# A QUANTITATIVE ISOPERIMETRIC INEQUALITY FOR FRACTIONAL PERIMETERS


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**Abstract.** Recently Frank & Seiringer have shown an isoperimetric inequality for nonlocal perimeter functionals arising from Sobolev seminorms of fractional order. This isoperimetric inequality is improved here in a quantitative form.


## 1. Introduction

Isoperimetric inequalities play a crucial role in many areas of mathematics such as geometry, linear and nonlinear PDEs, or probability theory. In the Euclidean setting, it states that among all sets of prescribed measure, balls have the least perimeter. More precisely, for any Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure,

$$N|B|^{1/N}|E|^{(N-1)/N} \leq P(E)\,, \tag{1.1}$$

where $B$ denotes the unit ball of $\mathbb{R}^N$ centered at the origin. Here $P(E)$ denotes the distributional perimeter of $E$ which coincides with the $(N-1)$-dimensional measure of $\partial E$ when $E$ has a (piecewise) smooth boundary. It is a well kown fact that inequality (1.1) is strict unless $E$ is a ball. Here the natural framework for studying the isoperimetric inequality is the theory of sets of finite perimeter. We briefly recall that a Borel set $E$ of finite Lebesgue measure is said to be of finite perimeter if its characteristic function $\chi_E$ belongs to $BV(\mathbb{R}^N)$, and then $P(E)$ is given by the total variation of the distributional derivative of $\chi_E$. Throughout this paper, we shall refer to the monograph [4] for the basic properties of sets of finite perimeter.

The *isoperimetric deficit* of a set $E$ of finite perimeter is defined as the scaling and translation invariant quantity

$$D(E) := \frac{P(E) - P(B_r)}{P(B_r)}\,,$$

where $B_r := rB$ is the ball having the same measure as $E$, *i.e.*, $r^N|B| = |E|$. By the characterization of the equality cases in (1.1), the isoperimetric inequality rewrites $D(E) \geq 0$, and $D(E) = 0$ if and only if $E$ is a translation of $B_r$. Hence the isoperimetric deficit measures in some sense how far is a set from being ball. Finding a quantitative version of (1.1) consists in proving that the isoperimetric deficit controls a more usual notion of "distance from the familly of the balls". To this aim is introduced the so-called *Fraenkel asymmetry* of the set $E$, and it is defined by

$$A(E) := \min\left\{ \frac{|E \triangle B_r(x)|}{|E|} : x \in \mathbb{R}^N \, , \; r^N|B| = |E| \right\} ,$$

where $B_r(x) := x + rB$, and $\triangle$ denotes the symmetric difference between sets. Note that asymmetry is also invariant under scaling and translations. We then look for a positive constant $C_N$ depending only on the dimension, and an exponent $\alpha > 0$ such that $A(E) \leq C_N\big(D(E)\big)^\alpha$, which can be rewritten as a quantitative form of (1.1),

$$P(E) \geq \left(1 + \left(\frac{A(E)}{C_N}\right)^{1/\alpha}\right) N|B|^{1/N}|E|^{(N-1)/N} .$$

We shall not attempt here to sketch the history of this problem, but simply refer to the recent paper by Fusco, Maggi, and Pratelli [17] (and references therein) where this inequality has been first proved with the optimal exponent $\alpha = 1/2$, and to Figalli, Maggi, and Pratelli [14] for anisotropic perimeter functionals (see also [12], and [19] for a survey).

The main goal of this paper is to prove a quantitative isoperimetric type inequality for nonlocal perimeter functionals arising from Sobolev seminorms of fractional order. First, let us introduce what we call the fractional $s$-perimeter of a set. For $s \in (0,1)$ and a Borel set $E \subset \mathbb{R}^N$, $N \geq 1$, we define the fractional $s$-perimeter of $E$ by

$$P_s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}}\, dxdy \, .$$

If $P_s(E) < \infty$, we observe that

$$P_s(E) = \frac{1}{2}[\chi_E]^p_{W^{\sigma,p}(\mathbb{R}^N)} , \tag{1.2}$$

for $p \geq 1$ and $\sigma p = s$, where $[\cdot]_{W^{\sigma,p}(\mathbb{R}^N)}$ denotes the Gagliardo $W^{\sigma,p}$-seminorm and $\chi_E$ the characteristic function of $E$. The functional $P_s(E)$ can be thought as a $(N-s)$-dimensional perimeter in the sense that $P_s(\lambda E) = \lambda^{N-s}P_s(E)$ for any $\lambda > 0$ (compare to the $(N-1)$-homogeneity of the standard perimeter), and $P_s(E)$ can be finite even if the Hausdorff dimension of $\partial E$ is strictly greater than $N-1$ (see *e.g.* [22]). It is also immediately checked from the definition that $P_s(E) < \infty$ for any set $E \subset \mathbb{R}$ of finite perimeter and finite measure.

The fractional $s$-perimeter has already been investigated by several authors, specially by Caffarelli, Roquejoffre, and Savin [7] who studied regularity for sets of minimal $s$-perimeter (see also [9]). Besides the fact that fractional Sobolev seminorms are naturally related to fractional diffusion processes, one motivation for studying $s$-perimeters appears when we look at the asymptotic $s \uparrow 1$. It turns out that $s$-perimeters give an approximation of the standard perimeter, and more precisely, it follows from [13] (see also [5]) that for any (bounded) set $E$ of finite perimeter,

$$\lim_{s\uparrow 1}(1-s)P_s(E) = K_N P(E) , \tag{1.3}$$

where $K_N$ is a positive constant depending only on the dimension. Analysis by $\Gamma$-convergence as $s \uparrow 1$ of $s$-perimeter functionals can be found in [20], and [3]. Concerning the behavior of $P_s(E)$ as $s \downarrow 0$, we finally mention that

$$\lim_{s \downarrow 0} sP_s(E) = N|B|\,|E|\,, \tag{1.4}$$

for any set $E$ of finite measure and finite $s$-perimeter for every $s \in (0,1)$, as a consequence of [21, Theorem 3].

An isoperimetric type inequality for $s$-perimeters has been recently proved by Frank & Seiringer [16], and it states that for any Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure,

$$|E|^{(N-s)/N} \leq C_{N,s} P_s(E)\,, \tag{1.5}$$

for a suitable constant $C_{N,s}$, with equality holding if and only if $E$ is a ball. Actually, inequality (1.5) can be deduced from a symmetrization result due to Almgren & Lieb [2], and the cases of equality have been determined in [16]. The constant $C_{N,s}$ is given in [16, formula (4.2)], and we notice that $C_{N,s}$ is of order $(1-s)$ as $s \uparrow 1$, and of order $s$ as $s \downarrow 0$ by (1.3) and (1.4) respectively.

Inequality (1.5) is of course equivalent to say that

$$P_s(B_r) \leq P_s(E) \tag{1.6}$$

for any Borel set $E \subset \mathbb{R}^N$ such that $|E| = |B_r|$. In this paper we prove a quantitative version of inequality (1.6). To this purpose we introduce the following scaling and translation invariant quantity extending the standard isoperimetric deficit to the fractional setting. For a Borel set $E \subset \mathbb{R}^N$ of finite measure and $B_r$ such that $|E| = |B_r| > 0$, we define the *s-isoperimetric deficit* as

$$D_s(E) := \frac{P_s(E) - P_s(B_r)}{P_s(B_r)}\,.$$

We have the following result.

**Theorem 1.1.** *Let $N \geq 1$ and $s \in (0,1)$. There exists a constant $\mathcal{C}_{N,s}$ depending only on $N$ and $s$ such that for any Borel set $E \subset \mathbb{R}^N$ with $0 < |E| < \infty$,*

$$A(E) \leq \mathcal{C}_{N,s} D_s(E)^{s/4}\,. \tag{1.7}$$

We emphasize that, as in the standard perimeter case, the exponent appearing in (1.7) does not depend on the dimension. However we strongly suspect that the optimal exponent should be $1/2$ as for the classical isoperimetric inequality (see [17,14,12]). The dependence on $s$ of the constant $\mathcal{C}_{N,s}$ remains unclear since our method does not yield any precise control as $s \uparrow 1$ or $s \downarrow 0$, but some information can be deduced from (1.3) and (1.4).

We conclude with a few comments on the proof of Theorem 1.1. The key tool used here is a local representation due to Caffarelli & Silvestre [8] of the $H^{s/2}$-seminorm. It allows us to rewrite the $s$-perimeter $P_s(E)$ as a Dirichlet type energy of a suitable (inhomogeneous) harmonic extension of the characteristic function of $E$ in $\mathbb{R}^{N+1}_+$ (see Remark 2.2). With such a representation in hands, we can adapt some symmetrization techniques developed in [17,11].

## 2. Preliminary results

Throughout the paper, given $s \in (0,1)$, we shall consider functions belonging to the following weighted Sobolev space

$$\mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+) := \left\{ u \in W^{1,1}_{\rm loc}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla u|^2 \, dxdz < +\infty \right\},$$

where $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0,+\infty)$, and $\partial\mathbb{R}^{N+1}_+ \simeq \mathbb{R}^N$. It can be easily checked that each $u \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ has a trace belonging to $L^2_{\rm loc}(\mathbb{R}^N)$ that we shall denote by $u(\cdot,0)$.

In the proof of Theorem 1.1, a key point is given by the following extension lemma which is a consequence of a recent result by Caffarelli & Silvestre [8, Formula (3.7)], and a well known representation of the $H^{s/2}$-seminorm in Fourier space (see *e.g.* [15, Lemma 3.1]). Note that for our purposes we restrict ourselves to $s \in (0,1)$, but Lemma 2.1 actually holds for any $s \in (0,2)$.

**Lemma 2.1.** *Let $s \in (0,1)$. There exists a constant $\gamma_{N,s} > 0$ such that for any function $g \in H^{s/2}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^2}{|x-y|^{N+s}} \, dxdy = \gamma_{N,s} \int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla u|^2 \, dxdz \,, \tag{2.1}$$

*where $u$ is the unique solution in $\mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ of*

$$\begin{cases} \operatorname{div}(z^{1-s}\nabla u) = 0 & \text{in } \mathbb{R}^{N+1}_+ \,, \\ u = g & \text{on } \mathbb{R}^N \,. \end{cases}$$

*Moreover, $u$ is explicitely given by the Poisson formula,*

$$u(x,z) = \lambda_{N,s} \int_{\mathbb{R}^N} \frac{z^s g(y)}{(|x-y|^2 + z^2)^{(N+s)/2}} \, dy \,, \tag{2.2}$$

*for a constant $\lambda_{N,s}$ only depending on $N$ and $s$, and $u$ can be characterized as the unique minimizer of*

$$\int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla v|^2 \, dxdz \,,$$

*among all functions $v \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ satisfying $v(\cdot,0) = g$.*

**Remark 2.1.** The constant $\lambda_{N,s}$ in (2.2) is precisely given by (see *e.g.* [1])

$$\lambda_{N,s} = \left( \int_{\mathbb{R}^N} \frac{1}{(|y|^2+1)^{(N+s)/2}} \, dy \right)^{-1} = \frac{\Gamma\big((N+s)/2\big)}{\pi^{N/2}\,\Gamma(s/2)} \,, \tag{2.3}$$

where $\Gamma$ is Euler's Gamma function.

**Remark 2.2.** As a consequence of Lemma 2.1 and (1.2), for any Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure and finite $s$-perimeter, one has

$$P_s(E) = \frac{\gamma_{N,s}}{2} \int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla u_E|^2 \, dxdz \,,$$

where $u_E$ is the unique solution in $\mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ of

$$\begin{cases} \operatorname{div}(z^{1-s}\nabla u_E) = 0 & \text{in } \mathbb{R}^{N+1}_+ \,, \\ u_E = \chi_E & \text{on } \mathbb{R}^N \,. \end{cases} \tag{2.4}$$

Note that formula (2.2) yields $u_E \in C^\infty(\mathbb{R}^{N+1}_+)$, $0 \le u_E \le 1$, and $u_E(x,z) \to 0$ as $|x| \to \infty$ for every $z > 0$. In particular, for every $z > 0$ and $t > 0$, the set $\{u_E(\cdot, z) > t\}$ is bounded in $\mathbb{R}^N$. In addition, it follows from (2.2)-(2.3) that $u_E(x,z) \to 1$ as $z \downarrow 0$ at every point $x$ of density 1 of $E$, and $u_E(x,z) \to 0$ as $z \downarrow 0$ at every point $x$ of density 0.

The proof of Theorem 1.1 also makes use of symmetric rearrangements, and we need to recall some well known facts. For a measurable function $g : \mathbb{R}^N \to [0,\infty)$ such that for all $t > 0$,

$$\mu(t) := \big|\{g > t\}\big| < \infty\,, \tag{2.5}$$

the *symmetric rearrangement* $g^\sharp$ of $g$ is defined as the unique radially symmetric decreasing function on $\mathbb{R}^N$ satisfying

$$\big|\{g^\sharp > t\}\big| = \mu(t) \qquad \text{for all } t > 0\,.$$

It is well known that if $g \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ then also $g^\sharp \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover (see *e.g.* [10, Lemma 3.2 and (3.19)]) for a.e. $t > 0$,

$$\mu'(t) = -\int_{\{g^\sharp = t\}} \frac{1}{|\nabla g^\sharp|}\, d\mathcal{H}^{N-1} \le -\int_{\{g=t\}} \frac{1}{|\nabla g|}\, d\mathcal{H}^{N-1}\,. \tag{2.6}$$

Pólya–Szegö Inequality states that the Dirichlet integral of $g$ decreases under symmetric rearrangement. Next proposition gives a quantitative version of this inequality in the special case where $g$ is an $N$*-symmetric function*, *i.e.*, a function symmetric with respect to all coordinate hyperplanes (see [11, Theorem 3] for a similar result).

**Proposition 2.1.** *Let $N \ge 1$. There exists a positive constant $C_N$ such that for any nonnegative, $N$-symmetric function $g \in H^1(\mathbb{R}^N)$, one has*

$$\int_{\mathbb{R}^N} |g - g^\sharp|\, dx \le C_N |\mathrm{supp}\, g|^{\frac{N+2}{2N}} \left( \int_{\mathbb{R}^N} |\nabla g|^2\, dx - \int_{\mathbb{R}^N} |\nabla g^\sharp|^2\, dx \right)^{1/2}.$$

**Proof.** Without loss of generality we may assume that $|\mathrm{supp}\, g| < +\infty$ so that (2.5) holds for all $t > 0$. By the coarea formula, Hölder's inequality, and (2.6) we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla g|^2\, dx &= \int_0^\infty dt \int_{\{g=t\}} |\nabla g|\, d\mathcal{H}^{N-1} \\ &\ge \int_0^\infty \frac{\big(\mathcal{H}^{N-1}(\{g=t\})\big)^2}{\int_{\{g=t\}} \frac{1}{|\nabla g|}\, d\mathcal{H}^{N-1}}\, dt \ge \int_0^\infty \frac{\big(\mathcal{H}^{N-1}(\{g=t\})\big)^2}{-\mu'(t)}\, dt\,. \end{aligned} \tag{2.7}$$

Since $|\nabla g^\sharp|$ is constant on $\{g^\sharp = t\}$ for a.e. $t > 0$, we obtain in the same way,

$$\int_{\mathbb{R}^N} |\nabla g^\sharp|^2\, dx = \int_0^\infty \frac{\big(\mathcal{H}^{N-1}\big(\{g^\sharp = t\}\big)\big)^2}{-\mu'(t)}\, dt\,. \tag{2.8}$$

Recalling that $P\big(\{g > t\}\big) = \mathcal{H}^{N-1}\big(\{g = t\}\big)$ for a.e. $t > 0$, and that $\{g^\sharp > t\}$ is a ball, we infer from (2.7), (2.8), and the classical isoperimetric inequality that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla g|^2 - |\nabla g^\sharp|^2\, dx &\ge \int_0^\infty \frac{P^2\big(\{g>t\}\big) - P^2\big(\{g^\sharp > t\}\big)}{-\mu'(t)}\, dt \\ &\ge 2\int_0^\infty \frac{P\big(\{g>t\}\big) - P\big(\{g^\sharp > t\}\big)}{P\big(\{g^\sharp > t\}\big)} \cdot \frac{P^2\big(\{g^\sharp > t\}\big)}{-\mu'(t)}\, dt\,. \end{aligned} \tag{2.9}$$

Assume now that $N \geq 2$. Since $\{g > t\}$ is an $N$-symmetric set, and $\{g^\sharp > t\}$ is the ball with the same measure centered at the origin, the quantitative isoperimetric inequality proved in [17] and Lemma 2.2 below yield

$$\frac{P\big(\{g > t\}\big) - P\big(\{g^\sharp > t\}\big)}{P\big(\{g^\sharp > t\}\big)} \geq C A\left(\{g > t\}\right)^2 \geq \frac{C}{9}\left(\frac{\big|\{g > t\}\triangle\{g^\sharp > t\}\big|}{|\{g^\sharp > t\}|}\right)^2 , \qquad (2.10)$$

where $C$ denotes a positive constant depending only on $N$. Next we notice that (2.10) is trivially true for $N = 1$.

Observing that for $N \geq 1$,

$$P\big(\{g^\sharp > t\}\big) = N|B|^{1/N}\left|\{g^\sharp > t\}\right|^{\frac{N-1}{N}} = N|B|^{1/N}\mu(t)^{\frac{N-1}{N}} \quad \text{for all } 0 < t < \operatorname{ess\,sup} g\,,$$

and

$$\int_0^\infty \mu(t)^{2/N}(-\mu'(t))\,dt = |\operatorname{supp} g|^{\frac{N+2}{N}}\,,$$

we infer from (2.9) and (2.10) that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla g|^2 - |\nabla g^\sharp|^2\,dx &\geq C\int_0^\infty \frac{\big|\{g > t\}\triangle\{g^\sharp > t\}\big|^2}{\mu(t)^{2/N}(-\mu'(t))}\,dt \\ &\geq \frac{C}{\int_0^\infty \mu(t)^{2/N}(-\mu'(t))\,dt}\left(\int_0^\infty \big|\{g > t\}\triangle\{g^\sharp > t\}\big|\,dt\right)^2 \\ &\geq \frac{C}{|\operatorname{supp} g|^{\frac{N+2}{N}}}\left(\int_0^\infty \big|\{g > t\}\triangle\{g^\sharp > t\}\big|\,dt\right)^2 , \qquad (2.11)\end{aligned}$$

where we have used Jensen's inequality, and $C$ still denotes a positive constant depending only on $N$, possibly changing from line to line.

Finally we estimate

$$\begin{aligned}\int_{\mathbb{R}^N} |g - g^\sharp|\,dx &= \int_{\mathbb{R}^N}\left|\int_0^\infty \chi_{\{g>t\}}(x) - \chi_{\{g^\sharp>t\}}(x)\,dt\right|\,dx \\ &\leq \int_0^\infty dt\int_{\mathbb{R}^N}\big|\chi_{\{g>t\}}(x) - \chi_{\{g^\sharp>t\}}(x)\big|\,dx = \int_0^\infty \big|\{g > t\}\triangle\{g^\sharp > t\}\big|\,dt\,, \quad (2.12)\end{aligned}$$

and the conclusion follows gathering (2.11) and (2.12). □

In the proof of Proposition 2.1, we have used the following simple lemma which is proved in [19, Lemma 5.2].

**Lemma 2.2.** *Let $E \subset \mathbb{R}^N$ be an $N$-symmetric Borel set of finite Lebesgue measure, with $|E| = |B_r|$. Then,*

$$A(E) \leq \frac{|E\triangle B_r|}{|B_r|} \leq 3A(E)\,.$$

We continue by showing that the Dirichlet type energy in (2.1) decreases under "horizontal" symmetric rearrangement. More precisely, we have the following result.

**Lemma 2.3.** *Let $s \in (0,1)$ and $u \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ be a nonnegative function such that $u(\cdot,z)$ is measurable and satisfies (2.5) for every $z \in (0,\infty)\setminus N$ for a (possibly empty) set $N$ of vanishing Lebesgue measure. Let $u^* : \mathbb{R}^{N+1}_+ \to \mathbb{R}_+$ be the function defined by*

$$u^*(x,z) := \left(u(\cdot,z)\right)^\sharp(x) \qquad \textit{for every } z \in (0,+\infty)\setminus N \textit{ and } x \in \mathbb{R}^N.$$

*Then* $u^* \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$,

$$\int_{\mathbb{R}^{N+1}_+} z^{1-s}|\partial_z u|^2 \, dxdz \geq \int_{\mathbb{R}^{N+1}_+} z^{1-s}|\partial_z u^*|^2 \, dxdz \,, \tag{2.13}$$

*and*

$$\int_{\mathbb{R}^{N+1}_+} z^{1-s}|\nabla_x u|^2 \, dxdz \geq \int_{\mathbb{R}^{N+1}_+} z^{1-s}|\nabla_x u^*|^2 \, dxdz \,. \tag{2.14}$$

**Proof.** First, observe that inequality (2.14) immediately follows by applying Pólya–Szegö inequality to each function $u(\cdot, z)$.

To prove (2.13) we need to recall that, given a nonnegative measurable function $g : \mathbb{R}^N \to [0, \infty)$ satisfying $|\{g > t\}| < \infty$ for all $t > 0$ and $\nu \in \mathbb{S}^{N-1}$, the Steiner rearrangement of $g$ in the direction $\nu$ is the unique function $g^\nu$ such that $\{g^\nu > t\}$ is the Steiner symmetral in the direction $\nu$ of $\{g > t\}$ for all $t > 0$.

Let $u \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ be a nonnegative function such that $u(\cdot, z) \in C^\infty_c(\mathbb{R}^N_+)$ for a.e. $z > 0$. Given a sequence of directions $\{\nu_k\} \subset \mathbb{S}^{N-1} \times \{0\}$ dense in $\mathbb{S}^{N-1} \times \{0\}$, we define by induction the following sequence of iterated Steiner rearrangements:

$$u_1 := u^{\nu_1} \,, \qquad u_{k+1} := (u_k)^{\nu_{k+1}} \,.$$

From the Pólya–Szegö inequality for Steiner symmetrization, we infer that the sequence $\{u_k\}$ is equibounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ and that for a.e. $z > 0$

$$\text{the sequence } \{u_k(\cdot, z)\} \text{ is equibounded in } W^{1,p}(\mathbb{R}^N) \text{ for all } p \geq 1. \tag{2.15}$$

Therefore, up to a (not relabeled) subsequence, $u_k$ converges weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ to a function $v$ which is symmetric with respect to all directions $\nu_k$. From (2.15) we have that for a.e. $z > 0$, $v(\cdot, z) \in W^{1,p}(\mathbb{R}^N)$ for all $p \geq 1$. Hence, by continuity, for all such $z$ it turns out that $v(\cdot, z)$ is symmetric with respect to all direction $\nu \in \mathbb{S}^{N-1} \times \{0\}$. By construction we have

$$\big|\{x \in \mathbb{R}^N : u_k(x, z) > t\}\big| = \big|\{x \in \mathbb{R}^N : u(x, z) > t\}\big| \,, \quad \text{for all } k \in \mathbb{N},\, z \in \mathbb{R},\, t > 0 \,,$$

which yields

$$\big|\{x \in \mathbb{R}^N : v(x, z) > t\}\big| = \big|\{x \in \mathbb{R}^N : u(x, z) > t\}\big| \,, \quad \text{for all a.e. } z,\, t > 0 \,.$$

Hence $v(\cdot, z) = (u(\cdot, z))^\sharp$ for a.e. $z > 0$. Since (see *e.g.* [6, Theorem 1]) for all $k \in \mathbb{N}$

$$\int_{\mathbb{R}^{N+1}_+} z^{1-s}|\partial_z u|^2 \, dxdz \geq \int_{\mathbb{R}^{N+1}_+} z^{1-s}|\partial_z u_k|^2 \, dxdz \,,$$

we deduce (2.13) by lower semicontinuity, letting $k \to \infty$ in the above inequality. The general case follows by approximating any nonnegative $u \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ as in the statement of the lemma with a sequence $\{u_n\} \subset \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ of nonnegative functions such that $u_n(\cdot, z) \in C^\infty_c(\mathbb{R}^N)$ for a.e. $z > 0$, $u_n \to u$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$, and

$$\int_{\mathbb{R}^{N+1}_+} z^{1-s}|\partial_z u_n|^2 \, dxdz \to \int_{\mathbb{R}^{N+1}_+} z^{1-s}|\partial_z u|^2 \, dxdz$$

as $n \to \infty$. $\square$

Applying the symmetrization procedure of Lemma 2.3 to the function $u_E$ defined by (2.4), we find that the exceptional set $N$ is empty by Remark 2.2, and that $u^*_E \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$. We now check that the trace of $u^*_E$ on $\mathbb{R}^N$ coincides with the characteristic function of the symmetrized set.

**Lemma 2.4.** *For any Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure, $u_E^* \in \mathcal{W}_s^{1,2}(\mathbb{R}_+^{N+1})$ and $u_E^*(\cdot, 0) = \chi_{B_r}$ with $r^N|B| = |E|$.*

**Proof.** The first assertion directly follows from Lemma 2.3. Fix now $\varepsilon > 0$ and $z > \varepsilon$. Then for every $x \in \mathbb{R}^N$ we may estimate

$$\begin{aligned} |u_E(x,z)| &\le |u_E(x,\varepsilon)| + \int_\varepsilon^z |\partial_z u_E(x,t)|\, dt \\ &\le |u_E(x,\varepsilon)| + \frac{z^{s/2}}{\sqrt{s}} \left( \int_\varepsilon^z t^{1-s} |\partial_z u_E(x,t)|^2\, dt \right)^{1/2} . \end{aligned}$$

Letting $\varepsilon \to 0+$ and recalling Remark 2.2, we deduce

$$|u_E(x,z)| \le \chi_E(x) + \frac{z^{s/2}}{\sqrt{s}} \left( \int_0^\infty t^{1-s} |\nabla u_E(x,t)|^2\, dt \right)^{1/2} \tag{2.16}$$

for a.e. $x \in \mathbb{R}^N$. Since the function on the right-hand side of (2.16) belongs to $L^2(\mathbb{R}^N)$, recalling again Remark 2.2, by the Dominated Convergence Theorem we have $u_E(\cdot, z) \to \chi_E$ in $L^2(\mathbb{R}^N)$ as $z \to 0^+$. Recall now that the map $f \mapsto f^\sharp$ is continuous in $L^2(\mathbb{R}^N)$. Hence, we may conclude that $u_E^*(\cdot, z) = (u_E(\cdot, z))^\sharp \to \chi_E^\sharp = \chi_{B_r}$ in $L^2(\mathbb{R}^N)$ as $z \to 0^+$, which finishes the proof of the lemma. □

## 3. Proof of Theorem 1.1

As in the case of the quantitative isoperimetric inequality for the standard perimeter proved in [17], the strategy consists in reducing the proof of (1.7) to the case of $N$-symmetric sets, *i.e.*, sets symmetric with respect to the $N$ coordinate hyperplanes. To this aim, we start by proving the following continuity lemma which is needed in the proof of Proposition 3.1.

**Lemma 3.1.** *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subset \mathbb{R}^N$ is a Borel set of finite Lebesgue measure satisfying $A(E) \le 3/2$ and $D_s(E) \le \delta$, then $A(E) \le \varepsilon$.*

**Proof.** We argue by contradiction assuming that there exists a sequence of Borel sets $E_n \subset \mathbb{R}^N$ such that $|E_n| = |B|$, $A(E_n) \le 3/2$, and

$$D_s(E_n) \to 0 \quad \text{with} \quad A(E_n) \ge \varepsilon\,,$$

for some $\varepsilon > 0$. We now apply the concentration-compactness Lemma I.1 of [18] to deduce that there exists a (not relabeled) subsequence $\{E_n\}$ such that the following three possible cases may occur:

*(i)* (up to translations) the sets $\{E_n\}$ have the property that for every $\delta > 0$ there exists $R_\delta > 0$ such that $|E_n \cap B_{R_\delta}| \ge |B| - \delta$ for all $n$;

*(ii)* for all $R > 0$, $\sup\limits_{x \in \mathbb{R}^N} |E_n \cap B_R(x)| \to 0$ as $n \to +\infty$;

*(iii)* there exists $\lambda \in (0, |B|)$ such that for all $\sigma > 0$, there exist $n_0 \in \mathbb{N}$, $E_n^1 \subset E_n$, and $E_n^2 \subset E_n$ satisfying for all $n \ge n_0$,

$$\begin{cases} \big|E_n \setminus (E_n^1 \cup E_n^2)\big| < \sigma\,, \quad \big||E_n^1| - \lambda\big| < \sigma\,, \quad \big||E_n^2| - (|B| - \lambda)\big| < \sigma\,, \\ \operatorname{dist}(E_n^1, E_n^2) \to +\infty\,. \end{cases}$$

Notice that though Lemma I.1 in [18] is stated in a seemingly different form, a quick inspection of the proof shows that it is in fact equivalent to the above statement.

We analyse separately the three cases.

*Case (i).* By the compact embedding of $H^{s/2}(\mathbb{R}^N)$ into $L^1_{\rm loc}(\mathbb{R}^N)$, up to a subsequence, there exists a set $F$ such that $\chi_{E_n} \to \chi_F$ in $L^1_{\rm loc}(\mathbb{R}^N)$. Hence, for every $\delta > 0$ there exists $R_\delta$ such that $|F \cap B_{R_\delta}| > |B| - \delta$, and thus $|F| = |B|$. By the assumption $D_s(E_n) \to 0$ and the lower semicontinuity of the $s$-perimeter, we infer that $D_s(F) = 0$, *i.e.*, $F$ is a ball of radius 1. Hence $A(E_n) \le |E_n \triangle F| \to 0$, which contradicts $A(E_n) \ge \varepsilon$ for all $n$.

*Case (ii).* We observe that this case can not occur since the assumption $A(E_n) \le 3/2$ implies that, up to suitable translation of each $E_n$, $|E_n \triangle B| \le 3|B|/2$. In particular we have $|E_n \cap B| \ge |B|/4$ for every $n$.

*Case (iii).* Let us fix an arbitrary constant $\eta > 0$. We introduce the regularized kernel $K_\eta : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ defined by

$$K_\eta(x,y) := \begin{cases} \eta^{-(N+s)} & \text{if } |x-y| < \eta\,, \\ \dfrac{1}{|x-y|^{N+s}} & \text{if } \eta \le |x-y| \le \eta^{-1}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We observe that

$$\begin{aligned} P_s(E_n) &\ge \int_{E_n}\int_{E_n^c} K_\eta(x,y)\,dxdy \\ &\ge \int_{E_n^1}\int_{E_n^c} K_\eta(x,y)\,dxdy + \int_{E_n^2}\int_{E_n^c} K_\eta(x,y)\,dxdy \\ &\ge \int_{E_n^1}\int_{(E_n^1)^c} K_\eta(x,y)\,dxdy + \int_{E_n^2}\int_{(E_n^2)^c} K_\eta(x,y)\,dxdy - \mathscr{R}_n^1 - \mathscr{R}_n^2\,, \end{aligned} \tag{3.1}$$

where for $i = 1, 2$,

$$\mathscr{R}_n^i := \int_{E_n^i}\int_{E_n \setminus E_n^i} K_\eta(x,y)\,dxdy\,.$$

Since $K_\eta(x,y) = 0$ whenever $|x-y| > \eta^{-1}$ and $\operatorname{dist}(E_n^1, E_n^2) \to +\infty$, we have for $n$ large enough

$$\mathscr{R}_n^i = \int_{E_n^i}\int_{E_n \setminus (E_n^1 \cup E_n^2)} K_\eta(x,y)\,dxdy \le \frac{|B|\sigma}{\eta^{N+s}}\,. \tag{3.2}$$

On the other hand, by Lemma A.2 in [16], we have

$$\begin{aligned} \int_{E_n^i}\int_{(E_n^i)^c} K_\eta(x,y)\,dxdy &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\chi_{E_n^i}(x) - \chi_{E_n^i}(y)| K_\eta(x,y)\,dxdy \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\chi_{B_{r_n^i}}(x) - \chi_{B_{r_n^i}}(y)| K_\eta(x,y)\,dxdy = \int_{B_{r_n^i}}\int_{(B_{r_n^i})^c} K_\eta(x,y)\,dxdy\,, \end{aligned}$$

where $(r_n^i)^N|B| = |E_n^i|$. From this last inequality, (3.1), (3.2), and the assumption $D_s(E_n) \to 0$, letting $n \to +\infty$ and then $\sigma \to 0$, we deduce that

$$P_s(B) \ge \int_{B_{r^1}}\int_{(B_{r^1})^c} K_\eta(x,y)\,dxdy + \int_{B_{r^2}}\int_{(B_{r^2})^c} K_\eta(x,y)\,dxdy\,, \tag{3.3}$$

where $(r^1)^N|B| = \lambda$ and $(r^2)^N|B| = |B| - \lambda$.

Finally, letting $\eta \to 0$ in (3.3), we conclude that

$$P_s(B) \geq P_s(B_{r^1}) + P_s(B_{r^2}) = \left[\left(\frac{\lambda}{|B|}\right)^{(N-s)/N} + \left(1 - \frac{\lambda}{|B|}\right)^{(N-s)/N}\right] P_s(B)\,,$$

which is impossible by strict concavity. $\square$

The following proposition shows that we can reduce the proof of (1.7) to the case of $N$-symmetric sets. Its proof is almost entirely similar to the proof of Theorem 2.1 in [17] except for a few changes indicated below.

**Proposition 3.1.** *There exists a constant $C_{N,s} > 0$, depending only on $N$ and $s$, such that for every Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure there is a $N$-symmetric Borel set $F \subset \mathbb{R}^N$ satisfying $|E| = |F|$, $A(E) \leq C_{N,s} A(F)$, and $D_s(F) \leq 2^N D_s(E)$.*

**Proof.** Without loss of generality, assume that $|E| = |B|$. Given a direction $\nu \in \mathbb{S}^{N-1}$ and $\alpha \in \mathbb{R}$, let us set $H_\nu^{\pm} = \{x \in \mathbb{R}^N : x \cdot \nu \gtrless \alpha\}$ be two half spaces orthogonal to $\nu$ such that $|E_\nu^{\pm}| = |E|/2$, where $E_\nu^{\pm} := E \cap H_\nu^{\pm}$. Up to a translation we may assume that $\alpha = 0$, *i.e.*, $H_\nu = \partial H_\nu^+$ passes through the origin. We also set

$$F_\nu^+ := E_\nu^+ \cup \mathcal{R}_\nu(E_\nu^+), \quad F_\nu^- := E_\nu^- \cup \mathcal{R}_\nu(E_\nu^-)\,,$$

where $\mathcal{R}_\nu : \mathbb{R}^N \to \mathbb{R}^N$ denotes the reflection with respect to $H_\nu$. We claim that

$$P_s(E) \geq \frac{P_s(F_\nu^+) + P_s(F_\nu^-)}{2}\,. \tag{3.4}$$

Indeed, let $u_E$ be the function defined in Remark 2.2. We write $u_E$ as the sum of $\chi_{H_\nu^+ \times \mathbb{R}_+} u_E^+$ and $\chi_{H_\nu^- \times \mathbb{R}_+} u_E^-$, where

$$u_E^{\pm}(x,z) := \begin{cases} u_E(x,z) & \text{if } x \in H_\nu^{\pm}\,, \\ u_E(\mathcal{R}_\nu(x), z) & \text{otherwise}\,. \end{cases}$$

It is well known that $u_E^{\pm} \in \mathcal{W}_s^{1,2}(\mathbb{R}_+^{N+1})$, and that

$$\int_{\mathbb{R}_+^{N+1}} z^{1-s} |\nabla u_E^{\pm}|^2 \, dxdz = 2 \int_{H_\nu^{\pm} \times \mathbb{R}_+} z^{1-s} |\nabla u_E|^2 \, dxdz\,.$$

Since $u_E^{\pm}(\cdot, 0) = \chi_{F_\nu^{\pm}}$ we infer from Lemma 2.1 that

$$\begin{aligned} \int_{\mathbb{R}_+^{N+1}} z^{1-s} |\nabla u_E|^2 \, dxdz &= \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} z^{1-s} \big(|\nabla u_E^+|^2 + |\nabla u_E^-|^2\big) \, dxdz \\ &\geq \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} z^{1-s} \big(|\nabla u_{F_\nu^+}|^2 + |\nabla u_{F_\nu^-}|^2\big) \, dxdz\,, \end{aligned}$$

from which (3.4) follows.

Next we observe that the case $N = 1$ immediately follows from (3.4). In fact, given the $E \subset \mathbb{R}$ and denoting by $F^1$ and $F^2$ the set obtained by the construction above, inequality (3.4) yields

$$\frac{D_s(F^1) + D_s(F^2)}{2} \leq D_s(E)\,,$$

while

$$A(E) \leq \frac{|E \triangle (-1,1)|}{2} \leq \frac{1}{2}\Big(\frac{|F^1 \triangle (-1,1)|}{2} + \frac{|F^2 \triangle (-1,1)|}{2}\Big) \leq \frac{3}{2}\big(A(F^1) + A(F^2)\big)\,,$$

where the last inequality follows by Lemma 2.2. Hence the conclusion follows by taking $F^i$ for which $A(F^i) \geq A(E)/3$.

For $N \geq 2$, we follow the strategy used in [17] which is based on the following claim (see [17, Lemma 2.5]):

*Claim: There exist two constants $C$ and $\delta$, depending only on $N, s$, such that, given $E$ with $|E| = |B|$ and $D_s(E) \leq \delta$, and two orthogonal vectors $\nu_1$ and $\nu_2$ in $\mathbb{S}^{N-1}$, one can find $i \in \{1,2\}$ and $j \in \{+,-\}$ with the property that*

$$A(E) \leq C A(F^j_{\nu_i}), \quad D_s(F^j_{\nu_i}) \leq 2D_s(E).$$

Let us observe that the claim is easily proved when $A(E) \geq 3/2$. Indeed, in this case any of the four possible choices $F^j_{\nu_i}$ would work. In fact, given $i \in \{1,2\}$ and $j \in \{+,-\}$, from (3.4) we have that $D_s(F^j_{\nu_i}) \leq 2D_s(E)$. Moreover, by the assumption $A(E) \geq 3/2$ we have $|E \cap B(x)| \leq |B|/4$ for all $x \in \mathbb{R}^N$, and thus $|F^j_{\nu_i} \cap B(x)| \leq |B|/2$ for all $x \in \mathbb{R}^N$. Therefore $A(F^j_{\nu_i}) \geq 1$.

If instead $A(E) \leq 3/2$, the proof of the claim follows exactly as the proof of Lemma 2.5 in [17] with the obvious observation that the continuity Lemma 2.3 in [17] must be replaced here by our Lemma 3.1, which holds since $A(E) \leq 3/2$ by assumption.

Once the claim above is proved, the argument used in the proof of Theorem 2.1 in [17] can be reproduced here word for word, thus leading to the conclusion. □

**Proof of Theorem 1.1.** Without loss of generality we may assume that $|E| = |B|$, and that $E$ has finite $s$-perimeter. Moreover, we may also assume that $D_s(E) \leq 1$, and that $E$ is an $N$-symmetric set thanks to Proposition 3.1 .

By Lemma 2.4 we have $u^*_E \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$ and $u^*_E = \chi_B$ on $\mathbb{R}^N$, and we infer from Lemma 2.1 and Remark 2.2 that

$$P_s(B) = \frac{\gamma_{N,s}}{2} \int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla u_B|^2 \, dxdz \leq \frac{\gamma_{N,s}}{2} \int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla u^*_E|^2 \, dxdz\,.$$

From Lemma 2.3 and Fubini's theorem, we also deduce that

$$\begin{aligned} \frac{2P_s(B)}{\gamma_{N,s}} D_s(E) &\geq \int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla u_E|^2 \, dxdz - \int_{\mathbb{R}^{N+1}_+} z^{1-s} |\nabla u^*_E|^2 \, dxdz \\ &\geq \int_{\mathbb{R}^{N+1}_+} z^{1-s} \big( |\nabla_x u_E|^2 - |\nabla_x u^*_E|^2 \big) \, dxdz \\ &\geq \int_0^1 z^{1-s} \left( \int_{\mathbb{R}^N} |\nabla_x u_E|^2 - |\nabla_x u^*_E|^2 \, dx \right) dz\,. \end{aligned} \tag{3.5}$$

Let us now set $v_E := \big(u_E - \frac{1}{2}\big)^+$. It is standard to check that $v_E \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$, $v_E(\cdot, 0) = \frac{1}{2}\chi_E$, and that

$$\nabla v_E = \chi_{\{u_E > 1/2\}} \nabla u_E \quad \text{a.e. in } \mathbb{R}^{N+1}_+\,.$$

By Remark 2.2, $v_E(\cdot, z)$ has compact support and belongs to $H^1(\mathbb{R}^N)$ for all $z > 0$, and

$$\nabla_x v_E(\cdot, z) = \chi_{\{u_E(\cdot,z) > 1/2\}} \nabla_x u_E(\cdot, z) \quad \text{a.e. in } \mathbb{R}^N\,. \tag{3.6}$$

Then we observe that $v^*_E = \big(u^*_E - \frac{1}{2}\big)^+$, so that $v^*_E \in \mathcal{W}^{1,2}_s(\mathbb{R}^{N+1}_+)$, $v^*_E(\cdot, 0) = \frac{1}{2}\chi_B$, and

$$\nabla v^*_E = \chi_{\{u^*_E > 1/2\}} \nabla u^*_E \quad \text{a.e. in } \mathbb{R}^{N+1}_+\,.$$

In addition, by the Pólya-Szegö inequality we have $v^*_E(\cdot, z) \in H^1(\mathbb{R}^N)$ for all $z > 0$, and

$$\nabla_x v^*_E(\cdot, z) = \chi_{\{u^*_E(\cdot,z) > 1/2\}} \nabla_x u^*_E(\cdot, z) \quad \text{a.e. in } \mathbb{R}^N\,. \tag{3.7}$$

Squaring both sides of (2.16), and integrating over $\mathbb{R}^N$, we infer

$$\int_{\mathbb{R}^N} |u_E(x,z)|^2\, dx \le 2|E| + \frac{2z^s}{s}\int_{\mathbb{R}^{N+1}_+} t^{1-s}\, |\nabla u_E(x,t)|^2\, dxdt$$
$$\le 2|B| + \frac{4}{s\gamma_{N,s}} P_s(E) \le 2|B| + \frac{8P_s(B)}{s\gamma_{N,s}} =: \beta(s)\,, \quad (3.8)$$

where we have used the fact that $D_s(E)\le 1$ in the last inequality. As a consequence, for all $z\in(0,1)$ we have by Chebyshev's inequality,

$$|\operatorname{supp} v_E(\cdot,z)| = \left|\left\{x\in\mathbb{R}^N\,:\, u_E(x,z)\ge\frac{1}{2}\right\}\right| \le 4\beta(s)\,. \quad (3.9)$$

Since the set $E$ is $N$-symmetric, it follows from (2.2) that $u_E$ and $v_E$ inherit the same symmetry with respect to $x$. Using Proposition 2.1 and (3.9), we may now estimate for all $z\in(0,1)$,

$$\int_{\mathbb{R}^N} |v_E(x,z) - v_E^*(x,z)|\, dx$$
$$\le c_N\beta(s)^{\frac{N+2}{2N}}\left(\int_{\mathbb{R}^N} |\nabla_x v_E(x,z)|^2 - |\nabla_x v_E^*(x,z)|^2\, dx\right)^{1/2}, \quad (3.10)$$

for a suitable constant $c_N>0$ depending only on $N$.

Next we claim that for all $z>0$,

$$\int_{\{u_E(\cdot,z)=t\}} |\nabla_x u_E(x,z)|\, d\mathcal{H}^{N-1}_x - \int_{\{u_E^*(\cdot,z)=t\}} |\nabla_x u_E^*(x,z)|\, d\mathcal{H}^{N-1}_x \ge 0 \quad \text{for a.e. } t>0\,. \quad (3.11)$$

Indeed, given $z>0$, we may argue as in the proof of (2.7) to obtain for a.e. $t>0$,

$$\int_{\{u_E(\cdot,z)=t\}} |\nabla_x u_E(x,z)|\, d\mathcal{H}^{N-1}_x \ge \frac{\big(\mathcal{H}^{N-1}(\{u_E(\cdot,z)=t\})\big)^2}{\int_{\{u_E(\cdot,z)=t\}}\frac{1}{|\nabla_x u_E(x,z)|}\, d\mathcal{H}^{N-1}_x}$$
$$\ge \frac{\big(\mathcal{H}^{N-1}(\{u_E^*(\cdot,z)=t\})\big)^2}{\int_{\{u_E^*(\cdot,z)=t\}}\frac{1}{|\nabla_x u_E^*(x,z)|}\, d\mathcal{H}^{N-1}_x} = \int_{\{u_E^*(\cdot,z)=t\}} |\nabla_x u_E^*(x,z)|\, d\mathcal{H}^{N-1}_x\,,$$

using (2.6), the standard isoperimetric inequality, and the fact that $|\nabla_x u_E^*(\cdot,z)|$ is constant on $\{u_E^*(\cdot,z)=t\}$.

Then we derive from (3.11), (3.6), (3.7), and the coarea formula that for all $z>0$,

$$\int_{\mathbb{R}^N} |\nabla_x u_E(x,z)|^2 - |\nabla_x u_E^*(x,z)|^2\, dx = \int_{\mathbb{R}^N} |\nabla_x v_E(x,z)|^2 - |\nabla_x v_E^*(x,z)|^2\, dx$$
$$+\int_0^{1/2} dt\left(\int_{\{u_E(\cdot,z)=t\}} |\nabla_x u_E(x,z)|\, d\mathcal{H}^{N-1}_x - \int_{\{u_E^*(\cdot,z)=t\}} |\nabla_x u_E^*(x,z)|\, d\mathcal{H}^{N-1}_x\right)$$
$$\ge \int_{\mathbb{R}^N} |\nabla_x v_E(x,z)|^2 - |\nabla_x v_E^*(x,z)|^2\, dx\,. \quad (3.12)$$

By an argument similar to the one used in the proof of (2.16) and (3.8), we may estimate for all $z>0$,

$$\frac{1}{2}\int_B |u_E(x,0) - u_E^*(x,0)|\, dx = \int_B |v_E(x,0) - v_E^*(x,0)|\, dx$$
$$\le \int_B |v_E(x,z) - v_E^*(x,z)|\, dx + \frac{|B|^{1/2}z^{s/2}}{\sqrt{s}}\left(\int_{B\times\mathbb{R}_+} t^{1-s}|\nabla(v_E - v_E^*)|^2\, dxdt\right)^{\frac{1}{2}}.$$

From the above inequality, (3.10), and (3.12) we deduce that for all $z \in (0,1)$,

$$
\begin{aligned}
|B \setminus E| &= \int_B |u_E(x,0) - u_E^*(x,0)| \, dx \\
&\le 2c_N \beta(s)^{\frac{N+2}{2N}} \left( \int_{\mathbb{R}^N} |\nabla_x v_E(x,z)|^2 - |\nabla_x v_E^*(x,z)|^2 \, dx \right)^{1/2} \\
&\quad + \frac{2|B|^{1/2} z^{s/2}}{\sqrt{s}} \left( \int_{\mathbb{R}^{N+1}_+} t^{1-s} |\nabla v_E(x,t) - \nabla v_E^*(x,t)|^2 \, dxdt \right)^{1/2} \\
&\le 2c_N \beta(s)^{\frac{N+2}{2N}} \left( \int_{\mathbb{R}^N} |\nabla_x u_E(x,z)|^2 - |\nabla_x u_E^*(x,z)|^2 \, dx \right)^{1/2} \\
&\quad + \frac{2\sqrt{2}|B|^{1/2} z^{s/2}}{\sqrt{s}} \left( \int_{\mathbb{R}^{N+1}_+} t^{1-s} \left( |\nabla u_E|^2 + |\nabla u_E^*|^2 \right) dxdt \right)^{1/2}
\end{aligned}
$$

Let us fix $\tau \in (0,1]$ to be chosen. Squaring the first and last sides of the inequality above, multiplying by $z^{1-s}$, and integrating in $(0,\tau)$ with respect to $z$ yields

$$
\begin{aligned}
\frac{|B \setminus E|^2}{2-s} \tau^{2-s} \le 8c_N^2 \beta(s)^{\frac{N+2}{N}} \int_0^1 z^{1-s} \left( \int_{\mathbb{R}^N} |\nabla_x u_E(x,z)|^2 - |\nabla_x u_E^*(x,z)|^2 \, dx \right) dz \\
+ \frac{8|B|\tau^2}{s} \int_{\mathbb{R}^{N+1}_+} z^{1-s} \left( |\nabla u_E|^2 + |\nabla u_E^*|^2 \right) dxdz \, .
\end{aligned}
$$

Using the Pólya–Szegö inequality, the assumption $D_s(E) \le 1$, and (3.5), we derive that

$$
\begin{aligned}
|B \setminus E|^2 &\le 32 c_N^2 \beta(s)^{\frac{N+2}{N}} \frac{P_s(B)}{\gamma_{N,s}} D_s(E) \tau^{s-2} + \frac{64|B|P_s(B)}{s\gamma_{N,s}} \tau^s \\
&\le C^*_{N,s} \left( D_s(E) \frac{\tau^{s-2}}{2-s} + \frac{\tau^s}{s} \right) , \qquad (3.13)
\end{aligned}
$$

with

$$
C^*_{N,s} := \frac{64 P_s(B)}{\gamma_{N,s}} \max \left\{ c_N^2 \beta(s)^{\frac{N+2}{N}}, |B| \right\}
$$

which only depends on $s$ and $N$. Next we observe that, among all values of $\tau \in (0,1]$, the right handside of (3.13) is minimized for $\tau = \sqrt{D_s(E)}$. Hence,

$$
|B \setminus E| \le \left( \frac{2C^*_{N,s}}{s(2-s)} \right)^{1/2} D_s(E)^{s/4} \, .
$$

Finally we observe that $2|B \setminus E| = |B \triangle E| \ge |B| A(E)$ since $|E| = |B|$, and the proof is complete. □

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