Dynamics of multiple degree Ginzburg-Landau vortices

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Abstract

For the two dimensional complex parabolic Ginzburg-Landau equation we prove that, asymptotically, vortices evolve according to a simple ordinary differential equation, which is a gradient flow of the Kirchhoff-Onsager functional. This convergence holds except for a finite number of times, corresponding to vortex collisions and splittings, which we describe carefully. The only assumption is a natural energy bound on the initial data.

This paper, together with [3, 4], is companion to [2] where the higher dimensional case is considered.

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1 Introduction

This work closes a series of papers [3, 4] devoted to the study of the two-dimensional complex-valued parabolic Ginzburg-Landau equation

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \qquad \text{on } \mathbb{R}^2 \times \mathbb{R}^+.$$

Our focus is put on the description of the asymptotic behavior of sequences of solutions as $\varepsilon \to 0$, under the only assumption that the initial datum $u_\varepsilon^0(\cdot) \equiv u_\varepsilon(\cdot, 0)$ verifies

$$(\mathrm{H}_0) \qquad E_\varepsilon(u_\varepsilon^0) = \int_{\mathbb{R}^2} e_\varepsilon(u_\varepsilon^0) = \int_{\mathbb{R}^2} [\frac{|\nabla u_\varepsilon^0|^2}{2} + \frac{(1-|u_\varepsilon^0|^2)^2}{4\varepsilon^2}] \le M_0 |\log \varepsilon| \, ,$$

where $M_0 > 0$ is some fixed constant.

This problem has received a lot of attention in the last decade, as our historical review below will show. In particular it has been recognized that the energy regime given by (H_0) allows for the formation of topological defects called vortices. It has also been recognized that the dynamics of these vortices is non trivial accelerating time by a factor $|\log \varepsilon|$, that is considering the functions

$$\mathfrak{u}_\varepsilon(z, s) = u_\varepsilon(z, s|\log \varepsilon|), \qquad z = x + iy \equiv (x, y) \in \mathbb{R}^2.$$

The next theorem, proved in [3, 4], gives a precise meaning to the notion of vortices in this setting, as well as a first description of their dynamics.

Theorem 1 ([3, 4]). *Assume $(u_\varepsilon)_{\varepsilon>0}$ verifies $(PGL)_\varepsilon$ and (H_0). Then, for a subsequence $\varepsilon_n \to 0$ we have*

$$\mathfrak{u}_{\varepsilon_n}(z, s) \to \mathfrak{u}_*(z, s) = \prod_{i=1}^{l(s)} \left(\frac{z - a_i(s)}{|z - a_i(s)|} \right)^{d_i(s)} \exp[i(\langle \vec{c}(s), z \rangle + b(s)], \tag{1}$$

where, for $i = 1, \dots, \ell(s)$, $a_i(s) \in \mathbb{R}^2$, $d_i(s) \in \mathbb{Z}$, $b(s) \in [0, 2\pi)$ *and* $\vec{c} : \mathbb{R}^+ \to \mathbb{R}^2$ *is a lipschitz function. The convergence in* (1) *is uniform on every compact subset of* $\mathbb{R}^2 \times \mathbb{R}^+ \setminus \Sigma_{\mathfrak{v}}$, *where*

$$\Sigma_{\mathfrak{v}} = \cup_{s>0} \cup_{i=1}^{l(s)} \{(a_i(s), s)\}.$$

Moreover, the trajectory set $\Sigma_{\mathfrak{v}}$ *is a closed, 1-dimensional rectifiable subset of* $\mathbb{R}^2$.

We proved moreover that the numbers $l(s)$ and $d_i(s)$ are uniformly bounded by a constant depending only on M_0, and that, except for a finite number of times[1],

$$d_i(s) \neq 0. \tag{2}$$

It is worthwile to notice that the limiting map $\mathfrak{u}_*(\cdot, s)$ has modulus 1, hence with values in the circle S^1, but is singular at the points $a_i(s)$ when $d_i(s) \neq 0$. In this case, it also has diverging local Dirichlet energy. The points $a_i(s)$ are called the vortices at time s, and the integers $d_i(s)$ their degrees: they correspond to the winding numbers of the limiting map $\mathfrak{u}_*(., s)$ around the vortices $a_i(s)$. The number $b(s) \in \mathbb{R}$ corresponds to a constant phase shift, and the vector $\vec{c}(s) \in \mathbb{R}^2$ is reminiscent of a wavenumber.[2]

The set $\Sigma_{\mathfrak{v}}$ describes the evolution in time of the set of vortices, and therefore we refer to it as the trajectory set. The main results of this paper provide a complete description of the trajectory set $\Sigma_{\mathfrak{v}}$. We first have

Theorem 2. *There exists a finite number of times* $0 = \tau_0 < \tau_1 < \cdots < \tau_q < \tau_{q+1} = +\infty$ *such that*

i) The number of vortices $\ell(s) \equiv \ell_k$ *is constant on each interval* (τ_k, τ_{k+1}), *for* $k = 0, \dots, q$.

ii) The restriction of $\Sigma_{\mathfrak{v}}$ *to* $\mathbb{R}^2 \times (\tau_k, \tau_{k+1})$ *is a disjoint union of* ℓ_k *smooth one dimensional graphs. More precisely, relabelling possibly the points* $a_1(s), \dots, a_{\ell_k}(s)$, *their degrees* $d_i(s) = d_i$ *are constant in* (τ_k, τ_{k+1}), *and their trajectories are given by the system of ordinary differential equations*

$$d_i^2 \frac{da_i}{ds}(s) = -\nabla_{a_i} W(a_1, \dots, a_{\ell_k}) + d_i c(s)^\perp, \qquad i = 1, \dots, \ell_k\,, \tag{3}$$

where W *is the Kirchhoff-Onsager function defined as*

$$W(a_1, \dots, a_{\ell_k}) = -2 \sum_{i \neq j = 1}^{\ell_k} d_i d_j \log |a_i - a_j|\,. \tag{4}$$

[1] Which are among the times $\tau_1, \dots, \tau_q$ of Theorem 2

[2] Notice that functions $\vec{c}$ and b depend only on the time variable s, but not on the space variable z. The function $\vec{c}$ can be directly deduced from the initial value, for instance by Fourier transform. It accounts for persistence of low frequency oscillations in the phase over the diverging time period considered, namely $t = s|\log \varepsilon|$. The possible presence of low frequencies is of course related to the fact that the domain $\mathbb{R}^2$ is unbounded.

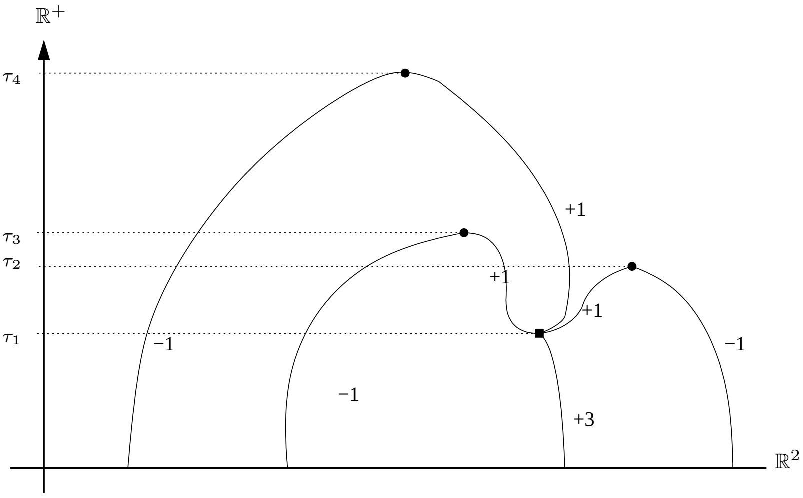


Figure 1 : An example of trajectory set.

The times $\tau_1, \ldots, \tau_q$ were already identified in [4] as the only times of dissipation in an appropriate asymptotic sense. More precisely, the following was proved there (Theorem 4 and Corollary 3.1).

Theorem 3 ([4]). *For $s \notin \{\tau_0, \ldots, \tau_q\}$,*

$$\mathfrak{v}^s_{\varepsilon_n}(x) \equiv \frac{e_{\varepsilon_n}(\mathfrak{u}_{\varepsilon_n}(x,s))}{|\log \varepsilon_n|}\, dx \rightharpoonup \mathfrak{v}^s_* = \pi \sum_{i=1}^{\ell(s)} d_i^2(s) \delta_{a_i(s)} \tag{5}$$

in the sense of measures on $\mathbb{R}^2$, and

$$|\partial_t u_{\varepsilon_n}|^2 dx ds \rightharpoonup \omega_* = \pi \sum_{k=0}^{q} \sum_{i=1}^{\ell(\tau_k)} \beta_i^k \delta_{(a_i(\tau_k),\tau_k)}, \tag{6}$$

in the sense of measures on $\mathbb{R}^2 \times \mathbb{R}^+$, where $\beta_i^k \in \mathbb{N}$.

Since the total energy $\pi \sum d_i^2(s)$ is quantized and non increasing (see [4]), it is also piecewise constant. The times $\tau_1, \ldots, \tau_q$ correspond therefore to the times of energy loss, where dissipation concentrates. The points $(a_i(\tau_k), \tau_k)$ for which $\beta_i^k \neq 0$ are called the *dissipation points.*

At this stage, we have completely described the trajectories inside the intervals (τ_k, τ_{k+1}). The next step is to understand the behavior of the trajectories accross the dissipation times. Since we already know by Theorem 1 that $\Sigma_{\mathfrak{v}}$ is closed, the points $(a_i(\tau_k), \tau_k)$ are the only possible endpoints of the trajectories in (τ_{k-1}, τ_k) and (τ_k, τ_{k+1}). For a given point $(a_i(\tau_k), \tau_k)$, let $\mathcal{C}_1^-, \ldots, \mathcal{C}^-_{l_i^-}$ and $\mathcal{C}_1^+, \ldots, \mathcal{C}^+_{l_i^+}$ denote the vortices trajectories respectively in $\mathbb{R}^2 \times (\tau_{k-1}, \tau_k)$ and $\mathbb{R}^2 \times (\tau_k, \tau_{k+1})$ for which $(a_i(\tau_k), \tau_k)$ is an endpoint. Accordingly, let $d_1^-, \ldots, d^-_{l_i^-}$ and $d_1^+, \ldots, d^+_{l_i^+}$ be the degrees of the corresponding vortices. It follows from (5), (6) and the formula for the evolution of the energy in localized form (see e.g. [4], Proposition 3.3), that

$$\beta_i^k = \sum_{j=1}^{l_i^-} (d_j^-)^2 - \sum_{j=1}^{l_i^+} (d_j^+)^2. \tag{7}$$

We say that a point $(a_i(\tau_k),\tau_k))$ is a *regular point* of the trajectory set if $\Sigma_{\mathfrak{v}}$ is a lipschitz graph over s in the neighborhood of $(a_i(\tau_k),\tau_k)$, or equivalently if $l_i^- = l_i^+ = 1$. If not, we say that $(a_i(\tau_k),\tau_k)$ is a *branching point.*

Theorem 4. *A point $(a_i(\tau_k),\tau_k)$ is a branching point if and only if it is a dissipation point. In this case, we have*

$$\sum_{j=1}^{l_i^-} d_j^- = d_i(\tau_k) = \sum_{j=1}^{l_i^+} d_j^+ \qquad \textit{(Conservation of the degree)} \tag{8}$$

$$\sum_{j=1}^{l_i^-} (d_j^-)^2 \geq d_i^2(\tau_k) \geq \sum_{j=1}^{l_i^+} (d_j^+)^2 \quad \textit{(Energy decrease)} \tag{9}$$

where the first (resp. second) inequality in (9) *is strict whenever $l_i^- \geq 2$ (resp. $l_i^+ \geq 2$). In particular,*

$$\sum_{j=1}^{l_i^-} (d_j^-)^2 > \sum_{j=1}^{l_i^+} (d_j^+)^2.$$

Part of the statements in the previous theorems have already been known in the case $|d_i| = 1$ (see the historical review below). An important a novelty here is that there are no restriction on the d_i's. One may wonder whether multiple degrees may be really observed as limits of solutions to $(\mathrm{PGL})_\varepsilon$. This question is positively answered in Section 5.3.

For a given M_0, the number of integer solutions to (8) - (9) is quite large. We may classify them into four different classes of increasing complexity. First, there are collisions with annihilation, corresponding to $l_i^+ = 0$ (an example of such a collision is provided by Figure 2a). There are also collisions without annihilation nor splitting: here $l_i^+ = 1$, and $l_i^- \geq 2$ (see Figure 2b). Next, there are splittings of single multiple degree vortices, for which $l_i^- = 1$ and $l_i^+ \geq 2$ (see Figure 2c). The remaining solutions to (8) - (9) correspond to simultaneous collisions and splittings, for which $l_i^+ \geq 2$ and $l_i^- \geq 2$ (see Figure 2d). In section 5.3, we will show examples in the classes a, b and c which may be realized through limits of solutions to $(\mathrm{PGL})_\varepsilon$.

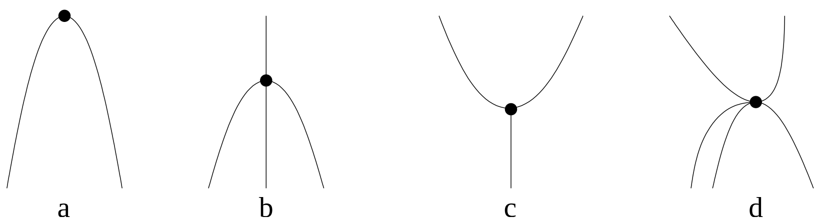


Figure 2 : Four classes of dissipation points.

In a related direction, a natural question is to know if the positions and degrees of the vortices at some time s_0 completely determine their positions and degrees at future times. Whereas collisions are determined by singularities in the ordinary differential equation (3), splittings are not, and clearly Theorem 2 does not settle the question of the occurrence of such splittings. More precisely, the dissipation times $\tau_1,\ldots,\tau_q$ as well as the number of vortices and their degrees involved after splittings are not inferred in a constructive way.

Notice in particular that the algebraic relations (8) and (9) do not have in general a unique solution.[3] As a matter of fact, there is no hope to determine the complete future trajectories by knowing the positions and the degrees at some fixed time: an important part of the relevant information is lost in the limiting procedure. This issue will be discussed in more details in Section 5.3, when we construct an example of splitting.

In Section 6, we study the asymptotic behavior of the trajectories near a branching point. We show that, after a suitable parabolic rescaling centered at the collision point, vortices converge to the set of critical points of W restricted to the manifold

$$\mathcal{M} = \left\{ \sum d_j^2 z_j \bar{z}_j = \mp 4\Gamma_i^\pm \right\},$$

that is those for which $\nabla_j W = -\frac{1}{2} d_j^2 z_j$. Such critical points have already attracted attention in the context of fluid dynamics, in particular for rotating stationary configurations of vortices (see e.g. Palmore [16]).

A brief historical review. The first works on the dynamics of Ginzburg-Landau vortices, in particular by Neu [13, 14], Peres & Rubinstein [17], Pismen & Rubinstein [18], and E [7], were based on formal matched asympotics. In these works, the vortices are described as the nodal set of the complex field u_ε.

Most of the first rigorous results deal with $(\mathrm{PGL})_\varepsilon$ on a simply connected bounded domain with fixed Dirichlet or Neumann boundary conditions: in that case the interaction energy W has to be modified appropriately in order to take into account the boundary datum. In [19], Rubinstein & Sternberg rigorously studied the dynamics of a single vortex, under the a priori assumption on the full solution that the nodal set consists of exactly one point for all positive time: they proved that the vortex speed is of order $|\log\varepsilon|^{-1}$ in the original time scale. Lin [11] extended their result to the case of ℓ distinct vortices of equal degree $+1$, removing the technical a priori assumption of [19] by a set of more natural assuptions on the initial data, among which the energy bound

$$\int_\Omega e_\varepsilon(u_\varepsilon^0) \le \pi\ell|\log\varepsilon| + O(1) \qquad \text{as } \varepsilon \to 0. \tag{10}$$

Concerning the dynamical law (3), the first mathematical proofs are due Jerrard & Soner [9] and Lin [11] independently. In [12], Lin established (3) under the assumptions of [11]: in this case $\vec{c} = 0$, $d_i = +1$, and it is therefore easily seen that the solutions of (3) exist for all time. In [9], Jerrard & Soner were able to handle the case of ℓ vortices of degrees $+1$ and -1, under a set of assumptions on the initial data involving as in [11, 12] the energy bound (10). In this case, $\vec{c}$ still vanishes, but (3) has a finite life span (corresponding to our first dissipation time τ_1) if at least two vortices have different signs. The convergence and the dynamical law was there established up to τ_1. In these papers, vortices are identified as concentration points for the energy density. Upper and lower energy estimates are crucial in their approach, an important point being that the energy is bounded off the vorticity set. In [20, 21], Sandier & Serfaty and Serfaty proposed a somehow different proof of the result of [9], introducing an abstract theory for $\Gamma-$convergence of gradient flows. In [21], it is also shown that if $E_\varepsilon(u_\varepsilon^0) \le \pi\ell|\log\varepsilon| + |\log\varepsilon|(\log|\log\varepsilon|)^{-\beta}$ for some $\beta > 1$, then (10) is met after translating time by some $T_\varepsilon \le C\log|\log\varepsilon|$.

[3] Except for collisions involving clusters of ±1-vortices, with total degree equal to 0 or ±1.

Concerning collisions and annihilations of vortices, the first rigorous result was obtained by Baumann, Chen, Phillips & Sternberg [1], who showed that for fixed ε and under some natural energy bounds, the solution to $(\mathrm{PGL})_\varepsilon$ on $\mathbb{R}^2$ converges to a unitary constant as time goes to infinity: in particular, if vortices exist they annihilate in finite time. A localized quantitative version of annihilation was obtained in [3] for confined clusters of vortices of total degree zero, and independently in [21] for confined ± 1 dipoles.

Concerning technical ingredients, in this context the important algebraic relation (24), as well as the energy quantization of energy which may be derived from it, was first introduced in [6] in the elliptic case. It was then extended independently in [4] and [21] to the parabolic setting.

Although it might not be obvious at first sight, inequality (10) is a rather strong well-preparedness assumption, since the minimal energy necessary for the creation of a $+1$ or a -1 vortex (which plays the role of the topological ground state) is exactly $\pi|\log\varepsilon| + O(1)$. A first restriction related to inequality (10) is that it does not allow for initial data with multiple degree vortices, nor diverging phase energy. A second important restriction was already pointed out in [9] Remark 2.2: inequality (10) relates the energy to the degrees, and it is not clear at first that this relation remains after collisions. The assumption (H_0), which was our framework in [3, 4] and here, was motivated by the possibility to encompass the previous restrictions. We also decided to study $(\mathrm{PGL})_\varepsilon$ on the whole $\mathbb{R}^2$ mainly for two reasons. First, this avoids some technicalities related to the boundary, and allows to use more freely the scale invariance of the equation. Second, it permits the phase and the vortices to interact, as shown by the presence of the $\vec{c}$ term in (3).

In [3], we proved Theorem 1, which gives a precise meaning to the notion of vortices in our framework. In contrast to the results in [11, 9, 20], our statements require the use of subsequences. This intrinsic restriction is related to the wider class of possible initial data that (H_0) allows for, as well as the fact that the occurence of splittings cannot be inferred once the limit in ε has been taken. [3] also contains the previously mentioned result on annihilation.

In [4], we proved Theorem 3 and established the results in Theorem 2 on (τ_k, τ_{k+1}) under the additional assumption that $d_i(s) = \pm 1$ for all $i = 1, \ldots, \ell(s)$ and some $s \in (\tau_k, \tau_{k+1})$. As emphasized in [4], the main obstacle on the way to Theorem 2 is the possible splitting and recombination of multiple degree vortices without energy loss, i.e. outside the dissipation times. The main point in our proof of Theorem 2 is to show that they do not occur.

Strategy for the proofs. The arguments involved in the proofs of Theorem 2 and 4 do not rely on additional results about $(\mathrm{PGL})_\varepsilon$, but instead on properties of $\Sigma_{\mathfrak{v}}$ established in [4] as well as new algebraic properties of W.

For the proof of Theorem 2 i), we need to show that for $s_0 \notin \{\tau_1, \ldots, \tau_k\}$ and $(a_i(s_0, s_0)) \in \Sigma_{\mathfrak{v}}$, for s close to s_0 and for some neighborhood B_i of $a_i(s_0)$, B_i contains only a single vortex. In order to analyze the size and the spreading of the possible cluster of vortices emanating from $(a_i(s_0), s_0)$, we consider the variance

$$f_i(s) = \frac{\sum_{a_j(s)\in B_i} d_j^2(s)|a_j(s) - \hat{a}_i(s)|^2}{\sum_{a_j(s)\in B_i} d_j^2(s)},$$

where $\hat{a}_i(s)$ denotes the barycenter of the cluster of vortices in B_i, with weights given by the energy densities $d_j^2(s)$, namely

$$\hat{a}_i(s) = \frac{\sum_{a_j(s)\in B_i} d_j^2(s) a_j(s)}{\sum_{a_j(s)\in B_i} d_j^2(s)}.$$

Our goal is to prove that $f_i(s)$ vanishes identically in a neighborhood of s_0, by mean of a Gronwall type inequality. In order to define more precisely B_i, we recall that in [4] (Theorem 5 and identity (9)), we have shown that given $s_0 > 0$ and $i \in \{1, \dots, \ell(s_0)\}$, there exists some $\Delta s_0 > 0$ and $r_i(s_0) > 0$ such that

$$\Sigma_{\mathfrak{v}}^s \cap B(a_i(s_0), r_i(s_0)) \setminus B(a_i(s_0), r_i(s_0)/2) = \emptyset \tag{11}$$

for all s in $[s_0 - \Delta s_0, s_0 + \Delta s_0]$, where $\Sigma_{\mathfrak{v}}^s = \{a_1(s), \dots, a_{\ell(s)}(s)\}$. If $s_0 \notin \{\tau_1, \dots, \tau_q\}$, we may assume, decreasing possibly Δs_0, that $[s_0 - \Delta s_0, s_0 + \Delta s_0]$ contains none of the dissipation times τ_k. In this case, the degrees $d_i(s)$ of vortices emanating from or colliding at $a_i(s_0)$ have been shown (see Theorem 5) to be constrained by the algebraic equilibrium relation

$$\sum_{a_j(s)\in B_i} d_j^2(s) = \Big(\sum_{a_j(s)\in B_i} d_j(s)\Big)^2 = d_i^2(s_0) \tag{12}$$

for all s in $[s_0 - \Delta s_0, s_0 + \Delta s_0]$, where $B_i = B(a_i(s_0), r_i(s_0))$.

If $s \notin \{\tau_1, \dots, \tau_q\}$, the computation of $f_i'(s)$ follows from the identities

$$\frac{d}{ds}\int_{B_i} |x|^2 \, d\mathfrak{v}_*^s \;=\; 4\pi \sum_{\substack{a_k(s)\notin B_i \\ a_j(s)\in B_i}} d_k(s) d_j(s) \mathcal{R}e\Big(\frac{a_j(s)}{a_k(s) - a_j(s)}\Big) + 2\pi \sum_{a_j(s)\in B_i} d_j(s) \langle a_j(s), \vec{c}(s)^\perp \rangle \tag{13}$$

and

$$\frac{d}{ds}\hat{a}_i(s) = \frac{1}{d_i(s_0)}[\vec{c}(s)^\perp + \sum_{\substack{a_k(s)\notin B_i \\ a_j(s)\in B_i}} 2 d_k(s) \nabla_{a_j}(\log|a_j(s) - a_k(s)|)] \tag{14}$$

for $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$, which we proved in [4]. In view of the expression for $\mathfrak{v}_*^s$ in (5), we are therefore led to

$$\begin{aligned} &\frac{d}{ds}\Big(\sum_{a_j(s)\in B_i} \pi d_j^2(s) |a_j(s) - \hat{a}_i(s)|^2\Big) \\ &= 4\pi \sum_{\substack{a_k(s)\notin B_i \\ a_j(s)\in B_i}} d_k(s) d_j(s) \mathcal{R}e\Big(\frac{a_j(s) - \hat{a}_i(s)}{a_k(s) - a_j(s)}\Big) + 2\pi d_i(s_0) \langle \check{a}_i(s) - \hat{a}_i(s), \vec{c}(s)^\perp \rangle, \end{aligned} \tag{15}$$

where $\check{a}_i(s)$ denotes a second type of barycenter, namely

$$\check{a}_i(s) = \frac{\sum_{a_j(s)\in B_i} d_j(s) a_j(s)}{\sum_{a_j(s)\in B_i} d_j(s)}\cdot \tag{16}$$

We refer to $\hat{a}_i(s)$ as the center of mass of the cluster and to $\check{a}_i(s)$ as the topological center of mass of the cluster. The identity (15) leads to

Proposition 1. *The function $f_i(s)$ is lipschitzian and verifies*

$$|f_i'(s)| \le C(M_0, s_0)\,(|\hat{a}_i(s) - \check{a}_i(s)| + f_i(s)) \qquad \text{on } [s_0 - \Delta s_0, s_0 + \Delta s_0], \tag{17}$$

where $C(M_0, s_0)$ depends only on s_0 and M_0.

In order to integrate the differential inequality (17), we need some control on the term $|\check{a}_i(s) - \hat{a}_i(s)|$. For arbitrary configurations of points and degrees, there is no reason that $\hat{a}$ and $\check{a}$ should be close. However, for simple examples of critical points of W we noticed that they are equal. This observation led us to the following identity.

Lemma 1. *Consider ℓ points $z_1, \dots, z_\ell \in \mathbb{C}$, and ℓ real numbers $d_1, \dots, d_l$ whose sum is non zero. Then the following identity holds:*

$$\frac{\sum d_j z_j}{\sum d_j} = \frac{\sum d_j^2 z_j}{(\sum d_j)^2} + \frac{\sum \nabla_{z_j} W(z_1, \dots, z_\ell) z_j^2}{2(\sum d_j)^2}\,, \tag{18}$$

where the sums are meant for j ranging from 1 to ℓ.

Specifying formula (18) with $z_j = a_j(s) - \hat{a}_i(s)$, and in view of (12), we obtain

Proposition 2. *It holds*

$$|\check{a}_i(s) - \hat{a}_i(s)| \le C(M_0)|\nabla W(\{a_j(s)\}_{j \in I(s)})|\, f_i(s), \tag{19}$$

where $C(M_0)$ depends only on M_0 and where $I(s) = \{j \in \{1, \dots, \ell(s)\},\ a_j(s) \in B_i\}$.

Combining (17) and (19), we finally derive

$$|f_i'(s)| \le C(M_0, s_0)(1 + |\nabla W(\{a_j(s)\}_{j \in I(s)})|) f_i(s)\,. \tag{20}$$

Since $f_i(s_0) = 0$, Gronwall's lemma would then allow to conclude that $f_i(s) \equiv 0$ on a neighborhood I of s_0 provided that

$$\int_I |\nabla W(\{a_j(s)\}_{j \in I(s)})|\, ds < +\infty. \tag{21}$$

As a matter of fact, we will even prove that

$$\int_I |\nabla W(\{a_j(s)\}_{j \in I(s)})|^2\, ds < +\infty, \tag{22}$$

using the gradient-flow type properties of the ode (3). This last statement may seem rather odd at first reading, since the ode (3) is precisely what we wish to show. Our actual argument is by induction on $d_i(s_0)$. Indeed, when $|d_i(s_0)| = 2$, the splitting may only create ± 1 vortices, for which we already established (3) in [4]. Similarly, if $|d_i(s_0)| = k$, the splitting may only involve vortices of degree at most $k - 1$ in absolute value, which are handled by the inductive argument. To establish (22) in view of the gradient-flow properties, we invoke

Proposition 3. *We have, for any $s > 0$,*

$$|W(\{a_j(s)\}_{j \in I(s)})| \le C(M_0)\Big(|\log \operatorname{dist}(s, \{\tau_0, \dots, \tau_q\})| + 1\Big). \tag{23}$$

The proof of Proposition 3 relies only on the special form of W and on the κ-confinement result of [4] which we recall next.

Theorem 5 ([4] Theorem 5). *Let $s > 0$, $a \in \mathbb{R}^2$, $r > 0$ and $0 < \kappa \leq \frac{1}{2}$ be such that*

$$\emptyset \neq \Sigma_{\mathfrak{v}}^s \cap B(a, r) \subset B(a, \kappa r).$$

There exist constants $0 \leq \kappa_1 \leq \frac{1}{4}$ and $\gamma_1 > 0$,depending only on M_0, such that if $0 < \kappa \leq \kappa_1$ and

$$\mathrm{dist}(s, \{\tau_0, \ldots, \tau_q\}) \geq \gamma_1 \kappa^2 r^2,$$

then

$$\sum_{i \in I(s)} d_i^2(s) = \Big(\sum_{i \in I(s)} d_i(s) \Big)^2 \tag{24}$$

where we have set $I(s) = \{i \in 1, ..., \ell(s) \mid a_i(s) \in B(a, r)\}$.

Once it is proved that $f_i(s) \equiv 0$, statement i) of Theorem 2 follows straightforwardly. Statement ii) of Theorem 2 is a direct consequence of the equation (14) for the center of mass $\hat{a}_i(s)$ and the fact that there is no splitting.

Remark. One may consider more generally the class of ode's

$$\frac{d}{ds} m_i a_i(s) = -\nabla_{a_i} W(a_1, \ldots, a_{\ell_k}) + d_i c(s)^{\perp}, \qquad i = 1, \ldots, \ell_k\,, \tag{25}$$

where the coefficients $m_i > 0$ may be thought as masses. In our case, the masses and the degrees are constrained by the relation $m_i = d_i^2$. Defining for the general case the center of mass $\hat{a}(s)$ as

$$\hat{a}(s) = \frac{\sum m_j a_j(s)}{\sum m_j},$$

then (17) still holds, with $\check{a}(s)$ being as before the topological center of mass. However, inequality (19) in Proposition 2 does not hold in general. In particular, we do not know if a cluster verifying the algebraic equilibrium relation $(\sum d_i^2) = \sum d_i^2$ may expand or not for the ode (25). Our arguments therefore heavily rely on the quatization of energy $m_i = d_i^2$.

Concerning the proof of Theorem 4, it relies essentially on refinements of the arguments involved in the proof of Theorem 2.

The outline of this paper is as follows. In Section 2, we derive some important properties concerning the Kirchhoff-Onsager functional. In Section 3, we analyze the growth of cluster of vortices, and in particular give the proofs of Proposition 1, 2 and 3. Section 4 contains the proof of Theorem 2 and 4. In Section 5, we provide examples of branching points which are limits of solutions to $(\mathrm{PGL})_\varepsilon$. Finally in Section 6 we describe the behavior of trajectories near branching points.

2 Some properties of the interaction energy

We consider, for ℓ distinct points $z_1, \ldots, z_\ell \in \mathbb{C}$ and ℓ real numbers $d_1, \ldots, d_\ell \neq 0$, the interaction energy[4]

$$W(z_1, \ldots, z_\ell) = -\sum_{i \neq j} d_i d_j \log |z_i - z_j|^2 = -\sum_{i \neq j} \log (z_i - z_j)(\overline{z_i - z_j}).$$

This functional appears in several topics of mathematical physics. In this section, we will derive various properties of W, most of which will be used later in this paper.

2.1 Properties of the gradient of W

For fixed $d_1, \ldots, d_\ell \neq 0$, the function W is clearly well defined and smooth on the open subset of $\mathbb{C}^\ell$ consisting of ℓ-tuples of distinct points. It possesses some elementary symmetry properties, namely

Lemma 2.1. (Invariance by rotations, translations and dilations). *We have*

$$\nabla W(\alpha(z_1 - \beta), \ldots, \alpha(z_\ell - \beta)) = \frac{1}{\bar{\alpha}} \nabla W(z_1, \ldots, z_\ell)$$

for any $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$.

The gradient ∇W has a simple expression in complex notation

Lemma 2.2. *For any $k \in \{1, \cdots, \ell\}$ we have*

$$\frac{1}{2} \nabla_{z_k} W = \frac{\partial W}{\partial \bar{z}_k} = -d_k \sum_{j \neq k} \frac{d_j}{\overline{z_j - z_k}}\,. \tag{2.1}$$

As a consequence,

$$\sum_{k=1}^{\ell} \langle \nabla_k W, z_k \rangle = \mathrm{Re}\Big(\sum_{k=1}^{\ell} \nabla_k W \bar{z}_k \Big) = \sum_{j \neq k} d_k d_j. \tag{2.2}$$

Proof. We observe that for a smooth function h on $\mathbb{R}^+$, we have

$$\nabla_z h(|z|^2) = 2h'(|z|^2) z = 2h'(z\bar{z}) \frac{\partial}{\partial \bar{z}} (z\bar{z}) = 2 \frac{\partial}{\partial \bar{z}} h(z\bar{z})\,, \tag{2.3}$$

and the conclusion (2.1) follows. □

The next formula is the starting point for the proof of Lemma 1.

Lemma 2.3. *We have*

$$\Big(\sum_{k=1}^{\ell} \frac{d_k}{z - z_k} \Big)^2 = \sum_{k=1}^{\ell} \frac{d_k^2}{(z - z_k)^2} + \sum_{k=1}^{\ell} \frac{\overline{\nabla_{z_k} W}}{z - z_k}\,. \tag{2.4}$$

[4]Clearly W depends also on the d_i's, although this is not reflected in our notation $W(z_1, \cdots, z_\ell)$. In most places the d_i's are implicit from the context, i.e. the degrees of the vortices. However, in case of possible ambiguity, we will write $W(\{(z_i, d_i)\}_{1 \leq i \leq \ell})$.

Proof. Expanding the l.h.s. of (2.4), we obtain

$$\left(\sum_{k=1}^{\ell} \frac{d_k}{z - z_k}\right)^2 = \sum_{k=1}^{\ell} \frac{d_k^2}{(z - z_k)^2} + \sum_{j \neq k} \frac{d_k d_j}{(z - z_k)(z - z_j)} \,. \tag{2.5}$$

On the other hand, we have the identity

$$\frac{1}{(z - z_k)(z - z_j)} = \frac{1}{z_k - z_j}\left(\frac{1}{z - z_k} - \frac{1}{z - z_j}\right). \tag{2.6}$$

Inserting (2.6) into (2.5) we therefore derive

$$\left(\sum_{k=1}^{\ell} \frac{d_k}{z - z_k}\right)^2 = \sum_{k=1}^{\ell} \frac{d_k^2}{(z - z_k)^2} + 2\sum_{k=1}^{\ell}\left(\sum_{j \neq k} \frac{d_j}{z_k - z_j}\right) \frac{d_k}{z - z_k} \,, \tag{2.7}$$

and (2.4) follows.

□

2.2 Proof of Lemma 1

We expand each of the terms involved in equality (2.4) as power series of $\frac{1}{z}$.

$$\begin{aligned}
\left(\sum \frac{d_k}{z - z_k}\right)^2 &= \frac{1}{z^2}\left(\sum \frac{d_k}{1 - (z_k/z)}\right)^2 \\
&= \frac{1}{z^2}\left(\sum d_k + \frac{1}{z}\sum d_k z_k + O(\frac{1}{z^2})\right)^2 \\
&= \frac{1}{z^2}\left(\sum d_k\right)^2 + \frac{2}{z^3}(\sum d_k)(\sum d_k z_k) + O(\frac{1}{z^4}), \\
\sum \frac{d_k^2}{(z - z_k)^2} &= \frac{1}{z^2}\sum \frac{d_k^2}{(1 - (z_k/z))^2} \\
&= \frac{1}{z^2}(\sum d_k^2) + \frac{2}{z^3}(\sum d_k^2 z_k) + O(\frac{1}{z^4}), \\
\sum \frac{\overline{\nabla_{z_k} W}}{z - z_k} &= \frac{1}{z}\left(\sum \frac{\overline{\nabla_{z_k} W}}{1 - (z_k/z)}\right) \\
&= \frac{1}{z}(\sum \overline{\nabla_{z_k} W}) + \frac{1}{z^2}(\sum \overline{\nabla_{z_k} W} z_k) + \frac{1}{z^3}(\sum \overline{\nabla_{z_k} W} z_k^2) + O(\frac{1}{z^4}),
\end{aligned}$$

as $|z| \to \infty$. Identifying the coefficients of the expansion, in view of (2.4), we are led to

$$\sum_{k=1}^{\ell} \overline{\nabla_{z_k} W} = 0 \,, \tag{2.8}$$

$$(\sum_{k=1}^{\ell} d_k)^2 = \sum_{k=1}^{\ell} d_k^2 + \sum_{k=1}^{\ell} \overline{\nabla_{z_k} W} z_k \,, \tag{2.9}$$

$$2(\sum_{k=1}^{\ell} d_k)(\sum_{k=1}^{\ell} d_k z_k) = 2\sum_{k=1}^{\ell} d_k^2 z_k + \sum_{k=1}^{\ell} \overline{\nabla_{z_k} W} z_k^2 . \tag{2.10}$$

The first equality is a result of the symmetry properties of W, whereas the last one immediately yields (18). □

2.3 Critical points of W

In this subsection we study critical points of W, i.e. configurations $\{(z_i, d_i)\}_{1\leq i\leq \ell}$ verifying

$$\nabla_{z_k} W(z_1, \ldots, z_\ell) = 0 . \tag{2.11}$$

Alternatively, it follows from Lemma 2.3 that the configuration is critical if and only if

$$\sum_{j\neq k=1}^{\ell} \frac{d_j}{z_k - z_j} = 0 \qquad \forall\, k = 1, \ldots, \ell . \tag{2.12}$$

We refer to (2.12) as the **geometric equilibrium condition**. Although they do not enter directly in the proofs Theorem 2 and 4, critical configurations are useful for the understanding of the splitting phenomenon (see Section 5. They are also interesting by themselves, since they are stationary solutions for the dynamical law (3). They play a role in different topics. For instance, in fluid dynamics they describe stationary vortex solutions for the 2D Euler equation on the whole plane, and related examples may be found in electrostatics.

As a consequence of identities (2.9) and (2.10), we have

Proposition 2.1. *Assume that W possesses a critial point $(z_1, \ldots, z_\ell)$, then necessarily*

$$\sum d_k^2 = \Big(\sum d_k\Big)^2 \tag{2.13}$$

and

$$\frac{\sum d_k z_k}{\sum d_k} = \frac{\sum d_k^2 z_k}{\sum d_k^2}. \tag{2.14}$$

Identity (2.13), which we refer to as the algebraic equilibrium relation, was already found by Kirchhoff [10] in the context of vortex solutions for the Euler equation. In the framework of Ginzburg-Landau theory, the same relation was derived by Comte and Mironescu [6] for critical points of the Ginzburg-Landau energy E_ε, and exploited in [21, 4] in the asymptotics for $(\mathrm{PGL})_\varepsilon$. It expresses also the conformal invariance of W, that is

$$W(z_1, \ldots, z_\ell) = W(\lambda z_1, \ldots, \lambda z_\ell) \qquad \forall \lambda \in \mathbb{C}^* \tag{2.15}$$

if and only if (2.13) is verified.

Identity (2.14) can be interpreted as the equality of the center of mass and the topological center of mass, as defined in the introduction, in other words we have

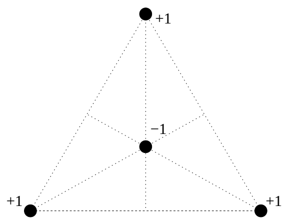
$$\hat{z} = \check{z} \tag{2.16}$$

for critical points $(z_1, \ldots, z_\ell)$.

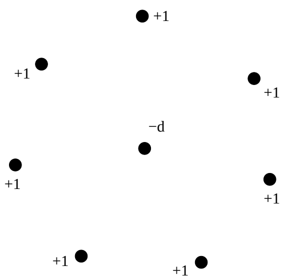
Solutions to the algebraic equilibrium relation. Set $D = \sum_{i=1}^{\ell} d_i$. Notice first that (2.13) implies $\ell \leq D^2$, so that there is no solution to (2.13) for $D = 0$, and only the trivial single point solution for $D = 1$. More generally, let $n(D)$ be the number of different solutions to (2.13). We observe that $n(D+1) > n(D)$, i.e. $n(D)$ is strictly increasing. Indeed, there is of course the trivial single degree D solution, and for any solution with total degree D one can construct a solution with total degree $D+1$ by adding $D+1$ degree $+1$ vortices, and D degree -1 vortices. In particular, for $D \geq 2$, (2.13) has at least one solution, namely $(D^2+D)/2$ vortices of degree $+1$ and $(D^2-D)/2$ vortices of degree -1.

In order to illustrate the previous discussion, notice for instance that for $D = 2$, the only solutions to (2.13) are the trivial solution $(+2)$, and, up to permutations, $(-1, +1, +1, +1)$. For $D = 3$, besides the already mentioned solution $(-1, -1, -1, +1, +1, +1, +1, +1, +1)$ and trivial solution $(+3)$, there is also the solution $(-1, +2, +2)$ and, splitting one of the $+2$ degrees, we obtain additionally $(-1, -1, +1, +1, +1, +2)$. For general D, the number of solutions to (2.13) may be quite large. Indeed, the previous example $D = 3$ shows that the set of solutions has a sort of tree structure. Once solutions to (2.13) are found, it remains to determine the corresponding geometric equilibrium configurations, i.e. the solutions to (2.12).

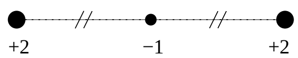
Solutions to the geometric equilibrium equation. For $D = 2$, the following solution to (2.12) is known, where the largest triangle is equilateral.



This construction can be generalized to arbitrary $D \geq 2$ as follows



where the origin is a vortex of degree $-d$ and where $2d+1$ vortices of degree $+1$ are located at the vertices of a regular polygon, so that $D = d+1$. For $D = 3$, besides the above solution, there is also the colinear configuration.



Notice also that given a solution $\{(z_i, d_i)\}_{1 \leq i \leq \ell}$ to (2.12), one obtains further solutions of the form $\{(z_i, kd_i)\}_{1 \leq i \leq \ell}$ for any $k \in \mathbb{Z}^*$. The number of solutions for large D is presumably large, however, for $D = 2$ we have

Proposition 2.2. *Assume $D = 2$, $\ell \neq 1$, and that $z_1, \ldots, z_\ell$ is a critical point of $W = -\sum d_i d_j \log |z_i - z_j|^2$. Then necessarily $\ell = 4$, and up to a rigid motion and a dilation (and to a permutation of indices), $d_1 = -1$, $d_2 = d_3 = d_4 = +1$, and*

$$(z_1, z_2, z_3, z_4) = (0, 1, \exp(2i\pi/3), \exp(4i\pi/3)) .$$

Proof. In view of the discussion in the previous section, we may assume that $(d_1, d_2, d_3, d_4) = (-1, +1, +1, +1)$ and, up to translations, rotations and dilations, $z_1 = 0$ and $z_2 = 1$. The system (2.12) is then reduced to

$$\frac{1}{z_3} + \frac{1}{z_4} = -1 \qquad \& \qquad \frac{1}{1 - z_3} + \frac{1}{1 - z_4} = +1.$$

In particular, $z_3^2 + z_3 + 1 = 0$, so that either $z_3 = \exp(2i\pi/3)$ and $z_4 = \exp(4i\pi/3)$ or viceversa. The proof is therefore complete. □

2.4 Clusterization and computation of W

In this subsection we present two elementary lemma of somewhat combinatorial nature, which, combined with Theorem 5, will yield the proof of Proposition 3. They already appeared in [4]. We provide their proofs for completeness.

Lemma 2.4. *Let X be a metric space, and consider ℓ distinct points $a_1, \ldots, a_\ell$ in X. Let $\delta_0 > 0$ and $0 < \kappa \leq \frac{1}{2}$ be given. Then there exists $\delta > 0$ such that*

$$\delta_0 \leq \delta \leq (\frac{\kappa}{2})^{-\ell} \delta_0 \tag{2.17}$$

and a subset $\{a_j\}_{j \in J}$ of $\{a_i\}_{1 \leq i \leq \ell}$ such that

$$\cup_{i=1}^{\ell} B(a_i, \delta_0) \subset \cup_{j \in J} B(a_i, \delta) \tag{2.18}$$

and

$$\mathrm{dist}(a_i, a_j) \geq \kappa^{-1} \delta \qquad \forall i \neq j \text{ in } J. \tag{2.19}$$

Proof. The proof is by iteration, in at most ℓ steps. First, consider the collection $\{a_i\}_{1 \leq i \leq \ell}$. If (2.18), (2.19) is verified with $\delta = \delta_0$ there is nothing else to do. Otherwise, take two points, say a_1, a_2 such that $\mathrm{dist}(a_1, a_2) \leq \kappa^{-1}\delta_0$, consider the collection $a_2, a_3, \ldots, a_l$, and set $\delta = 2\kappa^{-1}\delta_0$. If (2.18) is verified, we stop. Otherwise we go on in the same way. If the process does not stop in $\ell - 1$ steps, at the ℓ^{th} step we are left with one single ball of radius $\delta = (\frac{\kappa}{2})^{-\ell}\delta_0$, and (2.18) is void. □

Lemma 2.5. *Let X be a metric space, let $0 < \kappa < 1/2$ and $0 < r < R_{max}$ be given. Consider m distinct points $b_1, \ldots, b_m$ in X such that*

$$\mathrm{dist}(b_i, b_j) \geq \kappa^{-1} r, \qquad \text{for } i \neq j.$$

Then one of the following two situations holds

i) $\inf_{i \neq j} \mathrm{dist}(b_i, b_j) \geq R_{max}$

ii) There exists a partition $\{1,\dots,m\} = \cup_{i=1}^n J_i$ *with* $n < m$, *for each* $i \in \{1,\dots,n\}$ *some* $\tilde{b}_i \in \cup_{j\in J_i}\{b_j\}$ *and* $r < R \le (\frac{\kappa}{2})^{-m/2} R_{max}$ *such that*

$$\cup_{i=1}^m B(b_i, r) \subset \cup_{i=1}^n B(\tilde{b}_i, R) \tag{2.20}$$

$$\mathrm{dist}(\tilde{b}_i, \tilde{b}_j) \ge \kappa^{-1} R, \qquad \textit{for any } i \ne j \in \{1,\dots,n\}, \tag{2.21}$$

and, for every $d_1,\dots,d_m \in \mathbb{R}^m$,

$$\Big| \sum_{i\ne j=1}^{m} d_i d_j \log \mathrm{dist}(b_i, b_j) - \sum_{i\ne j=1}^{n} D_i D_j \log \mathrm{dist}(\tilde{b}_i, \tilde{b}_j) - \sum_{p=1}^{n} \sum_{i\ne j\in J_p} d_i d_j \log R \Big| \le C |\log\kappa| \tag{2.22}$$

where $D_i = \sum_{j\in J_i} d_j$, *and where the constant* C *depends only on* m *and* $\sup_i |d_i|^2$.

Proof. If i) holds there is nothing left to prove. Otherwise set

$$\delta_0 = \inf_{i\ne j} \mathrm{dist}(b_i, b_j).$$

Applying Lemma 2.4 we obtain a subset $\{\tilde{b}_1,\dots,\tilde{b}_n\}$ of $\{b_1,\dots,b_m\}$ and $\delta_0 \le \delta \le (\kappa/2)^{-m}\delta_0$ such that

$$\cup_{i=1}^m B(b_i, \delta_0) \subset \cup_{i=1}^n B(\tilde{b}_i, \delta) \tag{2.23}$$

and

$$\mathrm{dist}(\tilde{b}_i, \tilde{b}_j) \ge \kappa^{-1}\delta \qquad \forall i \ne j. \tag{2.24}$$

We choose $R = \delta$. It follows from the definition of δ_0 and (2.24) that $n < m$, whereas (2.20) and (2.21) follow directly from (2.23) and (2.24) respectively. We set, for $i = 1,\dots,n$,

$$J_i = \{j \ : \ \ b_j \in B(\tilde{b}_i, R)\}$$

and turn finally to (2.22). For $i \ne j$ in $\{1,\dots,m\}$ we distinguish two cases:

- i, j belong to the same J_p, for some $p \in \{1,\dots,n\}$: then

$$|\log \mathrm{dist}(b_i, b_j) - \log R| \le C|\log\kappa|$$

which follows from the fact that $\delta_0 \le \mathrm{dist}(b_i, b_j) \le 2R \le 2(\kappa/2)^{-m}\delta_0$.

- $i \in J_p$ and $j \in J_q$, $p \ne q$. Then

$$|\log \mathrm{dist}(b_i, b_j) - \log \mathrm{dist}(\tilde{b}_p, \tilde{b}_q)| \le C\kappa.$$

The proof of (2.22) follows then by summation. □

3 Properties for cluster of vortices

The purpose of this section is to give the proofs of Proposition 1, Proposition 2 and Proposition 3, which mainly involve properties for clusters of vortices of $(\mathrm{PGL})_\varepsilon$. We also propose a local version of Proposition 3 which will be used for the proof of Theorem 4.

We begin with

Proof of identity (15). We write,

$$|a_j(s) - \hat{a}_i(s)|^2 = |a_j(s)|^2 - 2\langle a_j(s), \hat{a}_i(s)\rangle + |\hat{a}_i(s)|^2\,. \tag{3.1}$$

We have

$$\begin{aligned}\frac{d}{ds}\sum_{a_j(s)\in B_i} \pi d_j^2(s)\langle a_j(s), \hat{a}_i(s)\rangle &= \sum_{a_j(s)\in B_i} \langle \frac{d}{ds} d_j^2(s) a_j(s),\, \hat{a}_i(s)\rangle - \sum_{a_j(s)\in B_i} \langle d_j^2(s) a_j(s),\, \frac{d}{ds}\hat{a}_i(s)\rangle \\ &= d_i(s_0)\Big[\langle \frac{d\hat{a}_i(s)}{ds},\, \hat{a}_i(s)\rangle - \langle \hat{a}_i(s),\, \frac{d\hat{a}_i(s)}{ds}\rangle\Big] \\ &= 0,\end{aligned} \tag{3.2}$$

whereas, in view of (12) and (14),

$$\begin{aligned}\frac{d}{ds}\sum_{a_j(s)\in B_i} d_j^2(s)|\hat{a}_i(s)|^2 &= 2d_i(s_0)\langle \hat{a}_i(s),\, \frac{d\hat{a}_i(s)}{ds}\rangle \\ &= 2d_i(s_0)\langle \hat{a}_i(s),\, \vec{c}(s)^\perp + 2\sum_{\substack{a_k(s)\notin B_i \\ a_j(s)\in B_i}} d_k(s)\nabla_{a_j}\log|a_k(s) - a_j(s)|\rangle.\end{aligned} \tag{3.3}$$

Combining (13), (3.1), (3.2) and (3.3) we deduce

$$\begin{aligned}&\frac{d}{ds}\Big(\sum_{a_j(s)\in B_i} \pi d_j^2(s)|a_j(s) - \hat{a}_i(s)|^2\Big) \\ &= 4\pi \sum_{\substack{a_k(s)\notin B_i \\ a_j(s)\in B_i}} d_k(s)d_j(s)\mathcal{R}e\Big(\frac{a_j(s) - \hat{a}_i(s)}{a_k(s) - a_j(s)}\Big) + 2\pi\sum_{a_j(s)\in B_i} d_j(s)\langle a_j(s) - \hat{a}_i(s), \vec{c}(s)^\perp\rangle\,,\end{aligned} \tag{3.4}$$

and the conclusion follows from the fact that $d_i(s_0)\check{a}_i(s) = \sum d_j(s)a_j(s)$. $\square$

Proof of Proposition 1. By (11), we have for $s \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$, for $a_k(s) \notin B_i$ and for $a_j(s) \in B_i$,

$$\Big|\mathcal{R}e\Big(\frac{a_j(s) - \hat{a}_i(s)}{a_k(s) - a_j(s)}\Big) - \mathcal{R}e\Big(\frac{a_j(s) - \hat{a}_i(s)}{a_k(s) - \hat{a}_i(s)}\Big)\Big| \leq C|a_j(s) - \hat{a}_i(s)|^2.$$

Therefore, we obtain, for the first term in the right hand side of (15),

$$\sum_{\substack{a_k(s)\notin B_i \\ a_j(s)\in B_i}} d_k(s)d_j(s)\mathcal{R}e\Big(\frac{a_j(s) - \hat{a}_i(s)}{a_k(s) - a_j(s)}\Big) = d_i(s_0)\langle \check{a}_i(s) - \hat{a}_i(s), \vec{\gamma}(s)\rangle + R(s)\,,$$

where

$$\vec{\gamma}(s) = \sum_{a_k(s)\notin B_i} d_k(s)\nabla_{a_k}\log|a_k(s) - \hat{a}_i(s)|$$

and

$$|R(s)| \leq C(M_0, s_0)\sup_{a_j(s)\in B_i}|a_j(s) - \hat{a}_i(s)|^2 \leq \frac{C(M_0, s_0)}{\pi} f_i(s)\,.$$

Hence,

$$\frac{d}{ds}\Big(\sum_{a_j\in B_i}\pi d_j^2(s)|a_j(s)-\hat{a}_i(s)|^2\Big) = 2\pi d(s_0)\langle \check{a}_i(s)-\hat{a}_i(s),\, \vec{c}(s)^\perp + 2\vec{\gamma}(s)\rangle + 4\pi R(s)$$

and the conclusion then follows. □

Proof of Proposition 2. In view of Lemma 1 and the translation invariance of W we have, omitting to write the dependence in s,

$$\frac{\sum d_j(a_j-\hat{a}_i)}{\sum d_j} - \frac{\sum d_j^2(a_j-\hat{a}_i)}{(\sum d_j)^2} = \frac{\sum \nabla_{a_j} W(\{a_j\}_{j\in I(s)})(a_j-\hat{a}_i)^2}{2(\sum d_j)^2}\,, \tag{3.5}$$

where the sums are taken over points $a_j(s)\in B_i$. On the other hand, by (12) we obtain

$$\frac{\sum d_j(s)(a_j(s)-\hat{a}_i(s))}{\sum d_j(s)} = \check{a}_i(s)-\hat{a}_i(s) \qquad \text{and} \qquad \frac{\sum d_j^2(s)(a_j(s)-\hat{a}_i(s))}{(\sum d_j(s))^2} = 0,$$

so that (3.5) implies

$$|\check{a}_i(s)-\hat{a}_i(s)| \le C|\nabla W(\{a_j(s)\}_{j\in I(s)})| \sup_{a_j(s)\in B_i} |a_j(s)-\hat{a}_i(s)|^2$$

and the conclusion follows from the definition of $f_i(s)$. □

In order to prove Proposition 3, we are led to consider the following situation. Let κ_1 and γ_1 be given by Theorem 5. Let $b\in\mathbb{R}^2$, $0<\kappa\le\kappa_1$, $0<R_1\le R_2$ and let $b_1,\dots,b_m$ be a collection of m points in $\mathbb{R}^2$ such that

$$\Sigma_{\mathfrak{v}}^s \cap B(b,\kappa^{-1}R_2) \subset \cup_{j=1}^m B(b_j,R_1) \subset B(b,R_2) \tag{3.6}$$

$$|b_i-b_j| \ge \kappa^{-1}R_1 \tag{3.7}$$

$$\mathrm{dist}(s,\{\tau_1,\dots,\tau_q\}) \ge \gamma_1 R_2^2. \tag{3.8}$$

We set, for $j=1,\dots,m$, $D_j=\sum_{a_k(s)\in B(b_j,R_1)} d_k(s)$. In view of (3.6) and the uniform bound on the number of vortices, we may always assume that $m\le C(M_0)$.

We first have

Lemma 3.1. *If* (3.6), (3.7) *and* (3.8) *are satisfied, then*

$$\sum_{i\neq j=1}^m D_iD_j = 0. \tag{3.9}$$

Proof. We apply Theorem 5 in two situations: first with $a=b$ and $r=R_2$, then with $a=b_j$ and $r=R_1$. The first yields

$$\sum_{a_i(s)\in B(b,R_2)} d_i^2(s) = \Big(\sum_{a_i(s)\in B(b,R_2)} d_i(s)\Big)^2 = \Big(\sum_{j=1}^m D_j\Big)^2$$

whereas the second gives, for $j = 1, \dots, m$

$$\sum_{a_i(s)\in B(b_j,R_1)} d_i^2(s) = \Big(\sum_{a_i(s)\in B(b_j,R_1)} d_i(s)\Big)^2 = D_j^2.$$

Since

$$\sum_{a_i(s)\in B(b,R_2)} d_i^2(s) = \sum_{j=1}^m \sum_{a_i(s)\in B(b_j,R_1)} d_i^2(s)$$

we deduce

$$\sum_{j=1}^m D_j^2 = \Big(\sum_{j=1}^m D_j\Big)^2,$$

which is equivalent to the conclusion (3.9). □

Lemma 3.2. *There exist constants $C(M_0)$ and $0 < \kappa(M_0) \le \kappa_1$ such that if* (3.6), (3.7) *and* (3.8) *hold and if $\kappa \le \kappa(M_0)$ then*

$$\Big| \sum_{i\neq j=1}^m D_i D_j \log|b_i - b_j| \Big| \le C(M_0). \tag{3.10}$$

Proof. We proceed by induction on m, the number of interior balls. When $m = 1$, there is nothing to prove, whereas when $m = 2$, the r.h.s. of (3.10) is zero by (3.9), so that (3.10) follows. By induction, we assume that there exist some constants $0 < \kappa(m-1) \le \kappa_1$ and $C(m-1)$ such that, for any collection of at most $m-1$ balls satisfying (3.6), (3.7) and (3.8) with $\kappa \le \kappa(m-1)$, (3.10) holds with r.h.s equal to $C(m-1)$. For a collection of m interior balls satisfying (3.6), (3.7) and (3.8), we apply Lemma 2.5 with $b_1, \dots, b_m$, κ, $r = R_1$ and $R_{max} = R_2$. This yields a new collection of $n \le m-1$ disjoint interior balls $B(\tilde b_j, R)$ which satisfy

$$R_1 \le R \le (\frac{\kappa}{2})^{-m/2} R_2 \tag{3.11}$$

$$|\tilde b_i - \tilde b_j| \ge \kappa^{-1} R \tag{3.12}$$

and moreover

$$\Big|\sum_{i\neq j=1}^m d_i d_j \log|b_i - b_j| - \sum_{i\neq j=1}^n D_i D_j \log|\tilde b_i - \tilde b_j| - \sum_{p=1}^n \sum_{i\neq j\in J_p} d_i d_j \log R\Big| \le C|\log\kappa|.$$

Since $\kappa \le \kappa_1$, $\sum_{i\neq j\in J_p} d_i d_j = 0$ by Lemma 3.1, so that

$$\Big|\sum_{i\neq j=1}^m d_i d_j \log|b_i - b_j| - \sum_{i\neq j=1}^n D_i D_j \log|\tilde b_i - \tilde b_j|\Big| \le C|\log\kappa|. \tag{3.13}$$

Since the balls $B(\tilde b_j, R)$ are disjoints and since $\tilde b_j \in B(b, R_2)$, it follows that $R \le R_2$. Set $\tilde b = b$, $\tilde R_1 = R$, $\tilde R_2 = 2R_2$ and $\tilde\kappa = 2\kappa$. One verifies in view of (3.11) and (3.12) that (3.6), (3.7) and (3.8) are satisfied for the point $\tilde b$, the collection $\tilde b_1, \dots, \tilde b_n$, $\tilde r_1$, $\tilde r_2$ and $\tilde\kappa$. If $\tilde\kappa \le \kappa(m-1)$, we may apply the inductive hypothesis, since $n \le m-1$, so that

$$\Big| \sum_{i\neq j=1}^n D_i D_j \log|\tilde b_i - \tilde b_j| \Big| \le C(m-1). \tag{3.14}$$

We choose $\kappa(m) = 2^{-m}\kappa_1$ and $C(m) = Cm|\log \kappa(m)|$. Combining (3.13) and (3.14) we obtain, if $\kappa \le \kappa(m)$,

$$\Big|\sum_{i\neq j=1}^{m} d_i d_j \log|b_i - b_j|\Big| \le C(m-1) + C|\log \kappa(m)| \le C(m)$$

and the proof is complete. □

Proof of Proposition 3. We wish to divide $\{a_j(s) \in B_i\}$ into subclusters of maximal size for which we may apply Lemma 3.2 with $\kappa = \kappa(M_0)$. For that purpose, we apply Lemma 2.4 to the points $\{a_1, \dots, a_\ell\} \equiv \{a_j(s) \in B_i\}$, $\kappa = \kappa(M_0)$ and δ_0 such that

$$\Big(\frac{\kappa(M_0)}{2}\Big)^{-\ell}\delta_0 = \sqrt{\frac{\mathrm{dist}(s, \{\tau_1, \dots, \tau_q\})}{\gamma_1}}. \tag{3.15}$$

For $j \in J$ we set $I_j = \{k \in \{1, \dots, \ell\},\ a_k \in B(a_j, \delta)\}$, and one verifies that (3.6), (3.7) and (3.8) are satisfied with $\kappa = \kappa(M_0)$, $b = a_j$, $\{b_k\} = \{a_k\}_{k\in I_j}$, $R_2 = \delta$ and R_1 sufficiently close to zero. We may therefore apply Lemma 3.2 to obtain, for any $j \in J$,

$$\Big|\sum_{p\neq q\in I_j} d_p(s)d_q(s)\log|a_p(s) - a_q(s)|\Big| \le C(M_0).$$

For $p \in I_j$ and $q \in I_{j'}$ with $j \neq j'$, we have

$$|a_p(s) - a_q(s)| \ge (\kappa(M_0)^{-1} - 2)\delta \ge \delta$$

so that

$$\log|a_p(s) - a_q(s)| \ge \log\delta_0 \ge \frac{1}{2}\log|\mathrm{dist}(s, \{\tau_1, \dots, \tau_q\})| - C(M_0).$$

On the other hand, we clearly have

$$\log|a_p(s) - a_q(s)| \le \log(2r(s_0)) \le 1.$$

Since

$$\sum_{p\neq q=1}^{\ell} = \sum_{j\in J}\sum_{p\neq q\in I_j} + \sum_{j\neq j'\in J}\ \sum_{p\in I_j,\, q\in I_{j'}}$$

inequality (23) follows by summation. □

For the proof of Theorem 4, we need also to consider the case of vortices at the dissipation times. We have the following variant of Proposition 3.

Proposition 3.1. *Let $(a_i(\tau_k), \tau_k) \in \Sigma_{\mathfrak{v}}$ and set*

$$\Gamma_i^{\pm} = \sum_{j\in I(s)} d_j^2(s) - d_i^2(\tau_k) \qquad \textit{for } s \to \tau_k^{\pm}. \tag{3.16}$$

Then we have

$$W(\{a_j(s)\}_{j\in I(s)}) = \Gamma_i^{\pm}\log|s - \tau_k| + O(1) \qquad \textit{as } s \to \tau_k^{\pm}. \tag{3.17}$$

Proof. We use exactly the same construction as in the proof of Proposition 3 above. For $j \in J$, we obtain, by the same argument,

$$\Big| \sum_{p \neq q \in I_j} d_p(s) d_q(s) \log |a_p(s) - a_q(s)| \Big| \leq C(M_0). \tag{3.18}$$

For the remaining terms, we expand further the computation in the proof of Proposition 3. For $p \in I_j$ and $q \in I_{j'}$ with $j \neq j'$, we have, by the parabolic cone property proved in [3], the definition (3.15) of δ_0 and (2.17)

$$|a_p(s) - a_q(s)| \leq C(M_0)\delta$$

provided s is sufficiently close to τ_k. Since on the other hand

$$|a_p(s) - a_q(s)| \geq (\kappa(M_0)^{-1} - 2)\delta \geq \delta$$

we obtain

$$|\log |a_p(s) - a_q(s)| - \log \delta| \leq C(M_0). \tag{3.19}$$

Since $\sum_{p \neq j \in I_j} d_p(s) d_q(s) = 0$ for all i, we are led to

$$\sum_{j \neq j'} \sum_{p \in I_j,\, q \in I_{j'}} d_p(s) d_q(s) = (d_i^2(\tau_k) - \sum_{j \in I(s)} d_j^2(s)) = -\Gamma_i^{\pm}. \tag{3.20}$$

In view of (3.15) and (2.17), we have $\log \delta = \frac{1}{2} \log |s - \tau_s| + O(1)$. The conclusion then follows from (3.18), (3.19),(3.20) and the definition of W. □

4 Non occurence of splittings without dissipation

The main purpose of this section is to provide the proofs for Theorem 2 and 4. As mentioned in the introduction, the main point is

Lemma 4.1. *For any $(a_i(s_0), s_0) \in \Sigma_{\mathfrak{v}}$, with $s_0 \notin \{\tau_1, \dots, \tau_q\}$ and Δs_0 and $r_i(s_0)$ as in* (11), *there exists a neighborhood I of s_0 such that*

$$f_i(s) \equiv \frac{1}{d_i^2(s_0)} \sum_{a_j(s) \in B_i} d_j^2(s) |a_j(s) - \hat{a}_i(s)|^2 = 0 \tag{4.1}$$

for all $s \in I$, or equivalently

$$\sharp\Big(\Sigma_{\mathfrak{v}}^s \cap B_i\Big) = 1 \tag{4.2}$$

for all $s \in I$.

Proof. The proof is by induction. More precisely, for $k \in \mathbb{N}^*$ let $(\mathcal{P}_k)$ be the statement that Lemma 4.1 holds for $|d_i(s_0)| \leq k$. First notice that $(\mathcal{P}_1)$ holds. Indeed, for a vortex $a_i(s_0)$ with $|d_i(s_0)| = 1$, (4.2) has already been established in [4] Lemma 5.3, and is actually an immediate consequence of (12). For $k \geq 2$, we next prove that $(\mathcal{P}_k)$ holds, assuming $(\mathcal{P}_{k-1})$. Let $(a_i(s_0), s_0) \in \Sigma_{\mathfrak{v}}^{s_0}$ with $s_0 \notin \{\tau_1, \dots, \tau_q\}$ and $|d_i(s_0)| = k$, and assume by contradiction that for any interval $I \ni s_0$, (4.2) does not hold on the whole I. Therefore, there exists $s_1 \in [s_0 - \Delta s_0, s_0 + \Delta s_0]$ such that

$$\sharp\Big(\Sigma_{\mathfrak{v}}^{s_1} \cap B_i\Big) \geq 2. \tag{4.3}$$

Assume first that $s_1 > s_0$.

Step 1. There exists a time $s_0' \in [s_0, s_1)$ such that

$$\sharp\{\Sigma_{\mathfrak{v}}^{s} \cap B_i\} = \sharp\{\Sigma_{\mathfrak{v}}^{s_1} \cap B_i\}, \qquad \forall\, s \in (s_0', s_1]$$

and

$$\sharp\{\Sigma_{\mathfrak{v}}^{s_0'} \cap B_i\} < \sharp\{\Sigma_{\mathfrak{v}}^{s_1} \cap B_1\}.$$

Indeed, first notice that, by identities (12) and (4.3), all vortices in $\Sigma_{\mathfrak{v}}^{s_1} \cap B_i$ have degrees strictly less than k in absolute value, so that the induction hypothesis $(\mathcal{P}_{k-1})$ can be used for these points. In view of (4.2) and conservation of energy (12), the number $\sharp\{\Sigma_{\mathfrak{v}}^{s} \cap B_i\}$ is locally constant, whereas $1 = \sharp\{\Sigma_{\mathfrak{v}}^{s_0} \cap B_i\} < \sharp\{\Sigma_{\mathfrak{v}}^{s_1} \cap B_i\}$. Therefore, s_0' is the end point of the largest open interval containing s_1 and on which the value of $\sharp\{\Sigma_{\mathfrak{v}}^{s} \cap B_i\}$ equals $\sharp\{\Sigma_{\mathfrak{v}}^{s_1} \cap B_i\}$.

In view of Step 1, $I(s)$ may be choosen independently of s, and the vortices $a_j(s)$ with $j \in I(s)$ may be unambiguously labelled on $(s_0', s_1]$. Therefore, without loss of generality we may write

$$\{a_j(s)\}_{j \in I(s)} = \{a_1(s), \ldots, a_m(s)\}, \quad \forall\, s \in (s_0', s_1]$$

for some $m \in \mathbb{N}^*$. Since $d_j(s)$ is constant, we also write $d_j \equiv d_j(s)$ for $j \in \{1, \ldots, m\}$ and $s \in (s_0', s_1]$.

Step 2. We have

$$\int_{s_0'}^{s_1} |\nabla W(a_1(s), \ldots, a_m(s))|^2 \, ds < +\infty. \tag{4.4}$$

In view of Lemma 5.2 of [4], we have, for any $s \in (s_0', s_1)$ and any $j \in \{1, \ldots, m\}$,

$$\frac{d}{ds} a_j(s) = \frac{1}{d_j}[\vec{c}(s)^{\perp} + 2 \sum_{k \neq j = 1}^{\ell(s)} d_k(s) \nabla_{a_k}(\log |a_k(s) - a_j(s)|)]. \tag{4.5}$$

In particular, for $j \in \{1, \ldots, m\}$,

$$\frac{d}{ds} a_j(s) = -\frac{1}{d_j^2} \nabla W(a_1(s), \ldots, a_m(s)) + R_j(s) \tag{4.6}$$

where R_j is a continuous function bounded by a constant depending only on s_0. The system of m ordinary differential equations (4.6) would be a pseudo-gradient flow for W if R_j were equal to zero. Here, we have

$$\begin{aligned}
\frac{d}{ds} W(a_1(s), \ldots, a_m(s)) &= \sum_{j=1}^{m} \nabla_{a_j} W(a_1(s), \ldots, a_m(s)) \frac{d}{ds} a_j(s) \\
&= -\sum_{j=1}^{m} \frac{1}{d_j^2} |\nabla_{a_j} W(a_1(s), \ldots, a_m(s))|^2 + \nabla_{a_j} W(a_1(s), \ldots, a_m(s)) \\
&\leq -C(|\nabla W(a_1(s), \ldots, a_m(s))|^2 - 1),
\end{aligned}$$

where $C > 0$. Integrating on (s_0', s_1) we are led to

$$\begin{aligned}\int_{s_0'}^{s_1} |\nabla W(a_1(s), \dots, a_m(s))|^2 \, ds &\leq C(|\int_{s_0'}^{s_1} \frac{d}{ds} W(a_1(s), \dots, a_m(s)) \, ds| + 1) \\ &\leq C(\overline{\lim_{s \to s_0'^+}} |W(a_1(s_1), \dots, a_m(s_1)) - W(a_1(s), \dots, a_m(s))| + 1) \\ &\leq C(2 \sup_{s \in (s_0', s_1]} |W(a_1(s), \dots, a_m(s))| + 1).\end{aligned}$$

The conclusion (4.4) then follows from Proposition 3.

Step 3. The set $\Sigma_{\mathfrak{v}}^{s_0'} \cap B_i$ is reduced to a single point. In particular $f_i(s_0') = 0$.

Indeed, assume by contradiction that $\Sigma_{\mathfrak{v}}^{s_0'} \cap B_i$ contains at least two points. In view of the conservation of energy (12), their degrees would be, in absolute value, strictly less than k, so that using the induction hypothesis $\sharp(\Sigma_{\mathfrak{v}}^{s} \cap B_i)$ would be constant equal to $\sharp(\Sigma_{\mathfrak{v}}^{s_0'} \cap B_i)$ in some neighborhood of s_0'. This would contradict Step 1.

Step 4. We claim that $\sharp(\Sigma_{\mathfrak{v}}^{s_1} \cap B_i) = 1$, which yields the desired contradiction to (4.3) when $s_1 > s_0$.

In view of (20) we have

$$|f_i'(s)| \leq C(1 + |\nabla W(a_1(s), \dots, a_m(s))|) \, f_i(s), \quad \forall \, s \in (s_0', s_1).$$

By Gronwall's lemma, we obtain

$$f_i(s) \leq f_i(s_0') \exp(C \int_{s_0'}^{s_1} (1 + |\nabla W(a_1(s), \dots, a_m(s))|) \, ds), \quad \forall \, s \in (s_0', s_1].$$

Hence $f_i(s) = 0$ on $(s_0', s_1]$ in view of Step 2 and Step 3. In particular $f_i(s_1) = 0$ and the conclusion follows.

The case $s_1 < s_0$ is treated with very similar arguments. □

Proof of Theorem 2. Lemma 4.1 implies that the number of vortices is locally constant out of the dissipation times. By continuation, it is therefore constant on each interval (τ_k, τ_{k+1}). Hence statement i) is proved. Once the total number of vortices is known to be constant, statement ii) follows from the evolution law for the center of mass (14), which we proved in Lemma 5.2 of [4]. □

For the proof of Theorem 4, we extend the result of Lemma 4.1 to vortices $(a_i(\tau_k), \tau_k)$ as follows.

Lemma 4.2. *Let $(a_i(\tau_k), \tau_k) \in \Sigma_{\mathfrak{v}}$ and assume that*

$$\sum_{j=1}^{\ell_i^+} (d_j^+)^2 = d_i^2(\tau_k) \qquad (\textit{resp.} \ \sum_{j=1}^{\ell_i^-} (d_j^-)^2 = d_i^2(\tau_k)).$$

Then

$$\ell_i^+ = 1 \qquad (\textit{resp.} \ \ell_i^- = 1).$$

Proof. Assume by contradiction that $\ell_i^+ \geq 2$. In view of Theorem 2, relabelling possibly the vortices, we have for $j \in \{1, \dots, \ell_i^+\}$

$$\frac{d}{ds}a_j(s) = -\frac{1}{d_j^2}\nabla W(a_1(s), \dots, a_{\ell_i^+}(s)) + R_j(s),$$

for $s \in (\tau_k, \tau_k + \Delta\tau_k)$, where $R_j(s)$ is a continuous and bounded function. Arguing as in Lemma 4.1, we obtain

$$\int_{\tau_k}^{\tau_k+\Delta\tau_k} |\nabla W(a_1(s), \dots, a_{\ell_i^+}(s))|^2\, ds \leq C(\sup_{s\in(\tau_k,\tau_k+\Delta\tau_k)} |W(a_1(s), \dots, a_{\ell_i^+}(s))| + 1).$$

In view of Proposition 3.1, the right hand side of the last inequality is finite, so that

$$\int_{\tau_k}^{\tau_k+\Delta\tau_k} |\nabla W(a_1(s), \dots, a_{\ell_i^+}(s))|\, ds < +\infty,$$

and we may then finish the proof as in Lemma 4.1 using Gronwall's lemma. □

Proof of Theorem 4. We first notice that (8), which was already obtained in [3] is, as mentioned, a consequence of the homotopy invariance of the degree, the convergence stated in Theorem 1, and the already established regularity properties of $\Sigma_{\mathfrak{v}}$.

Concerning (9), we turn to Proposition 3.1, and claim that

$$\Gamma_i^+ \leq 0\,, \qquad \Gamma_i^- \geq 0\,. \tag{4.7}$$

This is a rather direct consequence of expansion (3.17) and the gradient-flow type property of (3). Inequalities (4.7) then follow from (9).

We now turn to the first statement in Theorem 4, namely the identification of dissipation points and branching points. It is straightforward to show that a regular point is not a dissipation point. Indeed, in this case $\ell_i^+ = \ell_i^- = 1$, $d_i^+ = d_i^-$, so that $\beta_i^k = 0$ in view of (7) and (8).

We next prove that a point which is not a dissipation point is a regular point. By Lemma 4.2, if $\Gamma_i^+ = 0$ (resp. $\Gamma_i^- = 0$) then $\ell_i^+ = 1$ (resp. $\ell_i^- = 1$). In particular, if $(a_i(\tau_k), \tau_k)$ is not a dissipation point, then $\Gamma_i^+ - \Gamma_i^- = \beta_i^k = 0$. In view of (4.7) it follows in this case that $\Gamma_i^+ = \Gamma_i^- = 0$, so that $\ell_i^+ = \ell_i^- = 1$ and therefore $(a_i(\tau_k), \tau_k)$ is a regular point. □

5 Examples of branching points in the trajectory set

In the introduction we classified branching points by their complexity as illustrated in Figure 2. In this section we will show that cases a), b) and c) may be observed as limits of solutions to $(\mathrm{PGL})_\varepsilon$. Case d) would require more efforts.

5.1 Collisions with annihilation

We consider a well-prepared initial datum, having two vortices of degree +1 and −1 of the form

$$u_\varepsilon^0(z) = f(\frac{|z-1|}{\varepsilon})f(\frac{|z+1|}{\varepsilon})\frac{(z-1)(\bar{z}+1)}{|(z-1)(\bar{z}+1)|}\,,$$

where f is a smooth nonnegative function on $\mathbb{R}^+$ such that $f(0) = 0$, $f \equiv 1$ outside a compact set. In particular

$$u_\varepsilon^0 \to u_*^0 = \frac{(z-1)(\bar z + 1)}{|(z-1)(\bar z + 1)|} \qquad \text{as } \varepsilon \to 0\,,$$

i.e. u_*^0 has a vortex of degree $+1$ located at $a_+(0) = 1$, and a vortex of degree -1 located at $a_-(0) = -1$. The solution to the ordinary differential equation (3) with initial datum as above is given explicitely as

$$a_\pm(s) = \pm\sqrt{1-s}\,, \qquad \text{for } 0 \le s < 1\,.$$

Let u_ε be the solution of $(\mathrm{PGL})_\varepsilon$ with inital datum u_ε^0, and $a_i(s)$ be the points given by Theorem 1. It follows from [9] that for $0 \le s < 1$, $\ell(s) = 2$ and, after a possible relabelling, $a_1(s) = a_+(s)$, $a_2(s) = a_-(s)$.

It follows from Theorem 4 that for $s > 1$, $\ell(s) = 0$, so that vortices have disappeared after $s = 1$. This provides an example for Figure 2*a*.

5.2 Collisions without annihilation

We consider here a well-prepared initial datum of the form

$$u_\varepsilon^0(z) = v_\varepsilon \prod_{k=1}^{3} f\left(\frac{|z-a_k(0)|}{\varepsilon}\right)\left(\frac{z-a_k(0)}{|z-a_k(0)|}\right)^{d_i}\,,$$

where f is as in Section 5.1, v_ε is defined as

$$v_\varepsilon(z) = f\left(\frac{|z-\varepsilon^{-1}|}{\varepsilon}\right)\frac{\bar z - \varepsilon^{-1}}{|\bar z - \varepsilon^{-1}|}\,,$$

and $a_k(0) = k-2$, $d_k = (-1)^{k+1}$, for $k = 1, 2, 3$. Since the total degree near the origin is different from zero, we added a "vortex at infinity" (by superposing v_ε) in order to have total degree zero at infinity. In particular, (H_0) is verified, with say $M_0 = 6\pi$. The solution of the ordinary differential equation (3) with initial datum $a_k(0) = k-2$ and $d_k = (-1)^{k+1}$, for $k = 1, 2, 3$, is given by

$$a_k(s) = (2-k)\sqrt{1-2s}\,, \qquad \text{for } 0 \le s < \frac{1}{2} \qquad \text{for } k = 1, 2, 3.$$

It follows again by the argument of [9] that $a_k(s)$ are the points provided in Theorem 1. It follows from Theorem 4, identity (8) and inequality (9) that, for $s > 1/2$, a_2 and a_3 have disappeared and that

$$a_1(s) = 0 \qquad \text{for } s \ge \frac{1}{2}.$$

In particular, the branching point $(0, \frac{1}{2})$ is as in Figure 2*b*.

5.3 On the persistency of multiple degree vortices

In view of (5), a multiple degree vortex, say of degree $d \ge 2$, is energetically less favourable than d vortices of degree one. One may therefore ask wether multiple degree vortices may arise and survive as limits of solutions to the gradient flow $(\mathrm{PGL})_\varepsilon$. The next construction, which in this respect complements Theorem 2 and Theorem 4 by an example, provides a positive answer to that question for $d = 2$.

Theorem 5.1. *Let $s_0 > 0$ be given. There exists M_0 and a sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ of $(PGL)_\varepsilon$ such that $\Sigma^s_{\mathfrak{v}}$ is reduced to a single vortex located at the origin and of degree $d = +2$, for $s < s_0$, and which splits at time $s = s_0$ into two distinct vortices of degree $+1$.*

The idea of the proof is to approximate the trivial multiple degree solution of (3) given by $\ell(s) = 1$, $a_1(s) \equiv 0$ for $s \in (0, s_0)$ by solutions of (3) involving only vortices of degree ± 1, for which two vortices collapse at time s_0. Since ± 1-vortices do not split, these solutions may be well approximated by $(\mathrm{PGL})_\varepsilon$, in view of [4], Proposition 1.

In order to construct these solutions, the idea is to consider as initial value the stationary solution of (3) presented in Proposition 2.2 to perturb it slightly so that the solution to the corresponding ordinary differential equation eventually breaks up, and finally to scale the whole construction down so that the configuration asymptotically appears as a single vortex of degree $+2$.

More precisely, for $\delta \in (0, 1)$, we consider as initial values for equation (3) the configuration

$$a_1^\delta(0) = \delta, \qquad a_i^\delta(0) = z_i\,, \quad \text{for } i = 1, 2, 3\,. \tag{5.1}$$

with $d_1 = -1$, $d_2 = d_3 = d_4 = +1$, and denote $a_i^\delta(s)$, for $i = 1, 2, 3, 4$, the corresponding solution to (3). We have

Lemma 5.1. *The maximal time of existence $T^*(\delta)$ of (3) with initial data (5.1) is finite. Moreover, the function $\delta \to T^*(\delta)$ is nonincreasing, continuous and satisfies*

$$T^*(\delta) \to +\infty \qquad \textit{as } \delta \to 0 \tag{5.2}$$

$$T^*(\delta) \to 0 \qquad \textit{as } \delta \to 1 \tag{5.3}$$

Proof. By symmetry,

$$\mathrm{Im}(a_i^\delta(s)) = 0, \qquad \forall\, s \in [0, T^*(\delta)\,), \qquad i = 1, 2 \tag{5.4}$$

$$a_3^\delta(s) = a_4^\delta(s) \qquad \forall\, s \in [0, T^*(\delta)\,)\,. \tag{5.5}$$

It is an elementary (although not straightforward) exercise to check, in view of (3), that $\forall s \in [0, T^*(\delta)\,)$,

$$\frac{d}{ds}\mathrm{Re}(a_1^\delta(s)) \geq c(\delta) > 0, \qquad \frac{d}{ds}\mathrm{Re}(a_2^\delta(s)) \leq -c(\delta) < 0, \tag{5.6}$$

$$\frac{d}{ds}\mathrm{Re}(a_3^\delta(s)) = \frac{d}{ds}\mathrm{Re}(a_4^\delta(s)) \leq 0, \qquad \frac{d}{ds}\mathrm{Im}(a_3^\delta(s)) = -\frac{d}{ds}\mathrm{Im}(a_4^\delta(s)) \geq 0. \tag{5.7}$$

In view of (5.7) the points a_1^δ and a_2^δ are moving towards each other at a speed bounded below by a positive constant so that they collide in finite time, in particular $T^*(\delta)$ is finite. On the other hand, one checks that, for any $s \in [0, T^*(\delta)\,)$,

$$\frac{d}{d\delta}(a_1^\delta(s)) > 0\,, \qquad \frac{d}{d\delta}(a_2^\delta(s)) \leq 0\,, \tag{5.8}$$

which shows that $T^*(\delta)$ is nonincreasing. Property (5.3) is straightforward, whereas property (5.2) follows from the continuous dependence with respect to initial data and the fact that (z_1, z_2, z_3, z_4) is a stationary solution to (3). $\square$

Lemma 5.2. *For every $\delta \in (0,1)$ and every $s \in [0, T^*(\delta)\,)$,*

$$\sup_{i=1,..,4} |a_i^\delta(s)| \leq 2 \tag{5.9}$$

Proof. We have

$$\frac{d}{ds} \sum_{i=1}^{4} |a_i^\delta(s)|^2 = 0\,. \tag{5.10}$$

Indeed, by (2.2),

$$\frac{1}{2}\frac{d}{ds} \sum_{i=1}^{4} |a_i^\delta(s)|^2 = \sum_{i=1}^{4} a_i^\delta(s) \nabla_{a_i} W(a_1^\delta(s), a_2^\delta(s), a_3^\delta(s), a_4^\delta(s)) = \sum d_i d_j = 0. \tag{5.11}$$

The conclusion follows from (5.10). □

At time $T^*(\delta)$ the point a_1^δ and a_2^δ collide and hence we remove them from the collection, whereas the functions a_3^δ and a_4^δ can be uniquely continued beyond time $T^*(\delta)$ according to (3), so that $\forall\, s \geq T^*(\delta)$, for $i = 3, 4$,

$$\mathrm{Re}(a_i^\delta(s)) = \mathrm{Re}(a_i^\delta(T^*(\delta))),$$

$$\mathrm{Im}(a_i^\delta(s)) = (-1)^{i+1} \left(s - T^*(\delta) - \frac{|\mathrm{Im}(a_i^\delta(s))|}{4} \right)^{1/2}.$$

We set

$$\Sigma^\delta = \cup_{s>0} \{a_i^\delta(s)\}\,.$$

Proposition 5.1. *Let $0 < \delta < 1$ be given. There exists a family $(u_\varepsilon^\delta)_{0<\varepsilon<1}$ of solutions of $(PGL)_\varepsilon$ satisfying (H_0) with $M_0 = 10\pi$ and such that $\Sigma_{\mathfrak{v}} = \Sigma^\delta$.*

Proof. In view of Theorem 2 and the fact that all the degrees involved here are either +1 or −1, it suffices to construct a family for which

$$\Sigma_{\mathfrak{v}}^0 = \{a_1^\delta(0), z_2, z_3, z_4\}\,.$$

This can be done as in the two previous subsections, one may take

$$u_\varepsilon^\delta(z, 0) = v_\varepsilon \prod_{i=1}^{4} f(\frac{|z - a_i^\delta(0)|}{\varepsilon}) \left(\frac{z - a_i^\delta(0)}{|z - a_i^\delta(0)|} \right)^{d_i} \qquad \forall\, z \in \mathbb{R}^2\,,$$

where f is a smooth nonnegative function on $\mathbb{R}^+$ such that $f(0) = 0$, $f \equiv 1$ outside a compact set, and v_ε is defined as

$$v_\varepsilon(z) = f(\frac{|z - \varepsilon^{-1}|}{\varepsilon}) \left(\frac{z - \varepsilon^{-1}}{|z - \varepsilon^{-1}|} \right)^{-2}.$$

One then argues as in Section 5.1 or Section 5.2. □

Proof of Theorem 5.1 For $n \in \mathbb{N}$, we denote by δ_n the value for which $T^*(\delta_n) = n$. In view of Proposition 5.1, Theorem 1 and 3, there exists ε_n such that

$$\left| \frac{1}{|\log \varepsilon_n|} \int_{B(a_i^{\delta_n}(s), 1/n^2)} - \pi \right| \leq \frac{1}{n^2}$$

for every $s \in [0, n^2] \setminus [n - 1/n, n + 1/n]$. Set $\tilde{\varepsilon}_n = s_0 \varepsilon_n / n$, and define

$$\mathfrak{u}_{\tilde{\varepsilon}_n}(z, s) = \mathfrak{u}_{\varepsilon_n}^{\delta_n}\left(\sqrt{\frac{n}{s_0}} z, \frac{n}{s_0} s\right)$$

so that $\mathfrak{u}_{\tilde{\varepsilon}_n}$ verifies $(\mathrm{PGL})_\varepsilon$ with $\varepsilon = \tilde{\varepsilon}_n$, and

$$\left| \frac{1}{|\log \tilde{\varepsilon}_n|} \int_{B(b_i^n(s), 1/n^{3/2})} e_{\tilde{\varepsilon}_n}(u_{\tilde{\varepsilon}_n}) - \pi \right| \leq C\left(\frac{1}{n^2} + \frac{\log n}{|\log \varepsilon_n|}\right) \tag{5.12}$$

for $s \in [0, n s_0] \setminus [s_0 - \frac{s_0}{n^2}, s_0 + \frac{s_0}{n^2}]$, where

$$b_i^n(s) = \sqrt{\frac{s_0}{n}} a_i^{\delta_n}\left(\frac{n}{s_0} s\right), \qquad i = 1, 2, 3, 4.$$

Notice that, for $i = 1, 2, 3, 4$,

$$b_i^n(s) \to 0 \qquad \text{for every } s \in [0, s_0) \tag{5.13}$$

and that, for $i = 3, 4$,

$$b_i^n(s) \to 2\sqrt{s - s_0} \qquad \text{for } s \in (s_0, +\infty) \tag{5.14}$$

It follows from (5.12), (5.13) and (5.14) that the set $\Sigma_{\mathfrak{v}}$ corresponding to $(u_{\tilde{\varepsilon}_n})_{n \in \mathbb{N}}$ fulfills the requirements of Theorem 5.1. □

Remark 5.1. *1. Theorem 5.1 provides an example for Figure* 2c.
2. Persistency of multiple degree vortices with $d > 2$ *could presumably be obtained by a similar construction, provided one knows a critical point of* W *of toal degree* d *involving only* $+1$ *and* -1 *vortices.*

6 Behavior near branching points

The parabolic nature of a branching point $(a_i(\tau_k), \tau_k)$ is suggested by the formula

$$\frac{d}{ds} \int_{B_i} |x - a_i(\tau_k)|^2 \, d\mathfrak{v}_*^s = 2\Gamma_i^\pm + O(|s - \tau_k|^{\frac{1}{4}}),$$

where $\Gamma_i^\pm$ are defined in (3.16). Therefore, in order to analyze the limiting behavior of trajectories near a branching point, we perform the parabolic change of variables, for[5] $s > \tau_k$,

$$\tilde{s} = -\log(s - \tau_k), \qquad \tilde{a}_j(\tilde{s}) = \frac{a_j(s) - a_i(\tau_k)}{\sqrt{s - \tau_k}}.$$

[5] One argues similarly for $s < \tau_k$.

The equation for $\tilde a_j$ becomes

$$\frac{d}{d\tilde s}\tilde a_j = \frac{1}{d_j^2}\nabla_j W(\tilde a_1,\cdots,\tilde a_{\ell_k}) - \frac{1}{d_j}\exp(-\tilde s)c^\perp + \frac{1}{2}\tilde a_j, \tag{6.1}$$

for which we have to consider the limit $\tilde s \to +\infty$, which corresponds to $s \to \tau_k^+$. In view of the change of variables, the vortices $\tilde a_j$ for $j \notin \{1,\cdots,\ell_i^+\}$ are sent at infinity, whereas in view of the parabolic cone property (see [3] Theorem 2) the points $\tilde a_j$ for $j \in \{1,\cdots,\ell_i^+\}$ remain in a bounded set. Equation (6.1) therefore reads, for $j \in \{1,\cdots,\ell_i^+\}$,

$$\frac{d}{d\tilde s}\tilde a_j = \frac{1}{d_j^2}\nabla_j W(\tilde a_1,\cdots,\tilde a_{\ell_i^+}) + \frac{1}{2}\tilde a_j + O(\exp(-\frac{\tilde s}{2})) \tag{6.2}$$

as $\tilde s \to +\infty$.

Lemma 6.1. *We have*

$$\Big|4\Gamma_i^+ + \sum_{k=1}^{\ell_i^+} d_k^2 \tilde a_k^2\Big| \le C\exp(-\frac{\tilde s}{4}).$$

Proof. In view of (6.2) and (2.2), we have

$$\frac{d}{d\tilde s}\sum_{k=1}^{\ell_i^+} d_k^2\tilde a_k^2 = 4\Gamma_i^+ + \sum_{k=1}^{\ell_i^+} d_k^2\tilde a_k^2 + O(\exp(-\frac{\tilde s}{2})).$$

Set $E(\tilde s) = (4\Gamma_i^+ + \sum_{k=1}^{\ell_i^+} d_k^2\tilde a_k^2)^2$. It follows from (6.1) that

$$\frac{d}{d\tilde s}E(\tilde s) = E(\tilde s) + O(\exp(-\frac{\tilde s}{2})).$$

Integrating from $\frac{\tilde s}{2}$ to $\tilde s$ we obtain

$$E(\frac{\tilde s}{2})\exp(\frac{\tilde s}{2}) \le E(\tilde s) + C\int_{\frac{\tilde s}{2}}^{\tilde s}\exp(\tilde s - t)\exp(-\frac{t}{2})\,dt \le C\exp(\frac{\tilde s}{4}),$$

so that $E(\tilde s) \le C\exp(-\frac{\tilde s}{2})$ and the conclusion follows. ◻

Arguing as in Lemme 6.1, one may prove[6] similarily that

$$\Big|\sum_{k=1}^{\ell_i^+} d_k^2\tilde a_k(\tilde s)\Big| \le C\exp(-\frac{\tilde s}{4}).$$

Lemma 6.2. *We have*

$$\limsup_{\tilde s\to+\infty}\Big|W(\tilde a_1(\tilde s),\cdots,\tilde a_{\ell_i^+}(\tilde s))\Big| < +\infty.$$

[6] We will not make use of this fact in our subsequent arguments.

Proof. By (3.17), we have, as $s \to \tau_k^+$,

$$W(a_1(s), \cdots, a_{\ell_i^+}(s)) = \Gamma_i^+ \log|s - \tau_k| + O(1) = -\Gamma_i^+ \tilde{s} + O(1).$$

On the other hand, by definition of $\tilde{a}_j$ and W,

$$W(\{\tilde{a}_j(\tilde{s})\}) = W(\{\frac{a_j(s)}{\sqrt{s - \tau_k}}\}) = W(\{a_j(s)\}) - \Gamma_i^+ \log(s - \tau_k) = W(\{a_j(s)\}) - \Gamma_i^+ \tilde{s}.$$

Combining the two identities we obtain the conclusion. □

Up to the exponentially decreasing error term, (6.2) represents a gradient flow of W under the constraint $\sum_{j=1}^{\ell_i^+} d_j^2 a_j^2 = -4\Gamma_i^+$.

Lemma 6.3. *For $\tilde{s} > 0$, set $\mathcal{L}(\tilde{s}) = W(\tilde{a}_1(\tilde{s}), \cdots, \tilde{a}_{\ell_i^+}(\tilde{s})) + \frac{1}{4}\sum_{j=1}^{\ell_i^+} d_j^2 \tilde{a}_j^2(\tilde{s})$. Then we have*

$$\frac{d}{d\tilde{s}}\mathcal{L}(\tilde{s}) \geq \frac{1}{2}\sum_{j=1}^{\ell_i^+} d_j^2 \Big(\frac{1}{d_j^2}\nabla_j W + \frac{1}{2}\tilde{a}_j\Big)^2 - C\exp(-\tilde{s}).$$

Proof. We have, by definition of $\mathcal{L}$ and (6.2),

$$\frac{d}{d\tilde{s}}\mathcal{L}(\tilde{s}) \geq \sum_{j=1}^{\ell_i^+}\Big(\nabla_j W + \frac{1}{2}d_j^2 \tilde{a}_j\Big)\frac{d}{d\tilde{s}}\tilde{a}_j = \sum_{j=1}^{\ell_i^+} d_j^2 \Big(\frac{1}{d_j^2}\nabla_j W + \frac{1}{2}\tilde{a}_j\Big)\Big(\frac{1}{d_j^2}\nabla_j W + \frac{1}{2}\tilde{a}_j + O(\exp(-\frac{\tilde{s}}{2}))\Big)$$

and the conclusion follows by Young's inequality. □

It follows from Lemma 6.1 and 6.2 that $\mathcal{L}$ is uniformly bounded, and therefore we deduce from Lemma 6.3

Corollary 6.1. *We have*

$$\int_0^{+\infty} \Big|\nabla_j W(\{\tilde{a}_j(\tilde{s})\}) + \frac{d_j^2}{2}\tilde{a}_j(\tilde{s})\Big|^2 d\tilde{s} < +\infty. \tag{6.3}$$

Inequality (6.3) is typical of gradient flows, where it is used to prove convergence towards critical points. In this context, we introduce the configuration set

$$\mathcal{C} = \left\{ \begin{array}{l} \{(a_j, d_j)\}_{j=1,\cdots,\ell},\ a_j \in \mathbb{C},\ a_j \neq a_k \text{ for } j \neq k, \\ d_j \in \mathbb{Z}^*, \sum d_j = d_i,\ \sum d_j^2 - (\sum d_j)^2 = \Gamma_i^+ \end{array} \right\},$$

the constrained subset of $\mathcal{C}$,

$$\mathcal{M} = \Big\{\{(a_j, d_j)\} \in \mathcal{C},\ \sum d_j^2 |a_j|^2 = -4\Gamma_i^+\Big\},$$

and the set of critical points of W restricted to $\mathcal{M}$,

$$\mathcal{K} = \Big\{\{(a_j, d_j)\} \in \mathcal{M},\ \nabla_k W(\{(a_j, d_j)\}) = -\frac{d_k^2}{2}a_k\Big\}.$$

Notice that the configuration sets $\mathcal{C}$ and $\mathcal{M}$ have a stratified structure, each leaf corresponding to fixed values of the total number of vortices. The notion of criticality for an element in $\mathcal{K}$ is meant here for the restriction of W to the leaf to which it belongs.

We define a distance on $\mathcal{C}$ by

$$\operatorname{dist}\Big(\{(a_j,d_j)\}_{1\le j\le \ell},\{(a'_j,d'_j)\}_{1\le j\le \ell'}\Big)=\sup_{|\nabla\xi|\le 1}\left|\sum_{j=1}^{\ell} d_j\xi(a_j)-\sum_{j=1}^{\ell'} d'_j\xi(a'_j)\right|.$$

Notice that this distance represents the flat norm (see [8]) of the current $\sum_{j=1}^{\ell} d_j\delta_{a_j}-\sum_{j=1}^{\ell'} d'_j\delta_{a'_j}$. It is known (see [5]) to be equivalent to the minimal connection between the points $\{(a_j,d_j)\}$ and $\{(a'_j,d'_j)\}$.

Before we proceed to the main result of this section, we gather some properties of $\mathcal{K}$ which will be used later.

Lemma 6.4. *The set $\mathcal{K}$ is compact.*

Proof. We consider a sequence $(\{(a_j^n,d_j^n)\}_{1\le j\le \ell_n})_{n\in\mathbb{N}}$ in $\mathcal{K}$, and show that a subsequence converges to an element of $\mathcal{K}$. Without loss of generality, we may assume that $\ell_n\equiv\ell$ is constant, and that $d_j^n=d_j$ is independent of n. Since $\mathcal{M}$ is bounded, so is $\mathcal{K}$, and we may therefore assume passing possibly to a subsequence that $a_j^n\to a_j$ as $n\to+\infty$, for $j=1,\cdots,\ell$. If all the points a_j are distinct, then we are done by continuity of ∇W. To complete the proof, it remains to consider the situation where several points converge to the same limit. In this case, denote by $\{b_j\}_{j\in J}$ the set of limit points, $J_j=\{k\in 1,\cdots,\ell,\ a_k^n\to b_j\}$, and set, for $j\in J$, $D_j=\sum_{k\in J_j} d_j$. Our aim is to prove that the configuration $\{(b_j,D_j)\}_{j\in J}$ belongs to $\mathcal{K}$. The fact that $\nabla_k W(\{a_j^n\})=-\frac{d_k^2}{2}a_k^n$ reads, in view of (2.1),

$$4\sum_{m\ne k}\frac{d_kd_m}{a_m^n-a_k^n}=d_k^2\overline{a_k^n},\qquad k=1,\cdots,\ell$$

that is, for $k\in J_j$,

$$4\sum_{\substack{m\ne k\\ m\in J_j}}\frac{d_kd_m}{a_m^n-a_k^n}=-4\sum_{\substack{m\ne k\\ m\notin J_j}}\frac{d_kd_m}{a_m^n-a_k^n}+d_k^2\overline{a_k^n},$$

and therefore

$$4\sum_{k\in J_j}\sum_{\substack{m\ne k\\ m\in J_j}}\frac{d_kd_m}{a_m^n-a_k^n}=-4\sum_{k\in J_j}\sum_{\substack{m\ne k\\ m\notin J_j}}\frac{d_kd_m}{a_m^n-a_k^n}+\sum_{k\in J_j}d_k^2\overline{a_k^n}.$$

The left hand side of the previous equality is zero, by antisymmetry. Passing to the limit as $n\to+\infty$ in the right hand side, we are led to

$$4\sum_{\substack{m\ne j\\ m\in J}}\frac{D_mD_j}{b_m-b_j}=\left(\sum_{k\in J_j}d_k^2\right)\overline{b_j}. \tag{6.4}$$

To conclude, we claim that for all $j\in J$,

$$\sum_{k\in J_j}d_k^2=D_j^2. \tag{6.5}$$

Indeed, by (2.9) we have

$$\sum_{k\in J_j} d_k^2 = \sum_{k\in J_j} d_k^2 + \sum_{k\in J_j} \nabla_{z_k} W(\{(a_k^n - b_j, d_k)\}_{k\in J_j})\,(a_k^n - b_j).$$

Since $\nabla_{z_k} W(\{(a_k^n - b_j, d_k)\}_{k\in J_j}) - \nabla_{z_k} W(\{(a_k^n - b_j, d_k)\}_{k=1,\cdots,\ell})$ is bounded independently of n, the conclusion (6.5) follows letting $n \to +\infty$. From (6.5) we then deduce that

$$\sum_{j\in J} D_j^2 - (\sum_{j\in J} D_j)^2 = \Gamma_i^+,$$

and from (6.4) and (6.5) that

$$\nabla_j W(\{(b_j, D_j)\}_{j\in J}) = -\frac{D_j^2}{2} b_j.$$

The proof is complete. □

In the spirit of the Palais-Smale condition, we have the following variant of Lemma 6.4.

Lemma 6.5. *Let $(a_j^n, d_j^n)_{1\le j\le \ell_n}$ be a sequence such that $a_j^n \in \mathbb{C}$, $a_j^n \neq a_k^n$ for $j \neq k$, $d_j^n \in \mathbb{Z}^*$,*

$$\sum d_j^n = d_i \qquad \text{and} \qquad \sum (d_j^n)^2 - (\sum d_j^n)^2 = \Gamma_i^+.$$

Assume moreover that for $1 \le j \le \ell_n$,

$$\nabla_j W(\{(a_k^n, d_k^n)\}) = -\frac{{d_j^n}^2}{2} a_j^n + o(1), \qquad \text{as } n \to +\infty.$$

Then, up to a subsequence $\{(a_j^n, d_j^n)\}$ converges towards an element in $\mathcal{K}$ as $n \to +\infty$.

The argument is exactly the same as in the proof of Lemma 6.4: it suffices in many identities to replace zero by $o(1)$. Therefore we omit the proof.

Corollary 6.2. *Let $\delta > 0$. There exists $\varepsilon > 0$ such that, for every configuration $(a_j, d_j)_{1\le j\le \ell}$ with $a_j \in \mathbb{C}$, $a_j \neq a_k$ for $j \neq k$, $d_j \in \mathbb{Z}^*$,*

$$\sum d_j = d_i \qquad \text{and} \qquad \sum (d_j)^2 - (\sum d_j)^2 = \Gamma_i^+,$$

if moreover

$$\operatorname{dist}(\{(a_j, d_j)\}, \mathcal{K}) \ge \delta \qquad \text{then} \qquad \sum_j |\nabla_j W(\{(a_k, d_k)\}) + \frac{d_j^2}{2} a_j| \ge \varepsilon.$$

The proof follows from Lemma 6.5 arguing by contradiction.

We come back now to the aymptotics of $\tilde{a}_j(\tilde{s})$. The main result of this section is

Theorem 6.1. *We have*

$$\operatorname{dist}(\{(\tilde{a}_j(\tilde{s}), d_j)\}, \mathcal{K}) \to 0$$

as $\tilde{s}$ tends to plus infinity.

Proof. This is again a standard argument for gradient flow type equations, once a Palais-Smale property has been established. Indeed, it follows from (6.3) and Lemma 6.5 that there exists a sequence $(\tilde{s}_n)_{n\in\mathbb{N}}$ such that $\tilde{s}_n \to +\infty$ and

$$\mathrm{dist}(\{(\tilde{a}_j(\tilde{s}_n), d_j)\}, \mathcal{K}) \to 0 \qquad \text{as } n \to +\infty.$$

It remains to show that convergence holds not only for a sequence but for $\tilde{s} \to +\infty$. To that aim, for a given $\delta > 0$, assume that $\tilde{s}_a < \tilde{s}_b$ are such that

$$\mathrm{dist}(\{(\tilde{a}_j(\tilde{s}_a), d_j)\}, \mathcal{K}) = \delta, \qquad \mathrm{dist}(\{(\tilde{a}_j(\tilde{s}_b), d_j)\}, \mathcal{K}) = 2\delta, \tag{6.6}$$

and for every $\tilde{s} \in (\tilde{s}_a, \tilde{s}_b)$ it holds $\mathrm{dist}(\{(\tilde{a}_j(\tilde{s}), d_j)\}, \mathcal{K}) \geq \delta$. Then, by Corollary 6.2 we have, for some $\varepsilon > 0$ depending on δ,

$$\sum_j |\nabla_j W(\{\tilde{a}_k(\tilde{s})\}) + \frac{d_j^2}{2}\tilde{a}_j(\tilde{s})| \geq \varepsilon \qquad \forall s \in (\tilde{s}_a, \tilde{s}_b).$$

Hence,

$$\begin{aligned}\int_{\tilde{s}_a}^{\tilde{s}_b} \sum_j |\nabla_j W(\{\tilde{a}_k(\tilde{s})\}) + \frac{d_j^2}{2}\tilde{a}_j(\tilde{s})|^2 \, d\tilde{s} &\geq \varepsilon \int_{\tilde{s}_a}^{\tilde{s}_b} \sum_j |\nabla_j W(\{\tilde{a}_k(\tilde{s})\}) + \frac{d_j^2}{2}\tilde{a}_j(\tilde{s})| \, d\tilde{s} \\ &\geq \varepsilon \sum_j \int_{\tilde{s}_a}^{\tilde{s}_b} |\frac{d}{d\tilde{s}}\tilde{a}_j(\tilde{s})| \, d\tilde{s} - C\exp(-\tilde{s}_a) \\ &\geq C(\varepsilon\delta - \exp(-\tilde{s}_a)).\end{aligned}$$

In particular,

$$\int_{\tilde{s}_a}^{+\infty} \sum_j |\nabla_j W(\{\tilde{a}_k(\tilde{s})\}) + \frac{d_j^2}{2}\tilde{a}_j(\tilde{s})|^2 \, d\tilde{s} \geq C(\varepsilon\delta - \exp(-\tilde{s}_a)). \tag{6.7}$$

In view of (6.3), the integral on the left hand side of (6.7) tends to zero as $\tilde{s}_a$ goes to $+\infty$, so that (6.6) may only happen for $\tilde{s}_a$ bounded by a constant depending only on δ and the conclusion follows. □

As a consequence of Theorem 6.1, there exists a sequence $\tilde{s}_n \to +\infty$ an a critical point $(b_j, D_j)_{j\in J}$ such that $(\tilde{a}_j(\tilde{s}_n), d_j) \to (b_j, D_j)_{j\in J}$ with $2 \leq \sharp J \leq \ell_i^+$. Asymptotic self-similarity of the trajectory set near the branching point $(a_i(\tau_k), \tau_k)$ would mean that the whole family $(\tilde{a}_j(\tilde{s}), d_j)$ converges to the same limit $(b_j, D_j)_{j\in J}$. However, we do not know if this holds.

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