# A quantitative Pólya–Szegö principle


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## 1 Introduction and main results

A very classical principle in the geometric theory of functions, going back, in its basic formulation, to Pólya and Szegö [PS], asserts that Dirichlet type integrals like

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \,,$$

with $p \geq 1$, do not increase under Schwarz radially decreasing symmetrization of $u$. More precisely, the Pólya–Szegö principle tells us that, given any real-valued weakly differentiable function $u$ in $\mathbb{R}^n$, $n \geq 1$, such that $|\nabla u| \in L^p(\mathbb{R}^n)$ and

$$\mathcal{L}^n\left(\{|u| > 0\}\right) < +\infty \,, \tag{1.1}$$

its Schwarz symmetral $u^\star : \mathbb{R}^n \to [0, +\infty]$ is weakly differentiable as well, and

$$\int_{\mathbb{R}^n} |\nabla u^\star|^p \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p \, dx \tag{1.2}$$

(see e.g. [BZ], [Ka1], [Ta1]). Here, $\mathcal{L}^n$ denotes the Lebesgue measure.

Inequality (1.2), together with its several variants (see e.g. [Ka1]), is a powerful key to a number of results in variational problems of geometric and functional nature, concerning remarkable extremal properties of domains and functions endowed with symmetries inherited from the data. Classical isoperimetric inequalities in mathematical physics, sharp eigenvalue inequalities, optimal Sobolev embeddings fall within these results; a priori estimates for solutions to elliptic problems in sharp form are also a closely related topic (we refer to [Ta2] and [Tro] for a survey of results on this matter).

Investigations on the uniqueness of the relevant extremals in this kind of problems are more recent, and typically require an additional delicate analysis, resting, in some instances, on the description of the cases of equality in (1.2) (see e.g. [CNV], [Ka1], [FP], [Mag]). Such a description has been the object of the series of papers [Ka1], [FM], [Ur], [BZ] appeared some twenty years ago, and has been recently extended and simplified by new contributions, including [Bur], [BG], [CF3], [FV1], [FV2], [ET]. Since equality can hold in (1.2) even if $u$ does not agree with any translate of $u^\bigstar$, the point was to find the weakest possible assumptions under which equality in (1.2) entails the symmetry of $u$. These were clarified in full generality in [BZ], where any nonnegative weakly differentiable function attaining equality in (1.2) for some $p > 1$ is shown to be symmetric, provided that

$$\mathcal{L}^n\big(\{\nabla u^\bigstar = 0\} \cap \{0 < u^\bigstar < \operatorname{esssup} u\}\big) = 0\,. \tag{1.3}$$

Condition (1.3), as well as the hypothesis $p > 1$, are indispensable to conclude about the symmetry of $u$, as demonstrated by suitable examples (see e.g. [BZ]). In particular, the simplest counterexamples in this connection involve functions $u$ whose graph contains a plateau at a level between 0 and $\operatorname{esssup} u$ - a situation prevented by assumption (1.3) - which are symmetric above and below the plateau level about different points.

The study of the stability - an issue of special interest in modern analysis - of the aforementioned variational problems is still largely open, although contributions have already appeared, especially as far as isoperimetric and isocapacitary inequalities are concerned ([BW], [BE], [Fu], [FMP], [Ha], [HHW], [HN]).
Motivated by applications to other of these problems - see e.g. [Ci], [CFMP] - the objective of the present paper is to strengthen inequality (1.2) by quantifying the gap between $\int_{\mathbb{R}^n} |\nabla u|^p dx$ and $\int_{\mathbb{R}^n} |\nabla u^\bigstar|^p dx$ in terms of the deviation of $u$ from $u^\bigstar$. In view of the picture sketched above, it is apparent that any remainder term, which accounts for the deficit between the right-hand side and the left-hand side of (1.2), has necessarily to depend not only on such a deviation, but also on $u^\bigstar$ itself, and, specifically, on its gradient. Our main result, contained in Theorem 1.1 below, ensures that, up to translations, the distance in $L^1(\mathbb{R}^n)$ of $u$ from either $u^\bigstar$ or $-u^\bigstar$ can be bounded in terms of the (normalized) excess

$$E(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\int_{\mathbb{R}^n} |\nabla u^\bigstar|^p dx} - 1\,,$$

by an estimate depending also on the (normalized complementary) distribution function of $|\nabla u^\bigstar|$ defined as

$$M_{u^\bigstar}(\sigma) = \frac{\mathcal{L}^n\left(\{|\nabla u^\bigstar| \le \sigma\} \cap \{0 < u^\bigstar < \operatorname{esssup} u\}\right)}{\mathcal{L}^n(\{|u| > 0\})} \qquad \text{for } \sigma \ge 0\,, \tag{1.4}$$

or on the function $M_u$, which is defined as in (1.4) on replacing $u^\bigstar$ by $|u|$. For simplicity of notations, we state our stability result for functions $u$ normalized and rescaled in such a way that

$$\mathcal{L}^n\left(\{|u| > 0\}\right) = 1\,, \tag{1.5}$$

and

$$\int_{\mathbb{R}^n} |\nabla u^\bigstar|^p dx = 1\,. \tag{1.6}$$

**Theorem 1.1** *Let $p > 1$ and let $n \geq 2$. Then positive constants $r$, $s$ and $C$, depending only on $p$ and $n$, exist such that, for every $u \in W^{1,p}(\mathbb{R}^n)$ satisfying (1.5) and (1.6),*

$$\min_{\pm} \inf_{x_0 \in \mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) \pm u^\bigstar(x + x_0)|\, dx \leq C \Big[ M_{u^\bigstar}\big(E(u)^r\big) + E(u) \Big]^s. \tag{1.7}$$

*Moreover, $\int_{\mathbb{R}^n} |\nabla u^\bigstar|^p dx$ and $M_{u^\bigstar}$ can be replaced by $\int_{\mathbb{R}^n} |\nabla u|^p dx$ and $M_u$, respectively, in (1.6) and (1.7).*

Conditions (1.5) and (1.6) can be easily removed via a rescaling and normalizing argument. An explicit form of the resulting estimate, in an even somewhat stronger version, can be found in Theorem 4.1, Section 4.

Theorem 1.1 recovers, in particular, Brothers and Ziemer's theorem corresponding to the case where $E(u) = 0$ and $M_{u^\bigstar}(0) = 0$. It reproduces (a special case of) [CF3] as well, since, under the sole assumption that $E(u) = 0$, inequality (1.7) enables us to estimate the distance in $L^1$ between $u$ and a suitable translated of $u^\bigstar$ in terms of $M_{u^\bigstar}(0)$, namely in terms of the left hand side of (1.3). Note that, in fact, Theorem 1.1 slightly improves these results in that, unlike [BZ] and [CF3], not necessarily positive functions are allowed.

Another consequence of (1.7) is that any sequence $\{u_k\}$ of nonnegative functions in $W^{1,p}(\mathbb{R}^n)$, with supports of equibounded measures, satisfying $\lim_{k\to\infty} E(u_k) = 0$, and such that $u_k^\bigstar \equiv v$ for some fixed symmetric function $v$ fulfilling $M_v(0) = 0$, converges (up to translations) to $v$ in $L^1(\mathbb{R}^n)$. This conclusion overlaps with a result of [BG], where the convergence of gradients is also shown, but no quantitative information is provided.

That the whole function $M_{u^\bigstar}$ comes into play in Theorem 1.1, instead of just $M_{u^\bigstar}(0)$ as in [CF3], is explained by the fact that large sets where $|\nabla u^\bigstar|$ and $|\nabla u|$ are small may allow $u$ to be very asymmetric when $E(u) > 0$, in spite of $\int_{\mathbb{R}^n} |\nabla u|^p dx$ being very close to $\int_{\mathbb{R}^n} |\nabla u^\bigstar|^p dx$. Apropos examples can be easily exhibited, even when the dimension $n$ equals one. For more details, we refer to [CF4], dealing with a parallel issue for Steiner symmetrization (see also [CCF, CF2] for related results). Incidentally, let us note that the approach of the present paper differs from that of [CF4], where essentially one-dimensional techniques, combined with slicing arguments, are employed.

Estimate (1.7) can be somewhat enhanced, on replacing the $L^1$ norm by a stronger norm on the left-hand side. Indeed, on exploiting standard multiplicative Gagliardo–Nirenberg inequalities (see e.g. [Ma]), one can easily infer from (1.7) that

$$\min_{\pm} \inf_{x_0 \in \mathbb{R}^n} \|u(\cdot) \pm u^\bigstar(\cdot + x_0)\|_{L^q} \leq C \Big[ M_{u^\bigstar}\big(E(u)^r\big) + E(u) \Big]^s, \tag{1.8}$$

where $q \in [1, \frac{np}{n-p})$ if $1 < p < n$, $q$ is any number $\geq 1$ if $p = n$, and $q = \infty$ if $p > n$ (and $r$, $s$, and $C$ are suitable constants).

Observe that, when $1 < p < n$ and $q = \frac{np}{n-p}$, the Sobolev conjugate of $p$, inequality (1.8) breaks down. Actually, any inequality like (1.8), with $q = \frac{np}{n-p}$, should be independent of $\mathcal{L}^n(\{|u| > 0\})$, because of dimensional reasons, and hence, in particular, functions not necessarily satisfying (1.1) should be admissible (in this connection, notice that (1.2) continues to hold even if (1.1) is replaced by the weaker

condition $\mathcal{L}^n(\{|u| > t\}) < \infty$ for every $t > 0$). On the other hand, if $\mathcal{L}^n(\{|u| > 0\}) = +\infty$, then $\mathcal{L}^n(\{|\nabla u^\star| \leq \sigma\} \cap \{0 < u^\star < \operatorname{esssup} u\}) = +\infty$ for every $\sigma > 0$, inasmuch as $|\nabla u^\star| \in L^p(\mathbb{R}^n)$, thus making definition (1.4) meaningless.
Whether a result in the spirit of Theorem 1.1 involving the distance in $L^{\frac{np}{n-p}}(\mathbb{R}^n)$ of $u$ from a translated of $u^\star$ can hold, and which kind of information on $|\nabla u^\star|$ should play a role, is still unclear at the moment.

Among other ingredients, our proof of Theorem 1.1 requires a study of mutual bounds between the functions $M_{u^\star}$ and $M_u$. This can be considered a problem of independent interest in the theory of symmetrization. Indeed, it is well known that the set of critical points of a Sobolev function generally shrinks under symmetrization. Namely, according to our notation, $M_{u^\star}(0) \leq M_u(0)$ for every $u \in W^{1,p}(\mathbb{R}^n)$ (see e.g. [CF1, Lemma 3.3]). On the other hand, $M_{u^\star}(\sigma)$ and $M_u(\sigma)$ are unrelated in general for $\sigma \geq 0$, in the sense that no estimate between the two functions can hold in either direction. To verify that $M_{u^\star}$ is not bounded by $M_u$ away from 0, consider the one-dimensional function $u_k(x)$ given by $1-(2k+1)|x|$ if $|x| < \frac{1}{2k+1}$, $k \in \mathbb{N}$, and extended periodically in $[-1, 1]$. Obviously, $u_k^\star(x) = 1 - |x|$ for $x \in [-1, 1]$. Therefore, fixed any $\sigma > 0$, one has $M_{u_k}(\sigma) = 0$ if $k$ is sufficiently large, whereas $M_{u_k^\star}(\sigma) = 1$ if $\sigma \geq 1$ for every $k \in \mathbb{N}$. Conversely, the function $M_u$ is not controlled by $M_{u^\star}$ even at 0, since functions $u$ can be exhibited such that $M_u(0)$ is arbitrarily close to 1, but $M_{u^\star}(0) = 0$ (see e.g. [AL]). Our next result shows that, nevertheless, $M_{u^\star}$ and $M_u$ can be estimated in terms of each other, if the extra information contained in $E(u)$ is exploited as well.

**Theorem 1.2** *Let $p > 1$ and let $n \geq 2$. Then there exist positive constants $r$ and $C$, depending only on $p$ and $n$, such that, if $u$ is any function as in Theorem 1.1, then*

$$M_u(\sigma) \leq C\left[M_{u^\star}(4\sigma) + E(u)^r + \frac{E(u)^r}{\sigma}\right] \tag{1.9}$$

*and*

$$M_{u^\star}(\sigma) \leq C\left[M_u(4\sigma) + E(u)^r + \frac{E(u)^r}{\sigma}\right] \tag{1.10}$$

*for every $\sigma > 0$.*

The paper is organized as follows. Section 2 is devoted to a study of relations between the level sets of $u$ and $u^\star$, and between the values of $|\nabla u|$ and $|\nabla u^\star|$ on their boundaries, depending on $E(u)$. The function $M_{u^\star}$ is irrelevant at this stage. Such a function comes into play in Section 3, where material from Section 2 is employed to prove Theorem 1.2. Finally, the proof of Theorem 1.1 is accomplished in Section 4, where, starting from the conclusions of Sections 2 and 3, the measure of the symmetric difference of the level sets of $u$ and of (a suitable translated of) $u^\star$ is estimated in terms of $E(u)$ and $M_{u^\star}$, thus enabling us to establish (1.7).

## 2 Estimates on level sets

Although alternative proofs of the Pólya–Szegö inequality are available in the literature (see e.g. [Bae], [Bur], [Ke]), the standard - and probably geometrically most transparent - approach relies upon two main tools: the coarea formula and the isoperimetric inequality.

The coarea formula states that, given any function $u \in W^{1,1}(\mathbb{R}^n)$ and any Borel function $\varphi : \mathbb{R}^n \to [0,+\infty]$,

$$\int_{\mathbb{R}^n} \varphi(x)|\nabla u|\,dx = \int_{-\infty}^{+\infty} \int_{\{u=t\}} \varphi(x)\,d\mathcal{H}^{n-1}(x)dt\,, \tag{2.1}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Here, and in the sequel, $u$ is assumed to agree with its precise representative $\widetilde{u}$, which is defined at $x$ as the unique real number $\widetilde{u}(x)$ such that $\lim_{r\to 0} \fint_{B_r(x)}|u(y)-\widetilde{u}(x)|dy = 0$ whenever such a number exists (and this occurs $\mathcal{H}^{n-1}$-a.e. in $\mathbb{R}^n$), and as 0 otherwise.

The classical isoperimetric inequality in $\mathbb{R}^n$ tells us that

$$P(E) \geq n\omega_n^{1/n}\mathcal{L}^n(E)^{1/n'} \tag{2.2}$$

for every measurable set having finite measure, and that equality holds in (2.2) if and only if $E$ is (equivalent to) a ball. Here $P(E)$ stands for the perimeter of $E$ defined according to geometric measure theory (see e.g. [AFP]), and $\omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2})$, the measure of the $n$-dimensional unit ball.

Inequality (1.2) can be derived quite easily from (2.1) and (2.2). We briefly sketch such a derivation for completeness, and, more importantly, because it is the starting point in our discussion of the quantitative version (1.7).

Let us begin by recalling a few definitions. Let $u$ be a function from $W^{1,p}(\mathbb{R}^n)$, for some $p > 1$, satisfying (1.1). Defined $\mu : [0,+\infty) \to [0,+\infty)$, the distribution function of $u$, as

$$\mu(t) = \mathcal{L}^n(\{x \in \mathbb{R}^n : |u(x)| > t\}) \qquad \text{for } t \geq 0\,,$$

the Schwarz symmetral $u^\bigstar$ of $u$ is given by

$$u^\bigstar(x) = \sup\{t > 0 : \mu(t) > \omega_n|x|^n\} \qquad \text{for } x \in \mathbb{R}^n.$$

Note that $u$ and $u^\bigstar$ share the same distribution function, a fact that will be used in the sequel without further mentioning. Let us suppose, in addition, that

$$u(x) \geq 0 \qquad \text{for a.e. } x \in \mathbb{R}^n. \tag{2.3}$$

Such an assumption, although by no means necessary, will be kept in force throughout this section for ease of presentation.

In order to prove (1.2), note that, owing to (2.1),

$$\mu(t) = \mathcal{L}^n\left(\{u > t\} \cap \{\nabla u = 0\}\right) + \int_t^{+\infty} \int_{\{u=\tau\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} d\tau \qquad \text{for } t > 0\,. \tag{2.4}$$

Hence,

$$-\mu'(t) \geq \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t > 0\,. \tag{2.5}$$

An application of (2.4) with $u$ replaced by $u^\bigstar$, plus an extra argument showing that

$$\frac{d}{dt}\mathcal{L}^n\big(\{u^\bigstar > t\} \cap \{\nabla u^\bigstar = 0\}\big) = 0 \qquad \text{for } \mathcal{L}^1\text{-a.e. } t > 0 \tag{2.6}$$

(see e.g. [CF1, Lemmas 2.4 and 2.6]), yield

$$-\mu'(t) = \frac{\mathcal{H}^{n-1}(\{u^\star = t\})}{|\nabla u^\star|_{|\{u^\star=t\}}} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t > 0. \tag{2.7}$$

Now, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u^\star|^p \, dx &= \int_0^{+\infty} \int_{\{u^\star=t\}} |\nabla u^\star|^{p-1} \, d\mathcal{H}^{n-1} dt && \text{(coarea formula (2.1))} \\
&= \int_0^{+\infty} \frac{\mathcal{H}^{n-1}(\{u^\star = t\})^p}{\left(\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\right)^{p-1}} \, dt \\
&= \int_0^{+\infty} \frac{\mathcal{H}^{n-1}(\{u^\star = t\})^p}{(-\mu'(t))^{p-1}} \, dt && \text{(equation (2.7))} \\
&\leq \int_0^{+\infty} \frac{\mathcal{H}^{n-1}(\{u = t\})^p}{(-\mu'(t))^{p-1}} \, dt && \text{(isoperimetric inequality (2.2))} \\
&\leq \int_0^{+\infty} \frac{\mathcal{H}^{n-1}(\{u = t\})^p}{\left(\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1}} \, dt && \text{(inequality (2.5))} \\
&\leq \int_0^{+\infty} \int_{\{u=t\}} |\nabla u|^{p-1} \, d\mathcal{H}^{n-1} dt && \text{(Hölder's inequality)} \\
&= \int_{\mathbb{R}^n} |\nabla u|^p \, dx && \text{((2.1) again),}
\end{aligned} \tag{2.8}$$

namely (1.2). Notice that, in applying (2.2) we have exploited the fact that

$$P(\{u > t\}) = \mathcal{H}^{n-1}(\{u = t\}) \qquad \text{for } \mathcal{L}^n\text{-a.e. } t > 0\,, \tag{2.9}$$

since $u$ is a Sobolev function (see e.g. [BZ]).

In the case where equality holds in (1.2), i.e. $E(u) = 0$, from chain (2.8) we infer that

$$\{u > t\} \text{ is (equivalent to) a ball for every } t > 0 \tag{2.10}$$

and

$$|\nabla u| = |\nabla u^\star|_{|\{u^\star=t\}} \ \ \mathcal{H}^{n-1}\text{-a.e. on } \{u = t\} \text{ for } \mathcal{L}^1\text{-a.e. } t > 0. \tag{2.11}$$

Of course, assertion (2.10) requires the characterization of the cases of equality in (2.2), whereas (2.11) rests upon the description of the cases of equality in Hölder's inequality. In fact, (2.10) and (2.11) are the first step in the available proofs of Brothers and Ziemer's theorem on the symmetry of extremals in (1.2) ([BZ], [FV1], [FV2]).

The task of the present section is, roughly speaking, to investigate on the stability of properties (2.10) and (2.11) under perturbations of equality in (1.2). More precisely, we provide estimates on the deviation of the level sets $\{u > t\}$ from balls, and of $|\nabla u|$ from the constant $|\nabla u^\star|_{|\{u^\star=t\}}$ on $\{u = t\}$, for every $t$ outside a small set, in terms of the deficit $E(u)$. Clearly, such estimates will require suitable quantitative versions of the isoperimetric inequality and of Hölder's inequality. Before going into details, let us preliminarily fix a few notations. We define

$$e(p) = \begin{cases} p & \text{if } p \geq 2 \\ p' & \text{if } 1 < p < 2 \end{cases} ,$$

and fix a positive number $\alpha$, smaller than $\dfrac{1}{8e(p)}$, to be chosen later. Moreover, we set

$$\varepsilon = \int_{\mathbb{R}^n} |\nabla u|^p \, dx - \int_{\mathbb{R}^n} |\nabla u^\star|^p \, dx \,, \tag{2.12}$$

the non normalized version of $E(u)$,

$$t_{\varepsilon,\alpha} = \sup\Big\{ t > 0 : \, \mathcal{L}^n(\{u > t\}) > \varepsilon^{\alpha n'} \Big\} \tag{2.13}$$

and

$$R_t = \left( \frac{\mathcal{L}^n(\{u > t\})}{\omega_n} \right)^{1/n} \qquad \text{for} \quad t \geq 0 \,.$$

We begin with the problem of measuring the distance of the level sets of $u$ from balls. In this connection, a key ingredient is a quantitative version of (2.2) to which we alluded above, stating that a positive constant $C_0(n)$ exists such that, for any measurable set $E \subset \mathbb{R}^n$ of finite measure and perimeter, there exists a ball $B$ satisfying $\mathcal{L}^n(B) = \mathcal{L}^n(E)$ and

$$\mathcal{L}^n(E \triangle B) \leq C_0 \mathcal{L}^n(E) \left( \frac{P(E) - P(B)}{P(B)} \right)^{1/2} . \tag{2.14}$$

Here, $\triangle$ stands for symmetric difference of sets. Inequality (2.14) is contained in ([Ha]) in a weaker form with the exponent $\frac{1}{2}$ replaced by $\frac{1}{4}$; the present sharp version is the object of the recent paper [FMP]. In view of (2.14) and (2.9), the problem of estimating the distance of $\{u > t\}$ from a suitable translated of the ball $\{u^\star > t\}$ is reduced to a bound for $\mathcal{H}^{n-1}(\{u = t\}) - \mathcal{H}^{n-1}(\{u^\star = t\})$. This is the content of our first lemma.

**Lemma 2.1** *There exists a set $I_1 \subset (0, \operatorname{esssup} u)$ satisfying*

$$\mathcal{L}^1(I_1) < \varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}} \mathcal{L}^n(\{u > 0\}) \,, \tag{2.15}$$

*and such that*

$$\mathcal{H}^{n-1}(\{u = t\}) \leq \left( 1 + \frac{\varepsilon^{\frac{1}{4p} - \alpha}}{n \omega_n^{1/n}} \right) \mathcal{H}^{n-1}(\{u^\star = t\}) \qquad \text{for every } t \in (0, t_{\varepsilon,\alpha}) \setminus I_1 \,. \tag{2.16}$$

**Proof.** From (2.8) we infer that

$$\int_0^{+\infty} \frac{\mathcal{H}^{n-1}(\{u = t\})^p - \mathcal{H}^{n-1}(\{u^\star = t\})^p}{(-\mu'(t))^{p-1}} \, dt \leq \varepsilon \,. \tag{2.17}$$

Define,

$$J_1 = \Big\{ t > 0 : \, \mathcal{H}^{n-1}(\{u = t\})^p - \mathcal{H}^{n-1}(\{u^\star = t\})^p > \sqrt{\varepsilon} (-\mu'(t))^{p-1} \Big\}$$

and observe that

$$\mathcal{L}^1(J_1) < \sqrt{\varepsilon} \,,$$

by (2.17). Moreover, on setting

$$J_2 = \Big\{ t > 0 : \, -\mu'(t) > \varepsilon^{-\frac{1}{4(p-1)}} \Big\} \tag{2.18}$$

we have

$$\mathcal{L}^1(J_2) < \varepsilon^{\frac{1}{4(p-1)}} \mathcal{L}^n(\{u > 0\}), \tag{2.19}$$

since

$$\varepsilon^{-\frac{1}{4(p-1)}} \mathcal{L}^1(J_2) < \int_0^{+\infty} (-\mu'(t))\, dt \le |D\mu|\,((0,+\infty)) = \mathcal{L}^n(\{u > 0\}).$$

The very definitions of $J_1$ and $J_2$ entail that

$$\mathcal{H}^{n-1}(\{u = t\})^p - \mathcal{H}^{n-1}(\{u^\star = t\})^p \le \varepsilon^{1/4} \qquad \text{for every } t \notin J_1 \cup J_2 .$$

Consequently,

$$\begin{aligned} \mathcal{H}^{n-1}(\{u = t\}) &\le \Big(\varepsilon^{1/4} + \mathcal{H}^{n-1}(\{u^\star = t\})^p\Big)^{1/p} \\ &\le \varepsilon^{1/(4p)} + \mathcal{H}^{n-1}(\{u^\star = t\}) = \mathcal{H}^{n-1}(\{u^\star = t\})\Big(1 + \frac{\varepsilon^{1/(4p)}}{n\omega_n R_t^{n-1}}\Big). \end{aligned} \tag{2.20}$$

Since $R_t > \frac{\varepsilon^{\alpha/(n-1)}}{\omega_n^{1/n}}$ if $0 < t < t_{\varepsilon,\alpha}$, inequality (2.16) follows from (2.20), on choosing $I_1 = J_1 \cup J_2$. □

The perturbed version of property (2.10) is a straightforward consequence of (2.14) and of Lemma 2.1, and is stated in the next lemma. In what follows, $B_R(x)$ denotes the ball of radius $R$ centered at $x$.

**Lemma 2.2** *Let $I_1$ be the set provided by Lemma 2.1. Then, for every $t \in (0, t_{\varepsilon,\alpha}) \setminus I_1$ there exists $x_t \in \mathbb{R}^n$ such that*

$$\mathcal{L}^n\left(\{u > t\} \triangle B_{R_t}(x_t)\right) \le C_0 \frac{\omega_n^{1-\frac{1}{2n}}}{\sqrt{n}} R_t^n \varepsilon^{\frac{1}{2}\left(\frac{1}{4p} - \alpha\right)} \tag{2.21}$$

*where $C_0$ is the constant appearing in (2.14).*

In the remaining part of this section we discuss the problem of quantifying the gap between $|\nabla u|$ and $|\nabla u^\star|_{|\{u^\star = t\}}$, or, more precisely, between $\frac{1}{|\nabla u|}$ and $\frac{1}{|\nabla u^\star|}_{|\{u^\star = t\}}$, on $\{u = t\}$. The clue to extract the necessary information contained in (2.8) is the following quantitative form of Hölder's inequality. Hereafter, given a finite positive measure space $(X, \nu)$ and a nonnegative integrable function $g$ in $(X, \nu)$, we set

$$\fint_X g d\nu = \frac{1}{\nu(X)} \int_X g d\nu .$$

**Proposition 2.3** *For every $p > 1$, there exists a positive constant $C_1(p)$ having the following property. Let $(X, \nu)$ be a finite positive measure space and let $f : X \to [0, \infty)$ be a $\nu$-measurable function such that $f > 0$ $\nu$-a.e. in $X$. Assume that $\int_X f^{p-1} d\nu < \infty$ and $\int_X \frac{1}{f} d\nu < \infty$, and set*

$$h = \left(\fint_X f^{p-1}\, d\nu\right)\left(\fint_X \frac{1}{f}\, d\nu\right)^{p-1} - 1, \qquad \gamma_1 = \left(\fint_X \frac{1}{f}\, d\nu\right)^{1/p'}, \qquad \gamma_2 = \left(\fint_X f^{p-1}\, d\nu\right)^{1/p}. \tag{2.22}$$

*If $p \ge 2$, then*

$$\fint_X \left|\frac{1}{f} - \frac{\gamma_1}{\gamma_2^{1/(p-1)}}\right| d\nu \le C_1 \frac{\gamma_1}{\gamma_2^{1/(p-1)}} h^{1/p} . \tag{2.23}$$

*If $1 < p < 2$, then*

$$\fint_X \left| \frac{1}{f} - \frac{\gamma_1}{\gamma_2^{1/(p-1)}} \right| d\nu \le C_1 \frac{\gamma_1}{\gamma_2^{1/(p-1)}} \left[ h^{p-1} + h^{1/(p-1)} \right]^{1/p} . \tag{2.24}$$

Results of this kind should be certainly available in the literature. However, we have not been able to find an appropriate reference. A proof of Proposition 2.3 will be thus provided at the end of this section, but we first go back to the main issue of the behavior of $\frac{1}{|\nabla u|}$ on $\{u = t\}$. This is dealt in the next three lemmas, where the notation

$$\beta_t = \frac{\left( \fint_{\{u=t\}} \frac{1}{|\nabla u|} d\mathcal{H}^{n-1} \right)^{1/p'}}{\left( \fint_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1} \right)^{\frac{1}{p(p-1)}}} \qquad \text{for } t > 0$$

is adopted.

**Lemma 2.4** *A positive constant $C_2(p, n) \ge 1$ exists having the following property. There exists a set $I_2 \subset (0, \operatorname{esssup} u)$ satisfying*

$$\mathcal{L}^1(I_2) < \varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}} \mathcal{L}^n(\{u > 0\}) \tag{2.25}$$

*and such that, if $\varepsilon \le 1$, then, for every $t \in (0, t_{\varepsilon,\alpha}) \setminus I_2$,*

$$\fint_{\{u=t\}} \left| \frac{1}{|\nabla u|} - \beta_t \right| d\mathcal{H}^{n-1} \le C_2 \beta_t \varepsilon^{\frac{1}{4e(p)} - \alpha} . \tag{2.26}$$

*Moreover, for every $t \in (0, t_{\varepsilon,\alpha}) \setminus I_2$, a Borel set $U_t \subset \{u = t\}$ exists such that*

$$\left| \frac{1}{|\nabla u(x)|} - \beta_t \right| \le C_2 \beta_t \varepsilon^{\frac{1}{8e(p)} - \alpha} \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \{u = t\} \setminus U_t , \tag{2.27}$$

*and*

$$\mathcal{H}^{n-1}(U_t) \le \varepsilon^{\frac{1}{8e(p)}} \mathcal{H}^{n-1}(\{u = t\}) . \tag{2.28}$$

**Proof.** Equation (2.8) ensures that

$$\int_0^{+\infty} \left[ \int_{\{u=t\}} |\nabla u|^{p-1} \, d\mathcal{H}^{n-1} - \frac{\mathcal{H}^{n-1}(\{u = t\})^p}{\left( \int_{\{u=t\}} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \right)^{p-1}} \right] dt \le \varepsilon .$$

Thus, on setting

$$J_3 = \Bigg\{ t > 0 : \mathcal{H}^{n-1}(\{\nabla u = 0\} \cap \{u = t\}) = 0 \text{ and}$$
$$\int_{\{u=t\}} |\nabla u|^{p-1} \, d\mathcal{H}^{n-1} - \mathcal{H}^{n-1}(\{u = t\})^p \left( \int_{\{u=t\}} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \right)^{1-p} > \sqrt{\varepsilon} \Bigg\} ,$$

one has

$$\mathcal{L}^1(J_3) < \sqrt{\varepsilon} \tag{2.29}$$

and

$$\left(\fint_{\{u=t\}} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}\right)\left(\fint_{\{u=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\right)^{p-1} \le \sqrt{\varepsilon}\frac{\left(\int_{\{u=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}\right)^{p-1}}{\mathcal{H}^{n-1}(\{u=t\})^p} + 1$$

for every $t \in (0, \operatorname{esssup} u) \setminus J_3$. Hence, by (2.5), by the isoperimetric inequality (2.2) and by definition (2.13),

$$\left(\fint_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}\right)\left(\fint_{\{u=t\}} \frac{1}{|\nabla u|} d\mathcal{H}^{n-1}\right)^{p-1} \le 1 + \frac{\varepsilon^{\frac{1}{4}-\alpha p}}{(n\omega_n^{1/n})^p} \quad \text{for every } t \in (0, t_{\varepsilon,\alpha}) \setminus (J_2 \cup J_3), \tag{2.30}$$

where $J_2$ is defined by (2.18). On choosing $I_2 = J_2 \cup J_3$, inequality (2.25) holds thanks to (2.19) and (2.29). Moreover, estimate (2.26) follows from (2.30), via Proposition 2.3. Finally, inequalities (2.27) and (2.28) are easy consequences of (2.26). □

**Lemma 2.5** *A set $I_3 \subset (0, \operatorname{esssup} u)$ exists satisfying*

$$\mathcal{L}^1(I_3) < \varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}} \mathcal{L}^n(\{u > 0\}), \tag{2.31}$$

*and such that*

$$\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} - \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \le C_3 \frac{\varepsilon^{\frac{1}{4}-\alpha p}}{(n\omega_n^{1/n})^p} \int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} \tag{2.32}$$

*for every $t \in (0, t_{\varepsilon,\alpha}) \setminus I_3$, where $C_3(p) = \max\{1, 1/(p-1)\}$.*

**Proof.** On interchanging the use of the isoperimetric inequality and of (2.5) in (2.8), we infer that

$$\int_{\mathbb{R}^n} |\nabla u^\star|^p\, dx = \int_0^{+\infty} \frac{\mathcal{H}^{n-1}(\{u^\star=t\})^p}{\left(\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\right)^{p-1}} dt \le \int_0^{+\infty} \frac{\mathcal{H}^{n-1}(\{u^\star=t\})^p}{\left(\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1}} dt \le \int_{\mathbb{R}^n} |\nabla u|^p\, dx\,.$$

Thus, a set $J_4 \subset (0, +\infty)$ exists such that

$$\mathcal{L}^1(J_4) < \sqrt{\varepsilon}\,, \tag{2.33}$$

and

$$\frac{\mathcal{H}^{n-1}(\{u^\star=t\})^p}{\left(\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1}} - \frac{\mathcal{H}^{n-1}(\{u^\star=t\})^p}{\left(\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\right)^{p-1}} \le \sqrt{\varepsilon} \qquad \text{for every } t \in (0, +\infty) \setminus J_4\,. \tag{2.34}$$

On modifying, if necessary, $J_4$ by a set of $\mathcal{L}^1$ measure zero, we may assume that (2.5) and (2.7) hold for every $t \in (0, +\infty) \setminus J_4$. Hence, by (2.34),

$$\left(\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\right)^{p-1} - \left(\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1} \le \frac{\sqrt{\varepsilon}(-\mu'(t))^{p-1}}{\mathcal{H}^{n-1}(\{u^\star=t\})^p}\left(\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1} \tag{2.35}$$

for every $t \in (0,+\infty) \setminus J_4$. Set

$$I_2 = J_2 \cup J_4$$

where $J_2$ is defined by (2.18). Then (2.31) holds owing to (2.19) and (2.33).
As far as (2.32) is concerned, if $p \geq 2$ one can make use of the fact that $s - r \leq \frac{s^{p-1}-r^{p-1}}{r^{p-2}}$ whenever $0 < r \leq s$ to deduce from (2.35) that

$$\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} - \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \leq \frac{\sqrt{\varepsilon}(-\mu'(t))^{p-1}}{\mathcal{H}^{n-1}(\{u^\star=t\})^p} \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \leq \frac{\sqrt{\varepsilon}(-\mu'(t))^{p-1}}{\mathcal{H}^{n-1}(\{u^\star=t\})^p} \int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}$$

for every $t \in (0,+\infty) \setminus J_4$. Notice that the last inequality holds since

$$\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \leq \int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,+\infty)\,, \tag{2.36}$$

by (2.5) and (2.7). Inasmuch as $\mathcal{H}^{n-1}(\{u^\star = t\}) = n\omega_n^{1/n}\mu(t)^{1/n'}$, inequality (2.32) follows on recalling definitions (2.13) and (2.18).
Assume now that $1 < p < 2$. Then $s - r \leq \frac{1}{p-1}\frac{s^{p-1}-r^{p-1}}{s^{p-2}}$ if $0 < r \leq s$, and inequality (2.35) implies that

$$\begin{aligned}\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} - \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} &\leq \frac{1}{p-1} \frac{\sqrt{\varepsilon}(-\mu'(t))^{p-1}\Big(\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\Big)^{p-1}}{\mathcal{H}^{n-1}(\{u^\star = t\})^p\Big(\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\Big)^{p-2}} \\ &\leq \frac{1}{p-1}\frac{\sqrt{\varepsilon}(-\mu'(t))^{p-1}}{\mathcal{H}^{n-1}(\{u^\star = t\})^p} \int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} \qquad \text{for every } t \in (0,+\infty) \setminus J_4\,.\end{aligned}$$

Hence, (2.32) follows as above. □

Henceforth, we set

$$I = I_1 \cup I_2 \cup I_3\,. \tag{2.37}$$

where $I_1$, $I_2$ and $I_3$ are the subsets of $(0,+\infty)$ appearing in Lemmas 2.1, 2.4, 2.5, respectively.

**Lemma 2.6** *A positive constant $C_4(p,n)$ exists such that, if*

$$\varepsilon^{\frac{1}{4p}-\alpha} \leq \min\Big\{1, \frac{n\omega_n^{1/n}}{(2C_3)^{1/p}}\Big\}\,, \tag{2.38}$$

*where $C_3$ is the constant defined in Lemma 2.5, then*

$$\left|\frac{1}{|\nabla u^\star|_{|\{u^\star=t\}}} - \beta_t\right| \leq C_4\beta_t\varepsilon^{\frac{1}{4e(p)}-\alpha} \qquad \textit{for every } t \in (0,t_{\varepsilon,\alpha}) \setminus I\,.$$

**Proof.** We have

$$\begin{aligned}\left|\frac{1}{|\nabla u^\star|_{|\{u^\star=t\}}} - \beta_t\right| \leq\; & \frac{1}{\mathcal{H}^{n-1}(\{u^\star = t\})}\left|\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} - \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right| \\ &+ \frac{1}{\mathcal{H}^{n-1}(\{u^\star = t\})}\left|\int_{\{u=t\}} \Big(\frac{1}{|\nabla u|} - \beta_t\Big)\, d\mathcal{H}^{n-1}\right| + \Big(\frac{\mathcal{H}^{n-1}(\{u = t\})}{\mathcal{H}^{n-1}(\{u^\star = t\})} - 1\Big)\beta_t\end{aligned} \tag{2.39}$$

for $\mathcal{L}^1$-a.e. $t > 0$. On exploiting Lemmas 2.5, 2.4 and 2.1 to estimate the first, the second and the third summand, respectively, on the right-hand side of (2.39) one gets

$$\left|\frac{1}{|\nabla u^\bigstar|_{|\{u^\bigstar=t\}}} - \beta_t\right| \le \frac{C_3\varepsilon^{\frac{1}{4}-\alpha p}}{(n\omega_n^{1/n})^p\mathcal{H}^{n-1}(\{u^\bigstar=t\})}\int_{\{u^\bigstar=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|} + \frac{C_2\varepsilon^{\frac{1}{4e(p)}-\alpha}\mathcal{H}^{n-1}(\{u=t\})}{\mathcal{H}^{n-1}(\{u^\bigstar=t\})}\beta_t + \frac{\varepsilon^{\frac{1}{4p}-\alpha}}{n\omega_n^{1/n}}\beta_t \tag{2.40}$$

for every $t \in (0, t_{\varepsilon,\alpha}) \setminus I$. By (2.38) and Lemma 2.1,

$$\mathcal{H}^{n-1}(\{u=t\}) \le 2\mathcal{H}^{n-1}(\{u^\bigstar=t\}) \qquad \text{if } t \in (0, t_{\varepsilon,\alpha}) \setminus I_1\,. \tag{2.41}$$

Since

$$\frac{1}{\mathcal{H}^{n-1}(\{u^\bigstar=t\})}\int_{\{u^\bigstar=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|} \le \left|\frac{1}{|\nabla u^\bigstar|_{\{u^\bigstar=t\}}} - \beta_t\right| + \beta_t\,,$$

from (2.40), (2.41) and (2.38) we get

$$\left|\frac{1}{|\nabla u^\bigstar|_{\{u^\bigstar=t\}}} - \beta_t\right| \le \frac{1}{2}\left|\frac{1}{|\nabla u^\bigstar|_{\{u^\bigstar=t\}}} - \beta_t\right| + \left(2C_2\varepsilon^{\frac{1}{4e(p)}-\alpha} + C\varepsilon^{\frac{1}{4p}-\alpha}\right)\beta_t$$

for some constant $C(n)$ and for every $t \in (0, t_{\varepsilon,\alpha}) \setminus I$. Hence (2.38) follows. □

**Proof of Proposition 2.3.** An application of Hölder's inequality yields

$$⨍_X\left|\frac{1}{f} - \frac{\gamma_1}{\gamma_2^{1/(p-1)}}\right| d\nu \le \frac{\gamma_1}{\gamma_2^{1/(p-1)}}\left(⨍_X\left|\gamma_1 f^{1/p'} - \frac{\gamma_2^{1/(p-1)}}{f^{1/p}}\right|^p d\nu\right)^{1/p}. \tag{2.42}$$

In order to estimate the right-hand side of (2.42), we distinguish the cases where $p \ge 2$ and $1 < p < 2$.
CASE $p \ge 2$. It is easily seen that

$$\frac{1}{p}|r-s|^p \le \frac{r^p}{p} + \frac{s^p}{p'} - rs^{p-1} \qquad \text{for every } r, s \ge 0\,.$$

Hence,

$$\frac{1}{p}⨍_X\left|\gamma_1 f^{1/p'} - \frac{\gamma_2^{1/(p-1)}}{f^{1/p}}\right|^p d\nu \le \frac{1}{p}\gamma_1^p\gamma_2^p + \frac{1}{p'}\gamma_1^{p'}\gamma_2^{p'} - \gamma_1\gamma_2\,. \tag{2.43}$$

Thus, owing to (2.43) and (2.22),

$$\begin{aligned}\frac{1}{p}⨍_X\left|\gamma_1 f^{1/p'} - \frac{\gamma_2^{1/(p-1)}}{f^{1/p}}\right|^p d\nu &\le \frac{1}{p}(1+h) + \frac{1}{p'}(1+h)^{1/(p-1)} - (1+h)^{1/p} \\ &\le \frac{1}{p}(1+h) + \frac{1}{p'}\Big(1 + \frac{h}{p-1}\Big) - 1 = \frac{2h}{p}\,.\end{aligned} \tag{2.44}$$

Note that the second inequality holds since $(1+h)^{1/(p-1)} \le 1 + (h/(p-1))$, for $1/(p-1) \le 1$. Inequality (2.23) follows from (2.42) and (2.44), with $C_1 = 2^{1/p}$.

Case $1 < p < 2$. Elementary considerations tell us that a positive constant $C(p)$ exists such that

$$C|r-s|^p \leq \left[\left(\frac{r^p}{p}+\frac{s^p}{p'}\right)^{\frac{1}{p-1}} - (rs^{p-1})^{\frac{1}{p-1}}\right]^{p-1} \qquad \text{for every } r,s \geq 0\,. \tag{2.45}$$

An application of (2.45) yields

$$C \fint_X \left|\gamma_1 f^{1/p'} - \frac{\gamma_2^{1/(p-1)}}{f^{1/p}}\right|^p d\nu \leq \fint_X \left[\left(\frac{\gamma_1^p f^{p-1}}{p}+\frac{\gamma_2^{p'}}{p' f}\right)^{\frac{1}{p-1}} - (\gamma_1\gamma_2)^{\frac{1}{p-1}}\right]^{p-1} d\nu\,. \tag{2.46}$$

Now, recall that a reverse triangle inequality holds for $L^q$ quasinorms when $0 < q < 1$. Hence, since $0 < p-1 < 1$, the right-hand side of (2.46) does not exceed

$$\left[\left(\fint_X \left(\frac{\gamma_1^p f^{p-1}}{p}+\frac{\gamma_2^{p'}}{p' f}\right) d\nu\right)^{\frac{1}{p-1}} - (\gamma_1\gamma_2)^{\frac{1}{p-1}}\right]^{p-1}.$$

Consequently,

$$C \fint_X \left|\gamma_1 f^{1/p'} - \frac{\gamma_2^{1/(p-1)}}{f^{1/p}}\right|^p d\nu \leq \left[\left(\frac{1}{p}\gamma_1^p\gamma_2^p + \frac{1}{p'}\gamma_1^{p'}\gamma_2^{p'}\right)^{\frac{1}{p-1}} - (\gamma_1\gamma_2)^{\frac{1}{p-1}}\right]^{p-1}. \tag{2.47}$$

From (2.47) and (2.22) one deduces that

$$C \fint_X \left|\gamma_1 f^{1/p'} - \frac{\gamma_2^{1/(p-1)}}{f^{1/p}}\right|^p d\nu \leq \left[\left(\frac{1+h}{p} + \frac{(1+h)^{\frac{1}{p-1}}}{p'}\right)^{\frac{1}{p-1}} - (1+h)^{\frac{1}{p(p-1)}}\right]^{p-1}. \tag{2.48}$$

On the other hand, it is easily seen that a constant $C'(p)$ exists such that the expression appearing on the right-hand side of (2.48) does not exceed $C'\left[h^{p-1}+h^{1/(p-1)}\right]$. Inequality (2.24) is thus a consequence of (2.42) and (2.48). □

# 3 On the distribution functions of $|\nabla u|$ and $|\nabla u^\star|$

The present section is devoted to the proof of Theorem 1.2. Since $E(u) = E(|u|)$ and $M_u = M_{|u|}$ for every Sobolev function $u$, throughout this section we may always deal, without loss of generality, with nonnegative functions. Thus, $u$ will denote a function from $W^{1,p}(\mathbb{R}^n)$, $p > 1$, satisfying (1.5) and (2.3).

The initial steps in the argument leading to inequality (1.9) consist in estimates for $M_u(0)$ and $\mathcal{L}^n(u^{-1}(I))$ in terms of $M_{u^\star}(\sigma)$, where $I$ is the set defined by (2.37), and are enucleated in Lemmas 3.1 and 3.2 below. They contain the only information playing a role in the proof of inequality (1.7). Of course, the last assertion in the statement of Theorem 1.1 requires also the use of inequality (1.10).

**Lemma 3.1** *A positive constant $C_5(p,n)$ exists such that*

$$M_u(0) \leq M_{u^\star}(0)+M_{u^\star}(\sigma)+C_5\left(\varepsilon^{\frac{1}{4}-\alpha p}+\varepsilon^{\alpha n'}+\frac{\varepsilon^{\frac{1}{2}}+\varepsilon^{\frac{1}{4(p-1)}}}{\sigma}\right) \tag{3.1}$$

*for every $\sigma > 0$.*

**Proof.** Equation (2.4) applied to $u$ and to $u^\bigstar$, with $t=0$, ensures that

$$\mathcal{L}^n(\{u=\operatorname{esssup} u\}) + M_u(0) + \int_0^{\operatorname{esssup} u} \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\, dt$$
$$= \mathcal{L}^n(\{u^\bigstar=\operatorname{esssup} u\}) + M_{u^\bigstar}(0) + \int_0^{\operatorname{esssup} u} \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\, dt\,.$$

Hence

$$\begin{aligned} M_u(0) - M_{u^\bigstar}(0) &= \int_0^{\operatorname{esssup} u} \Big( \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|} - \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \Big) dt \\ &\leq \int_{(0,t_{\varepsilon,\alpha})\setminus I_3} \Big( \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|} - \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \Big) dt \\ &\quad + \int_{(0,t_{\varepsilon,\alpha})\cap I_3} \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\, dt + \int_{t_{\varepsilon,\alpha}}^{\operatorname{esssup} u} \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\, dt, \end{aligned} \tag{3.2}$$

where $I_3$ is the set given by Lemma 2.5. By this lemma and by the coarea formula,

$$\begin{aligned} \int_{(0,t_{\varepsilon,\alpha})\setminus I_3} \Big( \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|} - \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \Big) dt &\leq \frac{C_3 \varepsilon^{\frac14 - \alpha p}}{(n\omega_n^{1/n})^p} \int_{(0,t_{\varepsilon,\alpha})\setminus I_3} \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\, dt \\ &\leq \frac{C_3 \varepsilon^{\frac14 - \alpha p}}{(n\omega_n^{1/n})^p}\,. \end{aligned} \tag{3.3}$$

By the coarea formula again,

$$\int_{(0,t_{\varepsilon,\alpha})\cap I_3} \int_{\{u^\bigstar=t\}\cap\{|\nabla u^\bigstar|\leq\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\, dt \leq M_{u^\bigstar}(\sigma)\,. \tag{3.4}$$

On the other hand, owing to (2.31),

$$\begin{aligned} \int_{(0,t_{\varepsilon,\alpha})\cap I_3} \int_{\{u^\bigstar=t\}\cap\{|\nabla u^\bigstar|>\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\, dt &\leq \mathcal{L}^1(I_3) \frac{n\omega_n R_0^{n-1}}{\sigma} \\ &\leq \Big( \varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}} \Big) \frac{n\omega_n^{1/n}}{\sigma}\,. \end{aligned} \tag{3.5}$$

Finally, the coarea formula and the definition (2.13) ensure that

$$\int_{t_{\varepsilon,\alpha}}^{\operatorname{esssup} u} \int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\, dt \leq \mathcal{L}^n(\{u > t_{\varepsilon,\alpha}\}) \leq \varepsilon^{\alpha n'}\,. \tag{3.6}$$

Combining (3.2)-(3.6) yields (3.1). □

**Lemma 3.2** *Let $I$ be the set defined by (2.37). Then*

$$\mathcal{L}^n\left(u^{-1}(I)\right) \leq M_u(0) + M_{u^\bigstar}(\sigma) + 3n\omega_n^{1/n} \frac{(\varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}})}{\sigma} \tag{3.7}$$

*for every $\sigma > 0$.*

**Proof.** We have

$$\mathcal{L}^1(I) < 3\big(\varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}}\big)\,. \tag{3.8}$$

Thus, by (2.36),

$$\begin{aligned}\mathcal{L}^n\left(u^{-1}(I)\right) &= \mathcal{L}^n\left(u^{-1}(I)\cap\{\nabla u=0\}\right) + \int_I\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt \le M_u(0) + \int_I\int_{\{u^\star=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\,dt\\ &\le M_u(0) + \int_I\left(\int_{\{u^\star=t\}\cap\{|\nabla u^\star|\le\sigma\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} + \int_{\{u^\star=t\}\cap\{|\nabla u^\star|>\sigma\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\right)dt\\ &\le M_u(0) + M_{u^\star}(\sigma) + \frac{3n\omega_n R_0^{n-1}}{\sigma}\left(\varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}}\right),\end{aligned}$$

namely (3.7). □

**Proof of Theorem 1.2.** Recall that we are assuming

$$\mathcal{L}^n(\{u>0\}) = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^n}|\nabla u^\star|^p\,dx = 1\,. \tag{3.9}$$

We shall establish (1.9) and (1.10) in Steps 1 and 2, respectively.
Step 1 Define

$$G = (0, t_{\varepsilon,\alpha})\setminus I\,. \tag{3.10}$$

Then

$$\begin{aligned}M_u(\sigma) &\le M_u(0) + \mathcal{L}^n(u^{-1}(I)) + \mathcal{L}^n\left((\{0<|\nabla u|\le\sigma\}\cap\{0<u<t_{\varepsilon,\alpha}\})\setminus u^{-1}(I)\right) + \mathcal{L}^n(\{u>t_{\varepsilon,\alpha}\})\\ &= M_u(0) + \mathcal{L}^n(u^{-1}(I)) + \mathcal{L}^n(\{u>t_{\varepsilon,\alpha}\}) + \int_0^{+\infty}\chi_G(t)\int_{\{u=t\}\cap\{0<|\nabla u|\le\sigma\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt\,.\end{aligned} \tag{3.11}$$

The terms $M_u(0)$, $\mathcal{L}^n(u^{-1}(I))$ and $\mathcal{L}^n(\{u>t_{\varepsilon,\alpha}\})$ will be estimated via Lemma 3.1, Lemma 3.2 and (3.6), respectively. Thus, our only task is to bound the last integral.
Under the assumption that $\varepsilon\le 1$, we have, by Lemma 2.4,

$$\begin{aligned}&\int_0^{+\infty}\chi_G(t)\int_{\{u=t\}\cap\{0<|\nabla u|\le\sigma\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt\\ &\le \int_0^{+\infty}\chi_G(t)\int_{\{u=t\}}\left|\frac{1}{|\nabla u|}-\beta_t\right|d\mathcal{H}^{n-1}\,dt + \int_0^{+\infty}\chi_G(t)\beta_t\mathcal{H}^{n-1}(\{u=t\}\cap\{0<|\nabla u|\le\sigma\})dt\\ &\le C_2\int_0^{+\infty}\chi_G(t)\varepsilon^{\frac{1}{4e(p)}-\alpha}\beta_t\mathcal{H}^{n-1}(\{u=t\})dt + \int_0^{+\infty}\chi_G(t)\beta_t\mathcal{H}^{n-1}(\{u=t\}\cap U_t)dt\\ &\qquad + \int_0^{+\infty}\chi_G(t)\beta_t\mathcal{H}^{n-1}((\{u=t\}\cap\{0<|\nabla u|\le\sigma\})\setminus U_t)dt\,.\end{aligned} \tag{3.12}$$

We now estimate the last three integrals. Lemma 2.4 ensures that, if $t\in G$ and

$$\varepsilon \le \left(\frac{1}{2C_2}\right)^{\frac{4e(p)}{1-4\alpha e(p)}}, \tag{3.13}$$

then

$$\beta_t\mathcal{H}^{n-1}(\{u=t\}) \le \int_{\{u=t\}}\Big|\beta_t - \frac{1}{|\nabla u|}\Big|\, d\mathcal{H}^{n-1} + \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \le \frac{1}{2}\beta_t\mathcal{H}^{n-1}(\{u=t\}) + \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,,$$

whence

$$\beta_t\mathcal{H}^{n-1}(\{u=t\}) \le 2\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,. \tag{3.14}$$

Thus,

$$\begin{aligned} C_2\int_0^{+\infty}\chi_G(t)\varepsilon^{\frac{1}{4e(p)}-\alpha}\beta_t\mathcal{H}^{n-1}(\{u=t\})dt \;&\le\; 2C_2\varepsilon^{\frac{1}{4e(p)}-\alpha}\int_0^{+\infty}\chi_G(t)\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt \\ &\le\; 2C_2\varepsilon^{\frac{1}{4e(p)}-\alpha}\,. \end{aligned} \tag{3.15}$$

Next, thanks to (2.28) and (3.14),

$$\beta_t\mathcal{H}^{n-1}(\{u=t\}\cap U_t) \le \varepsilon^{\frac{1}{8e(p)}}\beta_t\mathcal{H}^{n-1}(\{u=t\}) \le 2\varepsilon^{\frac{1}{8e(p)}}\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,,$$

provided that (3.13) is in force, whence

$$\int_0^{+\infty}\chi_G(t)\beta_t\mathcal{H}^{n-1}(\{u=t\}\cap U_t)dt \le 2\varepsilon^{\frac{1}{8e(p)}}\int_0^{+\infty}\chi_G(t)\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt \le 2\varepsilon^{\frac{1}{8e(p)}}\,. \tag{3.16}$$

Finally, from (2.27) one infers that, if

$$\varepsilon \le \left(\frac{1}{2C_2}\right)^{\frac{8e(p)}{1-8\alpha e(p)}} \tag{3.17}$$

and $t\in G$, then

$$\frac{1}{2|\nabla u(x)|} \le \beta_t \le \frac{2}{|\nabla u(x)|} \qquad\qquad \text{for every } x\in\{u=t\}\setminus U_t\,.$$

Hence, by Lemma 2.6, if $t\in G$ and $\varepsilon$ satisfies (2.38), (3.17) and

$$\varepsilon \le \left(\frac{1}{2C_4}\right)^{\frac{4e(p)}{1-4\alpha e(p)}}\,, \tag{3.18}$$

then

$$\frac{1}{|\nabla u^\bigstar|_{|\{u^\bigstar=t\}}} \ge \beta_t - C_4\beta_t\varepsilon^{\frac{1}{4e(p)}-\alpha} \ge \frac{\beta_t}{2} \ge \frac{1}{4|\nabla u(x)|} \qquad \text{for every } x\in\{u=t\}\setminus U_t\,. \tag{3.19}$$

Consequently, under the same assumptions on $\varepsilon$, one has $\frac{1}{|\nabla u^\bigstar|_{\{u^\bigstar=t\}}} \ge \frac{1}{4\sigma}$, or equivalently $\{u^\bigstar = t\} \subset \{|\nabla u^\bigstar| \le 4\sigma\}$, if $t\in G$ and satisfies $\mathcal{H}^{n-1}((\{u=t\}\cap\{0<|\nabla u|\le\sigma\})\setminus U_t) > 0$. Hence, by (2.41), for these values of $\varepsilon$ and $t$

$$\begin{aligned} \mathcal{H}^{n-1}\left((\{u=t\}\cap\{0<|\nabla u|\le\sigma\})\setminus U_t\right) \;&\le\; \mathcal{H}^{n-1}(\{u=t\}) \le 2\mathcal{H}^{n-1}(\{u^\bigstar=t\}) \\ &=\; 2\mathcal{H}^{n-1}(\{u^\bigstar=t\}\cap\{0<|\nabla u^\bigstar|\le 4\sigma\})\,. \end{aligned} \tag{3.20}$$

Thus, by (3.18), (3.19) and (3.20),

$$
\begin{aligned}
(3.21)\int_0^{+\infty} \chi_G(t)\beta_t \mathcal{H}^{n-1}((\{u=t\}\cap\{0<|\nabla u|\leq\sigma\})\setminus U_t)dt \leq \int_0^{+\infty}\int_{\{u^\star=t\}\cap\{0<|\nabla u^\star|\leq 4\sigma\}} \frac{4}{|\nabla u^\star|}d\mathcal{H}^{n-1}\,dt \\
\leq 4M_{u^\star}(4\sigma)\,.
\end{aligned}
$$

Combining (3.12), (3.15), (3.16) and (3.21) yields

$$
(3.22)\qquad \int_0^{+\infty}\int_{\{u=t\}\cap\{0<|\nabla u|\leq\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt \leq 2(C_2+1)\varepsilon^{\frac{1}{8e(p)}-\alpha} + 4M_{u^\star}(4\sigma)\,,
$$

provided that $\varepsilon$ does not exceed the minimum among 1 and the right-hand sides of (3.13), (3.17), (2.38) and (3.18). Let us denote by $\varepsilon_0(p,n)$ such a minimum. From (3.11), (3.6), (3.7) and (3.22) one gets

$$
(3.23)\qquad M_u(\sigma) \leq 2M_u(0) + \varepsilon^{\alpha n'} + M_{u^\star}(\sigma) + \frac{3n\omega_n^{1/n}}{\sigma}\left(\varepsilon^{1/2}+\varepsilon^{\frac{1}{4(p-1)}}\right) + 2(C_2+1)\varepsilon^{\frac{1}{8e(p)}-\alpha} + 4M_{u^\star}(4\sigma)\,,
$$

if $\varepsilon \leq \varepsilon_0$. Inequalities (3.23) and (3.1) yield (1.9) for $\varepsilon \leq \varepsilon_0$. Obviously, inequality (1.9) continues to hold, possibly with a different constant, also when $\varepsilon > \varepsilon_0$.

STEP 2 One has $M_{u^\star}(0) \leq M_u(0)$ (see e.g. [CF1, Lemma 3.3]). Thus, on setting

$$
L_\sigma = \left\{t>0:\ |\nabla u^\star| \leq \sigma \ \text{ on } \ \{u^\star = t\}\right\}\,,
$$

we get

$$
\begin{aligned}
(3.24)\qquad M_{u^\star}(\sigma) &= M_{u^\star}(0) + \int_0^{+\infty}\int_{\{u^\star=t\}\cap\{0<|\nabla u^\star|\leq\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\,dt \\
&\leq M_u(0) + \int_{I\cap L_\sigma}\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\,dt + \int_G\int_{\{u^\star=t\}\cap\{0<|\nabla u^\star|\leq\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\,dt \\
&\qquad + \int_{t_{\varepsilon,\alpha}}^{+\infty}\int_{\{u^\star=t\}\cap\{0<|\nabla u^\star|\leq\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\,dt\,.
\end{aligned}
$$

Set

$$
\mu_0(t) = \mathcal{L}^n(\{u>t\}\cap\{\nabla u = 0\}) \qquad \text{for } t \geq 0.
$$

From equation (2.4) applied to $u$ and $u^\star$, and from (2.6) we deduce that

$$
\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|} = \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} - \mu_0'(t)\,, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t>0\,.
$$

Thus, inequality (3.24) reads

$$
\begin{aligned}
(3.25)\qquad M_{u^\star}(\sigma) &\leq M_u(0) + \int_{I\cap L_\sigma}\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt + \int_I(-\mu_0'(t))dt \\
&\qquad + \int_G\int_{\{u^\star=t\}\cap\{0<|\nabla u^\star|\leq\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\star|}\,dt + \mu(t_{\varepsilon,\alpha})\,.
\end{aligned}
$$

The second summand on the right-hand side of (3.25) can be estimated as follows

$$
\begin{aligned}
(3.26)\quad \int_{I\cap L_\sigma}\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt &= \int_{I\cap L_\sigma}\int_{\{u=t\}\cap\{|\nabla u|\le\sigma\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt + \int_{I\cap L_\sigma}\int_{\{u=t\}\cap\{|\nabla u|>\sigma\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt\\
&\le M_u(\sigma)+\frac{1}{\sigma}\int_{I\cap L_\sigma}\mathcal{H}^{n-1}(\{u=t\})dt\\
&\le M_u(\sigma)+\frac{1}{\sigma}\int_{I\cap L_\sigma}\Big(\int_{\{u=t\}}|\nabla u|^{p-1}d\mathcal{H}^{n-1}\Big)^{1/p}\Big(\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\Big)^{1/p'}dt\\
&\le M_u(\sigma)+\frac{1}{\sigma}\Big(\int_{I\cap L_\sigma}\int_{\{u=t\}}|\nabla u|^{p-1}d\mathcal{H}^{n-1}\,dt\Big)^{1/p}\Big(\int_0^{+\infty}\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt\Big)^{1/p'}\\
&\le M_u(\sigma)+\frac{1}{\sigma}\Big[\int_{I\cap L_\sigma}\Big(\int_{\{u=t\}}|\nabla u|^{p-1}d\mathcal{H}^{n-1}-\int_{\{u^\star=t\}}|\nabla u^\star|^{p-1}d\mathcal{H}^{n-1}\Big)dt\\
&\qquad\qquad\qquad + \int_{I\cap L_\sigma}\int_{\{u^\star=t\}}|\nabla u^\star|^{p-1}d\mathcal{H}^{n-1}\,dt\Big]^{1/p}\\
&\le M_u(\sigma)+\frac{1}{\sigma}\Big[\int_0^{\infty}\int_{\{u=t\}}|\nabla u|^{p-1}d\mathcal{H}^{n-1}\,dt-\int_0^{\infty}\int_{\{u^\star=t\}}|\nabla u^\star|^{p-1}d\mathcal{H}^{n-1}\,dt\\
&\qquad\qquad\qquad +\sigma^{p-1}\mathcal{L}^1(I)n\omega_nR_0^{n-1}\Big]^{1/p}\\
&\le M_u(\sigma)+\frac{1}{\sigma}\Big[\varepsilon+3\sigma^{p-1}\Big(\varepsilon^{1/2}+\varepsilon^{\frac{1}{4(p-1)}}\Big)n\omega_n^{1/n}\Big]^{1/p}.
\end{aligned}
$$

Notice that in the last but one inequality we have made use of the fact that

$$
\int_{\{u^\star=t\}}|\nabla u^\star|^{p-1}d\mathcal{H}^{n-1}\le\int_{\{u=t\}}|\nabla u|^{p-1}d\mathcal{H}^{n-1}\qquad \text{for } \mathcal{L}^1\text{-a.e. } t>0\,,
$$

as shown by an inspection of (2.8), whereas (3.8) has been exploited in the last inequality.
On the other hand,

$$
(3.27)\qquad \int_I(-\mu_0'(t))\,dt\le|D\mu_0|(I)\le M_u(0)\,.
$$

Finally, Lemmas 2.4 and 2.6 entail that a positive number $\varepsilon_1(p,n)\in(0,1)$ exists such that, if $\varepsilon\le\varepsilon_1$, then

$$
(3.28)\qquad \frac{|\nabla u^\star|_{|\{u^\star=t\}}}{4}\le|\nabla u(x)|\le 4|\nabla u^\star|_{|\{u^\star=t\}}\qquad \text{for } x\in\{u=t\}\setminus U_t\,,
$$

provided that $t\in G$, and, in particular,

$$
(3.29)\qquad |\nabla u(x)|\le 4\sigma\qquad \text{for } x\in\{u=t\}\setminus U_t\,,
$$

if $t \in G \cap L_\sigma$. Thus, a positive constant $\varepsilon_2(p,n)$, not exceeding $\varepsilon_1$, exists such that if $\varepsilon \le \varepsilon_2$, then

$$\begin{aligned}
(3.30)\quad & \int_{G\cap L_\sigma}\int_{\{u^\bigstar=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^\bigstar|}\,dt \le \int_{G\cap L_\sigma} \frac{\mathcal{H}^{n-1}(\{u=t\})}{|\nabla u^\bigstar|_{|\{u^\bigstar=t\}}}dt && \text{(isoperimetric inequality (2.2))}\\
&\le \int_{G\cap L_\sigma}\int_{\{u=t\}\setminus U_t} \frac{4}{|\nabla u|}d\mathcal{H}^{n-1}\,dt + (1+C_4)\int_{G\cap L_\sigma}\beta_t\mathcal{H}^{n-1}(U_t)dt && \text{((3.28) and Lemma 2.6)}\\
&\le 4\int_0^{+\infty}\int_{\{u=t\}\cap\{|\nabla u|\le 4\sigma\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt + (1+C_4)\varepsilon^{\frac{1}{8e(p)}}\int_G \beta_t\mathcal{H}^{n-1}(\{u=t\})dt && \text{((3.29) and (2.28))}\\
&\le 4M_u(4\sigma) + 2(1+C_4)\varepsilon^{\frac{1}{8e(p)}}\int_G\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt && \text{(by 3.14)}\\
&\le 4M_u(4\sigma) + 2(1+C_4)\varepsilon^{\frac{1}{8e(p)}}\,.
\end{aligned}$$

Now, if (3.9) is in force, inequality (1.10) easily follows from (3.25), (3.26), (3.27), (3.30) and (3.6) provided that $\varepsilon \le \varepsilon_2$ and $\sigma \le 1$. On the other hand, if $\sigma > 1$, then

$$4^p\sigma^p\mathcal{L}^n\left(\{|\nabla u| \ge 4\sigma\}\right) \le \int_{\mathbb{R}^n}|\nabla u|^p\,dx = \int_{\mathbb{R}^n}|\nabla u^\bigstar|^p\,dx + \varepsilon = 1+\varepsilon,$$

whence

$$M_u(4\sigma) \ge 1 - \frac{1+\varepsilon}{4^p\sigma^p} \ge 1 - \frac{2}{4^p}\,,$$

and (1.10) follows also in this case. Obviously, an inequality of type (1.10) holds when $\varepsilon > \varepsilon_2$ as well. □

## 4 Proof of Theorem 1.1, continued

In this section we accomplish the proof of Theorem 1.1. In fact, inequality (1.7) will be derived as a specialization of a somewhat more flexible estimate, involving a free parameter, contained in the following statement. In what follows,

$$\|\nabla u^\bigstar\|_{\L^p} = \left(\frac{\int_{\mathbb{R}^n}|\nabla u^\bigstar|^p dx}{\mathcal{L}^n(\{|u|>0\})}\right)^{1/p}\,,$$

and $\|\nabla u\|_{\L^p}$ is defined analogously, with $\int_{\mathbb{R}^n}|\nabla u^\bigstar|^p dx$ replaced by $\int_{\mathbb{R}^n}|\nabla u|^p dx$.

**Theorem 4.1** *Let $p>1$ and let $n\ge 2$. Then, positive constants $r_1$, $r_2$, $r_3$ and $C$, depending only on $p$ and $n$, exist such that, for every $u \in W^{1,p}(\mathbb{R}^n)$ satisfying (1.1),*
(4.1)

$$\min_{\pm}\inf_{x_0\in\mathbb{R}^n}\int_{\mathbb{R}^n}|u(x)\pm u^\bigstar(x+x_0)|dx \le C\|\nabla u^\bigstar\|_{\L^p}\mathcal{L}^n(\{|u|>0\})^{1+\frac{1}{n}}\left[M_{u^\bigstar}(\sigma)+E(u)^{r_1}+\frac{\|\nabla u^\bigstar\|_{\L^p}}{\sigma}E(u)^{r_2}\right]^{r_3},$$

*for $\sigma > 0$. Moreover, $M_{u^\bigstar}$ and $\|\nabla u^\bigstar\|_{\L^p}$ can be replaced by $M_u$ and $\|\nabla u\|_{\L^p}$, respectively, on the right-hand side of (4.1).*

Even with the material of the preceding sections in place, the proof of Theorem 4.1 still requires some careful steps. An outline is as follows.
Let $u$ be any nonnegative function as in Theorems 1.1 and 4.1. The conclusions of Section 2 can be roughly summarized by saying that, if the right-hand side of (4.1) is small for some $\sigma > 0$, then the level

sets $\{u > t\}$ are almost balls, and $|\nabla u|$ is almost constant on $\{u = t\}$, for every $t$ outside a set of small measure. The point is now to conclude from this piece of information that, up to a suitable translation, the set $\{u > t\}$ is close to $\{u^\bigstar > t\}$ for every $t$ outside a small set, and hence that $u$ is close to $u^\bigstar$ in $L^1(\mathbb{R}^n)$. This is the most delicate part of the proof, and it is by no means straightforward even in the special case dealt with in [BZ], where $E(u) = 0$ and $M_{u^\bigstar}(0) = 0$ (and hence the right-hand side of (4.1) vanishes). The technique employed in [BZ] to accomplish this step rests upon sophisticated tools from geometric measure theory, and does not seem suitable for extensions to the general case. Our approach is instead inspired by a method recently employed in [FV2], where a partially different proof of Brothers and Ziemer's theorem is presented. The argument of [FV2] starts by observing that $u$ can be factorized as

$$u(x) = u^\bigstar\Big(\frac{x}{|x|}\Phi(x)\Big) \qquad \text{for a.e. } x \in \mathbb{R}^n\,,$$

where $\Phi : \mathbb{R}^n \to [0, \infty)$ is given by

$$\Phi(x) = \left(\frac{\mu(u(x))}{\omega_n}\right)^{1/n} \qquad \text{for } x \in \mathbb{R}^n\,,$$

and agrees with the radius of the ball $\{u > u(x)\}$ for a.e. $x \in \mathbb{R}^n$. The absolute continuity of $\mu$, which is equivalent to the assumption $M_{u^\bigstar}(0) = 0$, is then exploited to derive that $\Phi$ is a Sobolev function. Property (2.10), asserting that all the level sets $\{u > t\}$ are balls, plays a role in this derivation. Next, the fact that, by (2.11), $|\nabla u|$ is constant on $\{u = t\}$, enables one to show that $\Phi$ is Lipschitz continuous, with Lipschitz constant equal to 1. Hence, via a simple geometric argument exploiting (2.10) again, one can deduce that (up to translations) $\Phi(x) = |x|$ a.e., namely that $u$ is spherically symmetric.

Such a proof, relying upon strong analytic and geometric properties of $u$ which follow from the conditions $E(u) = 0$ and $M_{u^\bigstar}(0) = 0$, breaks down in the present setting. This notwithstanding, the general strategy can still be adapted to the situation at hand, where, however, several obstacles arise. We begin by defining the modified distribution function $\widetilde{\mu} : [0, +\infty) \to [0, +\infty)$ as

$$\widetilde{\mu}(t) = \mathcal{L}^n(\{u = \operatorname{esssup} u\}) + \int_t^{+\infty} \chi_G(\tau) \int_{\{u=\tau\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\, d\tau \qquad \text{for } t > 0,$$

where $G$ is given by (3.10). In a sense, such a function takes into account only the contributions to $\mu$ form the "good" levels of $u$. Accordingly, we call $\widetilde{\Phi} : \mathbb{R}^n \to [0, +\infty)$ the function defined as

$$\widetilde{\Phi}(x) = \left(\frac{\widetilde{\mu}(u(x))}{\omega_n}\right)^{1/n} \qquad \text{for } x \in \mathbb{R}^n\,.$$

The idea is vaguely to show that $\widetilde{\mu}$ and $\widetilde{\Phi}$ are close to $\mu$ and $\Phi$, respectively, provided that the right-hand side of (4.1) is small, and that, in this case, $\widetilde{\Phi}(x)$ nearly equals $|x|$. Among others, a major problem to be faced is that, unlike the case where $E(u) = 0$ and $M_{u^\bigstar}(0) = 0$, the function $\widetilde{\Phi}$ need not enjoy the crucial Lipschitz continuity property substantiating the argument of [FV2]. A key step here consists in the proof that, although $\widetilde{\Phi}$ may be not globally Lipschitz continuous, it is nevertheless Lipschitz continuous, with Lipschitz constant almost equal to 1, the restriction of $\widetilde{\Phi}$ to sufficiently many straight lines for one to conclude, again via geometric considerations, about the closeness of $\widetilde{\Phi}(x)$ to $|x|$.

The full proof of Theorem 4.1 is split in separate lemmas. In the sequel, we set

$$\eta = 2M_u(0) + M_{u^\bigstar}(\sigma) + 3n\omega_n^{1/n}\frac{\varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}}}{\sigma} + \varepsilon^{\alpha n'}$$

and

$$t_\eta = \inf\left\{t \geq 0 :\, \widetilde{\mu}(t) \leq (1-\eta^{1/2})\mu(t)\right\}\,.$$

Moreover, we assume, for the time being, that the sign condition (2.3) is in force, that (1.5) holds, namely that

$$\mathcal{L}^n(\{u>0\}) = 1\,,$$

and that

$$\varepsilon \leq \left(\frac{1}{2C_2}\right)^{\frac{8e(p)}{1-8\alpha e(p)}}, \tag{4.2}$$

where $C_2$ is the constant appearing in Lemma 2.4, and

$$\eta \leq \frac{1}{4}\,. \tag{4.3}$$

**Lemma 4.2** *Under assumption (4.3), one has*

$$t_\eta > 0 \tag{4.4}$$

*and*

$$\mu(t_\eta) \leq \sqrt{\eta}\,. \tag{4.5}$$

**Proof.** Fix any $t \in (0, t_{\varepsilon,\alpha})$. Then

$$\begin{aligned} \mu(t)-\widetilde{\mu}(t) \;&\leq\; M_u(0) + \int_t^{t_{\varepsilon,\alpha}} \chi_I(\tau)\int_{\{u=\tau\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,d\tau + \int_{t_{\varepsilon,\alpha}}^{+\infty}\int_{\{u=\tau\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,d\tau \\ &\leq\; M_u(0) + \mathcal{L}^n(u^{-1}(I)) + \mathcal{L}^n(\{u>t_{\varepsilon,\alpha}\}) \\ &\leq\; 2M_u(0) + 3n\omega_n^{1/n}\frac{\varepsilon^{1/2}+\varepsilon^{\frac{1}{4(p-1)}}}{\sigma} + M_{u^\star}(\sigma) + \varepsilon^{\alpha n'} = \eta < \eta^{1/2} = \eta^{1/2}\mu(0)\,, \end{aligned} \tag{4.6}$$

where we have exploited (3.7) in the third inequality. Thus, since $\mu$ is right-continuous, $\widetilde{\mu}(t) > (1-\sqrt{\eta})\mu(t)$ provided that $t$ is sufficiently small, whence (4.4) follows.
The continuity of $\widetilde{\mu}$ and the continuity of $\mu$ from the right ensure that $\widetilde{\mu}(t_\eta) \leq (1-\sqrt{\eta})\mu(t_\eta)$. Thus, if $t_\eta \leq t_{\varepsilon,\alpha}$, (4.6) entails that $\sqrt{\eta}\mu(t_\eta) \leq \mu(t_\eta) - \widetilde{\mu}(t_\eta) \leq \eta$, and (4.5) holds. Inequality (4.5) also holds if $t_\eta > t_{\varepsilon,\alpha}$, for $\mu(t_\eta) \leq \mu(t_{\varepsilon,\alpha}) \leq \eta \leq \sqrt{\eta}$ in this case, by (4.3). □

**Lemma 4.3** *Let $1 < q < \min\{p, n'\}$. Then $\widetilde{\Phi} \in W^{1,q}(\mathrm{I\!R}^n)$ and $\|\nabla\widetilde{\Phi}\|_q \leq C_6(q,p,n)$ for some positive constant depending only on $p, q, n$. Moreover*

$$|\nabla\widetilde{\Phi}(x)| = \frac{\chi_G(u(x))}{n\omega_n^{1/n}}\,(\widetilde{\mu}(u(x)))^{-1/n'}\,\chi_{\{\nabla u\neq 0\}}(x)|\nabla u(x)|\int_{\{u=u(x)\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|} \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathrm{I\!R}^n\,. \tag{4.7}$$

**Proof.** Fixed any $\rho > 0$, define $\widetilde{\mu}_\varrho : [0,+\infty) \to [0,+\infty)$ as

$$\widetilde{\mu}_\varrho(t) = \varrho + \mathcal{L}^n(\{u = \operatorname{esssup} u\}) + \int_t^{+\infty}\chi_G(\tau)\int_{\{u=\tau\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|+\varrho}d\tau \qquad \text{for } t>0\,,$$

and $\widetilde{\Phi}_\varrho : \mathbb{R}^n \to [0, +\infty)$ by

$$\widetilde{\Phi}_\varrho(x) = \left(\frac{\widetilde{\mu}_\varrho(u(x))}{\omega_n}\right)^{1/n} \qquad \text{for } x \in \mathbb{R}^n .$$

It is easily checked that $t \mapsto (\widetilde{\mu}_\varrho(t)/\omega_n)^{1/n}$ is a Lipschitz continuous function and that $\lim_{\rho\to 0^+} \widetilde{\Phi}_\varrho(x) = \widetilde{\Phi}(x)$, for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. Moreover, by the chain rule,

$$\nabla\widetilde{\Phi}_\varrho(x) = \frac{1}{n\omega_n^{1/n}} \left(\widetilde{\mu}_\varrho(u(x))\right)^{-1/n'} \chi_G(u(x))\nabla u(x) \int_{\{u=u(x)\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u| + \varrho} \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n . \tag{4.8}$$

On making use of (4.8), (2.30) and Lemma 2.1, and recalling (1.5) and (4.2), we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla\widetilde{\Phi}_\varrho|^q dx &= \frac{1}{(n\omega_n^{1/n})^q} \int_0^{+\infty} \widetilde{\mu}_\varrho(t)^{-\frac{q}{n'}} (-\widetilde{\mu}'_\varrho(t))\chi_G(t) \left(\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|+\varrho}\right)^{q-1} \left(\int_{\{u=t\}} |\nabla u|^{q-1} d\mathcal{H}^{n-1}\right) dt \\
&\le \frac{1}{(n\omega_n^{1/n})^q} \int_0^{+\infty} \widetilde{\mu}_\varrho^{-\frac{q}{n'}} (-\widetilde{\mu}'_\varrho)\chi_G(t) \left[\left(\fint_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1} \left(\fint_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}\right)\right]^{\frac{q-1}{p-1}} \mathcal{H}^{n-1}(\{u=t\})^q dt \\
&\le \frac{1}{(n\omega_n^{1/n})^q} \int_0^{+\infty} \widetilde{\mu}_\varrho(t)^{-\frac{q}{n'}} (-\widetilde{\mu}'_\varrho(t))\chi_G(t) \left(1 + \frac{\varepsilon^{\frac{1}{4}-\alpha p}}{(n\omega_n^{1/n})^p}\right)^{\frac{q-1}{p-1}} \left(1 + \frac{\varepsilon^{\frac{1}{4p}-\alpha}}{n\omega_n^{1/n}}\right)^q \mathcal{H}^{n-1}(\{u^\star=t\})^q dt \\
&\le C \int_0^{+\infty} \widetilde{\mu}_\varrho(t)^{-\frac{q}{n'}} (-\widetilde{\mu}'_\varrho(t))\chi_G(t) dt \le \frac{C}{1-q/n'} \widetilde{\mu}\varrho(0)^{1-\frac{q}{n'}} \le C' ,
\end{aligned}$$

for every $\varrho < 1$, and for some positive constants $C(p, q, n)$ and $C'(p, q, n)$. Consequently, the family $\{|\nabla\widetilde{\Phi}_\varrho|\}$ is equi-bounded in $L^q(\mathbb{R}^n)$ for $\varrho \in (0, 1)$, and hence $\widetilde{\Phi} \in W^{1,q}(\mathbb{R}^n)$ and $\|\nabla\widetilde{\Phi}\|_q^q \le C'$. Finally, (4.7) follows on letting $\varrho$ go to $0^+$ in (4.8). □

Given any $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ (the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ centered at 0) and $\delta > 0$, we denote by $B_\delta^{n-1,\nu}(x)$ the $(n-1)$-dimensional ball centered at $x$, having radius $\delta$ and contained in the hyperplane containing $x$ and orthogonal to $\nu$.

**Lemma 4.4** *Let $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $\delta > 0$. Let $d > 0$ and $0 < \Theta < 1$. Let $1 < q < \min\{p, n'\}$. Then, constants $C_7(n)$, $C_8(p, n)$ and $C_9(p, q, n)$ exist such that, for every $\mathcal{H}^{n-1}$-measurable subset $S$ of $B_\delta^{n-1,\nu}(x)$ satisfying*

$$\mathcal{H}^{n-1}(S) \le \Theta\mathcal{H}^{n-1}(B_\delta^{n-1,\nu}(x)) ,$$

*there exists $z \in B_\delta^{n-1,\nu}(x) \setminus S$ such that*

$$|\widetilde{\Phi}(z + \nu d) - \widetilde{\Phi}(z)| \le \frac{d(1 + C_7\eta^{1/2})^{1/n'}\left(1 + C_8\varepsilon^{\frac{1}{8e(p)}-\alpha}\right)}{1-\Theta} + C_9\frac{\left(\varepsilon^{\frac{1}{8e(p)}-\alpha} + \eta^{1/2}\right)^{1/q'}}{(1-\Theta)\delta^{n-1}} ,$$

*provided that $\varepsilon$ satisfies (4.2).*

**Proof.** Define

$$V = \left\{x \in \{u > 0\} : \chi_G(u(x))\left|\frac{1}{|\nabla u|} - \beta_{u(x)}\right| > C_2\varepsilon^{\frac{1}{8e(p)}-\alpha}\beta_{u(x)}\right\} , \tag{4.9}$$

where $C_2$ is the constant appearing in (2.26). The set $V$ is a Borel set, as a consequence of the fact that the precise representative of $u$ is a Borel function. Moreover,

$$
\begin{aligned}
(4.10)\ \ \mathcal{L}^n(V\setminus\{\nabla u=0\}) &= \int_0^{+\infty}\chi_G(t)\int_{\{u=t\}}\frac{\chi_V(x)}{|\nabla u|}d\mathcal{H}^{n-1}dt \le \int_0^{+\infty}\chi_G(t)\int_{\{u=t\}\cap U_t}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}dt \quad \text{(by (2.27))}\\
&\le \int_0^{+\infty}\chi_G(t)\int_{\{u=t\}}\left|\frac{1}{|\nabla u|}-\beta_t\right|d\mathcal{H}^{n-1}\,dt+\int_0^{+\infty}\chi_G(t)\beta_t\mathcal{H}^{n-1}(\{u=t\}\cap U_t)dt\\
&\le C_2\varepsilon^{\frac{1}{4e(p)}-\alpha}\int_0^{+\infty}\chi_G(t)\beta_t\mathcal{H}^{n-1}(\{u=t\})dt+\varepsilon^{\frac{1}{8e(p)}}\int_0^{+\infty}\chi_G(t)\beta_t\mathcal{H}^{n-1}(\{u=t\})dt\\
&\le 2(1+C_2)\varepsilon^{\frac{1}{8e(p)}-\alpha}\int_0^{+\infty}\int_{\{u=t\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\,dt \qquad \text{((4.2) and (2.26))}\\
&\le 2(1+C_2)\varepsilon^{\frac{1}{8e(p)}-\alpha} \qquad \text{(coarea formula and (1.5)).}
\end{aligned}
$$

On the other hand, from (4.7) one can deduce that suitable constants $C(p,n), C_7(n)$ and $C_8(p,n)$ exist such that, for $\mathcal{L}^n$-a.e. $x\in u^{-1}(G\cap(0,t_\eta))\setminus V$,

$$
\begin{aligned}
(4.11)\ |\nabla\widetilde{\Phi}(x)| &= \left(\frac{\widetilde{\mu}(u(x))}{\mu(u(x))}\right)^{-1/n'}\frac{\mu(u(x))^{-1/n'}}{n\omega_n^{1/n}}|\nabla u(x)|\chi_{\{\nabla u\neq 0\}}(x)\int_{\{u=u(x)\}}\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\\
&\le \frac{1}{(1-\sqrt{\eta})^{1/n'}}\frac{|\nabla u(x)|}{\mathcal{H}^{n-1}(\{u^\bigstar=u(x)\})}\left[\int_{\{u=u(x)\}}\left|\frac{1}{|\nabla u|}-\beta_{u(x)}\right|d\mathcal{H}^{n-1}+\beta_{u(x)}\mathcal{H}^{n-1}(\{u=u(x)\})\right]\\
&\le \frac{1}{(1-\sqrt{\eta})^{1/n'}}\frac{|\nabla u(x)|}{\mathcal{H}^{n-1}(\{u^\bigstar=u(x)\})}\left[C_2\varepsilon^{\frac{1}{4e(p)}-\alpha}+1\right]\beta_{u(x)}\mathcal{H}^{n-1}(\{u=u(x)\})\\
&\le \frac{\left(1+C\varepsilon^{\frac{1}{4e(p)}-\alpha}\right)^2}{(1-\sqrt{\eta})^{1/n'}}|\nabla u(x)|\beta_{u(x)} \le \frac{\left(1+C\varepsilon^{\frac{1}{4e(p)}-\alpha}\right)^2}{(1-\sqrt{\eta})^{1/n'}\left(1-C_2\varepsilon^{\frac{1}{8e(p)}-\alpha}\right)} \le (1+C_7\sqrt{\eta})^{\frac{1}{n'}}\left(1+C_8\varepsilon^{\frac{1}{8e(p)}-\alpha}\right),
\end{aligned}
$$

where the first inequality holds since $u(x)\le t_\eta$, the second one is due to Lemma 2.4, the third one to Lemma 2.1, and the fourth one follows from (4.9).
Now, denote by $C^\nu_{\delta,d}(x)$ the cylinder whose basis is $B^{n-1,\nu}_\delta(x)$, whose axis is parallel to $\nu$ and whose height is $d$. Then,

$$
\begin{aligned}
(4.12)\qquad \int_{B^{n-1,\nu}_\delta(x)}|\widetilde{\Phi}(z+\nu d)-\widetilde{\Phi}(z)|\,d\mathcal{H}^{n-1}(z) &= \int_{B^{n-1,\nu}_\delta(x)}\left|\int_0^d\nabla\widetilde{\Phi}(z+\nu s)\cdot\nu\,ds\right|d\mathcal{H}^{n-1}(z)\\
&\le \int_{C^\nu_{\delta,d}(x)}|\nabla\widetilde{\Phi}(w)|\,dw\,.
\end{aligned}
$$

Set

$$\widetilde{V}=V\cup\{t_\eta\le u\le t_{\varepsilon,\alpha}\}\,.$$

From (4.12), (4.11), (4.10), Lemma 4.3 and (4.5) we get

$$
\begin{aligned}
(4.13)\qquad \int_{B^{n-1,\nu}_\delta(x)}|\widetilde{\Phi}(z+\nu d)-\widetilde{\Phi}(z)|d\mathcal{H}^{n-1}(z) &\le \int_{C^\nu_{\delta,d}(x)\setminus\widetilde{V}}|\nabla\widetilde{\Phi}(w)|dw+\int_{\widetilde{V}}|\nabla\widetilde{\Phi}(w)|dw\\
&\le (1+C_7\sqrt{\eta})^{1/n'}\left(1+C_8\varepsilon^{\frac{1}{8e(p)}-\alpha}\right)\mathcal{L}^n(C^\nu_{\delta,d}(x))+\|\nabla\widetilde{\Phi}\|_q\mathcal{L}^n(\widetilde{V}\setminus\{\nabla u=0\})^{1/q'}\\
&\le (1+C_7\sqrt{\eta})^{1/n'}d\left(1+C_8\varepsilon^{\frac{1}{8e(p)}-\alpha}\right)\mathcal{H}^{n-1}(B^{n-1,\nu}_\delta(x))+C'\left(\varepsilon^{\frac{1}{8e(p)}-\alpha}+\eta^{1/2}\right)^{1/q'}
\end{aligned}
$$

for some positive constant $C'(p,q,n)$. The conclusion easily follows from (4.13). $\square$

**Lemma 4.5** *There exist positive constants $\alpha_0(p,n)$, $\varepsilon_3(p,n)$, $\vartheta(p,n)$, $C_{10}(p,n)$ and $C_{11}(n)$ having the following property. If $0<\alpha<\alpha_0$ and $0<\varepsilon<\varepsilon_3$, then for every $t_0 \in G$ and $t \in G \cap [t_0,+\infty)$ there exist $\overline{x}, \overline{y} \in \mathbb{R}^n$ such that*

$$u(\overline{x}) \geq t, \qquad u(\overline{y}) \leq t_0$$

*and a ball $B_t$ such that*

$$\mathcal{L}^n(\{u>t\}) = \mathcal{L}^n(B_t)\,, \tag{4.14}$$

$$B_t \subset B_{t_0}\,, \tag{4.15}$$

$$\mathcal{L}^n(\{u>t\}\triangle B_t) \leq C_{11}\mathcal{L}^n(B_t)\varepsilon^{\left(\frac{1}{8p}-3\alpha\right)\frac{2}{n+2}}\,, \tag{4.16}$$

*and*

$$|\widetilde{\Phi}(\overline{x}) - \widetilde{\Phi}(\overline{y})| \leq \operatorname{dist}(\partial B_t, \partial B_{t_0})(1 + C_{10}\eta^{\vartheta}) + C_{10}\frac{\eta^{2\vartheta}}{R_t^{n-1}}\,. \tag{4.17}$$

**Proof.** Fix any $t_0 \in G$ and any $t \in G \cap [t_0,+\infty)$. By Lemma 2.2, a positive constant $C(n)$ exists such that

$$\mathcal{L}^n(\{u>t\}\triangle B'_t) \leq C\mathcal{L}^n(\{u>t\})\varepsilon^{\frac{1}{2}\left(\frac{1}{4p}-\alpha\right)} \tag{4.18}$$

for some ball $B'_t$ satisfying $\mathcal{L}^n(\{u>t\}) = \mathcal{L}^n(B'_t)$. Choose a positive number $\varepsilon_3(p,n)$ smaller than the right-hand side of (4.2). Since

$$B'_t \setminus B'_{t_0} = [(B'_t \setminus B'_{t_0}) \cap \{u>t\}] \cup [(B'_t \setminus B'_{t_0}) \cap \{u \leq t\}] \subset [\{u>t_0\} \setminus B'_{t_0}] \cup [B'_t \setminus \{u>t\}]\,,$$

by (4.18) one has

$$\begin{aligned} \mathcal{L}^n(B'_t \setminus B'_{t_0}) &\leq C\varepsilon^{\frac{1}{2}\left(\frac{1}{4p}-\alpha\right)}\Big[\mathcal{L}^n(\{u>t_0\}) + \mathcal{L}^n(\{u>t\})\Big] = C\omega_n\varepsilon^{\frac{1}{2}\left(\frac{1}{4p}-\alpha\right)}R_t^n\left[1+\left(\frac{R_{t_0}}{R_t}\right)^n\right] \\ &\leq C\omega_n\varepsilon^{\frac{1}{2}\left(\frac{1}{4p}-\alpha\right)}R_t^n\left[1+\frac{1}{\varepsilon^{\alpha n'}}\right] \leq 2C\omega_n\varepsilon^{\frac{1}{8p}-3\alpha}R_t^n\,. \end{aligned} \tag{4.19}$$

Notice that we have exploited (1.5) and the fact that $t \leq t_{\varepsilon,\alpha}$ in the second inequality, and (4.2) in the third one.

Choose $B_{t_0} = B'_{t_0}$. Elementary geometric considerations ensure that, if $\varepsilon$ is smaller than a suitable constant depending only on $n$, there exist a positive constant $C'(n)$ and a translated $B_t$ of $B'_t$ along $x_t - x_0$ such that $B_t \subset B_{t_0}$ and

$$\mathcal{L}^n(B'_t \setminus B_t) \leq C'\mathcal{L}^n(B'_t)\left(\frac{\mathcal{L}^n(B'_t \setminus B_{t_0})}{R_t^n}\right)^{\frac{2}{n+1}}\,. \tag{4.20}$$

By (4.18)-(4.20), a constant $C_{11}(n)$ exists such that

$$\mathcal{L}^n(\{u>t\}\triangle B_t) = 2\mathcal{L}^n(\{u>t\}\setminus B_t) \le 2\left[\mathcal{L}^n(\{u>t\}\setminus B'_t) + \mathcal{L}^n(B'_t\setminus B_t)\right]$$
$$= 2C\mathcal{L}^n(B'_t)\varepsilon^{\frac{1}{2}\left(\frac{1}{4p}-\alpha\right)} + 2C'\mathcal{L}^n(B'_t)\Big(2C\omega_n\varepsilon^{\frac{1}{8p}-3\alpha}\Big)^{\frac{2}{n+2}} \le C_{11}\mathcal{L}^n(B'_t)\varepsilon^{\left(\frac{1}{8p}-3\alpha\right)\frac{2}{n+2}},$$

whence (4.16) follows.
Given any $x'\in\partial B_t$ and any $s\in(0,1)$, and denoted by $c_t$ the center of $B_t$, let us set

$$B^{n-1}_{t,s} = \left\{z\in B_t : \langle z-(1-s)x'-sc_t, x'-c_t\rangle = 0\right\},$$

the intersection of $B_t$ with the hyperplane orthogonal to $x'-c_t$ whose distance from $x'$ is $sR_t$. We claim that, if

(4.21) $$\lambda \in \Big(\varepsilon^{\varrho}, \frac{1}{2}\Big)$$

for some $\varrho \in \left(0, \frac{2}{n+1}\Big[\frac{1}{4p(n+2)} - \Big(\frac{6}{n+2}+1\Big)\alpha\Big]\right)$, then there exists

(4.22) $$\overline{s}\in(\lambda, 2\lambda)$$

satisfying

(4.23) $$\mathcal{H}^{n-1}(B^{n-1}_{t,\overline{s}}\cap\{u\le t\}) \le C'_{12}\varepsilon^{\alpha}\mathcal{H}^{n-1}(B^{n-1}_{t,\overline{s}})$$

for some positive constant $C'_{12}(n)$. Indeed, assume, by contradiction, that (4.23) fails for any constant $C'_{12}$ and every $s\in(\lambda,2\lambda)$. Then, since

$$\mathcal{H}^{n-1}(B^{n-1}_{t,s}) = \omega_{n-1}\left(R_t\sqrt{2s-s^2}\right)^{n-1} \ge \omega_{n-1}s^{\frac{n-1}{2}}R_t^{n-1},$$

we have

$$\mathcal{L}^n(B_t\cap\{u\le t\}) > \int_{\lambda}^{2\lambda} R_t\mathcal{H}^{n-1}(B^{n-1}_{t,s}\cap\{u\le t\})\,ds$$
$$\ge C'_{12}\varepsilon^{\alpha}R_t\int_{\lambda}^{2\lambda}\mathcal{H}^{n-1}(B^{n-1}_{t,s})\,ds \ge C'_{12}\omega_{n-1}\varepsilon^{\alpha}R_t^n\lambda^{\frac{n+1}{2}}\frac{2(2^{\frac{n+1}{2}}-1)}{n+1}.$$

Hence, by (4.16),

$$C_{11}\mathcal{L}^n(B_t)\varepsilon^{\left(\frac{1}{8p}-3\alpha\right)\frac{2}{n+2}} \ge C'_{12}\frac{\omega_{n-1}}{\omega_n}\frac{2}{n+1}(2^{\frac{n+1}{2}}-1)\varepsilon^{\alpha}\mathcal{L}^n(B_t)\lambda^{\frac{n+1}{2}}.$$

The last inequality contradicts (4.21), if $C'_{12}\ge\frac{C_{11}\omega_n(n+1)}{2\omega_{n-1}(2^{\frac{n+1}{2}}-1)}$.
Now, let $x'\in\partial B_t$ and $y'\in\partial B_{t_0}$ be such that

(4.24) $$|x'-y'| = \mathrm{dist}(\partial B_t, \partial B_{t_0}),$$

let $\overline{\delta} = R_t\sqrt{2\overline{s}-\overline{s}^2}$, the radius of the $(n-1)$-dimensional ball $B^{n-1}_{t,\overline{s}}$, and let $\overline{\nu} = \frac{x'-y'}{|x'-y'|}$. Property (4.24) implies that $\overline{\nu}$ is parallel to $x'-c_t$; consequently $B^{n-1,\overline{\nu}}_{\overline{\delta}}(y')$ and $B^{n-1}_{t,\overline{s}}$ are contained in parallel hyperplanes. An analogous (and actually simpler) argument as in the proof of (4.23), calling into play a cylinder, lying outside $B_{t_0}$, whose basis is $B^{n-1,\overline{\nu}}_{\overline{\delta}}(y')$ and whose axis is parallel to $\overline{\nu}$, implies that, if

$$0 < \varrho < \frac{2}{n+1}\left(\frac{1}{8p} - \frac{3}{2}\alpha - \alpha n'\right),$$

then there exists $y'' \in \mathbb{R}^n$ having the following properties: $B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'') \subset \mathbb{R}^n \setminus B_{t_0}$, $y'' - y'$ is parallel to $\overline{\nu}$,

$$\lambda R_t \le |y' - y''| \le 2\lambda R_t\,, \tag{4.25}$$

and

$$\mathcal{H}^{n-1}(B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'') \cap \{u > t_0\}) \le C''_{12}\varepsilon^{\alpha}\mathcal{H}^{n-1}(B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'')) \tag{4.26}$$

for some constant $C''_{12}(n)$. Note that such an implication makes use of the fact that, by (1.5) and (2.13), $\frac{R^n_{t_0}}{R^n_t} \le \varepsilon^{-\alpha n'}$ and that (2.21) holds with $B_{R_{t_0}} = B_{t_0}$. Now, if $\varepsilon$ satisfies (4.2), we may apply Lemma 4.4 to $B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'')$, with

$$S = \big(B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'') \cap \{u > t_0\}\big) \cup \big\{y \in B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'') \colon \text{the projection of } y \text{ on } B^{n-1}_{t,\overline{s}} \text{ along } \overline{\nu} \text{ belongs to } \{u \le t\}\big\}.$$

By (4.23) and (4.26),

$$\mathcal{H}^{n-1}(S) \le 2C_{12}\varepsilon^{\alpha}\mathcal{H}^{n-1}(B^{n-1,\overline{\nu}}_{\overline{\delta}}(y''))\,,$$

where $C_{12}(n) = \max\{C'_{12}, C''_{12}\}$. Moreover, on denoting by $\bar{d}$ the distance between the centers of $B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'')$ and $B^{n-1}_{t,\overline{s}}$, one has

$$\bar{d} \le 4\lambda R_t + |x' - y'|\,,$$

thanks to (4.22) and (4.25). Hence, there exist $\overline{y} \in B^{n-1,\overline{\nu}}_{\overline{\delta}}(y'') \cap \{u \le t_0\}$ and $\overline{x} \in B^{n-1}_{t,\overline{s}} \cap \{u > t\}$ such that

$$|\widetilde{\Phi}(\overline{x}) - \widetilde{\Phi}(\overline{y})| \le \frac{(|x'-y'| + 4\lambda R_t)(1 + C_7\sqrt{\eta})^{1/n'}\left(1 + C_8\varepsilon^{\frac{1}{8e(p)}-\alpha}\right)}{1 - 2C_{12}\varepsilon^{\alpha}} + C_9\frac{\left(\varepsilon^{\frac{1}{8e(p)}-\alpha} + \eta^{1/2}\right)^{1/q'}}{(1 - 2C_{12}\varepsilon^{\alpha})\lambda^{\frac{n-1}{2}}R^{n-1}_t}\,. \tag{4.27}$$

Thus, if $\alpha$ is chosen so small that $\min\{\frac{1}{8e(p)} - \alpha, \rho\} > \alpha n'$, then

$$\varepsilon^{\frac{1}{8e(p)}-\alpha} < \varepsilon^{\alpha n'} < \eta\,. \tag{4.28}$$

By (4.27) and (4.28), if $\varepsilon_3(p,n)$ is so small that $\frac{1}{1-2C_{12}\varepsilon^{\alpha}} \le 1 + \varepsilon^{\alpha/2}$ whenever $\varepsilon < \varepsilon_3$, one has

$$|\widetilde{\Phi}(\overline{x}) - \widetilde{\Phi}(\overline{y})| \le (|x'-y'| + 4\lambda R_t)(1 + C_7\sqrt{\eta})^{\frac{1}{n'}}(1 + C_8\eta)\left(1 + \eta^{\frac{1}{2n'}}\right) + 2C_9\frac{\eta^{\frac{1}{2q'}}\left(1 + \eta^{\frac{1}{2n'}}\right)}{\lambda^{\frac{n-1}{2}}R^{n-1}_t} \tag{4.29}$$

for $\varepsilon < \varepsilon_3$. Notice that, fixed any $q \in (1, \min\{p, n'\})$, we may choose $\lambda = \dfrac{1}{2}\eta^{\frac{1}{2q'(n-1)}}$ in (4.29), since

$$\varepsilon^{\rho} < \varepsilon^{\alpha n'} < \eta \le \frac{1}{2}\eta^{\frac{1}{2}} < \frac{1}{2}\eta^{\frac{1}{2q'(n-1)}} < \frac{1}{2}\,,$$

Hence, (4.17) easily follows with a sufficiently small $\vartheta(p,n)$. □

**Proof of Theorem 4.1, concluded.** Assume that $u$ satisfies the additional assumption (2.3). Moreover, suppose that (1.5), (1.6) and (4.3) are in force and that $\varepsilon < \varepsilon_3$, where $\varepsilon_3$ is the constant provided by Lemma 4.5. We begin by observing that, if $x \in \mathbb{R}^n$ satisfies $u(x) < t_\eta$, then, by the very definition of $t_\eta$,

$$|\Phi(x) - \widetilde{\Phi}(x)| = \frac{1}{\omega_n^{1/n}}\Big[\mu(u(x))^{1/n} - \widetilde{\mu}(u(x))^{1/n}\Big] \leq \frac{1}{\omega_n^{1/n}}\Big[\mu(u(x)) - \widetilde{\mu}(u(x))\Big]^{1/n} \leq \eta^{\frac{1}{2n}}\Phi(u(x))\,. \tag{4.30}$$

Next, set

$$\bar{\vartheta} = \min\Big\{\Big(\frac{1}{8p} - 3\alpha\Big)\frac{2}{n+1}, \vartheta\Big\}\,,$$

where $\vartheta$ is the number appearing in Lemma 4.5, and

$$t_1 = \sup\Big\{t \geq 0:\ \mathcal{L}^n(\{u > t\}) > \eta^{\bar{\vartheta}n'}\Big\}\,.$$

Since $\bar{\vartheta}n' < 1$ and $\eta \leq 1/4$, by (4.28)

$$\eta^{\bar{\vartheta}n'} > \eta > \varepsilon^{\alpha n'}\,,$$

whence $t_1 < t_{\varepsilon,\alpha}$. On the other hand, by (3.8), $t_0 \in G$ can be chosen in such a way that

$$t_0 \leq 4\Big(\varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}}\Big)\,. \tag{4.31}$$

Thus, Lemma 4.5 tell us that, if $t_0 < t_1$, then for every $t \in G \cap [t_0, t_1]$, there exist $\overline{x}, \overline{y} \in \mathbb{R}^n$ satisfying $u(\overline{x}) \geq t$, $u(\overline{y}) \leq t_0$, and a ball $B_t$ fulfilling (4.14)-(4.16) and

$$|\widetilde{\Phi}(\overline{x}) - \widetilde{\Phi}(\overline{y})| \leq \mathrm{dist}(\partial B_t, \partial B_{t_0})(1 + C_{10}\eta^{\bar{\vartheta}}) + C_{10}\omega_n^{\frac{n-1}{n}}\eta^{\bar{\vartheta}}\,. \tag{4.32}$$

Moreover, since $\bar{\vartheta}n' < 1/2$ and $\eta \leq 1/4$ one has $\eta^{\bar{\vartheta}n'} > \eta^{1/2}$, and hence, by (4.5) and by the very definition of $t_1$, one has $t_1 \leq t_\eta$. Consequently, from (4.30) and (4.32) we deduce that, for every $t \in [t_0, t_1] \cap G$,

$$\begin{aligned} R_{t_0} - R_t \ &\leq\ R_{u(\overline{y})} - R_{u(\overline{x})} = \Phi(\overline{y}) - \Phi(\overline{x}) \leq (\Phi(\overline{y}) - \widetilde{\Phi}(\overline{y})) + (\widetilde{\Phi}(\overline{y}) - \widetilde{\Phi}(\overline{x})) \\ &\leq \eta^{\frac{1}{2n}}\Phi(\overline{y}) + \mathrm{dist}(\partial B_t, \partial B_{t_0})(1 + C_{10}\eta^{\bar{\vartheta}}) + C_{10}\omega_n^{\frac{n-1}{n}}\eta^{\bar{\vartheta}} \\ &\leq \mathrm{dist}(\partial B_t, \partial B_{t_0})(1 + C_{10}\eta^{\bar{\vartheta}}) + C\eta^{\bar{\vartheta}}\,, \end{aligned} \tag{4.33}$$

for some positive constant $C(p,n)$. Note that we have made use of the fact that $\widetilde{\Phi} \leq \Phi$ in the second inequality, and that $\bar{\vartheta} < 1/(2n)$ in the last one.
We may assume, up to translations, that $B_{t_0}$ is centered at the origin. Thus, since $c_t$ is the center of $B_t$ and $c_{t_0} = 0$, we have from (4.33) that

$$|c_t| = |c_{t_0} - c_t| = R_{t_0} - R_t - \mathrm{dist}(\partial B_t, \partial B_{t_0}) \leq (C_{10} + C)\eta^{\bar{\vartheta}} \qquad \text{if } t \in [t_0, t_1] \cap G\,. \tag{4.34}$$

Furthermore, if $t_1 < \operatorname{esssup} u$,

$$\mathcal{L}^n(\{u \geq t_1\}) \leq \eta^{\bar{\vartheta}n'}\,. \tag{4.35}$$

The following chain holds

$$
\begin{aligned}
(4.36)\int_{\mathbb{R}^n}|u-u^\bigstar|dx &= \int_{\mathbb{R}^n}\left|\int_0^{\operatorname{esssup}u}(\chi_{\{u>t\}}(x)-\chi_{\{u^\bigstar>t\}}(x))dt\right|dx\\
&\le \int_0^{\operatorname{esssup}u}\int_{\mathbb{R}^n}|\chi_{\{u>t\}}(x)-\chi_{\{u^\bigstar>t\}}(x)|dx\,dt=\int_0^{\operatorname{esssup}u}\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\})dt\\
&= \int_0^{t_0}\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\})dt+\int_{t_0}^{t_1}\chi_G(t)\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\})dt\\
&\qquad+\int_{t_0}^{t_1}\chi_I(t)\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\})dt+\int_{t_1}^{\operatorname{esssup}u}\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\})dt\\
&\le 2t_0+2\mathcal{L}^1(I)+\int_{t_0}^{t_1}\chi_G(t)\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\})dt+2\int_{\{u>t_1\}}|u|dx\\
&\le 14\big(\varepsilon^{1/2}+\varepsilon^{\frac{1}{4(p-1)}}\big)+\int_{t_0}^{t_1}\chi_G(t)\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\})dt+2\|u\|_p\mathcal{L}^n(\{u>t_1\})^{\frac{1}{p'}}\,,
\end{aligned}
$$

where the second inequality is due to (1.5), and the third one to (4.31) and (3.8). Now,

$$
\varepsilon^{1/2}+\varepsilon^{\frac{1}{4(p-1)}}\le 2\varepsilon^{\alpha n'}<2\eta^{\bar\vartheta} \tag{4.37}
$$

provided that $\alpha$ is sufficiently small. On the other hand, inequality (4.34) entails that a constant $C'(p,n)$ exists such that

$$
\mathcal{L}^n(\{u^\bigstar>t\}\triangle B_t)\le C'\eta^{\bar\vartheta}\qquad\text{if } t\in G\cap[t_0,t_1]\,,
$$

whence, by (4.16)

$$
\begin{aligned}
\mathcal{L}^n(\{u>t\}\triangle\{u^\bigstar>t\}) &\le \mathcal{L}^n(\{u>t\}\triangle B_t)+\mathcal{L}^n(\{u^\bigstar>t\}\triangle B_t)\\
&\le C_{11}\varepsilon^{\left(\frac{1}{8p}-3\alpha\right)\frac{2}{n+1}}+C'\eta^{\bar\vartheta}\le(C_{11}+C')\eta^{\bar\vartheta}
\end{aligned}\tag{4.38}
$$

if $t\in G\cap[t_0,t_1]$. From (4.36), (4.37), (4.38) and (4.35) we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^n}|u-u^\bigstar|\,dx &\le 28\eta^{\bar\vartheta}+(C_{11}+C')t_1\eta^{\bar\vartheta}+2\|u\|_p\eta^{\frac{\bar\vartheta n'}{p'}}\\
&\le 28\eta^{\bar\vartheta}+(C_{11}+C')\eta^{\bar\vartheta}\int_0^{t_1}\mathcal{L}^n(\{u>t\})dt+2\|u\|_p\eta^{\frac{\bar\vartheta n'}{p'}}\\
&\le 28\eta^{\bar\vartheta}+(C_{11}+C')\eta^{\bar\vartheta}\|u\|_1+2\|u\|_p\eta^{\frac{\bar\vartheta n'}{p'}}\,.
\end{aligned}\tag{4.39}
$$

Notice that a similar inequality trivially holds also when $t_0\ge t_1$.
Inequality (4.39) has been established under assumptions (2.3), (1.5), (1.6), (4.3) and $\varepsilon<\varepsilon_3$. However, since

$$
\int_{\mathbb{R}^n}|u-u^\bigstar|\,dx\le 2\|u\|_1\le 8\eta\|u\|_1\,,
$$

if $\eta\ge 1/4$, inequality (4.39) continues to hold even if (4.3) is dropped. Similarly, since $\varepsilon^{\alpha n'}<\eta$, an inequality of type (4.39) is true even without the assumption $\varepsilon<\varepsilon_3$.

Owing to (1.5) and (1.6), by the Poincaré inequality, a constant $C''(p,n)$ exists such that

$$\|u\|_{L^1(\mathbb{R}^n)} = \|u^\bigstar\|_{L^1(\mathbb{R}^n)} \le C'' \qquad \text{and} \qquad \|u\|_{L^p(\mathbb{R}^n)} = \|u^\bigstar\|_{L^p(\mathbb{R}^n)} \le C'' .$$

Hence, from (4.39) and Lemma 3.1, we deduce that positive constants $C_{13}$, $r_1, r_2, r_3$, depending on $p, n$, exist such that

$$\int_{\mathbb{R}^n} |u - u^\bigstar|\, dx \le C_{13}\Big[M_{u^\bigstar}(\sigma) + \varepsilon^{r_1} + \frac{\varepsilon^{r_2}}{\sigma}\Big]^{r_3} \tag{4.40}$$

whenever (2.3), (1.5) and (1.6) are in force.

Our next task is to remove the sign assumption (2.3). Given $u$ as in the statement, satisfying (1.5) and (1.6), define $u_1 = \frac{|u|+u}{2}$ and $u_2 = \frac{|u|-u}{2}$, the positive and the negative parts of $u$, respectively, and set $\mu_1(t) = \mathcal{L}^n(\{u_1 > t\})$ and $\mu_2(t) = \mathcal{L}^n(\{u_2 > t\})$ for $t \ge 0$. Observe that the quantities $\varepsilon$ and $t_{\varepsilon,\alpha}$ defined in (2.12) and (2.13), remain unchanged after replacing $u$ by $|u|$. Thus, by Lemma 2.1 applied to $|u|$, a set $I_1$ satisfying (2.15) exists such that

$$\mathcal{H}^{n-1}(\{|u| = t\}) \le \Big(1 + \frac{\varepsilon^{\frac{1}{4p}-\alpha}}{n\omega_n^{1/n}}\Big)\mathcal{H}^{n-1}(\{u^\bigstar = t\}) \qquad \text{for every } t \in (0, t_{\varepsilon,\alpha}) \setminus I_1 . \tag{4.41}$$

Obviously, $\mu(t) = \mu_1(t) + \mu_2(t)$ for $t > 0$. Moreover, a property of the Hausdorff measure entails that

$$\mathcal{H}^{n-1}(\{|u| = t\}) = \mathcal{H}^{n-1}(\{u_1 = t\}) + \mathcal{H}^{n-1}(\{u_2 = t\}) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t > 0 ,$$

since $\{u_1 = t\}$ and $\{u_2 = t\}$ are disjoint Borel sets. Hence, from (4.41) and the isoperimetric inequality (2.2), one gets that

$$\mu_1(t)^{1/n'} + \mu_2(t)^{1/n'} \le (\mu_1(t) + \mu_2(t))^{1/n'} + \frac{\varepsilon^{\frac{1}{4p}-\alpha}}{n\omega_n^{1/n}}(\mu_1(t) + \mu_2(t))^{1/n'} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, t_{\varepsilon,\alpha}) \setminus I_1. \tag{4.42}$$

It is easily verified that, if $\varsigma \in (0,1)$,

$$a^\varsigma + b^\varsigma - (a+b)^\varsigma \ge (1-\varsigma)\frac{(ab)^\varsigma}{(a+b)^\varsigma} \qquad \text{for } a, b \ge 0 . \tag{4.43}$$

From (4.42), via (4.43) with $\varsigma = 1/n'$, we obtain that

$$\frac{\varepsilon^{\frac{1}{4p}-\alpha}}{\omega_n^{1/n}}(\mu_1(t) + \mu_2(t))^{2/n'} \ge (\mu_1(t)\mu_2(t))^{1/n'} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, t_{\varepsilon,\alpha}) \setminus I_1 . \tag{4.44}$$

By the very definition of $t_{\varepsilon,\alpha}$, one has $\mu_1(t) + \mu_2(t) > \varepsilon^{\alpha n'}$ if $t < t_{\varepsilon,\alpha}$. Therefore, since $\mu_1$ and $\mu_2$ are nonincreasing functions,

$$\text{either} \qquad \mu_1(t) \ge \frac{\varepsilon^{\alpha n'}}{2} \qquad \text{or} \qquad \mu_2(t) \ge \frac{\varepsilon^{\alpha n'}}{2} \qquad \text{for every } t < t_{\varepsilon,\alpha} . \tag{4.45}$$

To fix ideas, assume that the former situation occurs. Thus, by (4.44),

$$\frac{\varepsilon^{\frac{1}{4p}-\alpha}}{\omega_n^{1/n}}(\mu_1(t) + \mu_2(t))^{2/n'} \ge \mu_2(t)^{1/n'}\frac{\varepsilon^\alpha}{2^{1/n'}} ,$$

whence

$$\mu_2(t) \le 2\frac{\varepsilon^{\left(\frac{1}{4p}-2\alpha\right)n'}}{\omega_n^{1/(n-1)}}\mu(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,t_{\varepsilon,\alpha}) \setminus I_1 . \tag{4.46}$$

Owing to (4.46), we have

$$\begin{aligned} \|u_2\|_{L^1(\mathbb{R}^n)} &= \int_0^{+\infty} \mu_2(t)\,dt \le \int_{I_1} \mu_2(t)\,dt + \int_{(0,t_{\varepsilon,\alpha})\setminus I_1} \mu_2(t)\,dt + \int_{t_{\varepsilon,\alpha}}^{+\infty} \mu_2(t)\,dt \\ &\le \mathcal{L}^1(I_1) + 2\frac{\varepsilon^{\left(\frac{1}{4p}-2\alpha\right)n'}}{\omega_n^{1/(n-1)}} \int_0^{+\infty} \mu(t)\,dt + \int_{\{|u|>t_{\varepsilon,\alpha}\}} |u|\,dx \\ &\le \varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}} + 2\frac{\varepsilon^{\left(\frac{1}{4p}-2\alpha\right)n'}}{\omega_n^{1/(n-1)}} \|u\|_{L^1(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\varepsilon^{\frac{\alpha n'}{p'}} . \end{aligned} \tag{4.47}$$

Thus, a constant $\widetilde{C}(p,n)$ exists such that

$$\begin{aligned} \|u-u^\bigstar\|_{L^1(\mathbb{R}^n)} &\le \|u_2\|_{L^1(\mathbb{R}^n)} + \||u|-u^\bigstar\|_{L^1(\mathbb{R}^n)} \\ &\le \varepsilon^{1/2} + \varepsilon^{\frac{1}{4(p-1)}} + \widetilde{C}\varepsilon^{\left(\frac{1}{4p}-2\alpha\right)n'} + \widetilde{C}\varepsilon^{\frac{\alpha n'}{p'}} + C_{13}\Big[M_{u^\bigstar}(\sigma) + \varepsilon^{r_1} + \frac{\varepsilon^{r_2}}{\sigma}\Big]^{r_3} . \end{aligned} \tag{4.48}$$

Note that the first inequality is just a consequence of the triangle inequality, whereas the second one relies upon (4.47), (1.5), (1.6) and (4.40) applied with $u$ replaced by $|u|$. On distinguishing (similarly as above) the case where $\varepsilon \le 1$ from the case where $\varepsilon > 1$, one can easily deduce from (4.48) that inequality (4.40) continues to hold, for suitable $r_1, r_2, r_3$, and for any $u$ as in the statement satisfying (1.5), (1.6) and the first inequality in (4.45). If the second inequality in (4.45) is in force, inequality (4.40), with $u$ replaced by $-u$ on the left-hand side, follows via an analogous argument.In conclusion, we have shown that (4.1) holds under conditions (1.5) and (1.6).

In order to remove these additional conditions, notice that if $u$ is any function as in the statement, then the function $v : \mathbb{R}^n \to [0,+\infty)$ given by

$$v(x) = a\,u\Big(\frac{x}{b}\Big) \qquad \text{for } x \in \mathbb{R}^n ,$$

where $a = \dfrac{\mathcal{L}^n(\{u>0\})^{\frac{1}{p}-\frac{1}{n}}}{\big(\int_{\mathbb{R}^n}|\nabla u^\bigstar|^p dx\big)^{1/p}}$ and $b = \mathcal{L}^n(\{u>0\})^{-1/n}$, does satisfy (1.5) and (1.6). Furthermore,

$$\int_{\mathbb{R}^n} |v \pm v^\bigstar|\,dx = ab^n \int_{\mathbb{R}^n} |u \pm u^\bigstar|\,dx , \qquad E(v) = E(u) ,$$

and

$$M_v\Big(\frac{a}{b}\sigma\Big) = b^n M_u(\sigma), \qquad M_{v^\bigstar}\Big(\frac{a}{b}\sigma\Big) = b^n M_{u^\bigstar}(\sigma) \qquad \text{for } \sigma \ge 0 .$$

Thus, an application of (4.1) to $v$, with $\sigma$ replaced $b\sigma/a$, yields the same inequality also for $u$.

Finally, on applying (1.10) to the function $v$, one easily gets (4.1) with $M_{u^\bigstar}$ replaced by $M_u$ and $\|\nabla u^\bigstar\|_{L^p}$ replaced by $\|\nabla u\|_{L^p}$. □

**Proof of Theorem 1.1.** In order to deduce (1.7) from (4.1), let us first assume that $0 < E(u) \le 1$ and choose

$$\sigma = E(u)^r ,$$

for some $0 < r < r_2$. Since $M_{u^\star}(\sigma) \le 1$ for every $\sigma > 0$,

$$\left[M_{u^\star}(E(u)^r) + E(u)^{r_1} + E(u)^{r_2 - r}\right]^{r_3} \le C_{15}\left[M_{u^\star}(E(u)^r) + E(u)\right]^s , \tag{4.49}$$

where $s = \min\{1, r_3 \min\{r_1, r_2 - r\}\}$. Thus, (1.7) follows from (4.1) and (4.49). On the other hand, if $E(u) \ge 1$, inequality (1.7) is a straightforward consequence of the Poincaré inequality. Finally, (1.7) continues to hold even when $E(u) = 0$, owing to the continuity of $M_{u^\star}$ from the right. □

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