e., $|\nu_\varepsilon|(\Omega) \le c$, for every $\varepsilon > 0$, and hence there exists a signed measure ν on $\mathcal{B}(\Omega)$ such that, up to a subsequence,

$$\nu_\varepsilon \overset{*}{\rightharpoonup} \nu \qquad \text{as } \varepsilon \to 0.$$

We can show that $\nu = \mu(Y)\overline{u}\mathcal{L}^n$. To this end, it is enough to compute the limit of $\nu_\varepsilon(A)$ for every open set $A \subset \Omega$ with Lipschitz boundary. Given A, we set $A_\varepsilon = \cup\{Y^i_\varepsilon : Y^i_\varepsilon \subset A\}$. We have

$$\nu_\varepsilon(A) = \int_A u_\varepsilon d\mu_\varepsilon = \int_{A_\varepsilon} u_\varepsilon d\mu_\varepsilon + \int_{A\setminus A_\varepsilon} u_\varepsilon d\mu_\varepsilon,$$

where

$$\Big| \int_{A\setminus A_\varepsilon} u_\varepsilon d\mu_\varepsilon \Big| \le c\mu_\varepsilon(A \setminus A_\varepsilon)^{1-\frac{1}{p}},$$

and $\mu_\varepsilon(A \setminus A_\varepsilon) \to 0$, as $\varepsilon \to 0$. By the definition of u^i_ε

$$\int_{A_\varepsilon} u_\varepsilon d\mu_\varepsilon = \mu(Y) \int_{A_\varepsilon} \overline{u}_\varepsilon dx = \mu(Y) \Big(\int_A \overline{u}_\varepsilon dx - \int_{A\setminus A_\varepsilon} \overline{u}_\varepsilon dx \Big) \to \mu(Y) \int_A \overline{u} dx.$$

Since

$$\Big| \int_{A\setminus A_\varepsilon} \overline{u}_\varepsilon dx \Big| \le c\mathcal{L}^n(A \setminus A_\varepsilon)^{1-\frac{1}{p}},$$

and $\mathcal{L}^n(A \setminus A_\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence we obtain that

$$\nu_\varepsilon(A) \to \mu(Y) \int_A \overline{u} dx, \qquad \text{as } \varepsilon \to 0,$$

which means that $\nu(A) = \mu(Y) \int_A \overline{u} dx$, i.e. $\nu = \mu(Y)\overline{u}\mathcal{L}^n$.

Proof of (b). We have to prove that $\overline{u} \in W^{1,p}(\Omega)$. To this end we note that, by (H_1) and (14), we have

$$\sum_{i,j \in I_\varepsilon, |i-j|=1} \varepsilon^n \Big| \frac{u^i_\varepsilon - u^j_\varepsilon}{\varepsilon} \Big|^p \le c \int_\Omega |D_{\mu_\varepsilon} u_\varepsilon|^p d\mu_\varepsilon \le K, \tag{20}$$

for a positive constant K and for every $\varepsilon > 0$, with $I_\varepsilon = \{i \in \mathbb{Z} : \varepsilon i + (-\varepsilon\delta, \varepsilon\delta)^n \subset \Omega\}$. To prove that $\overline{u} \in W^{1,p}(\Omega)$ we fix $l \in \{1, \dots, n\}$ and show that $\frac{\partial u}{\partial x_l} \in L^p(\Omega)$. More precisely, for every $\Omega' \subset\subset \Omega$ we construct a function $v^l_\varepsilon \in L^p(\Omega')$ which is piecewise affine in the variable (x_l), and such that

$$\|\overline{u}_\varepsilon - v^l_\varepsilon\|_{L^p(\Omega')} \leq c\varepsilon \tag{21}$$

and

$$\int_{\Omega'} \Big|\frac{\partial v^l_\varepsilon}{\partial x_l}\Big|^p dx \leq c \sum_{i,j\in I_\varepsilon, |i-j|=1} \varepsilon^n \Big|\frac{u^i_\varepsilon - u^j_\varepsilon}{\varepsilon}\Big|^p \tag{22}$$

for every $\varepsilon > 0$. Clearly, from (21) and (19), up to a subsequence $v^l_\varepsilon \rightharpoonup \overline{u}$ weakly in $L^p(\Omega')$. Moreover, (20), (22) imply that $\frac{\partial \overline{u}}{\partial x_l} \in L^p(\Omega)$. Since l is arbitrary, then $u, \overline{u} \in W^{1,p}(\Omega)$. By summing over l in (22), (20) and passing to the limit, also (16) follows.

Now we prove (21) and (22). Let $i \in \mathbb{Z}^n$, and let

$$S^{l,i}_\varepsilon = \cup\{Y^j_\varepsilon : Y^j_\varepsilon \subset \Omega, j - i = \lambda e_l, \lambda \in \mathbb{Z}\}.$$

In this region $S^{l,i}_\varepsilon$ we define v^l_ε, depending only on x_l, as the affine interpolation of the values u^j_ε. In this way, when $Y^j_\varepsilon \subset S^{l,i}_\varepsilon$ we have

$$\int_{Y^j_\varepsilon} |\overline{u}_\varepsilon - v^l_\varepsilon|^p dx \leq \sum_{|k-j|=1} |u^k_\varepsilon - u^j_\varepsilon|^p \varepsilon^n.$$

Then, given $\Omega' \subset\subset \Omega$, for ε small enough we have $\Omega' \subset \cup_i S^{l,i}_\varepsilon$ and then

$$\begin{aligned}
\int_{\Omega'} |\overline{u}_\varepsilon - v^l_\varepsilon|^p dx &\leq \sum_i \int_{S^{l,i}_\varepsilon} |\overline{u}_\varepsilon - v^l_\varepsilon|^p dx \\
&= \sum_i \sum_{Y^j_\varepsilon \subset S^{l,i}_\varepsilon} \sum_{|k-j|=1} |u^k_\varepsilon - u^j_\varepsilon|^p \varepsilon^n \\
&\leq c_0 \varepsilon^p \sum_i \sum_{Y^j_\varepsilon \subset S^{l,i}_\varepsilon} \sum_{|k-j|=1} \int_{Y^j_\varepsilon \cup Y^k_\varepsilon + (-\varepsilon\delta,\varepsilon\delta)^n} |D_{\mu_\varepsilon} u|^p d\mu_\varepsilon \leq K\varepsilon^p,
\end{aligned}$$

which implies (21). On the other hand, by construction, if $Y^j_\varepsilon \subset S^{l,i}_\varepsilon$, then

$$\int_{Y^j_\varepsilon} \Big|\frac{\partial v^l_\varepsilon}{\partial x_l}\Big|^p dx = \sum_{|k-j|=1} \Big|\frac{u^k_\varepsilon - u^j_\varepsilon}{\varepsilon}\Big|^p \frac{\varepsilon^n}{2},$$

so that, summing over i, j, by (20) also (22) follows.

Proof of (c). To show (17), we can use the Compactness Criterion by Fréchet and Kolmogorov (see, for instance, [B]) and prove that, for every $\omega \subset\subset \Omega$, and every $\eta > 0$, there exists $\delta > 0$, $\delta < \operatorname{dist}(\omega, \mathbb{R}^n \setminus \Omega)$, such that for every $h \in \mathbb{R}^n$, $|h| < \delta$, then

$$\|\tau_h \overline{u}_\varepsilon - \overline{u}_\varepsilon\|_{L^p(\omega)} < \eta, \tag{23}$$

where we have set $\tau_h \overline{u}_\varepsilon(x) = \overline{u}_\varepsilon(x+h)$. Let us start by assuming $h = \lambda e_1$ (or, more generally, $h = \lambda e_i$, $i \in \{1, \dots, n\}$). If we prove inequality (23) for such h, then by the triangle inequality we can obtain the same result for every $h \in \mathbb{R}^n$. Let us take a function $\overline{u}_\varepsilon$ and a point $x \in \omega$. Then, there exist i, l such that $x \in Y^i_\varepsilon$, and $x + h \in Y^l_\varepsilon$. Hence $\overline{u}_\varepsilon(x) = u^i_\varepsilon$ and $\overline{u}_\varepsilon(x+h) = u^l_\varepsilon$. By (H^*_1)

$$|\overline{u}_\varepsilon(x) - \overline{u}_\varepsilon(x+h)|^p = |u^i_\varepsilon - u^l_\varepsilon|^p \leq c_0 2^p 2n \varepsilon^{p-n} \int_{S(x,h)} |D_{\mu_\varepsilon} u_\varepsilon|^p d\mu_\varepsilon,$$

where we have denoted by $S(x,h)$ the set $S_{il}+(-\delta,\delta)^n$, which depends on the choice of x and h. If we integrate with $x\in Y^i_\varepsilon$, we have

$$\begin{aligned}\int_{Y^i_\varepsilon}|\overline{u}_\varepsilon(x)-\overline{u}_\varepsilon(x+h)|^p dx &\leq c_0 2^p 2n\varepsilon^{p-n}\int_{Y^i_\varepsilon}\Big(\int_{S(x,h)}|D_{\mu_\varepsilon}u_\varepsilon|^p d\mu_\varepsilon\Big)dx\\ &\leq c_0 2^p 2n\varepsilon^{p}\int_{S(Y^i_\varepsilon,h)}|D\mu_{\varepsilon u}\varepsilon|^p d\mu_\varepsilon\end{aligned}$$

where $S(x,h)\subset S(Y^i_\varepsilon,h)=\cup\{S(x,h):x\in Y^i_\varepsilon\}$.

Now, we may sum over $I_\varepsilon=\{i:\omega\subset\cup Y^i_\varepsilon\subset\Omega\}$, noting that the number of induces i,j such $S(Y^i_\varepsilon,h)\cap S(Y^j_\varepsilon,h)\neq\emptyset$ is of the order of $|h|\varepsilon^{-1}$ (the ratio between the size of $S(Y^i_\varepsilon,h)$ and the size of Y^i_ε), and we have

$$\sum_{i\in I_\varepsilon}\int_{Y^i_\varepsilon}|\overline{u}_\varepsilon(x)-\overline{u}_\varepsilon(x+h)|^p dx\leq\varepsilon^p|h|\varepsilon^{-1}c_0 2^p 2n\int_\Omega|D_{\mu_\varepsilon}u_\varepsilon|^p d\mu_\varepsilon.$$

By (31), then we have

$$\|\tau_h\overline{u}_\varepsilon-\overline{u}_\varepsilon\|_{L^p(\omega)}\leq|h|\varepsilon^{p-1}c,$$

and hence (23) follows. The compactness criterion implies that, up to a subsequence $\overline{u}_\varepsilon\to v$, but since we already know that $\overline{u}_\varepsilon\rightharpoonup\overline{u}$ then $v=\overline{u}$ and the proof of (17) is complete.

To conclude the proof, we have to show that (18) holds true. By applying (H_2) in each cell Y^i_ε

$$\int_{\Omega'}|u_\varepsilon-\overline{u}_\varepsilon|^p d\mu_\varepsilon\leq c\varepsilon^p\int_{\Omega'}|D_{\mu_\varepsilon}u_\varepsilon|^p d\mu_\varepsilon\leq K\varepsilon^p.$$

Hence, to prove (18) it is enough to show that

$$\int_{\Omega'}|\overline{u}_\varepsilon|^p d\mu_\varepsilon\to\int_{\Omega'}|\overline{u}|^p d\mathcal{L}^n$$

for a fixed open set $\Omega'\subset\subset\Omega$. Let us fix $\eta>0$ and Ω" such that $\Omega'\subset\subset\Omega''\subset\subset\Omega$ and

$$\int_{\Omega''}|\overline{u}|^p\,d\mathcal{L}^n<\int_{\Omega'}|\overline{u}|^p\,d\mathcal{L}^n+\eta. \tag{24}$$

Let $I'_\varepsilon=\{i:Y^i_\varepsilon\subset\Omega'\}$, $I''_\varepsilon=\{i:Y^i_\varepsilon\subset\Omega''\}$. From (17), there exists $\varepsilon_0>0$ such that for every $\varepsilon<\varepsilon_0$

$$\int_{\Omega'}|\overline{u}_\varepsilon|^p\,d\mathcal{L}^n-\eta<\sum_{i\in I'_\varepsilon}\int_{Y^i_\varepsilon}|\overline{u}_\varepsilon|^p\,d\mathcal{L}^n. \tag{25}$$

By our definitions, we have

$$\sum_{i\in I'_\varepsilon}\int_{Y^i_\varepsilon}|u^i_\varepsilon|^p\,d\mu_\varepsilon\leq\int_{\Omega'}|\overline{u}_\varepsilon|^p\,d\mu_\varepsilon\leq\sum_{i\in I''_\varepsilon}\int_{Y^i_\varepsilon}|u^i_\varepsilon|^p\,d\mu_\varepsilon. \tag{26}$$

By the definition of u^i_ε, for each $Y^i_\varepsilon\subset\Omega$ we have

$$\int_{Y^i_\varepsilon}|u^i_\varepsilon|^p\,d\mu_\varepsilon=|u^i_\varepsilon|^p\mu_\varepsilon(Y^i_\varepsilon)=\mu(Y)\varepsilon^n|u^i_\varepsilon|^p=\mu(Y)\int_{Y^i_\varepsilon}|u^i_\varepsilon|^p\,d\mathcal{L}^n.$$

Hence, from (26) we deduce that

$$\mu(Y)\sum_{i\in I'_\varepsilon}\int_{Y^i_\varepsilon}|\overline{u}_\varepsilon|^p\,d\mathcal{L}^n\leq\int_{\Omega'}|\overline{u}_\varepsilon|^p\,d\mu_\varepsilon\leq\mu(Y)\sum_{i\in I''_\varepsilon}\int_{Y^i_\varepsilon}|\overline{u}_\varepsilon|^p\,d\mathcal{L}^n$$

and so

$$\mu(Y)\int_{\Omega'}|\overline{u}_\varepsilon|^p\,d\mathcal{L}^n-\eta\le\int_{\Omega'}|\overline{u}_\varepsilon|^p\,d\mu_\varepsilon\le\mu(Y)\int_{\Omega'}|\overline{u}_\varepsilon|^p\,d\mathcal{L}^n.$$

Taking the limit as $\varepsilon\to 0$ we conclude that

$$\mu(Y)\int_{\Omega'}|\overline{u}|^p\,d\mathcal{L}^n-\eta\le\lim_{\varepsilon\to 0}\int_{\Omega'}|\overline{u}_\varepsilon|^p\,d\mu_\varepsilon\le\mu(Y)\int_{\Omega'}|\overline{u}|^p\,d\mathcal{L}^n+\eta.$$

Since η is arbitrary, we have shown (18). □

Remark 2 (compactness with boundary data)**.** If $u_\varepsilon\in W^{1,p}_{\rm loc}(\mathbb{R}^n,\mu_\varepsilon)$ are such that $u_\varepsilon=u_0\in W^{1,\infty}(\mathbb{R}^n)$ on $\mathbb{R}^n\setminus\Omega$ and $\sup_\varepsilon\int_\Omega|D_{\mu_\varepsilon}u_\varepsilon|^pd\mu_\varepsilon<+\infty$ then we deduce that (18) holds with Ω in place of Ω'.

In fact, from the scaled condition (H_1) and a discrete Poincaré inequality we deduce that $\overline{u}_\varepsilon$ are equibounded in $L^p_{\rm loc}(\mathbb{R}^n)$, from which in turn we deduce that also $\sup_\varepsilon\int_{\Omega''}|u_\varepsilon|^pd\mu_\varepsilon<+\infty$ from (H_2) for all Ω''. We may then apply Proposition 2.1 (c) with Ω'' strictly containing Ω.

3. **The main result.** In this section we state our main result concerning the asymptotic behaviour of the family of functionals $F_\varepsilon+G_\varepsilon$, in the sense of Γ-convergence. To this end, we note that Lemma 2.1 implies that the family of functionals $(F_\varepsilon+G_\varepsilon)_\varepsilon$ is equicoercive with respect to the convergence in (13). For this reason, we introduce a specific notation, setting

$$u_\varepsilon\xrightarrow{\mu_\varepsilon}u\qquad\Longleftrightarrow\qquad u_\varepsilon\mu_\varepsilon\rightharpoonup\mu(Y)u\mathcal{L}^n\tag{27}$$

as $\varepsilon\to 0^+$, for $u_\varepsilon\in W^{1,p}(\Omega,\mu_\varepsilon)$ and $u\in W^{1,p}(\Omega)$.

Our aim is now to compute the Γ-limit of the sequence $(F_\varepsilon+G_\varepsilon)_\varepsilon$ as $\varepsilon\to 0$, with respect to the convergence in (27). For the reader's convenience, we recall the definition of Γ-convergence adapted to the present context (see [10], [7]).

Definition 3.1. Let $F_\varepsilon:W^{1,p}(\Omega,\mu_\varepsilon)\to\mathbb{R}$, $F:W^{1,p}(\Omega)\to\mathbb{R}$. We say F_ε Γ-*converge* to F, or that

$$F=\Gamma(\mu_\varepsilon)\text{-}\lim_{\varepsilon\to 0}F_\varepsilon$$

if for every $u\in W^{1,p}(\Omega)$

(a) *(liminf inequality)* For every $u_\varepsilon\in W^{1,p}(\Omega,\mu_\varepsilon)$ such that $u_\varepsilon\xrightarrow{\mu_\varepsilon}u$

$$F(u)\le\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon).$$

(b) *(existence of a recovery sequence)* For every $u\in W^{1,p}(\Omega)$ there exists a sequence $u_\varepsilon\in W^{1,p}(\Omega,\mu_\varepsilon)$ such that $u_\varepsilon\xrightarrow{\mu_\varepsilon}u$ and

$$F(u)\ge\limsup_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon).$$

Theorem 3.2 (homogenization theorem)**.** *Let μ be a Y-periodic Radon measure on $\mathbb{R}^n$ satisfying conditions (H_1), (H_2), and let $F_\varepsilon,G_\varepsilon$ be defined by* (5)*. Let Ω be a bounded open set with Lipschitz boundary. Then $F_\varepsilon+G_\varepsilon$ Γ-converges on $W^{1,p}(\Omega)$ in the sense of Definition* 3.1 *to the functional $F+G:W^{1,p}(\Omega)\to\mathbb{R}$, defined by*

$$F(u)=\int_\Omega f_{\rm hom}(Du)dx,\quad G(u)=\mu(Y)\int_\Omega|u|^pdx$$

where $f_{\rm hom}:\mathbb{M}^{m\times n}\to\mathbb{R}$ *is a quasi-convex function given by*

$$f_{\rm hom}(\xi)=\lim_{T\to+\infty}\inf\Big\{\frac{1}{T^n}\int_{(0,T)^n}f(y,D_\mu\Phi)d\mu:\Phi(y)-\xi\cdot y\in W^{1,p}_0((0,T)^n,\,\mu)\Big\},\tag{28}$$

such that

$$k_1|\xi|^p\le f_{\rm hom}(\xi)\le k_2(1+|\xi|^p)\qquad \textit{for all } \xi\in\mathbb{M}^{m\times n},\tag{29}$$

with $0<k_1\le k_2$.

In the theorem above we have used the notation $v\in W^{1,p}_0(A,\,\mu)$ meaning that the extension by $v=0$ outside A gives a function in $W^{1,p}(\mathbb{R}^n,\,\mu)$.

Remark 3 (extensions and consequences)**.** (a) Note that the same result holds considering in place of F_ε the functionals

$$\int_\Omega f\Big(\frac{x}{\varepsilon},Du\Big)\,d\mu_\varepsilon\qquad u\in C^\infty(\Omega)$$

(where Du is the usual pointwise gradient), upon defining

$$f_{\rm hom}(\xi)=\lim_{T\to+\infty}\inf\Big\{\frac{1}{T^n}\int_{(0,T)^n}f(y,D\Phi)d\mu:\Phi(y)-\xi\cdot y\in C^\infty_0((0,T)^n)\Big\},\tag{30}$$

the proof being exactly the same.

(b) In place of (or in addition to) the term G_ε a boundary condition can be required restricting the domain of F_ε to those $u\in W^{1,p}_{\rm loc}(\mathbb{R}^n,\mu_\varepsilon)$ satisfying $u=w$ on $\mathbb{R}^n\setminus\Omega$, where w is a fixed function in $W^{1,\infty}(\mathbb{R}^n)$. In that case the Γ-limit is the restriction of F to $w+W^{1,p}_0(\Omega)$ (see Step 6 in the proof of Theorem 3.2).

(c) Both in the case of Theorem 3.2 and with boundary conditions, the functionals considered are equi-coercive with respect to the convergence $u_\varepsilon\xrightarrow{\mu_\varepsilon}u$, and thus the Γ-convergence result implies the corresponding convergence of minimum problems.

The following proposition and lemma will be useful in the proof of the lim inf inequality and in the construction of recovery sequences. Note that both results deal with Lipschitz target functions, which belong to all the Sobolev spaces we consider.

Proposition 1. *Assume that* μ *satisfies conditions* (H_1) *and* (H_2). *Let* $w\in W^{1,\infty}(\Omega)$, $w_\varepsilon\in W^{1,p}(\Omega,\mu_\varepsilon)$ *be such that* $w_\varepsilon\xrightarrow{\mu_\varepsilon}w$, *and*

$$\int_\Omega\Big(|w_\varepsilon|^p+|D_{\mu_\varepsilon}w_\varepsilon|^p\Big)d\mu_\varepsilon\le c,\qquad \textit{for all } \varepsilon>0.\tag{31}$$

Then, for every open set $\Omega'\subset\subset\Omega$

$$\int_{\Omega'}|w_\varepsilon-w|^pd\mu_\varepsilon\to 0,\qquad \text{as } \varepsilon\to 0.\tag{32}$$

Proof. It is enough to consider the case where $w=0$, since, in the general case the following arguments hold replacing w_ε by $w_\varepsilon-w$. Note that our assumptions imply that $\overline{w}_\varepsilon\rightharpoonup 0$ weakly in $L^p(\Omega)$ (see (12) in Lemma 2.1). Let $I_\varepsilon(\Omega)=\{i\in\mathbb{Z}^n:Y^i_\varepsilon\subset\Omega\}$. Given $\Omega'\subset\subset\Omega$, then for ε small enough we have $\Omega'\subset\cup\{Y^i_\varepsilon:i\in I_\varepsilon(\Omega)\}$, and by (H_2), we can estimate

$$\begin{aligned}\int_{\Omega'}|w_\varepsilon-\overline{w}_\varepsilon|^pd\mu_\varepsilon &\le \sum_{i\in I_\varepsilon(\Omega)}\int_{Y^i_\varepsilon}|w_\varepsilon-\overline{w}_\varepsilon|^pd\mu_\varepsilon\\ &\le c\varepsilon^p\sum_{i\in I_\varepsilon(\Omega)}\int_{Y^i_\varepsilon}|D_{\mu_\varepsilon}w_\varepsilon|^pd\mu_\varepsilon\le c\varepsilon^p\int_\Omega|D_{\mu_\varepsilon}w_\varepsilon|^pd\mu_\varepsilon.\end{aligned}$$

By (31) it follows that

$$\int_{\Omega'} |w_\varepsilon - \overline{w}_\varepsilon|^p d\mu_\varepsilon \to 0, \qquad \text{as } \varepsilon \to 0.$$

Since

$$\int_{\cup Y_i^\varepsilon} |\overline{w}_\varepsilon|^p\, dx = \mu(Y) \int_{\cup Y_i^\varepsilon} |\overline{w}_\varepsilon|^p\, d\mu_\varepsilon,$$

if we prove that $\overline{w}_\varepsilon \to 0$ also strongly in $L^p(\Omega)$, then (32) follows immediately, and the proof is complete. To this end, we can again use the Compactness Criterion by Fréchet and Kolmogorov and prove that, for every $\omega \subset\subset \Omega$, and every $\eta > 0$, there exists $\delta > 0$, $\delta < \operatorname{dist}(\omega, \mathbb{R}^n \setminus \Omega)$, such that for every $h \in \mathbb{R}^n$, $|h| < \delta$, then

$$\|\tau_h \overline{w}_\varepsilon - \overline{w}_\varepsilon\|_{L^p(\omega)} < \eta, \tag{33}$$

where we have set $\tau_h \overline{w}_\varepsilon(x) = \overline{w}_\varepsilon(x+h)$. We begin by assuming $h = \lambda e_1$ (or, more generally, $h = \lambda e_i$, $i \in \{1, \ldots, n\}$). If we prove inequality (33) for such h, then by the triangle inequality we can obtain the same result for every $h \in \mathbb{R}^n$. Let us take a function $\overline{w}_\varepsilon$ and a point $x \in \omega$. Then, there exist $i, l \in \mathbb{Z}^n$ such that $x \in Y_\varepsilon^i$, and $x + h \in Y_\varepsilon^l$. By (H_1^*)

$$|\overline{w}_\varepsilon(x) - \overline{w}_\varepsilon(x+h)|^p = |w_\varepsilon^i - w_\varepsilon^l|^p \le c_0 2^p 2n \int_{S^\varepsilon(x,h)} |D_{\mu_\varepsilon} w_\varepsilon|^p d\mu_\varepsilon,$$

where we have denoted by $S^\varepsilon(x,h)$ the set $\varepsilon S_{il} + (-\varepsilon\delta, \varepsilon\delta)^n$, which depends on ε and on the choice of x and h. If we integrate with $x \in Y_\varepsilon^i$, we have

$$\begin{aligned}\int_{Y_\varepsilon^i} |\overline{w}_\varepsilon(x) - \overline{w}_\varepsilon(x+h)|^p dx &\le c_0 2^p 2n \int_{Y_\varepsilon^i} \Big(\int_{S^\varepsilon(x,h)} |D_{\mu_\varepsilon} w_\varepsilon|^p d\mu_\varepsilon \Big) dx \\ &\le c_0 2^p 2n \int_{S(Y_\varepsilon^i,h)} \Big(\int_{S^\varepsilon(x,h)} |D_{\mu_\varepsilon} w_\varepsilon|^p d\mu_\varepsilon \Big) dx\end{aligned}$$

where $S^\varepsilon(x,h) \subset S(Y_\varepsilon^i, h) := \cup\{S^\varepsilon(x,h) : x \in Y_\varepsilon^i\}$.

After summing over $I_\varepsilon = \{i : \omega \subset \cup Y_\varepsilon^i \subset \Omega\}$, noting that the number of indices i, j such $S(Y_\varepsilon^i, h) \cap S(Y_\varepsilon^j, h) \neq \emptyset$ is of the order of $|h|\varepsilon^{-1}$ (the ratio between the size of $S(Y_\varepsilon^i, h)$ and the size of Y_ε^i), and we have

$$\sum_{i \in I_\varepsilon} \int_{Y_\varepsilon^i} |\overline{w}_\varepsilon(x) - \overline{w}_\varepsilon(x+h)|^p dx \le |h| \varepsilon^{n-1} c_0 2^p 2n \int_{Y_\varepsilon^i} \Big(\int_{S^\varepsilon(x,h)} |D_{\mu_\varepsilon} w_\varepsilon|^p d\mu_\varepsilon \Big) dx.$$

By (31), we then have

$$\|\tau_h \overline{w}_\varepsilon - \overline{w}_\varepsilon\|_{L^p(\omega)} \le |h| \varepsilon^{n-1} c,$$

and hence (33) follows. The compactness criterion implies that, up to a subsequence $\overline{w}_\varepsilon \to v$, but since we assume $\overline{w}_\varepsilon \rightharpoonup 0$ then $v = 0$ and the proof of the proposition is complete. □

Lemma 3.3. *Let $A \subset \Omega$ be an open set with Lipschitz boundary, $w_\varepsilon \in W^{1,p}(\Omega, \mu_\varepsilon)$, $w \in W^{1,\infty}(\Omega)$, $w_\varepsilon \xrightarrow{\mu_\varepsilon} w$, and $(F_\varepsilon + G_\varepsilon)(w_\varepsilon) \le c$ for every $\varepsilon > 0$. Then, for every $\varepsilon > 0$ there exists a function $\zeta_\varepsilon \in W^{1,p}(\Omega, \mu_\varepsilon)$ such that $\zeta_\varepsilon = w$ in a neighbourhood of ∂A, including $\Omega \setminus A$, and*

$$(F_\varepsilon + G_\varepsilon)(\zeta_\varepsilon, A) \le (F_\varepsilon + G_\varepsilon)(w_\varepsilon, A) + o(1)$$

as $\varepsilon \to 0$.

Proof. Let $B \subset\subset A$ be an open set with Lipschitz boundary, and let $\operatorname{dist}(B, \partial A) = \delta$. For every $N \in \mathbb{N}$, and every $j = 1, \ldots, N+1$, let $B_j = \{x \in A : \operatorname{dist}(x, \partial B) < \frac{j\delta}{N+2}\}$. Hence we have $B \subset\subset B_1 \subset\subset \ldots \subset\subset B_{N+1} \subset\subset A$. For every $j = 1, \ldots, N$ let φ_j be a cut-off function between B_j and B_{j+1}, i.e., $\varphi_j \in \mathcal{C}_0^\infty(B_{j+1})$, $\varphi_j \equiv 1$ in a neighbourhood of $\overline{B}_j$, $0 \le \varphi_j \le 1$. By construction $\operatorname{dist}(B_j, \partial B_{j+1}) = \frac{\delta}{N+2}$ for every j, and hence we may choose φ_j such that there exists a constant $c > 0$ such that $|D\varphi_j| \le c\frac{N+2}{\delta}$ for every j.

For a fixed $j = j(\varepsilon) \in \{1, \ldots, N\}$, to be chosen later, we set $\zeta_\varepsilon = \varphi_j w_\varepsilon + (1 - \varphi_j) w \in W^{1,p}(\Omega, \mu_\varepsilon)$. Note that

$$\zeta_\varepsilon = w_\varepsilon \quad \text{in a neighbourhood of } \overline{B}_j, \qquad \zeta_\varepsilon = w \quad \text{in } \Omega \setminus \overline{B}_{j+1},$$

and $D_{\mu_\varepsilon}\zeta_\varepsilon = \varphi_j D_{\mu_\varepsilon} w_\varepsilon + (1 - \varphi_j) D_{\mu_\varepsilon}\varphi_j (w_\varepsilon - w)$. In particular

$$\zeta_\varepsilon = w \quad \text{in } \Omega \setminus A, \quad \text{for all } \varepsilon > 0.$$

Now, we evaluate the energy $F_\varepsilon(\zeta_\varepsilon, A)$. We have

$$F_\varepsilon(\zeta_\varepsilon, A) = F_\varepsilon(\zeta_\varepsilon, \overline{B}_j) + F_\varepsilon(\zeta_\varepsilon, B_{j+1} \setminus \overline{B}_j) + F_\varepsilon(\zeta_\varepsilon, A \setminus \overline{B}_{j+1}) \tag{34}$$

First of all, we have

$$F_\varepsilon(\zeta_\varepsilon, \overline{B}_j) = F_\varepsilon(w_\varepsilon, \overline{B}_j) \le F_\varepsilon(w_\varepsilon, A).$$

As for the second term in (34), where, for simplicity we set $B_{j+1} \setminus \overline{B}_j = C_j$, by (4) we have

$$\begin{aligned} F_\varepsilon(\zeta_\varepsilon, C_j) &\le c\int_{C_j}\Big(1 + |D_{\mu_\varepsilon} w_\varepsilon|^p + |D_{\mu_\varepsilon} w|^p\Big)d\mu_\varepsilon + c\int_{C_j} |D_{\mu_\varepsilon}\varphi_j|^p |w_\varepsilon - w|^p d\mu_\varepsilon \\ &\le c\sigma_\varepsilon(C_j) + c\Big(\frac{N+2}{\delta}\Big)^p \int_{C_j} |w_\varepsilon - w|^p d\mu_\varepsilon, \end{aligned}$$

where we have used the notation

$$\sigma_\varepsilon(C) = \int_C \Big(1 + |D_{\mu_\varepsilon} w_\varepsilon|^p + |D_{\mu_\varepsilon} w|^p\Big)d\mu_\varepsilon.$$

Since $\cup_{i=1}^N C_j = B_{N+1}$, and $\sigma_\varepsilon(\cup_{i=1}^N C_j) = \sum_{i=1}^N \sigma_\varepsilon(C_j)$, then there exists $j \in \{1, \ldots, N\}$ such that

$$\sigma_\varepsilon(C_j) \le \frac{1}{N}\sigma_\varepsilon\Big(\bigcup_{i=1}^N C_j\Big) = \frac{1}{N}\int_{B_{N+1}}\Big(1 + |D_{\mu_\varepsilon} w_\varepsilon|^p + |D_{\mu_\varepsilon} w|^p\Big)d\mu_\varepsilon \le \frac{c}{N}$$

for every $\varepsilon > 0$. As for the second integral, by Proposition 1 we know that

$$\Big(\frac{N+2}{\delta}\Big)^p \int_{C_j} |w_\varepsilon - w|^p d\mu_\varepsilon \le \Big(\frac{N+2}{\delta}\Big)^p \int_{B_{N+1}} |w_\varepsilon - w|^p d\mu_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

As for the last term in (34) we have

$$\begin{aligned} F_\varepsilon(\zeta_\varepsilon, A \setminus \overline{B}_{j+1}) &\le c\int_{A\setminus\overline{B}} (1 + |D_{\mu_\varepsilon} w|^p) d\mu_\varepsilon \\ &\le c\int_{A\setminus\overline{B}} (1 + |Dw|^p) d\mu_\varepsilon = c\mu(Y)\int_{A\setminus\overline{B}} (1 + |Dw|^p) d\mathcal{L}^n + o(1) \end{aligned}$$

as $\varepsilon \to 0$. Up to now, we have shown that

$$F_\varepsilon(\zeta_\varepsilon, A) \le F_\varepsilon(w_\varepsilon, A) + \frac{c}{N} + c\Big(\frac{N+2}{\delta}\Big)^p \int_A |w_\varepsilon - w|^p d\mu_\varepsilon + c\int_{A\setminus\overline{B}} (1 + |Dw|^p) d\mu_\varepsilon.$$

Since for $G_\varepsilon(\zeta_\varepsilon, A)$ we have that

$$G_\varepsilon(\zeta_\varepsilon, A) = \int_A |\zeta_\varepsilon|^p d\mu_\varepsilon \leq \int_A |w_\varepsilon|^p d\mu_\varepsilon + 2^{p-1} \int_{A\setminus\overline{B}} (|w_\varepsilon|^p + |w|^p) d\mu_\varepsilon$$

and

$$\int_{A\setminus\overline{B}} (|w_\varepsilon|^p + |w|^p) d\mu_\varepsilon = 2\mu(Y) \int_{A\setminus\overline{B}} |w|^p d\mathcal{L}^n + o(1),$$

we conclude that

$$F_\varepsilon(\zeta_\varepsilon, A) \leq F_\varepsilon(w_\varepsilon, A) + \frac{c}{N} + c\int_{A\setminus\overline{B}} (1 + |D_{\mu_\varepsilon} w|^p) d\mathcal{L}^n + o(1),$$

and analogously

$$(F_\varepsilon + G_\varepsilon)(\zeta_\varepsilon, A) \leq (F_\varepsilon + G_\varepsilon)(w_\varepsilon, A) + \frac{c}{N} + c\mu(Y) \int_{A\setminus\overline{B}} (1 + |w|^p + |D_{\mu_\varepsilon} w|^p) d\mathcal{L}^n + o(1)$$

as $\varepsilon \to 0$. Since the last two terms in the above inequalities are arbitrarily small when $\delta \to 0$ and $N \to +\infty$, we have completed the proof. □

We now face the proof of the main result, which uses the localization methods of Γ-convergence. The main issue here is to relate the lower-semicontinuity and measure properties of the functionals defined on the varying Sobolev spaces with respect to the measures μ_ε to the analogous properties on the usual Sobolev spaces. On one hand we will identify those functionals with energies defined on a common Lebesgue space by a 'discretization' argument to deduce lower-semicontinuity properties, on the other hand we will use Lemma 2.1 and Proposition 1 to relate convergence and measure properties in the different spaces. Note however that some regularity on the set Ω must be imposed to obtain such a correspondence, as shown by the counterexample in Section 4.

Proof of Theorem 3.2. Our aim is to prove that $F_\varepsilon + G_\varepsilon \xrightarrow{\Gamma} F + G$ in the sense of Definition 3.1. This will be done through an adaptation of the localization techniques of Γ-convergence (see [10, 7, 5, 6]).

Step 1 (localization and compactness) Let $\mathcal{A}(\Omega)$ be the family of all open subsets of Ω. We set $F_\varepsilon, G_\varepsilon : W^{1,p}(\Omega, \mu_\varepsilon) \times \mathcal{A}(\Omega) \to [0, +\infty]$ as

$$F_\varepsilon(u, A) = \int_A f\Big(\frac{x}{\varepsilon}, D_{\mu_\varepsilon} u\Big) d\mu_\varepsilon, \quad G_\varepsilon(u, A) = \int_A |u|^p \, d\mu_\varepsilon$$

for every $u \in W^{1,p}(\Omega, \mu_\varepsilon)$ and every open set $A \subseteq \Omega$. If A is an open set with $A \subset\subset \Omega$, then by (18) it immediately follows that if we prove the Γ convergence of F_ε to F with respect to the convergence

$$u_\varepsilon \xrightarrow{\mu_\varepsilon} u \quad \text{ and } \sup_\varepsilon \|u_\varepsilon\|_{L^p(\Omega,\mu_\varepsilon)} < +\infty \tag{35}$$

then

$$(F_\varepsilon + G_\varepsilon)(\cdot, A) \xrightarrow{\Gamma} (F + G)(\cdot, A) \tag{36}$$

in the sense of Definition 3.1.

Since the topology induced by the convergence (27) satisfies the second countability axiom on bounded sets, then, up to a subsequence, there exists

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(\cdot, A) = \mathcal{F}(\cdot, A) \quad \text{ for all } A \in \mathcal{A}_0,$$

where $\mathcal{A}_0$ is a suitable countable dense family; e.g., the family of all finite unions of open rectangles with rational vertices compactly contained in Ω. Now, $\mathcal{F}(\cdot, A)$ can be extended to the family $\mathcal{A}(\Omega)$ of all open subsets of Ω by inner regularity, setting

$$\overline{\mathcal{F}}(\cdot, A) = \sup\{\mathcal{F}(\cdot, A') : A' \in \mathcal{A}_0, A' \subseteq A\}. \tag{37}$$

Step 2 (Fundamental Estimate). A crucial step is to proving a version of the so-called Fundamental Estimate in this context. More precisely, we have to show that for every $\eta > 0$, and every $A, A', B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$, there exist $M, \varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, and all $u, v \in W^{1,p}(\Omega, \mu_\varepsilon)$ we construct a function $\varphi \in \mathcal{C}_0^\infty(B)$, with $0 \leq \varphi \leq 1$, and $\varphi \equiv 1$ in A', such that

$$F_\varepsilon(\varphi u + (1-\varphi)v, A' \cup B) \leq (1+\eta)\Big(F_\varepsilon(u, A) + F_\varepsilon(v, B)\Big) + M\int_{B'} |u-v|^p d\mu_\varepsilon \tag{38}$$

where $B' \subset\subset (A \cap B) \setminus \overline{A'}$. Moreover, if we have sequences u_ε, v_ε with $\sup_\varepsilon(F_\varepsilon(u, A) + F_\varepsilon(v, B)) < +\infty$ we can choose B' independent of ε, up to subsequences.

This property can be proved adapting an argument orginally introduced by De Giorgi. The proof that we sketch is analogous to the one proposed in [7]. Given $\eta > 0$ and the open sets $A, A', B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$, define $\delta = \mathrm{dist}\,(A', \partial A)$, and fix $N \in \mathbb{N}$. Denote $A_j = \{x \in A : \mathrm{dist}\,(x, \partial A') < \frac{j\delta}{N+2}\}$, for every $j = 1, \dots, N+1$. In this way, we have $A' \subset\subset A_1 \subset\subset \dots \subset\subset A_{N+1} \subset\subset A$. For each j we choose $\varphi_j \in \mathcal{C}_0^\infty(A_{j+1})$, $0 \leq \varphi \leq 1$, and $\varphi_j \equiv 1$ in a neighbourhood of $\overline{A_j}$. Since $\mathrm{dist}\,(A_j, \partial A_{j+1}) = \frac{\delta}{N+2}$, then $|D\varphi_j| \leq c\frac{N+2}{\delta}$. It is clear that $\zeta = \varphi_j u + (1-\varphi_j)v \in W^{1,p}(\Omega, \mu_\varepsilon)$, $\zeta = u$ in a neighbourhood of $\overline{A_j}$, $\zeta = v$ in a neighbourhood of $\Omega \setminus \overline{A_{j+1}}$. Moreover, it can be checked that $D_{\mu_\varepsilon}\zeta = \varphi_j D_{\mu_\varepsilon} u + (1-\varphi_j) D_{\mu_\varepsilon} v + D_{\mu_\varepsilon}\varphi_j (u - v)$. Now, we estimate

$$F_\varepsilon(\zeta, A' \cup B) \leq F_\varepsilon(u, A) + F_\varepsilon(v, B) + F_\varepsilon(\zeta, C_j),$$

where $C_j = (A_{j+1} \setminus \overline{A_j}) \cap B$. Note that $\cup_{j=1}^N C_j = A_{N+1} \cap B \subseteq A \cap B$. To prove (38) we have to estimate $F_\varepsilon(\zeta, C_j)$. From (4) we have

$$\begin{aligned} F_\varepsilon(\zeta, C_j) &\leq c_2 \int_{C_j} \Big(1 + |D_{\mu_\varepsilon}\zeta|^p\Big) d\mu_\varepsilon \\ &\leq c_2 \int_{C_j} \Big(1 + |D_{\mu_\varepsilon} u|^p + |D_{\mu_\varepsilon} v|^p\Big) d\mu_\varepsilon + c_2 \int_{C_j} |D_{\mu_\varepsilon}\varphi_j|^p |u-v|^p d\mu_\varepsilon \\ &\leq c_2 \int_{C_j} \Big(1 + |D_{\mu_\varepsilon} u|^p + |D_{\mu_\varepsilon} v|^p\Big) d\mu_\varepsilon + c\Big(\frac{N+2}{\delta}\Big)^p \int_{C_j} |u-v|^p d\mu_\varepsilon. \end{aligned} \tag{39}$$

We denote

$$\sigma_\varepsilon(C) = \int_C \Big(1 + |D_{\mu_\varepsilon} u|^p + |D_{\mu_\varepsilon} v|^p\Big) d\mu_\varepsilon.$$

Since $\cup_{i=1}^N C_j = A_{N+1} \cap B$, and $\sigma_\varepsilon(\cup_{i=1}^N C_j) = \sum_{i=1}^N \sigma_\varepsilon(C_j)$, there exists j such that

$$\begin{aligned} \sigma_\varepsilon(C_j) &\leq \frac{1}{N}\sigma_\varepsilon(\cup_{j=1}^N C_j) = \frac{1}{N}\sigma_\varepsilon(A_{N+1} \cap B) \\ &\leq \frac{1}{N}\int_{A \cap B} \left(1 + |D_{\mu_\varepsilon} u|^p + |D_{\mu_\varepsilon} v|^p\right) d\mu_\varepsilon \\ &\leq \frac{1}{N}\mu_\varepsilon(A \cap B) + \frac{1}{c_1 N} F_\varepsilon(u, A) + \frac{1}{c_1 N} F_\varepsilon(v, B). \end{aligned}$$

In the following we shall choose N large enough, such that

$$\frac{c_2}{N} < \eta, \quad \frac{1}{c_1 N} < \eta.$$

By setting $B' = C_j$ (38) then follows.

Step 3 (identification with functionals on a Lebesgue space) It is convenient to identify the family of functionals $F_\varepsilon + G_\varepsilon$ with a family H_ε defined on the usual Lebesgue L^p spaces. To that end we define for all open sets A the functionals $H_\varepsilon(\cdot, A) : L^p_{\rm loc}(\Omega) \to [0, +\infty]$ by

$$H_\varepsilon(u,A) = \begin{cases} \inf\{(F_\varepsilon + G_\varepsilon)(v,A) : v \in W^{1,p}(\Omega,\mu_\varepsilon) \text{ such that } \overline{v}_\varepsilon = u\} \\ \hfill \text{if } u \text{ is piecewise constant in } A \\ +\infty \hfill \text{otherwise,} \end{cases} \tag{40}$$

where, for every $v \in W^{1,p}(\Omega, \mu_\varepsilon)$,

$$\overline{v}_\varepsilon = \sum_i v^i_\varepsilon \chi_{Y^i_\varepsilon}, \quad v^i_\varepsilon = \begin{cases} \dfrac{1}{\mu_\varepsilon(Y^i_\varepsilon)} \displaystyle\int_{Y^i_\varepsilon} v\, d\mu_\varepsilon & \text{if } Y^i_\varepsilon \subset \Omega \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

As done for the functional $\mathcal{F}$ above, we may suppose that the Γ-limit $\mathcal{H}$ of $H_\varepsilon(\cdot, A)$ exists for A in the same class $\mathcal{A}_0$ with respect to the convergence in $L^p_{\rm loc}(\Omega)$, and we extend such $\mathcal{H}$ to a functional $\overline{\mathcal{H}}$ by inner regularity.

We now show that $\overline{\mathcal{H}}(u,A) = (\overline{\mathcal{F}} + G)(u,A)$. Indeed, if $A \in \mathcal{A}_0$ and $v_\varepsilon \xrightarrow{\mu_\varepsilon} u$ then

$$\liminf_{\varepsilon\to 0}(F_\varepsilon(v_\varepsilon, A) + G_\varepsilon(u_\varepsilon, A)) \geq \liminf_{\varepsilon\to 0} H_\varepsilon(\overline{v}_\varepsilon, A)) \geq H(u,A),$$

which shows that $(\mathcal{F} + G)(u,A) \geq \overline{\mathcal{H}}(u,A)$; on the other hand, if $u_\varepsilon \to u$ in $L^p_{\rm loc}(\Omega)$ is such that $H(u,A) = \lim_{\varepsilon\to 0} H_\varepsilon(u,A)$ then we choose $\overline{v}_\varepsilon$ such that $F_\varepsilon(\overline{v}_\varepsilon, A) + G_\varepsilon(\overline{u}_\varepsilon, A) \leq H_\varepsilon(u_\varepsilon, A) + o(1)$. Note that by Proposition 2.1 we have $\overline{v}_\varepsilon \xrightarrow{\mu_\varepsilon} u$ locally in A, so that we get

$$H(u,A) \geq \liminf_{\varepsilon\to 0}(F_\varepsilon(\overline{v}_\varepsilon, A') + G_\varepsilon(\overline{v}_\varepsilon, A')) \geq (\mathcal{F} + G)(u, A')$$

for all $A' \subset\subset A$, which proves the converse inequality.

Step 4 (integral representation). Here we show that there exists a Caratheodory function $\varphi : \Omega \times \mathbb{M}^{m\times n} \to [0, +\infty[$, such that

$$k_1|\xi|^p \leq \varphi(\xi) \leq k_2(1 + |\xi|^p) \quad \text{ for all } \xi \in \mathbb{M}^{m\times n},$$

and

$$\overline{\mathcal{F}}(u,A) = \int_A \varphi(x, Du)dx \quad \text{ for all } u \in W^{1,p}(\Omega), A \in \mathcal{A}(\Omega).$$

To this end, we apply an integral representation result on Sobolev spaces, as Theorem 9.1 in [7]. We have to check that $\overline{\mathcal{F}}(u,A)$ satisfies properties (i)-(v) therein for all $u \in W^{1,p}(\Omega)$ and all $A \in \mathcal{A}(\Omega)$; i.e.,

(i) (locality) $\overline{\mathcal{F}}$ is local, i.e., $\overline{\mathcal{F}}(u,A) = \overline{\mathcal{F}}(v,A)$ if $u = v$ a.e. in $A \in \mathcal{A}(\Omega)$;

(ii) (growth condition) there exists $c > 0$ and $a \in L^1(\Omega)$ such that

$$\overline{\mathcal{F}}(u,A) \leq c\int_A (a(x) + |Du|^p) d\mathcal{L}^n$$

for all $u \in W^{1,p}(\Omega)$ and $A \in \mathcal{A}(\Omega)$;

(iii) (measure property) for all $u \in W^{1,p}(\Omega)$ the set function $\overline{\mathcal{F}}(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$;
(iv) (translation invariance in u) $\overline{\mathcal{F}}(u+z,A) = \overline{\mathcal{F}}(u,A)$ for all $z \in \mathbb{R}^m$, $u \in W^{1,p}(\Omega)$ and $A \in \mathcal{A}(\Omega)$;
(v) (lower semicontinuity) for all $A \in \mathcal{A}(\Omega)$ $\overline{\mathcal{F}}(\cdot,A)$ is lower semicontinuous with respect to the weak convergence in $W^{1,p}(\Omega)$.

First we note that (v) follows from the previous step, since $\mathcal{F}(\cdot,A)$ coincides with $\mathcal{H}(\cdot,A)-G(\cdot,A)$ for $A\in\mathcal{A}_0$, which by definition is lower semicontinuous with respect to the $L^p_{\rm loc}(\Omega)$-convergence. Hence $\overline{\mathcal{F}}(\cdot,A)$ is lower semicontinuous as the supremum of a family of lower semicontinuous function, for all $A\in\mathcal{A}(\Omega)$.

The proof of (i) is a direct consequence of the definition of Γ-convergence and the locality property of the functionals F_ε.

The proof of (ii), for $u\in W^{1,\infty}(\Omega)$, is a consequence of the growth conditions (4). In fact, by the pointwise inequality $|D_{\mu_\varepsilon}u|\le|Du|$, then for each $u\in W^{1,\infty}(\Omega)$, $A\in\mathcal{A}(\Omega)$

$$F_\varepsilon(u,A)\le c_2\int_A(1+|D_{\mu_\varepsilon}u|^p)\,d\mu_\varepsilon\le c_2\int_A(1+|Du|^p)\,d\mu_\varepsilon.$$

Note that if $A'\subset\subset A$ is Lipschitz then we have

$$\lim_{\varepsilon\to0}\int_{A'}(1+|Du|^p)d\mu_\varepsilon=\mu(Y)\int_{A'}(1+|Du|^p)\,dx\le\mu(Y)\int_A(1+|Du|^p)\,dx,$$

and hence for all $A\in\mathcal{A}(\Omega)$

$$\overline{\mathcal{F}}(u,A)\le c_2\mu(Y)\int_A(1+|Du|^p)\,dx. \tag{42}$$

By the lower semicontinuity of $\overline{\mathcal{F}}$ with respect to the $W^{1,p}$-convergence this inequality extends to $W^{1,p}(\Omega)$ by approximation.

Actually, $\overline{\mathcal{F}}$ satisfies also a lower bound on $W^{1,p}(\Omega)$. In fact, if we choose $u_\varepsilon\stackrel{\mu_\varepsilon}{\longrightarrow}u$ such that

$$\sup\|u_\varepsilon\|_{L^p(\Omega,\mu_\varepsilon)}<+\infty\qquad\text{and } \mathcal{F}(u,A)=\lim_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,A),$$

then, by (16), which is valid also with Ω replaced by any Lipschitz subset of A, we obtain that

$$\int_A|Du|^p\,dx\le\frac{k_0}{c_1}\overline{\mathcal{F}}(u,A). \tag{43}$$

Hence, we have proved that there exist k_1, k_2, such that $0<k_1\le k_2$ and

$$k_1\int_A|Du|^p\,dx\le\overline{\mathcal{F}}(u,A)\le k_2\int_A(1+|Du|^p)\,dx \tag{44}$$

for all $u\in W^{1,p}(\Omega)$, and every $A\in\mathcal{A}(\Omega)$.

The proof of (iii) is more delicate. According to the De Giorgi - Letta Criterion (see Theorem 10.2 in [7]) we have to show that, for every $u\in W^{1,p}(\Omega)$, the set function $\overline{\mathcal{F}}(u,\cdot)$ is additive, and inner regular. The superadditivity is obviuous. The proof of the subadditivity, which is based on the Fundamental Estimate (38), and the upper bound (44), can be obtained by arguing as in the proof of Proposition 11.6 in [7].

The proof of (iv) is a direct consequence of the definition of Γ-convergence and the translation invariance in u of the functionals F_ε.

A classical translation argument, independent from the functional setting, shows that indeed $\varphi(x,\xi) = \varphi(\xi)$ (see [7] Proposition 14.3). By Theorem 9.1 in [7] we then have that

$$\overline{\mathcal{F}}(u,A) = \int_A \varphi(Du)dx \quad \text{ for all } u \in W^{1,p}(\Omega), A \in \mathcal{A}(\Omega).$$

Step 5 (inner regularity) Note that

$$F^-(\cdot,A) := \Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(\cdot,A), \qquad F^+(\cdot,A) := \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(\cdot,A),$$

defined for every $A \in \mathcal{A}(\Omega)$ satisfy by definition

$$F^-(\cdot,A) = \mathcal{F}(\cdot,A) = F^+(\cdot,A), \quad \text{ for all } A \in \mathcal{A}_0.$$

If we prove that $F^-(\cdot,A)$, $F^+(\cdot,A)$ are inner regular on Lipschitz open sets, then it follows that

$$\overline{\mathcal{F}}(\cdot,A) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(\cdot,A)$$

on those sets, and in particular on Ω. Now, the inner regularity of $F^\pm(\cdot,A)$ follows from the Fundamental Estimate proved in Step 2 (see, for instance, [7], Chapter 11), upon remarking that the condition that A is Lipschitz is used when using the upper estimate.

Step 6 (convergence with boundary data) From Remark 2 and Lemma 3.3 we deduce that given w in $W^{1,\infty}(\mathbb{R}^n)$ the functionals given by

$$F^w_\varepsilon(u,A) = \begin{cases} F_\varepsilon(u,A) & \text{if } u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n,\mu_\varepsilon),\ u = w \text{ on } \mathbb{R}^n \setminus A, \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge to

$$F^w(u,A) = \begin{cases} F(u,A) & \text{if } u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n),\ u = w \text{ on } \partial A, \\ +\infty & \text{otherwise,} \end{cases}$$

for all A with Lipschitz boundary, and that F^w_ε are equicoercive.

Step 7 (homogenization formula) Another usual subadditivity argument shows that the limit defining f_{hom} exists ([7] Proposition 14.4). To prove the equality $\varphi = f_{\text{hom}}$, it suffices then to apply Step 4 with $w = \xi\cdot x$ and $A = (0,1)^n$, and the convergence of minimum problems for Γ-converging equi-coercive sequences to get

$$\begin{aligned} \varphi(\xi) &= \min\Big\{\int_{(0,1)^n} \varphi(\xi + Du)\,dx : u \in W^{1,p}_0((0,1)^n;\mathbb{R}^m)\Big\} \\ &= \lim_{\varepsilon\to 0}\inf\Big\{F_\varepsilon(\xi\cdot x + u,(0,1)^n) : u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n,\mu_\varepsilon),\ u = 0 \text{ on } \mathbb{R}^n\setminus(0,1)^n\Big\} \\ &= f_{\text{hom}}(\xi), \end{aligned}$$

where we have used the quasiconvexity of φ in the first equality. As a consequence, we have obtained that

$$\mathcal{F}(u,A) = F(u,A) \quad \text{ for all } u \in W^{1,p}(\Omega), \tag{45}$$

and A with Lipschitz boundary. Moreover, the convergence in (35) and (36) hold true as $\varepsilon \to 0$ independently on the subsequence.

Step 8 By Step 1 we have proved the Γ-convergence for $A \subset\subset \Omega$. We now conclude the proof by showing that it holds also on the whole Ω.

We first assume that $u \in W^{1,\infty}(\Omega)$. We prove the Γ-limsup inequality (b). The proof of Γ-liminf inequality (a) is similar. For every constant $\eta > 0$, there exists a regular set $A_\eta \subset\subset \Omega$ such that

$$\int_{\Omega\setminus A_\eta} (1 + |u|^p + |D_{\mu_\varepsilon} u|^p)\, d\mu_\varepsilon < \eta \quad \forall\varepsilon > 0. \tag{46}$$

Let $u_\varepsilon \in W^{1,p}(\Omega, \mu_\varepsilon)$ satisfy condition (b) in Definition 3.1 for the functionals $(F_\varepsilon + G_\varepsilon)(\cdot, A_\eta)$, $(F + G)(\cdot, A_\eta)$, i.e. $u_\varepsilon \xrightarrow{\mu_\varepsilon} u$ and

$$(F + G)(u, A_\eta) \geq \limsup_\varepsilon (F_\varepsilon + G_\varepsilon)(u_\varepsilon, A_\eta).$$

By Lemma 3.3 there exists $v_\varepsilon \in W^{1,p}(\Omega, \mu_\varepsilon)$ such that $v_\varepsilon = u$ in $\Omega \setminus A_\eta$ and

$$\limsup_\varepsilon (F_\varepsilon + G_\varepsilon)(u_\varepsilon, A_\eta) \geq \limsup_\varepsilon (F_\varepsilon + G_\varepsilon)(v_\varepsilon, A_\eta).$$

Now, by our assumption

$$\limsup_\varepsilon (F_\varepsilon + G_\varepsilon)(v_\varepsilon, A_\eta) \geq \limsup_\varepsilon (F_\varepsilon + G_\varepsilon)(v_\varepsilon, \Omega) - \eta$$

So we have obtained that

$$(F + G)(u, \Omega) \geq (F + G)(u, A_\eta) \geq \limsup_\varepsilon (F_\varepsilon + G_\varepsilon)(v_\varepsilon, \Omega) - \eta.$$

By the arbitrariness of η we have proved condition (b) in Definition 3.1 for $u \in W^{1,\infty}(\Omega)$ and $A = \Omega$.

Now we show that the Definition 3.1 is satisfied also when $u \in W^{1,p}(\Omega)$ and $A = \Omega$. In this case, the Γ-liminf inequality is trivial. To prove the Γ-limsup inequality, let $u \in W^{1,p}(\Omega)$ and $u_j \in W^{1,\infty}(\Omega)$ converge strongly to u in $W^{1,p}(\Omega)$. By the lower-semicontinuity of $\mathcal{H}$ (defined above, in Step 4), the continuity of $F+G$, and the fact that $\mathcal{H} = F + G$, we have

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} (F_\varepsilon + G_\varepsilon)(u, \Omega) = \mathcal{H}(u, \Omega) \leq \liminf_{j\to+\infty} (F + G)(u_j, \Omega) = (F + G)(u, \Omega)$$

as desired. □

4. **A counterexample.**

Remark 4. As for the case of functionals defined on measures dealt with in [2] and contrary to the usual homogenization results in the framework of ordinary Sobolev spaces, the hypothesis that Ω has a Lipschitz boundary cannot be removed from Theorem 3.2. To check this, we can use the same geometry as in the corresponding counterexample in [2], where $n = 2$ and

$$\Omega = \Big(\bigcup_{i=1}^\infty (q_i - 2^{-i-3}, q_i + 2^{-i-3}) \times (0, 1)\Big) \cup \Big(\bigcup_{i=1}^\infty (0, 1) \times (q_i - 2^{-i-3}, q_i + 2^{-i-3})\Big),$$

where (q_i) is a numbering of $\mathbb{Q}\cap(0, 1)$. Take as μ the measure of Example 1 and any f in Theorem 3.2. Note that, since the support of each measure $\mu_{1/k}$ is contained in Ω then the two spaces $W^{1,p}_{\mu_{1/k}}(\Omega\cap(0, 1)^2; \mathbb{R}^m)$ and $W^{1,p}_{\mu_{1/k}}((0, 1)^2; \mathbb{R}^m)$ are equivalent, and

$$F_{1/k}(u, \Omega \cap (0, 1)^2) = F_{1/k}(u, (0, 1)^2).$$

If the thesis of Theorem 3.2 were true, then we would easily conclude that for all $v \in W^{1,p}(\Omega \cap (0,1)^2; \mathbb{R}^m)$ with $F(v, \Omega \cap (0,1)^2) < +\infty$ there exists $u \in W^{1,p}((0,1)^2; \mathbb{R}^m)$ with $u = v$ on $\Omega \cap (0,1)^2$ and

$$F(v, \Omega \cap (0,1)^2) = F(u, (0,1)^2),$$

which gives a contradiction since $|\Omega \cap (0,1)^2| \neq |(0,1)^2|$.

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