# ON A HIERARCHY OF MODELS FOR ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES

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ABSTRACT. In this paper we derive a hierarchy of models for electrical conduction in a biological tissue, which is represented by a periodic array of period $\varepsilon$ of conducting phases surrounded by dielectric shells of thickness $\varepsilon\eta$ included in a conductive matrix. Such a hierarchy will be obtained from the Maxwell equations by means of a concentration process $\eta \to 0$, followed by a homogenization limit with respect to $\varepsilon$. These models are then compared with regard to their physical meaning and mathematical issues.

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## 1. INTRODUCTION

In this paper we deal with models of electrical conduction in composite media and, specifically, biological tissues. The classical governing equation is

$$- \operatorname{div}(\mathcal{A}\nabla u + \mathcal{B}\nabla u_t) = 0 \,, \tag{1.1}$$

which is derived from the Maxwell equation in the quasi-stationary approximation. Here, $u$ is the electrical potential. We look at a material which is a mixture of a conductive phase of conductivity $\mathcal{A}$ and a dielectric phase of permeability $\mathcal{B}$; we extend the conductivity $\mathcal{A}$ (respectively, the permeability $\mathcal{B}$) by zero in the dielectric phase (respectively, in the conductive phase). More precisely, we look at a conductive phase containing a finely mixed periodic array of conductive inclusions coated by dielectric shells. The typical structure of the periodic cell we have in mind is given in Figure 1. This leads to the assumption

$$\mathcal{A} = \mathcal{A}(x/\varepsilon_0) \,, \qquad \mathcal{B} = \mathcal{B}(x/\varepsilon_0) \,,$$

where $\varepsilon_0$ is the period of the physical structure.
On the other hand, a second small parameter appears in the models we look at, that is the width of the dielectric shell, which we denote by $\eta_0\varepsilon_0$, with $\eta_0 \ll 1$.
For example, in the case of a biological tissue, the conductive phases are given, respectively, by the extracellular fluid and the cell cytosol, while the dielectrical

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phase is given by the cell membrane. In this case, the diameter of the cell is of the order of tens of micrometers, while the width of the membrane is of the order of ten nanometers, so that $\eta_0 \sim 10^{-3}$.

A concentration of capacity is performed to replace the thin dielectric shell with a two dimensional surface, in order to simplify the model, and, possibly, get a better understanding of the effect of the geometrical properties of the microscopic structure. Then a homogenization limit is taken, replacing $\varepsilon_0$ with a parameter $0 < \varepsilon < \varepsilon_0$ which is sent to 0.

Material properties are usually rescaled in concentration processes. In particular, we will assume that in the dielectric $\mathcal{B} = \eta\alpha$, and $\alpha$ is the ratio between the physical permeability of the membrane and the constant $\eta_0$. The reason of this scaling is explained in Remark 2.1. On the other hand, we note that $\mathcal{A}$ and $\mathcal{B}$ are physical properties of the material and, in principle, should not change in the homogenization limit: a kind of stability which is standard in homogenization theory.

Letting $\eta \to 0$ yields

$$-\operatorname{div}(\mathcal{A}\nabla u_\varepsilon) = 0 \,, \qquad\qquad \text{in } \Omega_T^\varepsilon; \tag{1.2}$$

$$\mathcal{A}\nabla u_\varepsilon^{\text{int}} \cdot \nu = \mathcal{A}\nabla u_\varepsilon^{\text{out}} \cdot \nu \,, \qquad \text{on } \Gamma_T^\varepsilon; \tag{1.3}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] = \mathcal{A}\nabla u_\varepsilon^{\text{out}} \cdot \nu \,, \qquad \text{on } \Gamma_T^\varepsilon; \tag{1.4}$$

$$[u_\varepsilon](x,0) = S_\varepsilon(x) \,, \qquad\qquad \text{on } \Gamma^\varepsilon, \tag{1.5}$$

where $\Omega^\varepsilon$ denotes the union of the two disjoint conductive phases, $\Gamma^\varepsilon$ is the separating interface, $T$ is a positive time, $\Omega_T^\varepsilon = \Omega^\varepsilon \times (0,T)$, $\Gamma_T^\varepsilon = \Gamma^\varepsilon \times (0,T)$, and $u_\varepsilon^{\text{int}}$, $u_\varepsilon^{\text{out}}$ are the potential in the internal and the external conductive phase, respectively. Let $\nu$ be the normal unit vector to $\Gamma^\varepsilon$ pointing into the external conductive phase. We also denote

$$[u_\varepsilon] = u_\varepsilon^{\text{out}} - u_\varepsilon^{\text{int}} \,. \tag{1.6}$$

If the dielectric phase has also a conductive behaviour, which can be modeled simply by letting $\mathcal{A} = \eta\beta$ in the dielectric phase ($\beta$ is a non negative constant), condition (1.4) is replaced with

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + \frac{\beta}{\varepsilon}[u_\varepsilon] = \mathcal{A}\nabla u_\varepsilon^{\text{out}} \cdot \nu \,. \tag{1.7}$$

However for the sake of simplicity we consider the case $\beta = 0$ in the Sections below. Clearly, (1.2)–(1.5) should be complemented with boundary conditions for $u_\varepsilon$ on $\partial\Omega$. Since all our arguments are independent of such boundary conditions, we omit them in the following (see [2] for the Dirichlet problem).

This model, which has been investigated in [1] and [2] , is a special case ($k = 1$) of a hierarchy of models where the equation (1.4) is replaced by

$$\frac{\alpha}{\varepsilon^k}\frac{\partial}{\partial t}[u_\varepsilon] = \mathcal{A}\nabla u_\varepsilon^{\text{out}} \cdot \nu \,, \qquad \text{on } \Gamma^\varepsilon, \tag{1.8}$$

with $k \in \mathbf{Z}$. For example, the case $k = -1$ is used in [8] in order to obtain, in the homogenization limit, the well known bidomain model for the cardiac syncithial tissue, where however, in the left hand side of (1.8) an extra term depending on $[u_\varepsilon]$ appears, modelling the nonlinear conductive behaviour of the membrane. The case

$k = 0$ (more precisely, its stationary version where the term on the left-hand side of (1.8) is replaced by $\beta[u_\varepsilon]$) is considered in [10] in connection with a heat conduction problem in presence of thermal barrier resistance.

We show that this family of concentrated problems can be derived from equation (1.1) by means of the unified approach sketched above, when we rescale in $\varepsilon$ the physical constant $\mathcal{B}$, i.e., we take $\mathcal{B} = \eta\alpha/\varepsilon^{k-1}$.

We observe that the homogenized limit equation for the cases $k \leq -2$, $k = 0$, $k \geq 2$, do not preserve any trace of the permeability $\mathcal{B}$ of the dielectric phase, accounting for the capacitive behaviour of the dielectric shell. Conversely, the limit equations for the cases $k = -1$ and $k = 1$ keep trace of the permeability $\mathcal{B}$. However, these two models differ substantially from a mathematical point of view, since the former is described by a system of degenerate parabolic equations, whereas the latter consists of an elliptic equation with memory. These models have been employed to study electrical conduction in biological tissues in different frequency ranges ([1], [2], [8]). Our analysis presents a unified derivation of different schemes, thus allowing a comparison among their underlying physical assumptions, and could be useful to assess the viability of each scheme as a model for a specific experimental setting.

The paper is organized as follows: the problem to be concentrated and homogenized, together with the relevant geometrical assumptions, is stated in Section 2. The concentration limit $\eta \to 0$ is performed in Section 3, in a rigorous mathematical way, yielding the family of concentrated problems (4.1)–(4.4), depending on $k \in \mathbf{Z}$. These problems are then formally homogenized ($\varepsilon \to 0$) in Section 4, with the aim of drawing a comparison among them. The homogenization limit are rigorously performed for the case $k = 1$ in [2] and for the case $k = -1$ in [11]. Finally, in Section 5, the so obtained models are compared with regard to their physical meaning and mathematical issues.

## 2. Position of the problem

The typical periodic geometrical setting is displayed in Figure 1. Here we give a detailed formal definition of it.

Let $\Omega$ be an open connected bounded subset of $\mathbf{R}^N$, and let $\Omega = \Omega_{\text{int}}^\varepsilon \cup \Omega_{\text{out}}^\varepsilon \cup \Gamma^\varepsilon$, where $\Omega_{\text{int}}^\varepsilon$ and $\Omega_{\text{out}}^\varepsilon$ are two disjoint open subsets of $\Omega$, and $\Gamma^\varepsilon = \partial\Omega_{\text{int}}^\varepsilon \cap \Omega = \partial\Omega_{\text{out}}^\varepsilon \cap \Omega$. The region $\Omega_{\text{out}}^\varepsilon$ [respectively, $\Omega_{\text{int}}^\varepsilon$] corresponds to the outer conductive phase [respectively, to the conductive inclusions], while $\Gamma^\varepsilon$ is the dielectric interface. Let $\nu$ denote the normal unit vector to $\Gamma^\varepsilon$ pointing into $\Omega_{\text{out}}^\varepsilon$ and $[u_\varepsilon]$ be defined as in (1.6).

For $\eta > 0$, let us write $\Omega$ as $\Omega = \Omega^{\varepsilon,\eta} \cup \Gamma^{\varepsilon,\eta} \cup \partial\Gamma^{\varepsilon,\eta}$, where $\Omega^{\varepsilon,\eta}$ and $\Gamma^{\varepsilon,\eta}$ are two disjoint open subsets of $\Omega$, $\Gamma^{\varepsilon,\eta}$ is the tubular neighborhood of $\Gamma^\varepsilon$ with thickness $\varepsilon\eta$, and $\partial\Gamma^{\varepsilon,\eta}$ is the part of the boundary of $\Gamma^{\varepsilon,\eta}$ which intersects $\Omega$. Moreover, we assume also that $\Omega^{\varepsilon,\eta} = \Omega_{\text{int}}^{\varepsilon,\eta} \cup \Omega_{\text{out}}^{\varepsilon,\eta}$ and $\partial\Gamma^{\varepsilon,\eta} = (\partial\Omega_{\text{int}}^{\varepsilon,\eta} \cup \partial\Omega_{\text{out}}^{\varepsilon,\eta}) \cap \Omega$. Again, $\Omega_{\text{out}}^{\varepsilon,\eta}$, $\Omega_{\text{int}}^{\varepsilon,\eta}$ correspond to the conductive regions, and $\Gamma^{\varepsilon,\eta}$ to the dielectric shell. We assume that, for $\eta \to 0$ and $\varepsilon > 0$ fixed, $|\Gamma^{\varepsilon,\eta}| \sim \varepsilon\eta|\Gamma^\varepsilon|_{N-1}$, $\Omega^{\varepsilon,\eta} \to \Omega_{\text{out}}^\varepsilon \cup \Omega_{\text{int}}^\varepsilon$ and $\partial\Gamma^{\varepsilon,\eta} \to \Gamma^\varepsilon$.

More specifically, let us introduce a periodic open subset $E$ of $\mathbf{R}^N$, so that $E + z = E$ for all $z \in \mathbf{Z}^N$. For all $\varepsilon > 0$ define $\Omega_{\text{int}}^\varepsilon = \Omega \cap \varepsilon E$, $\Omega_{\text{out}}^\varepsilon = \Omega \setminus \overline{\varepsilon E}$. We assume that $\Omega$, $E$ have regular boundary, say of class $C^\infty$ for the sake of simplicity. We also

employ the notation $Y = (0,1)^N$, and $E_{\text{int}} = E \cap Y$, $E_{\text{out}} = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. As a simplifying assumption, we stipulate that $|\Gamma \cap \partial Y|_{N-1} = 0$.

For every $\eta > 0$, let $Y = E^\eta \cup \Gamma^\eta \cup \partial\Gamma^\eta$, where $E^\eta$ and $\Gamma^\eta$ are two disjoint open subsets of $Y$, $\Gamma^\eta$ is the tubular neighborhood of $\Gamma$ with thickness $\eta$, and $\partial\Gamma^\eta$ is the part of the boundary of $\Gamma^\eta$ which intersects $Y$. Moreover, $E^\eta = E^\eta_{\text{int}} \cup E^\eta_{\text{out}}$ (see Figure 1). For $\eta \to 0$, $E^\eta \to E_{\text{int}} \cup E_{\text{out}}$, $|\Gamma^\eta| \sim \eta|\Gamma|_{N-1}$ and $\partial\Gamma^\eta \to \Gamma$.
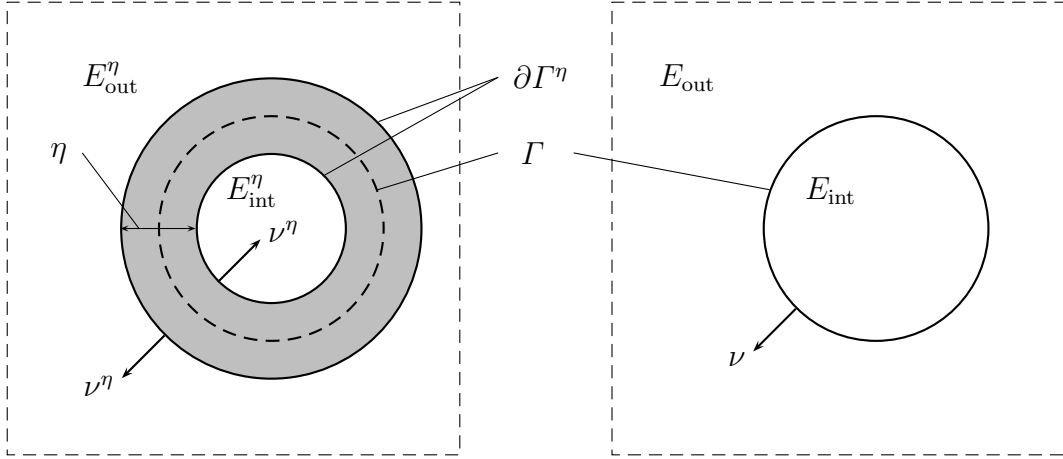


FIGURE 1. The periodic cell $Y$. *Left*: before concentration; $\Gamma^\eta$ is the shaded region, and $E^\eta = E^\eta_{\text{int}} \cup E^\eta_{\text{out}}$ is the white region. *Right*: after concentration; $\Gamma^\eta$ shrinks to $\Gamma$ as $\eta \to 0$.

Let $T > 0$ be a given time. For any spatial domain $G$, we denote by $G_T = G \times (0,T)$ the corresponding space–time cylindrical domain over the time interval $(0,T)$.

We start from the problem considered in the Introduction, i.e.,

$$-\operatorname{div}(A^\eta \nabla u_\varepsilon^\eta) = 0\,, \qquad\qquad \text{in } \Omega_T^{\varepsilon,\eta}; \qquad\qquad (2.1)$$

$$-\operatorname{div}(B^\eta \nabla u_{\varepsilon t}^\eta) = 0\,, \qquad\qquad \text{in } \Gamma_T^{\varepsilon,\eta}; \qquad\qquad (2.2)$$

$$A^\eta \nabla u_\varepsilon^\eta \cdot \nu^\eta = B^\eta \nabla u_{\varepsilon t}^\eta \cdot \nu^\eta\,, \qquad \text{on } \partial\Gamma_T^{\varepsilon,\eta}; \qquad\qquad (2.3)$$

$$\nabla u_\varepsilon^\eta(x,0) = \boldsymbol{\mathcal{S}}_\varepsilon^\eta(x)\,, \qquad\qquad \text{in } \Gamma^{\varepsilon,\eta}, \qquad\qquad (2.4)$$

where $u_\varepsilon^\eta(t) \in H^1(\Omega)$ at each time level $t$, $\boldsymbol{\mathcal{S}}_\varepsilon^\eta = \nabla\widetilde{S}_\varepsilon^\eta$, for some $\widetilde{S}_\varepsilon^\eta \in H^1(\Gamma^{\varepsilon,\eta})$ and $|\boldsymbol{\mathcal{S}}_\varepsilon^\eta| \sim 1/\eta$. Here $A^\eta$ ($\mathcal{A}$ in the previous section) satisfies $A^\eta(x) = \sigma_{\text{int}}$ in $\Omega_{\text{int}}^{\varepsilon,\eta}$, $A^\eta(x) = \sigma_{\text{out}}$ in $\Omega_{\text{out}}^{\varepsilon,\eta}$ and $A^\eta(x) = 0$ in $\Gamma^{\varepsilon,\eta}$; $B^\eta$ ($\mathcal{B}$ in the previous section) satisfies $B^\eta(x) = \alpha\eta$ in $\Gamma^{\varepsilon,\eta}$ and $B^\eta(x) = 0$ in $\Omega^{\varepsilon,\eta}$ (see Remark 2.1 and Subsection 5.1); $\nu^\eta$ is the unit normal vector to $\partial\Gamma^{\varepsilon,\eta}$ pointing into $\Omega^{\varepsilon,\eta}$ and $\sigma_{\text{int}}$, $\sigma_{\text{out}}$, $\alpha$ are positive constants. We also define $\sigma : \Omega \to \boldsymbol{R}$ as

$$\sigma = \sigma_{\text{int}} \quad \text{in } \Omega_{\text{int}}^\varepsilon, \qquad \sigma = \sigma_{\text{out}} \quad \text{in } \Omega_{\text{out}}^\varepsilon.$$

We note that (2.1)–(2.3) can be compactly written as

$$-\operatorname{div}\left(A^\eta \nabla u_\varepsilon^\eta + B^\eta u_{\varepsilon t}^\eta\right) = 0 \qquad \text{in } \Omega_T\,,$$

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coinciding with (1.1).

*Remark* 2.1. We are interested in preserving, in the limit $\eta \to 0$, the conduction *across* the membrane $\Gamma^\varepsilon$ instead of the *tangential* conduction on $\Gamma^\varepsilon$. To this purpose, we need to preserve the flux $B^\eta \nabla u^\eta_{\varepsilon t} \cdot \nu$ and the jump $[u^\eta_{\varepsilon t}]$ across the dielectric shells to be concentrated. Hence, we must rescale $B^\eta = \alpha\eta$, instead of scaling $B^\eta = \alpha/\eta$ in $\Gamma^{\varepsilon,\eta}$, as more usual in concentrated-capacity literature. $\qquad\square$

## 3. Derivation of the concentrated problems

For the sake of definiteness, we assume here zero Dirichlet boundary data on the boundary of $\Omega$. However, the argument is essentially local and could be reproduced once uniform $L^2$-estimates for $u^\eta_\varepsilon$ and $\nabla u^\eta_\varepsilon$ are available.

The rigorous formulation of the problem (2.1)–(2.3) is

$$\int_0^T \int_{\Omega^{\varepsilon,\eta}_{\text{out}} \cup \Omega^{\varepsilon,\eta}_{\text{int}}} A^\eta(x) \nabla u^\eta_\varepsilon \cdot \nabla \varphi \, dx \, d\tau = \int_0^T \int_{\Gamma^{\varepsilon,\eta}} \alpha\eta \nabla u^\eta_\varepsilon \cdot \nabla \varphi_\tau \, dx \, d\tau, \qquad (3.1)$$

where
$$u^\eta_\varepsilon \in L^2(0, T; H^1_o(\Omega)),$$
for all $\varphi \in C^2_o(\Omega \times (0, T))$.

The rigorous formulation of the concentrated problem (1.2)–(1.4) is moreover

$$\int_0^T \int_{\Omega^\varepsilon_{\text{out}} \cup \Omega^\varepsilon_{\text{int}}} \sigma(x) \nabla u_\varepsilon \cdot \nabla \phi \, dx \, d\tau = \int_0^T \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} [u_\varepsilon][\phi]_\tau \, d\sigma \, d\tau, \qquad (3.2)$$

where
$$u_{\varepsilon|\Omega^\varepsilon_{\text{out}}} \in L^2(0, T; H^1(\Omega^\varepsilon_{\text{out}})), \qquad u_{\varepsilon|\Omega^\varepsilon_{\text{int}}} \in L^2(0, T; H^1(\Omega^\varepsilon_{\text{int}})),$$
for all $\phi$ such that
$$\phi_{|\Omega^\varepsilon_{\text{int}}} \in C^2_o\left((\Omega^\varepsilon_{\text{int}} \cup \Gamma^\varepsilon) \times (0, T)\right), \qquad \phi_{|\Omega^\varepsilon_{\text{out}}} \in C^2_o\left((\Omega^\varepsilon_{\text{out}} \cup \Gamma^\varepsilon) \times (0, T)\right).$$

By an integration by parts, one can derive from (3.1) the energy inequality

$$\int_{t_0}^T \int_{\Omega^{\varepsilon,\eta}_{\text{out}} \cup \Omega^{\varepsilon,\eta}_{\text{int}}} |\nabla u^\eta_\varepsilon|^2 \, dx \, d\tau + \sup_{0 < t < T} \int_{\Gamma^{\varepsilon,\eta}(t)} \eta |\nabla u^\eta_\varepsilon|^2 \, dx \leq \gamma, \qquad (3.3)$$

where $\gamma$ does not depend on $\eta$.

As a consequence, the $L^2$-norm of $u^\eta_{\varepsilon|\Omega^\varepsilon_{\text{out}}}$ can be bounded by means of the usual Poincarè inequality and a standard extension technique. Then, the $L^2$-norm of the trace of $u^\eta_{\varepsilon|\Omega^\varepsilon_{\text{int}}}$ on $\partial\Gamma^{\varepsilon,\eta} \cap \Omega^\varepsilon_{\text{int}}$ can be bounded in terms of the $L^2$-norm of the trace of $u^\eta_{\varepsilon|\Omega^\varepsilon_{\text{out}}}$ and the bound in (3.3).

Hence, as $\eta \to 0$, we may assume, extracting a subsequence if needed,

$$u^\eta_\varepsilon \to u_\varepsilon, \qquad \text{weakly in } L^2_{\text{loc}}(\Omega \times (0, T)),$$

where
$$u_{\varepsilon|\Omega^\varepsilon_{\text{int}}} \in L^2_{\text{loc}}(0, T; W^2_1(\Omega^\varepsilon_{\text{int}})), \, u_{\varepsilon|\Omega^\varepsilon_{\text{out}}} \in L^2_{\text{loc}}(0, T; W^2_1(\Omega^\varepsilon_{\text{out}})).$$

Moreover, we may also assume

$$\nabla u_\varepsilon^\eta \to \nabla u_\varepsilon, \qquad \text{weakly in } L^2_{\text{loc}}(\Omega^\varepsilon_{\text{int}} \times (0, T)), \; L^2_{\text{loc}}(\Omega^\varepsilon_{\text{out}} \times (0, T)).$$

3.1. **Diffeomorphisms.** The following fact is well known.

**Theorem 3.1.** *There exists an $\eta_0 > 0$ such that for $\eta < \eta_0$, the application*

$$\psi : \Gamma^\varepsilon \times \left( -\frac{\varepsilon\eta}{2}, \frac{\varepsilon\eta}{2} \right) \to \mathbf{R}^N, \qquad \psi(\xi, r) = \xi + r\nu(\xi),$$

*is a diffeomorphism onto its image, which equals (by definition) $\Gamma^{\varepsilon,\eta}$.*

Locally, we can represent $\Gamma^\varepsilon$ as a graph

$$x_N = F(x'), \qquad x' = (x_1, \dots, x_{N-1}), \tag{3.4}$$

perhaps after relabelling the coordinates. In the following we always assume this representation, i.e., we choose a testing function whose support is contained in the region where (3.4) is valid. The general case can then be recovered by means of a standard partition of unity argument.

Let $\nu$ denote also the normal as a function of the local coordinates; then

$$\nu(x') = (-\nabla_{x'} F(x'), 1) g(x')^{-\frac{1}{2}}, \qquad g(x') = 1 + |\nabla_{x'} F(x')|^2.$$

Let us introduce the notation

$$\psi^{-1} : \Gamma^{\varepsilon,\eta} \to \Gamma^\varepsilon \times \left( -\frac{\varepsilon\eta}{2}, \frac{\varepsilon\eta}{2} \right), \qquad \psi^{-1}(y) = (\pi_0(y), F(\pi_0(y)), \rho(y)),$$

with

$$\pi_0 : \Gamma^{\varepsilon,\eta} \to \mathbf{R}^{N-1}, \qquad \rho : \Gamma^{\varepsilon,\eta} \to \left( -\frac{\varepsilon\eta}{2}, \frac{\varepsilon\eta}{2} \right).$$

We know that, for $y \in \Gamma^{\varepsilon,\eta}$, if we let $\mu(y)$ be the closest point of $\Gamma^\varepsilon$ to $y$, then $y - \mu(y)$ must be normal to $\Gamma^\varepsilon$. Since $\psi$ is a diffeomorphism, $(\pi_0(y), F(\pi_0(y)))$ is the only point with this property. It follows that in $\Gamma^{\varepsilon,\eta}$

$$\rho(y) = \text{(signed) distance from } \Gamma^\varepsilon.$$

Hence

$$\nabla \rho(y) = \nu(\pi_0(y)), \qquad \text{for } y = (\pi_0(y), F(\pi_0(y)) \in \Gamma^\varepsilon, \text{ i.e., for } \rho(y) = 0,$$

while in general, by virtue of the assumed regularity of $\Gamma^\varepsilon$,

$$\nabla \rho(y) = \nu(\pi_0(y)) + R_\eta(y), \qquad y \in \Gamma^{\varepsilon,\eta}, \tag{3.5}$$

where we denote by $R_\eta$ a quantity such that

$$|R_\eta| \leq \gamma\eta, \qquad 0 < \eta < \eta_0.$$

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3.2. **Piecewise smooth testing function.** Let

$$\varphi_1 \in C_o^2\big((\Omega_{\text{int}}^{\varepsilon} \cup \Gamma^{\varepsilon}) \times (0,T)\big)\,, \qquad \varphi_2 \in C_o^2\big((\Omega_{\text{out}}^{\varepsilon} \cup \Gamma^{\varepsilon}) \times (0,T)\big)\,.$$

Then define for $0 < \eta < \eta_0$, $y \in \Gamma^{\varepsilon,\eta}$, the functions $\varphi_{3\eta}$ and $\mathcal{J}$ from

$$\varphi_{3\eta}(y) = \left[\varphi_2\big(\psi(\pi_0(y), \varepsilon\eta/2), t\big) - \varphi_1\big(\psi(\pi_0(y), -\varepsilon\eta/2), t\big)\right]\frac{\rho(y) + \varepsilon\eta/2}{\varepsilon\eta}$$

$$+ \varphi_1\big(\psi(\pi_0(y), -\varepsilon\eta/2), t\big) =: \mathcal{J}\frac{\rho(y) + \varepsilon\eta/2}{\varepsilon\eta} + \varphi_1\big(\psi(\pi_0(y), -\varepsilon\eta/2), t\big)\,,$$

where for the sake of notational simplicity we use $\psi$ to denote the diffeomorphism written in local coordinates.

The function

$$\varphi_{\eta}(y,t) = \begin{cases} \varphi_1(y,t)\,, & y \in \Omega_{\text{int}}^{\varepsilon,\eta}\,, \\ \varphi_{3\eta}(y,t)\,, & y \in \Gamma^{\varepsilon,\eta}\,, \\ \varphi_2(y,t)\,, & y \in \Omega_{\text{out}}^{\varepsilon,\eta}\,, \end{cases}$$

is an admissible testing function in (3.1).

Owing to our smoothness assumptions, we have

$$\nabla\varphi_{3\eta}(y) = \big(\nabla\mathcal{J}\big)\frac{\rho(y) + \varepsilon\eta/2}{\varepsilon\eta} + \mathcal{J}(\varepsilon\eta)^{-1}\nabla\rho(y) + \nabla\varphi_1\big(\psi(\pi_0(y), -\varepsilon\eta/2), t\big)$$

$$= C_{\eta}(y) + \mathcal{J}(\varepsilon\eta)^{-1}\nabla\rho(y) = C_{\eta}(y) + \mathcal{J}(\varepsilon\eta)^{-1}\nu(\pi_0(y))\,, \quad (3.6)$$

where we have used (3.5) and denoted by $C_{\eta}$ any quantity that stays bounded for all $0 < \eta < \eta_0$.

3.3. **The limit $\eta \to 0$.** Write (3.1) for $\varphi_{\eta}$, i.e.,

$$\int\limits_0^T \int\limits_{\Omega_{\text{out}}^{\varepsilon,\eta} \cup \Omega_{\text{int}}^{\varepsilon,\eta}} A^{\eta}(x)\nabla u_{\varepsilon}^{\eta} \cdot \nabla\varphi_{\eta}\,\mathrm{d}x\,\mathrm{d}\tau = \int\limits_0^T \int\limits_{\Gamma^{\varepsilon,\eta}} \alpha\eta\nabla u_{\varepsilon}^{\eta} \cdot \nabla\varphi_{3\eta\tau}\,\mathrm{d}x\,\mathrm{d}\tau\,. \qquad (3.7)$$

The left hand side of (3.7) approaches as $\eta \to 0$

$$\int\limits_0^T \int\limits_{\Omega_{\text{out}}^{\varepsilon} \cup \Omega_{\text{int}}^{\varepsilon}} \sigma(x)\nabla u \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau\,,$$

where

$$\varphi(y,t) = \begin{cases} \varphi_1(y,t)\,, & y \in \Omega_{\text{int}}^{\varepsilon}\,, \\ \varphi_2(y,t)\,, & y \in \Omega_{\text{out}}^{\varepsilon}\,, \end{cases}$$

Indeed, this follows from the local convergence of $\nabla u_{\varepsilon}^{\eta}$ and the uniform boundedness given by (3.3).

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The right hand side of (3.7) equals, by virtue of (3.6),

$$\int\limits_0^T \int\limits_{\Gamma^{\varepsilon,\eta}} \alpha\eta\nabla u_\varepsilon^\eta \cdot \{C_\eta(y,\tau) + \mathcal{J}_\tau(\varepsilon\eta)^{-1}\nu(\pi_0(y))\}\, \mathrm{d}y\, \mathrm{d}\tau$$

$$= \eta \int\limits_0^T \int\limits_{\Gamma^{\varepsilon,\eta}} \alpha\nabla u_\varepsilon^\eta \cdot C_\eta(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau + \frac{\alpha}{\varepsilon} \int\limits_0^T \int\limits_{\Gamma^{\varepsilon,\eta}} \mathcal{J}_\tau\nabla u_\varepsilon^\eta \cdot \nu(\pi_0(y))\, \mathrm{d}y\, \mathrm{d}\tau = I_1 + I_2\,.$$

As $\eta \to 0$, owing to (3.3),

$$|I_1| \le \alpha\eta^{\frac{1}{2}}\Big(\eta \int\limits_0^T \int\limits_{\Gamma^{\varepsilon,\eta}} |\nabla u_\varepsilon^\eta|^2\, \mathrm{d}y\, \mathrm{d}\tau\Big)^{\frac{1}{2}} \Big(\int\limits_0^T \int\limits_{\Gamma^{\varepsilon,\eta}} |C_\eta(y,\tau)|^2\, \mathrm{d}y\, \mathrm{d}\tau\Big)^{\frac{1}{2}} \le \gamma\eta^{\frac{1}{2}} \to 0\,.$$

Let us next change coordinates in $I_2$, according to $y \mapsto (\pi_0(y), \rho(y)) = (x', r)$. The jacobian matrix of this change of coordinates is given by

$$J(x', r) = \begin{pmatrix} (e_1', \frac{\partial F}{\partial x_1}) + r\frac{\partial\nu(x')}{\partial x_1} \\ \cdots \\ (e_{N-1}', \frac{\partial F}{\partial x_{N-1}}) + r\frac{\partial\nu(x')}{\partial x_{N-1}} \\ \nu(x') \end{pmatrix}$$

where $e_k'$ is the standard basis in $\boldsymbol{R}^{N-1}$. Then the jacobian determinant is (recall the definition of $g$)

$$|\det J(x', r)| = |\det J(x', 0)| + R_\eta(x', r) = \sqrt{g(x')} + R_\eta(x', r)\,.$$

Hence

$$I_2 = \frac{\alpha}{\varepsilon} \int\limits_0^T \int\limits_{\boldsymbol{R}^{N-1}} \mathcal{J}_\tau\sqrt{g(x')} \int\limits_{-\frac{\varepsilon\eta}{2}}^{\frac{\varepsilon\eta}{2}} \nabla u_\varepsilon^\eta(\psi(x', r), \tau) \cdot \nu(x')\, \mathrm{d}r\, \mathrm{d}x'\, \mathrm{d}\tau$$

$$+ \frac{\alpha}{\varepsilon} \int\limits_0^T \int\limits_{\boldsymbol{R}^{N-1}} \mathcal{J}_\tau \int\limits_{-\frac{\varepsilon\eta}{2}}^{\frac{\varepsilon\eta}{2}} R_\eta(x', r)\nabla u_\varepsilon^\eta(\psi(x', r), \tau) \cdot \nu(x')\, \mathrm{d}r\, \mathrm{d}x'\, \mathrm{d}\tau = I_{21} + I_{22}\,.$$

Invoking again (3.3), we get from an application of Hölder's inequality

$$|I_{22}| \le C\eta \int\limits_0^T \int\limits_{\Gamma^{\varepsilon,\eta}} |\nabla u_\varepsilon^\eta|\, \mathrm{d}x\, \mathrm{d}\tau \le C\eta^{\frac{1}{2}} \to 0\,, \quad \text{as } \eta \to 0.$$

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Finally, by the bound on the traces mentioned above, it follows

$$
I_{21} = \frac{\alpha}{\varepsilon} \int\limits_0^T \int\limits_{\mathbf{R}^{N-1}} \left[ \varphi_{2\tau}\big(\psi(x', \varepsilon\eta/2), \tau\big) - \varphi_{1\tau}\big(\psi(x', -\varepsilon\eta/2), \tau\big) \right]
$$

$$
\sqrt{1 + |\nabla_{x'} F(x')|^2} \left[ u_\varepsilon^\eta\big(\psi(x', \varepsilon\eta/2), \tau\big) - u_\varepsilon^\eta\big(\psi(x', -\varepsilon\eta/2), \tau\big) \right] dx' \, d\tau
$$

$$
\rightarrow \frac{\alpha}{\varepsilon} \int\limits_0^T \int\limits_{\mathbf{R}^{N-1}} \left[ \varphi_{2\tau}\big(\psi(x', 0), \tau\big) - \varphi_{1\tau}\big(\psi(x', 0), \tau\big) \right]
$$

$$
\left[ u_\varepsilon^{\text{out}}\big((x', F(x')), \tau\big) - u_\varepsilon^{\text{int}}\big((x', F(x')), \tau\big) \right] \sqrt{1 + |\nabla_{x'} F(x')|^2} \, dx' \, d\tau
$$

$$
= \frac{\alpha}{\varepsilon} \int\limits_0^T \int\limits_{\Gamma^\varepsilon} [\varphi]_\tau [u_\varepsilon] \, d\sigma \, d\tau \, .
$$

Collecting the limiting relations above, we see that indeed (3.1) leads to (3.2) as $\eta \to 0$.

## 4. Homogenization of the concentrated problems

We look at the homogenization limit ($\varepsilon \to 0$) of the problem for $u_\varepsilon(x, t)$ stated in the Introduction. We give here a complete formulation for convenience (the operators div and $\nabla$ act only with respect to the space variable $x$):

$$
\begin{aligned}
-\operatorname{div}(\sigma \nabla u_\varepsilon) &= 0\,, & &\text{in } \Omega_{\text{int}}^\varepsilon,\ \Omega_{\text{out}}^\varepsilon; & &(4.1)\\
[\sigma \nabla u_\varepsilon \cdot \nu] &= 0\,, & &\text{on } \Gamma^\varepsilon; & &(4.2)\\
\frac{\alpha}{\varepsilon^k} \frac{\partial}{\partial t}[u_\varepsilon] &= \sigma_{\text{out}} \nabla u_\varepsilon^{\text{out}} \cdot \nu\,, & &\text{on } \Gamma^\varepsilon; & &(4.3)\\
[u_\varepsilon](x, 0) &= S_\varepsilon(x)\,, & &\text{on } \Gamma^\varepsilon. & &(4.4)
\end{aligned}
$$

We are interested in understanding how the limiting behaviour of the problem above when $\varepsilon \to 0$ depends on the parameter $k \in \mathbf{Z}$.

First, we have to determine the admissible order of $S_\varepsilon$ with respect to $\varepsilon$. Multiply (4.1) by $u_\varepsilon$ and integrate by parts. When, for the sake of simplicity, we look at the case of Dirichlet boundary conditions $u_\varepsilon = 0$ on $\partial\Omega$, we arrive, for all $0 < t < T$, to the energy estimate

$$
\int\limits_0^t \int\limits_\Omega \sigma |\nabla u_\varepsilon|^2 \, dx \, d\tau + \frac{\alpha}{2\varepsilon^k} \int\limits_{\Gamma^\varepsilon} [u_\varepsilon]^2(x, t) \, d\sigma = \frac{\alpha}{2\varepsilon^k} \int\limits_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, d\sigma \,. \tag{4.5}
$$

Since $|\Gamma^\varepsilon|_{N-1} \sim 1/\varepsilon$, we assume that

$$
S_\varepsilon = O(\varepsilon^{(k+1)/2})\,, \tag{4.6}
$$

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so that the right hand side of (4.5) is stable as $\varepsilon \to 0$. In fact (4.5), coupled with suitable Poincaré's inequalities, is a main tool in the rigorous proof of convergence of $u_\varepsilon$ to its limit ([9], [2]).

4.1. **The two-scale approach.** We summarize here, to establish the notation, some well known asymptotic expansions needed in the two-scale method (see, e.g., [5], [12]), when applied to stationary, or evolutive, problems involving second order partial differential equations. Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$u_\varepsilon = u_\varepsilon(x, y, t) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \dots . \qquad (4.7)$$

Note that $u_0$, $u_1$, $u_2$ are periodic in $y$, and $u_1$, $u_2$ are assumed to have zero integral average over $Y$. Recalling that

$$\operatorname{div} = \frac{1}{\varepsilon} \operatorname{div}_y + \operatorname{div}_x , \qquad \nabla = \frac{1}{\varepsilon} \nabla_y + \nabla_x , \qquad (4.8)$$

we compute

$$\Delta u_\varepsilon = \frac{1}{\varepsilon^2} A_0 u_0 + \frac{1}{\varepsilon} (A_0 u_1 + A_1 u_0) + (A_0 u_2 + A_1 u_1 + A_2 u_0) + \dots , \qquad (4.9)$$

Here

$$A_0 = \Delta_y , \quad A_1 = \operatorname{div}_y \nabla_x + \operatorname{div}_x \nabla_y , \quad A_2 = \Delta_x . \qquad (4.10)$$

Let us recall explicitly that

$$\nabla u_\varepsilon = \frac{1}{\varepsilon} \nabla_y u_0 + \left( \nabla_x u_0 + \nabla_y u_1 \right) + \varepsilon \left( \nabla_y u_2 + \nabla_x u_1 \right) + \dots . \qquad (4.11)$$

We also stipulate

$$S_\varepsilon = S_\varepsilon(x, y) = S_0(x, y) + \varepsilon S_1(x, y) + \varepsilon^2 S_2(x, y) + \dots . \qquad (4.12)$$

We consider here only expansions in integral powers of $\varepsilon$, though (4.6) would allow, in principle, for an expansion in powers of $\sqrt{\varepsilon}$.

4.2. **The two extreme cases $k = +\infty$, $k = -\infty$.** The following two cases, though they are essentially stationary problems, provide a frame of mind we find helpful to understand the overall picture of our class of models. Moreover they are strictly connected, from the technical point of view, to the homogenization process we develop.

4.2.1. *The Diffraction problem.* We look here at the case of perfectly conducting interfaces, i.e., to (4.1)–(4.4) with $S_\varepsilon \equiv 0$ and $k = +\infty$. More explicitly, we look at the problem obtained when (4.1), (4.2) are complemented with

$$[u_\varepsilon] = 0 , \qquad \text{on } \Gamma^\varepsilon,$$

which is assumed to hold for every $t \in [0, T]$, replacing (4.3) and (4.4). Then, time plays only the role of a parameter. On substituting in this formulation the expansion (4.7), and applying (4.8)–(4.11), one readily obtains by matching corresponding powers of $\varepsilon$, that $u_0$ solves $[u_0] = 0$ on $\Gamma$, and

$$\mathcal{P}_0[u_0] : \quad \begin{cases} -\sigma \Delta_y u_0 = 0 , & \text{in } E_{\text{int}}, E_{\text{out}}; \\ [\sigma \nabla_y u_0 \cdot \nu] = 0 , & \text{on } \Gamma. \end{cases}$$

Hence $u_0$ is independent of $y$, i.e., $u_0 = u_0(x, t)$.

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Moreover $u_1$ satisfies $[u_1] = 0$ on $\Gamma$, and

$$\boldsymbol{\mathcal{P}}_1[u_1] : \quad \begin{cases} -\sigma\,\Delta_y\,u_1 = 0\,, & \text{in } E_{\text{int}},\ E_{\text{out}}; \\ [\sigma\nabla_y u_1 \cdot \nu] = -[\sigma\nabla_x u_0 \cdot \nu]\,, & \text{on } \Gamma. \end{cases}$$

Finally, $u_2$ solves $[u_2] = 0$ on $\Gamma$, and

$$\boldsymbol{\mathcal{P}}_2[u_2] : \quad \begin{cases} -\sigma\,\Delta_y\,u_2 = \sigma\,\Delta_x\,u_0 + 2\sigma\dfrac{\partial^2 u_1}{\partial x_j \partial y_j}\,, & \text{in } E_{\text{int}},\ E_{\text{out}}; \\ [\sigma\nabla_y u_2 \cdot \nu] = -[\sigma\nabla_x u_1 \cdot \nu]\,, & \text{on } \Gamma. \end{cases}$$

Following a classical approach, one introduces the factorization

$$u_1(x,y,t) = -\chi^{\mathcal{D}}(y)\cdot\nabla_x u_0(x,t)\,, \tag{4.13}$$

for a vector function $\chi^{\mathcal{D}} : Y \to \boldsymbol{R}^N$, satisfying

$$-\sigma\,\Delta_y\,\chi_h^{\mathcal{D}} = 0\,, \qquad \text{in } E_{\text{int}},\ E_{\text{out}}; \tag{4.14}$$

$$[\sigma(\nabla_y\chi_h^{\mathcal{D}} - \boldsymbol{e}_h)\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \tag{4.15}$$

$$[\chi_h^{\mathcal{D}}] = 0\,, \qquad \text{on } \Gamma. \tag{4.16}$$

The components $\chi_h^{\mathcal{D}}$ are also required to be periodic functions in $Y$, with zero integral average on $Y$. The limiting equation for $u_0$ is finally obtained as a solvability condition for $\boldsymbol{\mathcal{P}}_2[u_2]$, and amounts to

$$-\operatorname{div}\left((\sigma_0 I + A^{\mathcal{D}})\nabla_x u_0\right) = 0\,, \qquad \text{in } \Omega_T, \tag{4.17}$$

where

$$(A^{\mathcal{D}}) = \int_{\Gamma} [\sigma]\nu \otimes \chi^{\mathcal{D}}\,\mathrm{d}\sigma\,, \qquad \sigma_0 = \sigma_{\text{int}}|E_{\text{int}}| + \sigma_{\text{out}}|E_{\text{out}}|\,. \tag{4.18}$$

To avoid repetition, we refer the reader to the classical texts (e.g., [5]) for the details of the proof of (4.17), or to Subsection 4.4 below, where similar calculations are carried out in a somehow more complex setting.

4.2.2. *The Neumann problem.* This is the case corresponding to $k = -\infty$, when the interfaces are perfectly insulating. Here (4.1), (4.2) are complemented with

$$\sigma_{\text{out}}\nabla u_\varepsilon^{\text{out}} \cdot \nu = 0\,, \qquad \text{on } \Gamma^\varepsilon,$$

so that, actually, we are solving two independent Neumann problems in $\Omega_{\text{int}}^\varepsilon$ and in $\Omega_{\text{out}}^\varepsilon$. Of course, this is meaningful if each one of the two phases is connected (see Remark 5.3). Again, time acts only as a parameter: the problem is in fact essentially stationary, and the initial condition (4.4) can no longer be assigned.
Reasoning as above, we find for $u_0$

$$\begin{cases} \boldsymbol{\mathcal{P}}_0[u_0]\,; \\ \sigma_{\text{out}}\nabla_y u_0^{\text{out}} \cdot \nu = 0\,, & \text{on } \Gamma. \end{cases}$$

In this case $[u_0] \neq 0$ on $\Gamma$, but $u_0$ is independent of $y$ in each phase, i.e.,

$$u_0(x,y,t) = \begin{cases} u_0^{\text{int}}(x,t)\,, & \text{in } E_{\text{int}}, \\ u_0^{\text{out}}(x,t)\,, & \text{in } E_{\text{out}}. \end{cases} \tag{4.19}$$

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The term $u_1$ solves
$$\begin{cases} \boldsymbol{\mathcal{P}}_1[u_1]\,; \\ \sigma_{\text{out}}\nabla_y u_1^{\text{out}} \cdot \nu + \sigma_{\text{out}}\nabla_x u_0^{\text{out}} \cdot \nu = 0\,, \quad \text{on } \Gamma. \end{cases}$$

Analogously, $u_2$ is given by
$$\begin{cases} \boldsymbol{\mathcal{P}}_2[u_2]\,; \\ \sigma_{\text{out}}\nabla_y u_2^{\text{out}} \cdot \nu + \sigma_{\text{out}}\nabla_x u_1^{\text{out}} \cdot \nu = 0\,, \quad \text{on } \Gamma. \end{cases}$$

We introduce again the factorization for $u_1$, which is given by
$$u_1(x,y,t) = \begin{cases} -\chi^{\mathcal{N}}(y) \cdot \nabla_x u_0^{\text{int}}(x,t) + \widetilde{u}_1^{\text{int}}(x,t) & y \in E_{\text{int}}\,, \\ -\chi^{\mathcal{N}}(y) \cdot \nabla_x u_0^{\text{out}}(x,t) + \widetilde{u}_1^{\text{out}}(x,t) & y \in E_{\text{out}}\,, \end{cases} \tag{4.20}$$

where $\chi^{\mathcal{N}}$ is given by
$$-\sigma\,\Delta_y\,\chi_h^{\mathcal{N}} = 0\,, \qquad \text{in } E_{\text{int}},\ E_{\text{out}}; \tag{4.21}$$
$$[\sigma(\nabla_y\chi_h^{\mathcal{N}} - \boldsymbol{e}_h) \cdot \nu] = 0\,, \qquad \text{on } \Gamma; \tag{4.22}$$
$$\sigma_{\text{out}}(\nabla_y\chi_{h,\text{out}}^{\mathcal{N}} - \boldsymbol{e}_h) \cdot \nu = 0\,, \qquad \text{on } \Gamma. \tag{4.23}$$

The components $\chi_h^{\mathcal{N}}$ are also required to be periodic functions in $Y$, with zero integral average on $Y$. This implies that only one of the two constants up to which $\chi_{\text{int}}^{\mathcal{N}}$ and $\chi_{\text{out}}^{\mathcal{N}}$ are determined by (4.21)–(4.23) is fixed, but this does not affect (4.24) and (4.25). Note that, in general, $[\chi_h^{\mathcal{N}}] \neq 0$ on $\Gamma$.

In the limit, we obtain (as compatibility conditions for the problem solved by $u_2$) two partial differential equations for the two components of $u_0$, i.e.,
$$-\operatorname{div}\left((\sigma_{\text{int}}|E_{\text{int}}|I + A_{\text{int}}^{\mathcal{N}})\nabla_x u_0^{\text{int}}\right) = 0\,, \qquad \text{in } \Omega_T; \tag{4.24}$$
$$-\operatorname{div}\left((\sigma_{\text{out}}|E_{\text{out}}|I + A_{\text{out}}^{\mathcal{N}})\nabla_x u_0^{\text{out}}\right) = 0\,, \qquad \text{in } \Omega_T, \tag{4.25}$$

where
$$A_{\text{int}}^{\mathcal{N}} = -\int_\Gamma \sigma_{\text{int}}\nu \otimes \chi_{\text{int}}^{\mathcal{N}}\,\mathrm{d}\sigma\,, \qquad A_{\text{out}}^{\mathcal{N}} = \int_\Gamma \sigma_{\text{out}}\nu \otimes \chi_{\text{out}}^{\mathcal{N}}\,\mathrm{d}\sigma\,. \tag{4.26}$$

**4.3. The case $k \geq 2$: essentially the same as $k = +\infty$.** According to (4.6), we stipulate $S_0 \equiv 0$, $S_1 \equiv 0$ in (4.12). If $k > 2$, we should further assume that $S_h \equiv 0$ for every $h$ up to the integer part of $(k+1)/2$. The problem for $u_0$ is given by
$$\begin{cases} \boldsymbol{\mathcal{P}}_0[u_0]\,; \\ \alpha\dfrac{\partial[u_0]}{\partial t} = 0\,, \quad [u_0]_{|t=0} = 0\,, \quad \text{on } \Gamma. \end{cases}$$

It follows that $[u_0] = 0$ for all $t$. Then, $\boldsymbol{\mathcal{P}}_0[u_0]$ implies as in Subsection 4.2 that $u_0$ is independent of $y$, i.e., $u_0 = u_0(x,t)$.

Next, one checks that $u_1$ solves
$$\begin{cases} \boldsymbol{\mathcal{P}}_1[u_1]\,; \\ \alpha\dfrac{\partial[u_1]}{\partial t} = \sigma_{\text{out}}\nabla_y u_0 \cdot \nu\,, \quad [u_1]_{|t=0} = 0\,, \quad \text{on } \Gamma. \end{cases}$$

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Since $\nabla_y u_0 \equiv 0$, we obtain $[u_1] = 0$ for all times. It is a simple matter to verify that $u_1$ may be represented as

$$u_1(x, y, t) = -\chi^{\mathcal{D}}(y) \cdot \nabla_x u_0(x, t) \,. \tag{4.27}$$

The problem for $u_2$ amounts to

$$\begin{cases} \mathcal{P}_2[u_2] \,; \\ \alpha \dfrac{\partial [u_2]}{\partial t} = \sigma_{\text{out}} \nabla_x u_0 \cdot \nu + \sigma_{\text{out}} \nabla_y u_1^{\text{out}} \cdot \nu \,, \quad [u_2]_{|t=0} = S_2 \,, \quad \text{on } \Gamma. \end{cases}$$

The compatibility condition for this problem is obtained integrating the partial differential equation by parts as in Subsection 4.4. On using also the expansion (4.27), we finally get in the limit

$$- \operatorname{div}\left( (\sigma_0 I + A^{\mathcal{D}}) \nabla_x u_0 \right) = 0 \,, \tag{4.28}$$

where the matrix $A^{\mathcal{D}}$ was defined in (4.18). Note that (4.28) is the same limiting equation we obtained in the Diffraction problem $k = +\infty$. Moreover, the limiting problem does not depend on the initial data $S_i$, $i \geq 2$, for $t > 0$: indeed, the dependence of $u_0$ on $t$ is merely parametrical. In other words, the homogenization limit in the case $k \geq 2$ is the same as in the case $k = \infty$.

4.4. **The case $k = 1$: a limiting equation with memory.** According to equation (4.6), we assume $S_0 \equiv 0$ in (4.12).

As a consequence, and reasoning as in Subsection 4.2, we find for $u_0$,

$$\begin{cases} \mathcal{P}_0[u_0] \,; \\ \sigma_{\text{out}} \nabla_y u_0^{\text{out}} \cdot \nu = \alpha \dfrac{\partial [u_0]}{\partial t} \,, \quad [u_0]_{|t=0} = 0 \,, \quad \text{on } \Gamma. \end{cases}$$

It follows (see [3]) that $[u_0] = 0$ for all times, and

$$u_0 = u_0(x, t) \,.$$

Next we find for $u_1$

$$\begin{cases} \mathcal{P}_1[u_1] \,; \\ \sigma_{\text{out}} \nabla_y u_1^{\text{out}} \cdot \nu + \sigma_{\text{out}} \nabla_x u_0^{\text{out}} \cdot \nu = \alpha \dfrac{\partial [u_1]}{\partial t} \,, \quad [u_1]_{|t=0} = S_1(x, y) \,, \quad \text{on } \Gamma. \end{cases}$$

We want to represent $u_1$ in a suitable way, in the spirit of (4.13), though the representation formula must be more complicated here. Let $s : \Gamma \to \mathbf{R}$ be a jump function, and consider the problem

$$-\sigma \, \Delta_y v = 0 \,, \qquad\qquad \text{in } E_{\text{int}}, \ E_{\text{out}}; \tag{4.29}$$

$$[\sigma \nabla_y v \cdot \nu] = 0 \,, \qquad\qquad \text{on } \Gamma; \tag{4.30}$$

$$\alpha \frac{\partial}{\partial t}[v] = \sigma_{\text{out}} \nabla_y v^{\text{out}} \cdot \nu \,, \qquad \text{on } \Gamma; \tag{4.31}$$

$$[v](y, 0) = s(y) \,, \qquad\qquad \text{on } \Gamma, \tag{4.32}$$

where $v$ is a periodic function in $Y$, such that $\int_Y v = 0$. Define the transform $\mathcal{T}$ by

$$\mathcal{T}(s)(y, t) = v(y, t) \,, \qquad y \in Y \,, t > 0 \,,$$

and extend the definition of $\mathcal{T}$ to vector (jump) functions, by letting it act componentwise on its argument.

Introduce also the function $\chi^1 : Y \to \boldsymbol{R}^N$ defined by

$$\alpha \chi_h^1 = \mathcal{T}\left(\sigma_{\text{out}}(\boldsymbol{e}_h - \nabla_y \chi_h^{\mathcal{D}}) \cdot \nu\right). \tag{4.33}$$

Straightforward calculations show that $u_1$ may be written in the form

$$u_1(x, y, t) = -\chi^{\mathcal{D}}(y) \cdot \nabla_x u_0(x, t) + \mathcal{T}(S_1(x, \cdot))(y, t)$$
$$- \int_0^t \nabla_x u_0(x, \tau) \cdot \chi^1(y, t - \tau) \, \mathrm{d}\tau. \tag{4.34}$$

The term $u_2$ in the expansion of $u_\varepsilon$ satisfies

$$\begin{cases} \mathcal{P}_2[u_2]; \\ \sigma_{\text{out}} \nabla_y u_2^{\text{out}} \cdot \nu + \sigma_{\text{out}} \nabla_x u_1^{\text{out}} \cdot \nu = \alpha \dfrac{\partial [u_2]}{\partial t}, \quad [u_2]_{|t=0} = S_2(x, y), \quad \text{on } \Gamma. \end{cases}$$

Let us find the solvability conditions for this problem. Integrating by parts the partial differential equations solved by $u_2$, both in $E_{\text{int}}$ and in $E_{\text{out}}$, and adding the two contributions, we get

$$\left[\int_{E_{\text{int}}} + \int_{E_{\text{out}}}\right]\left\{\sigma \Delta_x u_0(x, t) + 2\sigma \frac{\partial^2 u_1}{\partial x_j \partial y_j}\right\} \mathrm{d}y$$
$$= \int_\Gamma \left\{\sigma_{\text{out}} \nabla_y u_2^{\text{out}} \cdot \nu - \sigma_{\text{int}} \nabla_y u_2^{\text{int}} \cdot \nu\right\} \mathrm{d}\sigma = \int_\Gamma [\sigma \nabla_y u_2 \cdot \nu] \, \mathrm{d}\sigma = -\int_\Gamma [\sigma \nabla_x u_1 \cdot \nu] \, \mathrm{d}\sigma.$$

Thus for $\sigma_0$ defined as in (4.18), we get

$$\sigma_0 \Delta_x u_0 = 2 \int_\Gamma [\sigma \nabla_x u_1 \cdot \nu] \, \mathrm{d}\sigma - \int_\Gamma [\sigma \nabla_x u_1 \cdot \nu] \, \mathrm{d}\sigma = \int_\Gamma [\sigma \nabla_x u_1 \cdot \nu] \, \mathrm{d}\sigma. \tag{4.35}$$

We use next the expansion (4.34); namely, we recall that, in it, only the terms $\mathcal{T}(\dots)$ and $\chi^1$ have a non zero jump across $\Gamma$. Thus we infer from the equality above

$$\sigma_0 \Delta_x u_0 = \int_\Gamma [\sigma \nabla_x u_1 \cdot \nu] \, \mathrm{d}\sigma = -\int_\Gamma [\sigma] \chi_h^{\mathcal{D}}(y) \nu_j \, \mathrm{d}\sigma u_{0 x_h x_j}(x, t)$$

$$+ \frac{\partial}{\partial x_j} \int_\Gamma [\sigma \mathcal{T}(S_1(x, \cdot))](y, t) \nu_j \, \mathrm{d}\sigma - \int_0^t u_{0 x_h x_j}(x, \tau) \int_\Gamma [\sigma \chi_h^1](y, t - \tau) \nu_j \, \mathrm{d}\sigma \, \mathrm{d}\tau.$$

We finally write the partial differential equation for $u_0$ in $\Omega \times (0, T)$ as

$$-\operatorname{div}\left((\sigma_0 I + A^{\mathcal{D}})\nabla_x u_0 + \int_0^t A^1(t - \tau)\nabla_x u_0(x, \tau) \, \mathrm{d}\tau - \mathcal{F}\right) = 0. \tag{4.36}$$

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The constant matrix $A^{\mathcal{D}}$ is the same as in the limiting equation of the Diffraction problem (case $k = +\infty$; see (4.18)). The matrix $A^1$ is defined by

$$A^1(t) = \int_{\Gamma} [\sigma]\nu \otimes \chi^1(y, t)\, \mathrm{d}\sigma\,, \tag{4.37}$$

The matrices $A^{\mathcal{D}}$ and $A^1$ are symmetric, and $\sigma_0 I + A^{\mathcal{D}}$ is positive definite [2]. The vector $\mathcal{F}$ is defined by

$$\mathcal{F} = \int_{\Gamma} [\sigma \mathcal{T}(S_1(x, \cdot))](y, t)\nu\, \mathrm{d}\sigma\,. \tag{4.38}$$

4.5. **The case $k = 0$: a threshold case.** According to equation (4.6), we let $S_0 \equiv 0$ in (4.12).

The problem for $u_0$ is given by

$$\begin{cases} \mathcal{P}_0[u_0]\,; \\ \sigma_{\mathrm{out}}\nabla_y u_0^{\mathrm{out}} \cdot \nu = 0\,, \quad [u_0]_{|t=0} = 0\,, \quad \text{on } \Gamma. \end{cases}$$

It follows that $u_0(x, \cdot, t)$ solves two independent homogeneous Neumann problems in $E_{\mathrm{int}}$ and $E_{\mathrm{out}}$, as in Subsection 4.2.2, so that $u_0(x, \cdot, t)|_{E_{\mathrm{int}}}$ and $u_0(x, \cdot, t)|_{E_{\mathrm{out}}}$ are independent of $y$, and (4.19) holds.

The problem for $u_1$ is given by

$$\begin{cases} \mathcal{P}_1[u_1]\,; \\ \sigma_{\mathrm{out}}\nabla_y u_1^{\mathrm{out}} \cdot \nu + \sigma_{\mathrm{out}}\nabla_x u_0^{\mathrm{out}} \cdot \nu = \alpha\dfrac{\partial[u_0]}{\partial t}\,, \quad [u_1]_{|t=0} = S_1(x, y)\,, \quad \text{on } \Gamma. \end{cases} \tag{4.39}$$

On integrating by parts the partial differential equation in $\mathcal{P}_1[u_1]$, and using also (4.19), we have

$$|\Gamma|\left(\alpha\frac{\partial}{\partial t}[u_0]\right) = \int_{\Gamma} \sigma_{\mathrm{out}}\nabla_y u_1^{\mathrm{out}} \cdot \nu\, \mathrm{d}\sigma + \int_{\Gamma} \sigma_{\mathrm{out}}\nabla_x u_0^{\mathrm{out}} \cdot \nu\, \mathrm{d}\sigma = 0\,. \tag{4.40}$$

Hence, recalling $[u_0]_{|t=0} = 0$, we have $[u_0] = 0$ on $\Gamma$ for all $t$. As a consequence, in (4.19) the two components are actually equal, and $u_0 = u_0(x, t)$. Then, the problem for $u_1$ may be written as

$$\begin{cases} \mathcal{P}_1[u_1]\,; \\ \sigma_{\mathrm{out}}\nabla_y u_1^{\mathrm{out}} \cdot \nu + \sigma_{\mathrm{out}}\nabla_x u_0^{\mathrm{out}} \cdot \nu = 0\,, \quad [u_1]_{|t=0} = S_1(x, y)\,, \quad \text{on } \Gamma. \end{cases} \tag{4.41}$$

Apart from the initial data $S_1$, this is the same problem satisfied by $u_1$ in the Neumann problem ($k = -\infty$). Then we may represent $u_1$ as in (4.20), where $u_0^{\mathrm{int}} = u_0^{\mathrm{out}} =: u_0$ and $\chi^{\mathcal{N}}$ was defined in (4.21)–(4.23). The initial data for $[u_1]$ can not, in general, be satisfied, suggesting the onset of an initial layer as $\varepsilon \to 0$. One condition on the two functions $\widetilde{u}_1^{\mathrm{int}}$ and $\widetilde{u}_1^{\mathrm{out}}$ follows from the requirement that the integral average of $u_1$ is zero. A second condition will be found in (4.44) below.

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The problem for $u_2$ is given by

$$
\begin{cases}
\mathcal{P}_2[u_2]\,; \\
\sigma_{\mathrm{out}}\nabla_y u_2^{\mathrm{out}}\cdot\nu + \sigma_{\mathrm{out}}\nabla_x u_1^{\mathrm{out}}\cdot\nu = \alpha\dfrac{\partial[u_1]}{\partial t}\,, \quad [u_2]_{|t=0} = S_2(x,y)\,, \quad \text{on } \Gamma.
\end{cases}
$$

Since at this stage the terms containing $u_0$ and $u_1$ in the problem for $u_2$ are regarded as known, this reduces to two independent Neumann problems in $E_{\mathrm{int}}$ and in $E_{\mathrm{out}}$, respectively, so that two independent compatibility conditions will be enforced in the following.

First, integrating by parts the partial differential equation for $u_2$ both in $E_{\mathrm{int}}$ and in $E_{\mathrm{out}}$, adding the two contributions, and using (4.20), we obtain

$$
-\operatorname{div}\left((\sigma_0 I + A^{\mathcal{N}})\nabla_x u_0\right) = 0\,. \tag{4.42}
$$

The matrix $A^{\mathcal{N}}$ is defined as

$$
A^{\mathcal{N}} = \int_{\Gamma} [\sigma\nu \otimes \chi^{\mathcal{N}}]\,\mathrm{d}\sigma\,, \tag{4.43}
$$

that is, as matrix $A^{\mathcal{D}}$, with $\chi^{\mathcal{D}}$ formally substituted with $\chi^{\mathcal{N}}$. Note instead the definition (4.26) of the limiting diffusion matrices in the Neumann problem. Thus, we may see the case $k = 0$ as a threshold case in our sequence of problems: the insulation provided by the interface is sufficient to modify the limiting diffusion matrix, with respect to the Diffraction case, but it is not strong enough to force the existence of two different limit phases, as in the Neumann problem.

Second, on integrating the evolution equation for $u_2$ over $\Gamma$, and using again the elliptic differential equation for $u_2$, we obtain the second compatibility condition

$$
\begin{aligned}
\int_{\Gamma}&\left(-\alpha[\chi^{\mathcal{N}}]\cdot\frac{\partial}{\partial t}\nabla_x u_0\right)\mathrm{d}\sigma + |\Gamma|\left(\alpha\frac{\partial}{\partial t}[\widetilde{u}_1]\right) \\
&= \sigma_{\mathrm{out}}|E_{\mathrm{out}}|\,\Delta_x u_0 - \int_{\Gamma}\sigma_{\mathrm{out}}\nabla_x u_1^{\mathrm{out}}\cdot\nu\,\mathrm{d}\sigma \\
&= \sigma_{\mathrm{out}}|E_{\mathrm{out}}|\,\Delta_x u_0 + \int_{\Gamma}\sigma_{\mathrm{out}}\nabla_x(\chi_{\mathrm{out}}^{\mathcal{N}}\cdot\nabla_x u_0)\cdot\nu\,\mathrm{d}\sigma - \int_{\Gamma}\sigma_{\mathrm{out}}\nabla_x\widetilde{u}_1^{\mathrm{out}}\cdot\nu\,\mathrm{d}\sigma \\
&= \operatorname{div}\left((\sigma_{\mathrm{out}}|E_{\mathrm{out}}|I + A_{\mathrm{out}}^{\mathcal{N}})\nabla_x u_0\right),
\end{aligned} \tag{4.44}
$$

where the matrix $A_{\mathrm{out}}^{\mathcal{N}}$ was defined in (4.26). Equation (4.44) governs the evolution of $[\widetilde{u}_1]$ on $\Gamma$. The choice of the initial data for $[\widetilde{u}_1]$ is connected to the appearence of an initial layer. We observe that, differently from the case $k = -1$, equation (4.44) is not used to derive the homogenized problem for $u_0$.

4.6. **The case $k = -1$: a limit degenerate parabolic system.** Recalling (4.6), we remark that no one of the terms $S_0, S_1, \ldots,$ needs to vanish.

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The problem for $u_0$ is

$$\begin{cases} \boldsymbol{\mathcal{P}}_0[u_0]\,; \\ \sigma_{\text{out}}\nabla_y u_0^{\text{out}} \cdot \nu = 0\,, \quad [u_0]_{|t=0} = S_0\,, \quad \text{on } \Gamma. \end{cases} \tag{4.45}$$

As a consequence, $u_0$ is independent of $y$, separately in $E_{\text{int}}$ and in $E_{\text{out}}$, so that (4.19) is in force. It follows that the initial data $[u_0]_{|t=0} = S_0$ must be independent of $y$, or an initial layer must occur.

The term $u_1$ solves the same problem (4.41) as in case $k = 0$. Moreover, the representation (4.20) for $u_1$ is still valid. The argument exploited in Subsection 4.5 to infer that $[u_0] = 0$ for all times can not be repeated in the present case, since it relies on the formulation (4.39) which is no longer valid. Indeed, $u_\varepsilon$ does approach in the case $k \leq -1$ two different components $u_0^{\text{int}}$ and $u_0^{\text{out}}$, as $\varepsilon \to 0$ (clearly, in the two different compartments $\Omega_{\text{int}}^\varepsilon$ and $\Omega_{\text{out}}^\varepsilon$, respectively).

A further difference with the case $k = 0$ appears in the problem for $u_2$, which is now given by

$$\begin{cases} \boldsymbol{\mathcal{P}}_2[u_2]\,; \\ \sigma_{\text{out}}\nabla_y u_2^{\text{out}} \cdot \nu + \sigma_{\text{out}}\nabla_x u_1^{\text{out}} \cdot \nu = \alpha\dfrac{\partial[u_0]}{\partial t}\,, \quad [u_2]_{|t=0} = S_2(x,y)\,, \quad \text{on } \Gamma. \end{cases} \tag{4.46}$$

However, as in the case $k = 0$, this scheme reduces to two independent Neumann problems for $u_2$, in $E_{\text{int}}$ and in $E_{\text{out}}$, so that we have to enforce two independent compatibility conditions. Calculations similar to the ones outlined above, starting from (4.46), yield

$$-\operatorname{div}\left((\sigma_{\text{int}}|E_{\text{int}}|I + A_{\text{int}}^{\mathcal{N}})\nabla_x u_0^{\text{int}} + (\sigma_{\text{out}}|E_{\text{out}}|I + A_{\text{out}}^{\mathcal{N}})\nabla_x u_0^{\text{out}}\right) = 0\,. \tag{4.47}$$

The matrices $A_{\text{int}}^{\mathcal{N}}$ and $A_{\text{out}}^{\mathcal{N}}$ were defined in (4.26). This equation should be compared with the limiting equations (4.24)–(4.25) obtained for the Neumann problem $k = -\infty$: in that case, $u_0^{\text{int}}$ and $u_0^{\text{out}}$ solve two independent equations.

As in case $k = 0$, we get the second compatibility condition by integrating the evolution equation for $u_2$ over $\Gamma$, and using again the other equations of (4.46). We find

$$|\Gamma|\alpha\frac{\partial}{\partial t}(u_0^{\text{out}} - u_0^{\text{int}}) = \sigma_{\text{out}}|E_{\text{out}}|\,\Delta_x u_0^{\text{out}} - \int_\Gamma \sigma_{\text{out}}\nabla_x u_1^{\text{out}} \cdot \nu\,\mathrm{d}\sigma$$

$$= \sigma_{\text{out}}|E_{\text{out}}|\,\Delta_x u_0^{\text{out}} + \int_\Gamma \sigma_{\text{out}}\nabla_x(\chi_{\text{out}}^{\mathcal{N}} \cdot \nabla_x u_0^{\text{out}}) \cdot \nu\,\mathrm{d}\sigma - \int_\Gamma \sigma_{\text{out}}\nabla_x \tilde{u}_1^{\text{out}} \cdot \nu\,\mathrm{d}\sigma$$

$$= \operatorname{div}\left((\sigma_{\text{out}}|E_{\text{out}}|I + A_{\text{out}}^{\mathcal{N}})\nabla_x u_0^{\text{out}}\right)\,. \tag{4.48}$$

Equations (4.47) and (4.48) constitute a system of degenerate parabolic equations for the unknowns $u_0^{\text{int}}(x,t)$ and $u_0^{\text{out}}(x,t)$. They are complemented with the initial condition

$$u_0^{\text{out}}(x,0) - u_0^{\text{int}}(x,0) = S_0(x)\,, \qquad \text{on } \Omega\,, \tag{4.49}$$

which follows from the second of (4.45).

4.7. **The case $k \leq -2$: essentially the same as $k = -\infty$.** The expansion of $S_\varepsilon$ is the same as in the case $k = -1$.

The problems for $u_0$ and for $u_1$ are the same as in the case $k = -1$. Then, the representation (4.19) for $u_0$, and (4.20) for $u_1$ are still in force.

The problem for $u_2$ in this case reduces to

$$\begin{cases} \mathcal{P}_2[u_2]\,; \\ \sigma_{\text{out}} \nabla_y u_2^{\text{out}} \cdot \nu + \sigma_{\text{out}} \nabla_x u_1^{\text{out}} \cdot \nu = 0\,, \quad [u_2]_{|t=0} = S_2\,, \quad \text{on } \Gamma, \end{cases} \tag{4.50}$$

i.e., to the same scheme obtained in the Neumann problem $k = -\infty$. Therefore we obtain in the limit the two independent differential equations (4.24)–(4.25). In other words, the homogenization limit in the case $k \leq -2$ is the same as in the case $k = -\infty$.

## 5. Discussion

5.1. **$\varepsilon$-Scaling of the dielectric constant.** As remarked in the Introduction, we rescale in $\varepsilon$ the dielectric constant $B^\eta$ as follows

$$B^\eta = \frac{\eta \alpha}{\varepsilon^{k-1}}, \qquad k \in \mathbf{Z} \tag{5.1}$$

in order to derive the concentrated equations (4.1)–(4.4) from the equation (1.1).

We point out that different choices of $k$ keep different physical quantities constant in the homogenization limit $\varepsilon \to 0$. For instance, according to (5.1), the following physical quantities are preserved:

- the permeability of the dielectric phase, in the case $k = 1$;
- the capacity of the dielectric shell per unit of area, in the case $k = 0$;
- the total capacity of the intracellular phase with respect to the extracellular phase, or equivalently the total capacity per unit of volume, in the case $k = -1$.

5.2. **Comparison among the models.** Following the analysis in the previous sections, we obtained five models, which can be classified according to different criteria. As a first criterium, we could divide the models corresponding to $k \leq -1$ from the ones corresponding to $k \geq 0$. The former are known as bidomain models since they involve two unknown macroscopic functions $u_0^{\text{int}}$ and $u_0^{\text{out}}$; while the latter involve only one macroscopic function $u_0$ and henceforth they are known as monodomain models. A similar classification can be found in [4] in the case of static interface conditions.

A second criterium takes into account the mathematical structure of the involved cell functions and divides the models corresponding to $k \leq 0$ from the ones corresponding to $k \geq 1$. The former involve the cell function $\chi^{\mathcal{D}}$, while in the latter we find $\chi^{\mathcal{N}}$. Accordingly, we note that in the first class the model $k = 1$ is the most general, since it formally gives model $k \geq 2$ for $\alpha \to +\infty$; while in the second class the most general model is $k = -1$, since it formally gives models $k = 0$ and $k \leq -2$ on letting $\alpha \to +\infty$ or $\alpha \to 0$, respectively. Because of the different nature of $\chi^{\mathcal{D}}$ and $\chi^{\mathcal{N}}$ no continuous passage from one class to the other seems possible.

The last criterium, which is the most physical one, corresponds to single out the models which preserve in the limit the membrane properties (i.e., the constant $\alpha$). They are just two: the model $k = 1$ and the model $k = -1$, which, remarkably, were

the most general of their own class in the previous classification. These models are also the only ones able to take into account the influence or the initial data.

5.3. **Geometry of the problem.** In the cases $k \geq 1$, the matrix $\sigma_0 I + A^{\mathcal{D}}$, obtained by means of the cell function $\chi^{\mathcal{D}}$ and involved into the homogenized equations (4.17) and (4.36) is positive definite for any geometry (see [2]). On the contrary, for $k \leq 0$, the matrices involved into the limiting problems depend on the cell function $\chi^{\mathcal{N}}$ and may degenerate depending on the geometry of the media. Indeed, if either one of the phases $\Omega_{\mathrm{int}}^{\varepsilon}$ or $\Omega_{\mathrm{out}}^{\varepsilon}$ is not connected, it can be easily verified that the matrix $\sigma_{\mathrm{int}}|E_{\mathrm{int}}|I + A_{\mathrm{int}}^{\mathcal{N}}$ or $\sigma_{\mathrm{out}}|E_{\mathrm{out}}|I + A_{\mathrm{out}}^{\mathcal{N}}$, respectively, may degenerate or even vanish. For example, in the geometrical setting displayed in Figure 1, we have $\sigma_{\mathrm{int}}|E_{\mathrm{int}}|I + A_{\mathrm{int}}^{\mathcal{N}} = 0$. In a layered material, the two limiting diffusion matrices would degenerate along the direction orthogonal to the layers.

However, in the case $k = -1$, the two matrices cited above appear simultaneously in the homogenized equation (4.47). Hence the limiting problem is meaningful if at least one of the two matrices above does not vanish. Analogously, in the case $k = 0$, the homogenized equation (4.42) is not trivial when at least one of the two matrices is not zero, since their sum gives the matrix defining the elliptic operator in (4.42). On the other hand, in the cases $k \leq -2$, both matrices in (4.24) and (4.25) may not degenerate in order to obtain a meaningful limit problem. This is the case, for instance, when both phases are connected.

5.4. **Some simplified cases.** Many papers in the literature, devoted to the study of reconstruction of the interior of the human body from exterior electrical measurements, rely on the study of the Dirichlet-Neumann map for an elliptic equation. This fact corresponds, in the Maxwell equation (1.1), to set $\mathcal{B} = 0$ and to have $\mathcal{A}$ different from zero in the membrane. The choice $\mathcal{B} = 0$ amounts to neglect the effect of the displacement currents. Nevertheless, experimental data show that, for sufficiently high frequency range, displacement currents play a relevant role; i.e., memory effects appear (see [7]). This justify the study of models (4.36) (see [1] and [2]) and (4.47) (see [8], [11] and [6]).

However, taking into account, for instance, problem (4.36), it is worthwhile noticing that, for some simple geometries, it reduces to the study of an equation for the Laplacian. This occurs, for example, when the periodicity cell is invariant with respect to rotations of $\pi/2$, since the relevant matrix $A^{\mathcal{D}}$ and $A^1(t)$ are scalar multiples of the identity, which we denote by $a^{\mathcal{D}}$ and $a^1(t)$, respectively. In this case, assigning the current flux $h$ on the boundary corresponds to the equation

$$-(\sigma_0 + a^{\mathcal{D}})\frac{\partial u_0}{\partial n} - \int_0^t a^1(t - \tau)\frac{\partial u_0}{\partial n}(x, \tau)\, \mathrm{d}\tau = h(x, t)\,, \qquad (5.2)$$

which can be solved with respect $\frac{\partial u_0}{\partial n}$, thus obtaining a standard Neumann boundary condition. Accordingly, the study of the inverse problem reduces to the standard study of the Dirichlet-Neumann map, with a relevant physical difference; i.e., the value of the normal derivative is not given by $h(x, t)$ itself, but it can be obtained by means of a clever use of the Laplace transformation and its inverse applied to $h$

and $a^1$. This implies that the value of the normal derivative at time $t$ is influenced by the flux values at previous time and this fact in itself produces the appearance of physically relevant memory effects.

On the other hand, for general geometry, when we can not assume $\mathcal{B} = 0$ (as in the case of high frequency range) the study of the inverse problem for equation (4.36) is still an open problem.

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