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On a type of evolution of self-referred and hereditary phenomena

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Summary. In this note we establish some results of local existence and uniqueness for the equations

$$u(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau u(x,s)ds, \tau\right) d\tau, \quad t \geq 0, x \in \mathbb{R},$$

$$u(x,t) = u_0(x) + \int_0^t u\left(\frac{1}{\tau}\int_0^\tau u(x,s)ds, \tau\right) d\tau, \quad t \geq 0, x \in \mathbb{R}$$

and

$$u(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u(\varepsilon,s)d\varepsilon ds, \tau\right) d\tau, t \geq 0, x \in \mathbb{R}.$$

or, equivalently, for the initial value problem, respectively:

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = u\left(\displaystyle\int_0^t u(x,s)ds, t\right), \quad t \geq 0, x \in \mathbb{R} \\ u(x,0) = u_0(x), \quad x \in \mathbb{R} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = u\left(\dfrac{1}{t}\displaystyle\int_0^t u(x,s)ds, t\right), \quad t \geq 0, x \in \mathbb{R} \\ u(x,0) = u_0(x), \quad x \in \mathbb{R} \end{cases}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = u\left(\displaystyle\int_0^t \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u(\xi,\tau)d\xi d\tau, t\right), \quad t \geq 0, x \in \mathbb{R} \\ u(x,0) = u_0(x), \quad x \in \mathbb{R} \end{cases}$$

where u_0 e δ are given functions satisfying suitable conditions.

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1. Introduction

An important class of problems that have interested for a long time the mathematicians are the so called hereditarian problems, that is phenomena carrying on some memory of the past. Important studies have been developed starting from Volterra's works (see [9]), published at the beginning of 20th century.

The fundamental idea of Volterra consisted in recognizing that inside a given accepted model, that mathematically formalizes a constitutive law of a given physical phenomenon, it was sometimes necessary to consider also the possible action due to "memory" of the phenomenon itself. In general, then, Volterra proposed a further precisation of the used models in order to mathematically express the constitutive laws of the phenomenon.

Recently, there has been an increasing request of Mathematics in the study of phenomena for which, at today knowledge, there are not known (or it seems to be not possible) general principles or universally accepted constitutive laws, although it is possible to recognize evolutions of such phenomena depending on their past time history.

Phenomena with evolution depending also on the state of phenomenon has been studied in the framework of functional differential equations (see [6], [1], [7], [8]).

The proposed equations are one of the possible attempts to model the evolution of a phenomenon for which it is possible to omit the dependence from constitutive laws and such that it presents only some kind of self-reference with respect to its history.

A possible, qualitative, interpretation of the proposed equations can be the following. The equation

$$\frac{\partial}{\partial t}u(x,t) = u\left(\int_0^t u(x,s)ds, t\right)$$

can be interpreted as one of the possible evolutions of reasoning. If x represent a fact, $u(x,t)$ the reasoning on the fact x at time t, the given model assures that the evolution of reasoning in time of the fact x depends on reasonings at time t over all reasoning done up to time t.

The equation

$$\frac{\partial}{\partial t}u(x,t) = u\left(\int_0^t \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u(\xi,t)d\xi ds, t\right)$$

can be viewed again with the previous interpretation, when however we suppose that in the reasonings up to time t we have also considered facts "close" to x and we want "remember" the averages.

The equation

$$\frac{\partial}{\partial t}u(x,t) = u\left(\frac{1}{t}\int_0^t u(x,s)ds, t\right)$$

can be easily interpreted in a similar way, but also as a possible evolution of the price u of a product x at time t. The evolution depends on the price of the averages x until time t.

2. Some preliminary results

In this section we state and prove some technical preliminary results that we will use in the sequel. We denote by X the space of continuous functions $u:\mathbb{R}\times[0,+\infty)\to\mathbb{R}$; it makes then sense to consider the transformation $T:X\to X$, $u\mapsto Tu$ for which

$$Tu(x,t)=u_0(x)+\int_0^t u\left(\int_0^\tau u(x,s)ds,\tau\right)d\tau,$$

where $u_0\in\mathcal{C}(\mathbb{R},\mathbb{R})$. For this transformation we have the following properties.

Proposition 1. *Let $u_0\in\mathcal{C}(\mathbb{R},\mathbb{R})$ and let us suppose that*

$$\exists L_0>0 \text{ s.t. } \forall x_1,x_2\in\mathbb{R},\quad |u_0(x_1)-u_0(x_2)|\le L_0|x_1-x_2|. \tag{1}$$

If for $u\in X$ there exists a continuous function $L_u:(0,+\infty)\to[0,+\infty)$ such that

$$\forall x_1,x_2\in\mathbb{R},\quad |u(x_1,t)-u(x_2,t)|\le L_u(t)|x_1-x_2|, \tag{2}$$

then we have that for any $x_1,x_2\in\mathbb{R}$

$$|Tu(x_1,t)-Tu(x_2,t)|\le|x_1-x_2|\left(L_0+\frac{1}{2}\left(\int_0^t L_u(s)ds\right)^2\right).$$

Proof. First of all, we notice that

$$\begin{aligned}Tu(x_1,t)-Tu(x_2,t)\;=\;&u_0(x_1)-u_0(x_2)\\&+\int_0^t\left[u\left(\int_0^\tau u(x_1,s)ds,\tau\right)\right.\\&\left.-\,u\left(\int_0^\tau u(x_2,s)ds,\tau\right)\right]d\tau.\end{aligned}$$

Therefore, using hypotheses (1) and (2), we obtain that:

$$\begin{aligned}|Tu(x_1,t)-Tu(x_2,t)|\;\le\;&L_0|x_1-x_2|\\&+\int_0^t L_u(\tau)\left|\int_0^\tau(u(x_1,s)-u(x_2,s))\,ds\right|d\tau\\\le\;&L_0|x_1-x_2|\\&+\int_0^t L_u(\tau)\left(\int_0^\tau L_u(s)|x_1-x_2|ds\right)d\tau\\=\;&|x_1-x_2|\left[L_0+\int_0^t L_u(\tau)\left(\int_0^\tau L_u(s)ds\right)d\tau\right]\end{aligned}$$

$$= |x_1 - x_2| \left[L_0 + \int_0^t \frac{1}{2} \frac{d}{d\tau} \left(\int_0^\tau L_u(s) \right)^2 d\tau \right]$$
$$= |x_1 - x_2| \left[L_0 + \frac{1}{2} \left(\int_0^t L_u(s) ds \right)^2 \right],$$

and then the proposition follows. □

A second important property of the transformation T is given by the following proposition.

Proposition 2. *Let $u, v \in X$ such that*

1. *there exists a continuous function $L_v : (0, +\infty) \to [0, +\infty)$ satisfying the condition*

$$\forall x_1, x_2 \in \mathbb{R}, \forall t > 0, \quad |v(x_1, t) - v(x_2, t)| \leq L_v(t) |x_1 - x_2|; \tag{3}$$

2. *there exists a continuous function $A_{u,v} : (0, +\infty) \to [0, +\infty)$ such that*

$$\forall x \in \mathbb{R}, \forall t > 0., \quad |u(x, t) - v(x, t)| \leq A_{u,v}(t). \tag{4}$$

Then we get for every $t > 0$

$$|Tu(x, t) - Tv(x, t)| \leq \int_0^t \left(A_{u,v}(\tau) + L_v(\tau) \int_0^\tau A_{u,v}(s) ds \right) d\tau$$

for all $x \in \mathbb{R}$.

Proof. We start by considering the following quantity;

$$\begin{aligned} Tu(x, t) - Tv(x, t) &= \int_0^t \left[u \left(\int_0^\tau u(x, s) ds, \tau \right) - v \left(\int_0^\tau v(x, s) ds, \tau \right) \right] d\tau \\ &= \int_0^t \left[u \left(\int_0^\tau u(x, s) ds, \tau \right) - v \left(\int_0^\tau u(x, s) ds, \tau \right) \right] d\tau \\ &\quad + \int_0^t \left[v \left(\int_0^\tau u(x, s) ds, \tau \right) - v \left(\int_0^\tau v(x, s) ds, \tau \right) \right] d\tau. \end{aligned}$$

Therefore, using assumptions (3) and (4), it follows that

$$\begin{aligned} |Tu(x, t) - Tv(x, t)| &\leq \int_0^t \left(A_{u,v}(\tau) + L_v(\tau) \int_0^\tau |u(x, s) - v(x, s)| ds \right) d\tau \\ &\leq \int_0^t \left(A_{u,v}(\tau) + L_v(\tau) \int_0^\tau A_{u,v}(s) ds \right) d\tau, \end{aligned}$$

which finishes the proof. □

3. A local existence and uniqueness theorem on time

We denote again by X the space of continuous functions $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ and by T the transformation of X into itself defined by the formula

$$Tu(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau u(x,s)ds, \tau\right) d\tau \tag{5}$$

for some $u_0 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$; in this section we will always assume that the following conditions hold:

1. there exists $L_0 > 0$ such that $|u_0(x_1) - u_0(x_2)| \leq L_0 |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$;
2. $\|u_0\|_\infty < +\infty$.

With this type of initial data, we can state the following theorem that gives the local existence and uniqueness of solutions of equation (5).

Theorem 3. *For any given $u_0 \in Lip(\mathbb{R}, \mathbb{R}) \cap L^\infty(\mathbb{R}, \mathbb{R})$, there exist $\alpha > 0$ and a unique $u = u(x,t)$ defined, continuous and bounded in $\mathbb{R} \times [0, \alpha]$, Lipschitz in the first variable, uniformly with respect to the second one (u is, of course, Lipschitz in the second variable uniformly with respect to the first one), satisfying*

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\displaystyle\int_0^t u(x,\tau), t\right) & (x \in \mathbb{R}, 0 \leq t \leq \alpha) \\ u(x,0) = u_0(x), & (x \in \mathbb{R}), \end{cases}$$

or equivalently

$$u(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau u(x,s)ds, \tau\right) d\tau, \quad x \in \mathbb{R}, 0 \leq t \leq \alpha.$$

Proof. We construct the solution by successive iterations; we start from the initial datum u_0 and we consider the sequence $(u_n)_n$ defined by recurrence as

$$\begin{aligned} u_1 &= Tu_0, \\ u_{n+1} &= Tu_n, \quad n > 1. \end{aligned}$$

We are going to prove that the sequence $(u_n)_n$ is uniformly convergent to some function u_∞. To this end, we notice that

$$\begin{aligned} u_1(x,t) &= u_0(x) + \int_0^t u_0\left(\int_0^\tau u_0(x)ds\right) d\tau \\ &= u_0(x) + \int_0^t u_0(u_0(x)\tau) d\tau. \end{aligned}$$

Therefore we have

$$\forall x \in \mathbb{R}, t > 0, \quad |u_1(x,t) - u_0(x)| \leq \|u_0\|_\infty t.$$

On the other hand

$$\begin{aligned}|u_1(x_1,t)-u_1(x_2,t)| &\leq L_0|x_1-x_2|+\int_0^t L_0|x_1-x_2|L_0\tau d\tau\\ &= |x_1-x_2|\left(L_0+\int_0^t L_0^2\tau d\tau\right)\\ &= |x_1-x_2|\left(L_0+L_0^2\frac{t^2}{2}\right).\end{aligned}$$

We set

$$\begin{aligned}A_{1,0}(t) &= \|u_0\|_\infty t,\\ L_1(t) &= L_0+L_0^2\frac{t^2}{2}.\end{aligned}$$

Since $|u_1(x,t)| \leq \|u_0\|_\infty(1+t)$, if $\alpha > 0$ is a fixed number, it is possible by induction to prove that

$$\forall n\in\mathbb{N}, \forall t\in[0,\alpha], \forall x\in\mathbb{R}, \quad |u_n(x,t)| \leq \|u_0\|_\infty \sum_{i=0}^n \frac{t^i}{i!} \leq \|u_0\|_\infty e^\alpha;$$

hence

$$\forall n\in\mathbb{N}, \quad \|u_n\|_{L^\infty(\mathbb{R}\times[0,\alpha])} \leq e^\alpha\|u_0\|_\infty.$$

Let us now consider

$$u_2(x,t)-u_1(x,t)=\int_0^t\left[u_1\left(\int_0^\tau u_1(x,s)ds,\tau\right)-u_0\left(\int_0^\tau u_0(x)ds\right)\right]d\tau.$$

Using Proposition 2, we obtain that

$$|u_2(x,t)-u_1(x,t)| \leq \int_0^t\left(A_{1,0}(\tau)+L_0\int_0^\tau A_{1,0}(s)ds\right)d\tau;$$

similarly, using Proposition 1 we have that

$$|u_2(x_1,t)-u_2(x_2,t)| \leq |x_1-x_2|\left(L_0+\frac{1}{2}\left(\int_0^t L_1(s)ds\right)^2\right).$$

Therefore, we define

$$\begin{aligned}A_{2,1}(t) &:= \int_0^t\left(A_{1,0}(\tau)+L_0\int_0^\tau A_{1,0}(s)ds\right)d\tau,\\ L_2(t) &= L_0+\frac{1}{2}\left(\int_0^t L_1(s)ds\right)^2\end{aligned}$$

and, for arbitrary $n\in\mathbb{N}$

$$A_{n+1,n}(t)=\int_0^t\left(A_{n,n-1}(\tau)+L_{n-1}(\tau)\int_0^\tau A_{n,n-1}(s)ds\right)d\tau, \tag{6}$$

$$L_{n+1}(t) = L_0 + \frac{1}{2}\left(\int_0^t L_n(s)ds\right)^2. \tag{7}$$

We notice that $A_{n+1,n}$ and L_{n+1} are polynomials in the variable t, and, by induction on n, it is possible to prove that

$$\forall x \in \mathbb{R}, \forall t > 0, \forall n \in \mathbb{N}, \quad |u_{n+1}(x,t) - u_n(x,t)| \leq A_{n+1,n}(t).$$

For reasons that will be clear in the sequel, we assume $h \in (k_0/(k_0+L_0), 1)$. Let us fix $k_0 > 0$ and $\alpha > 0$ such that

$$(k_0+L_0)^2\alpha^2 \leq 2k_0 \quad \text{and} \quad \alpha + \frac{1}{2}(k_0+L_0)\alpha^2 \leq h < 1.$$

By setting $k_1 = k_0 + L_0$, we prove that for every $0 \leq t \leq \alpha$ and every $n \in \mathbb{N}$

$$\begin{aligned}
A_{n+1,n}(t) &\leq \int_0^t (\|A_{n,n-1}\|_{L^\infty([0,\alpha])} + k_1\tau\|A_{n,n-1}\|_{L^\infty([0,\alpha])})d\tau \\
&= \|A_{n,n-1}\|_{L^\infty([0,\alpha])}\int_0^t (1+k_1\tau)d\tau \\
&= \|A_{n,n-1}\|_{L^\infty([0,\alpha])}\left(t + k_1\frac{t^2}{2}\right),
\end{aligned}$$

and therefore

$$\|A_{n+1,n}\|_{L^\infty([0,\alpha])} \leq h\|A_{n,n-1}\|_{L^\infty([0,\alpha])}.$$

From the definition of $L_1(t)$, it follows that:

$$0 \leq L_1(t) - L_0 = \frac{1}{2}L_0^2t^2 \leq k_0,$$

Hence we obtain

$$\begin{aligned}
0 \leq L_2(t) - L_0 &= \frac{1}{2}\left(\int_0^t L_1(s)ds\right)^2 \\
&= \frac{1}{2}\left(\int_0^t (L_1(s) - L_0 + L_0)ds\right)^2 \\
&= \frac{1}{2}\left(\int_0^t (L_1(s) - L_0)ds + L_0t\right)^2 \\
&\leq \frac{1}{2}\left(\int_0^t k_0ds + L_0t\right)^2 = \frac{1}{2}(k_0+L_0)^2t^2 \leq k_0.
\end{aligned}$$

Therefore, it is easy to prove by recurrence that

$$\forall n \in \mathbb{N}, \forall t \in [0,\alpha], \quad 0 \leq L_n(t) - L_0 \leq k_0.$$

Hence we get

$$\exists k_1 > 0 : \forall t \in [0,\alpha], \forall n \in \mathbb{N} \quad 0 \leq L_n(t) \leq k_1. \tag{8}$$

From the previous inequality, for $t \in [0,\alpha]$, taking into account (6) and that $A_{n+1,n}(\cdot)$ are continuous functions, we deduce that

$$\begin{aligned} 0 \le A_{n+1,n}(t) &\le \int_0^t \left(\|A_{n,n-1}\|_{L^\infty([0,\alpha])} + L_{n-1}(\tau) \int_0^\tau \|A_{n,n-1}\|_{L^\infty([0,\alpha])} ds \right) d\tau \\ &\le \|A_{n,n-1}\|_{L^\infty([0,\alpha])} \left(t + k_1 \frac{t^2}{2} \right). \end{aligned}$$

With our choice of α we can assert:

$$\forall n \in \mathbb{N}, \quad \|A_{n+1,n}\|_{L^\infty([0,\alpha])} \le h \|A_{n,n-1}\|_{L^\infty([0,\alpha])}. \tag{9}$$

From this follows that

$$\sum_{n=0}^{\infty} \|A_{n+1,n}\|_{L^\infty([0,\alpha])}$$

is convergent and the same holds for the series

$$\sum_{n=0}^{\infty} (u_{n+1}(x,t) - u_n(x,t)), \quad (x \in \mathbb{R}, t \in [0,\alpha]).$$

If now we consider $X_\alpha = X \cap \mathcal{C}(\mathbb{R} \times [0,\alpha])$, we get that

$$\exists u \in X_\alpha : \quad u_n \to u \text{ uniformly in } \mathbb{R} \times [0,\alpha].$$

We notice that, since for every x and for every $t \in [0,\alpha]$ we have

$$|u_n(x,t) - u_n(y,t)| \le L_n(t)|x-y| \le k_1|x-y|,$$

also

$$|u(x,t) - u(y,t)| \le k_1|x-y|$$

holds for each x and for each $t \in [0,\alpha]$. Therefore

$$\begin{aligned} &\left| u_n\left(\int_0^t u_n(x,\tau) d\tau, t \right) - u\left(\int_0^t u(x,\tau) d\tau, t \right) \right| \\ &\le \left| u_n\left(\int_0^t u_n(x,\tau) d\tau, t \right) - u_n\left(\int_0^t u(x,\tau) d\tau, t \right) \right| \\ &\quad + \left| u_n\left(\int_0^t u(x,\tau) d\tau, t \right) - u\left(\int_0^t u(x,\tau) d\tau, t \right) \right| \\ &\le k_1 \int_0^t |u_n(x,\tau) - u(x,\tau)| d\tau + \|u_n - u\|_\infty \\ &\le k_1 \int_0^\alpha |u_n(x,\tau) - u(x,\tau)| d\tau + \|u_n - u\|_\infty \to 0 \end{aligned}$$

uniformly with respect to x and t. Moreover, we get that $Tu = u$, i.e.

$$u(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau (x,s) ds, \tau \right) d\tau, \quad \forall x \in \mathbb{R}, \forall t \in [0,\alpha].$$

Consequently, we can differentiate with respect to t and conclude that u satisfies, for all $t \in [0, \alpha]$ and for all $x \in \mathbb{R}$

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\displaystyle\int_0^t u(x,\tau)d\tau, t\right) \\ u(x,0) = u_0(x). \end{cases}$$

Moreover, we consider $\bar{u} = \bar{u}(x,t)$ (an element of X_α) such that

$$T\bar{u} = \bar{u}.$$

Since

$$\begin{aligned} T\bar{u}(x,t) - Tu(x,t) = & \int_0^t \left[\bar{u}\left(\int_0^\tau \bar{u}(x,s)ds, \tau\right) - u\left(\int_0^\tau \bar{u}(x,s)ds, \tau\right)\right. \\ & \left. + u\left(\int_0^\tau \bar{u}(x,s)ds, \tau\right) - u\left(\int_0^\tau u(x,s)ds, \tau\right)\right] d\tau, \end{aligned}$$

if $x \in K$ with K compact and $t \in [0, \alpha]$, then we have that

$$\begin{aligned} |T\bar{u}(x,t) - Tu(x,t)| \le & \int_0^t \left(\|\bar{u} - u\|_{L^\infty(K\times[0,\alpha])} + L_u(\tau)\int_0^\tau \|\bar{u} - u\|_{L^\infty(K\times[0,\alpha])}\right) d\tau \\ = & \|\bar{u} - u\|_{L^\infty(K\times[0,\alpha])} \int_0^t (1 + L_u(\tau)\tau) d\tau \\ \le & \|\bar{u} - u\|_{L^\infty(K\times[0,\alpha])} \int_0^t (1 + k_1\tau) d\tau \\ = & \|\bar{u} - u\|_{L^\infty(K\times[0,\alpha])} \left(t + k_1 \frac{t^2}{2}\right) \\ \le & h\|\bar{u} - u\|_{L^\infty(K\times[0,\alpha])}, \end{aligned}$$

which ends the proof. □

Using the same proof of the previous theorem, it is possible also to prove the following result.

Theorem 4. *For any given $u_0 \in Lip(\mathbb{R},\mathbb{R}) \cap L^\infty(\mathbb{R},\mathbb{R})$, there exist $\alpha > 0$ and a unique $u = u(x,t)$ defined, continuous and bounded on $\mathbb{R} \times [0,\alpha]$, Lipschitz in the first variable (uniformly with respect to the second one – u is, of course, Lipschitz in the second variable uniformly with respect to the first one), satisfying*

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\dfrac{1}{t}\displaystyle\int_0^t u(x,\tau), t\right) & x \in \mathbb{R}, 0 \le t \le \alpha \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

or equivalently

$$u(x,t) = u_0(x) + \int_0^t u\left(\frac{1}{t}\int_0^\tau u(x,s)ds, \tau\right) d\tau, \quad x \in \mathbb{R}, 0 \le t \le \alpha.$$

In this case formulae (6) and (7) become

$$A_{n+1,n}(t) = \int_0^t \left(A_{n,n-1}(\tau) + L_{n-1}(\tau)\frac{1}{\tau}\int_0^\tau A_{n,n-1}(s)ds \right) d\tau,$$

$$L_{n+1}(t) = L_0 + \int_0^t \left(L_n(\tau)\frac{1}{\tau}\int_0^\tau L_n(s)ds \right)^2,$$

and they imply (8) and (9) with $0 \leq t \leq \alpha$, for a suitable $\alpha > 0$. Therefore, repeating for Theorem 4 the proof of Theorem 3, we obtain the result.

4. An equation with averaged memory

Let again X be the space of continuous functions $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ and let us consider the transformation

$$Tu(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u(\xi,s)d\xi ds, \tau \right) d\tau \tag{10}$$

with u_0 and δ given real continuous functions.

Remark. Given a continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$, there holds:

$$\begin{aligned}
\left| \int_{x-\delta}^{x+\delta} f(\xi)d\xi - \int_{y-\delta}^{y+\delta} f(\xi)d\xi \right| &= \left| \int_{x-\delta}^{0} f(\xi)d\xi + \int_0^{x+\delta} f(\xi)d\xi \right. \\
&\qquad \left. - \int_{y-\delta}^{0} f(\xi)d\xi - \int_0^{y+\delta} f(\xi)d\xi \right| \\
&= \left| \int_{x-\delta}^{y-\delta} f(\xi)d\xi + \int_{y+\delta}^{x+\delta} f(\xi)d\xi \right| \\
&\leq 2\|f\|_\infty |x-y|.
\end{aligned}$$

Let us now suppose that

1. $\|u_0\|_{L^\infty(\mathbb{R},\mathbb{R})} < +\infty$;
2. there is $L_0 \geq 0$ such that $|u_0(x) - u_0(y)| \leq L_0|x-y|$ for every $x, y \in \mathbb{R}$;
3. $\delta : \mathbb{R} \to [0, +\infty)$ is chosen in such a way that $\forall t > 0$, $\int_0^t \frac{1}{\delta(s)} ds < +\infty$.

We consider the sequence of functions $(u_n)_n$ defined by recurrence,

$$u_1(x,t) = u_0(x) + \int_0^t u_0\left(\int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_0(\xi)d\xi ds \right) d\tau,$$

$$u_{n+1}(x,y) = u_0(x) + \int_0^t u_n\left(\int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_n(\xi,s)d\xi ds, \tau \right) d\tau.$$

Remark. $|u_1(x,t)| \le \|u_0\|_\infty(1+t)$ and in general (taking into account the definition of u_n) we can prove by induction that

$$\forall t > 0, x \in \mathbb{R}, \quad |u_n(x,t)| \le \|u_0\|_\infty \sum_{i=0}^{n} \frac{t^i}{i!}.$$

Therefore, given $\alpha > 0$, we can conclude that

$$\forall n \in \mathbb{N}, \quad \|u_n\|_{L^\infty(\mathbb{R}\times[0,\alpha))} \le \|u_0\|_\infty \sum_{i=0}^{n} \frac{\alpha^i}{i!} \le e^\alpha \|u_0\|_\infty.$$

We notice now that

$$\forall x \in \mathbb{R}, \forall t > 0, \quad |u_1(x,t) - u_0(x)| \le \|u_0\|_\infty t = A_1(t).$$

Then, since

$$\begin{aligned} u_1(x,t) - u_1(y,t) \;=\; & u_0(x) - u_0(y) + \int_0^t \left[u_0 \left(\int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_0(\xi) d\xi ds \right) \right. \\ & \left. - u_0 \left(\int_0^\tau \frac{1}{2\delta(s)} \int_{y-\delta(s)}^{y+\delta(s)} u_0(\xi) d\xi ds \right) \right] d\tau, \end{aligned}$$

we get, from the previous remark, that

$$\begin{aligned} & |u_1(x,t) - u_1(y,t)| \\ & \le L_0|x-y| + \int_0^t L_0 \int_0^\tau \left| \frac{1}{2\delta(s)} \left(\int_{x-\delta}^{x+\delta} u_0(\xi)d\xi - \int_{y-\delta}^{y+\delta} u_0(\xi)d\xi \right) \right| ds d\tau \\ & \le L_0|x-y| + \int_0^t L_0 \int_0^\tau \frac{1}{2\delta(s)} \|u_0\|_\infty 2|x-y| ds d\tau \\ & = \left(L_0 + \|u_0\|_\infty \int_0^t L_0 \int_0^\tau \frac{1}{\delta(s)} ds d\tau \right) |x-y| = L_1(t)|x-y|, \end{aligned}$$

where we have set

$$L_1(t) = L_0 + \|u_0\|_\infty \int_0^t L_0 \int_0^\tau \frac{1}{\delta(s)} ds d\tau.$$

Keeping in mind that

$$u_2(x,t) = u_0(x) + \int_0^t u_1 \left(\int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s) d\xi ds, \tau \right) d\tau,$$

we have that

$$u_2(x,t) - u_1(x,t) = \int_0^t \left[u_1 \left(\int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s) d\xi ds, \tau \right) \right.$$

$$
\begin{aligned}
&-u_0\left(\int_0^\tau \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s)d\xi ds\right)\\
&+u_0\left(\int_0^\tau \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s)d\xi ds\right)\\
&-u_0\left(\int_0^\tau \frac{1}{2\delta(s)}\int_{x\delta(s)}^{x+\delta(s)} u_0(\xi)d\xi ds\right)\Bigg] d\tau.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&|u_2(x,t)-u_1(x,t)|\\
&\leq \int_0^t \left(A_1(\tau)+L_0\int_0^\tau \left(\frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} |u_1(\xi,s)-u_0(\xi)|d\xi ds\right)\right)d\tau\\
&\leq \int_0^t \left(A_1(\tau)+L_0\int_0^\tau A_1(s)ds\right)d\tau = A_2(t).
\end{aligned}
$$

Moreover, we obtain

$$
\begin{aligned}
|u_2(x,t)-u_2(y,t)| \leq\ & |u_0(x)-u_0(y)|\\
&+\left|\int_0^t u_1\left(\int_0^\tau \frac{1}{2\delta(s)}\int_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s)d\xi ds,\tau\right)d\tau\right.\\
&\left.-\int_0^t u_1\left(\int_0^\tau \frac{1}{2\delta(s)}\int_{y-\delta(s)}^{y+\delta(s)} u_1(\xi,s)d\xi ds,\tau\right)d\tau\right|\\
\leq\ & L_0|x-y|+\int_0^t L_1(\tau)\int_0^\tau \frac{1}{2\delta(s)}\left|\int_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s)d\xi\right.\\
&\left.-\int_{y-\delta(s)}^{y+\delta(s)} u_1(\xi,s)d\xi\right|dsd\tau\\
\leq\ & L_0|x-y|+\int_0^t L_1(\tau)\int_0^\tau \frac{1}{\delta(s)}\|u_1\|_{L^\infty(\mathbb{R}\times[0,\alpha])}|x-y|dsd\tau\\
\leq\ & \left(L_0+e^\alpha\|u_0\|_{L^\infty}\int_0^t L_1(\tau)\int_0^\tau \frac{1}{\delta(s)}dsd\tau\right)|x-y|\\
=\ & L_2(t)|x-y|,
\end{aligned}
$$

where

$$
L_2(t) = L_0+e^\alpha\|u_0\|_{L^\infty}\int_0^t\left(L_1(\tau)\int_0^\tau \frac{1}{\delta(s)}ds\right)d\tau.
$$

It can be easily proved by induction that

$$
\forall x\in\mathbb{R}, 0\leq t\leq\alpha, \forall n\in N,\quad |u_{n+1}(x,t)-u_n(x,t)|\leq A_{n+1}(t),
$$

$$\forall x, y \in \mathbb{R}, 0 \le t \le \alpha, \forall n \in N, \quad |u_{n+1}(x,t) - u_{n+1}(y,t)| \le L_{n+1}(t)|x-y|,$$

where we set

$$A_{n+1}(t) = \int_0^t \left(A_n(\tau) + L_{n-1}(\tau) \int_0^\tau A_n(s)ds \right) d\tau \quad 0 \le t \le \alpha;$$
$$L_{n+1}(t) = L_0 + \|u_0\|_\infty e^\alpha \int_0^\tau \left(L_n(\tau) \int_0^\tau \frac{1}{\delta(s)} ds \right) d\tau \quad 0 \le t \le \alpha.$$

If we set $c^* = \int_0^\alpha \frac{1}{\delta(s)} ds$, we can choose $\alpha^* \in]0, \alpha]$ and $M_0 > 0$ in such a way that for every $0 \le t \le \alpha^*$

$$\|u_0\|_\infty e^\alpha c^* L_0 t \le M_0; \quad \|u_0\|_\infty e^\alpha c^* (M_0 + L_0) t \le M_0;$$
$$\left(t + \frac{1}{2}(M_0 + L_0)t^2 \right) \le h < 1.$$

holds. Since

$$\begin{aligned} 0 \le L_1(t) - L_0 &\le \|u_0\|_\infty e^\alpha \int_0^\tau \left(L_0 \int_0^\tau \frac{1}{\delta(s)} ds \right) d\tau \\ &\le \|u_0\|_\infty e^\alpha c^* \int_0^\tau L_0 d\tau \le M_0, \end{aligned}$$

and also

$$\begin{aligned} &0 \le L_2(t) - L_0 \le \|u_0\|_\infty e^\alpha \int_0^\tau \left(L_1(\tau) \int_0^\tau \frac{1}{\delta(s)} ds \right) d\tau \\ &\le \|u_0\|_\infty e^\alpha c^* \int_0^\tau L_1(\tau) d\tau \le \|u_0\|_\infty e^\alpha c^* \int_0^\tau (L_1(\tau) - L_0 + L_0) d\tau \\ &\le \|u_0\|_\infty e^\alpha c^* (M_0 + L_0) t \le M_0, \end{aligned}$$

we get then that

$$0 \le t \le \alpha^*, \forall n \in \mathbb{N}, \qquad 0 \le L_n(t) \le M_0 + L_0.$$

Keeping in mind the formula defining A_n, we obtain, by the choice of α^*, that

$$\begin{aligned} A_{n+1}(t) &\le \int_0^t \left(A_n(\tau) + (M_0 + L_0) \int_0^\tau A_n(s) ds \right) d\tau \le \\ &\le \|A_n\|_{L^\infty[0,\alpha^*]} \left(t + \frac{1}{2}(M_0 + L_0) t^2 \right) \le h \|A_n\|_{L^\infty[0,\alpha^*]}. \end{aligned}$$

In conclusion, the series

$$\sum_{n=0}^\infty \|A_n\|_{L^\infty[0,\alpha^*]}$$

is convergent and then we deduce that the sequence (u_n) is uniformly convergent to a function $\bar{u} \in X$ satisfying $T(\bar{u}) = \bar{u}$. Arguing in the same way as in the preceding section, we can prove that if there exists a continuous function $v = v(x,t)$ with $Tv = v$, then $\bar{u} = v$.

We have then proved the following theorem.

Theorem 5. *Let $u_0 \in Lip(\mathbb{R},\mathbb{R}) \cap L^\infty(\mathbb{R},\mathbb{R})$ and $\delta : \mathbb{R} \to [0,+\infty)$ be given functions such that for any $t > 0$*

$$\int_0^t \frac{1}{\delta(s)} ds < +\infty.$$

Then there exist $\alpha > 0$ and a unique $u = u(x,t) \in X$, with $x \in \mathbb{R}$ and $t \in [0,\alpha]$, continuous and bounded, Lipschitz in the first variable (uniformly with respect to the second one – u is of course Lipschitz in the second variable uniformly with respect to the first one) such that

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\displaystyle\int_0^t \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u(\xi,s)d\xi ds, t\right) & t \in [0,\alpha], x \in \mathbb{R} \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases}$$

that is,

$$u(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u(\xi,s)d\xi ds, \tau\right) d\tau,$$

for $x \in \mathbb{R}, 0 \le t \le \alpha$.

5. Some open problems

The previous results and the proposed type of equations can be investigated and generalized in many different ways. In what follows, we give some of the problems whose investigation seems to be interesting; some of them are suggested by well known applications of Mathematics.

A. We notice that the results contained in the theorems are a consequence of the fact that the equations we have considered allow the initial datum u_0 to develop maintaining, for a short time interval, the Lipschitzian character; a first problem would be to establish results of existence (and almost surely of non-uniqueness) for the equations with the only assumption that u_0 is a uniformly continuous and bounded function. The following step would be to consider the function u_0 only continuous and bounded. Moreover, it is not hard to establish some existence lemma for large time.

B. When the existence in large is guaranteed, we can consider the following problem. Assume that the datum u_0 is in $L^1(\mathbb{R})$ and let $u = u(x,t)$ be a solution of the equation. Is it true that $u = u(\cdot, t) \in L^1(\mathbb{R})$ for almost every $t > 0$? Furthermore, if the answer is positive and

$$\int_{\mathbb{R}} |u(x,t)|dx \neq 0, \quad a.e. \quad t > 0,$$

we can consider the following intriguing question: to study the behaviour of the real function

$$\phi(t) = \frac{\int_{\mathbb{R}} u^+(x,t)dx}{\int_{\mathbb{R}} |u(x,t)|dx}.$$

C. Other problems would be given by generalizing the equations, considering them as particular cases of equation of the type

$$\frac{\partial}{\partial t}u(x,t) = u(\psi(u,x,t,\lambda),t), \quad \lambda \in \mathbb{R}$$

where ψ is a given function. For instance, one can consider the equation

$$\frac{\partial}{\partial t}u(x,t) = u\left(\int_0^t \alpha(\tau)u(x,t-\tau)d\tau, t\right), \quad t > 0.$$

D. Much harder problems seem to arise when considering the equations of the second type with $\delta(s) = \delta$, for every $s \in [0, +\infty)$:

$$\frac{\partial}{\partial t}u(x,t) = u\left(\int_0^t \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} u(\xi,s)d\xi ds, t\right), \quad t > 0, \delta > 0.$$

In this case it would be interesting to consider the problem of studying the limit of the solutions as $\delta \to 0$ or as $\delta \to \infty$.

E. If we consider systems of equations, it would be interesting to study systems of the type

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = u(v(x,t)\displaystyle\int_0^t u(x,s)ds, t) \\ \dfrac{\partial}{\partial t}v(x,t) = v(u(x,t)\displaystyle\int_0^t v(x,s)ds, t), \end{cases}$$

or other analogous problems.

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