

# STABLE SOLUTIONS OF ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS WITH EUCLIDEAN COVERINGS

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ABSTRACT. We investigate the rigidity properties of stable, bounded solutions of semilinear elliptic partial differential equations in Riemannian manifolds that admit an Euclidean universal covering, finding conditions under which the level sets are geodesics or the solution is constant.

Let  $M$  be a complete, connected, Riemannian manifold, endowed with a metric  $g$ . Let  $f \in C^1(\mathbb{R})$ . We will consider classical, bounded solutions  $u \in C^2(M) \cap L^\infty(M)$  of the following semilinear partial differential equation:

$$(1) \quad \Delta_g u + f(u) = 0.$$

We say that  $u$  is stable if, for any  $\varphi \in C_0^\infty(M)$ , we have that

$$(2) \quad \int_M \left( |\nabla_g \varphi|^2 - f'(u)\varphi^2 \right) \geq 0.$$

The notion of stability is a classical topic in the calculus of variations and it may often be related with geometric features (such as monotonicity of the solution) and with quantities of physical relevance (such as energy minimization): we refer to [Dup11] for a throughout discussion on the notion of stability and its importance. Also, as usual, we say that the metric  $g$  is flat if its sectional curvature vanishes identically (see, e.g., [GHL90]).

The purpose of this note is to prove symmetry and rigidity properties for stable, bounded solutions  $u$  of (1) under several circumstances. First of all (see Theorem A), we will show that if  $M$  admits  $\mathbb{R}^2$  with the flat metric as universal covering, then the level sets of  $u$  are geodesics.

These results extend some previous works of the authors, such as Theorem 2 of [FSV08b].

The proof is based on some technology developed to study some questions related to a major problem posed by De Giorgi (see [DG79], and also [FV09] for a recent review on this topic).

Though we do not know whether our assumptions in Theorem A are optimal, we recall that, in general, there exist bounded, stable solutions in higher dimensional Euclidean spaces whose level sets are not totally geodesic (see [dPKW08]), so our hypotheses cannot be dropped completely.

Next (see Theorem B), we prove that if  $f \geq 0$  and  $M$  has a Euclidean covering space  $\mathbb{R}^n$  with  $n \leq 4$ , then  $u$  is constant. For results on compact manifolds, see the pioneer work of [Jim84]. For related results on manifolds, see [FSV08b]. See also [DF10], for the Euclidean case up to dimension 4, and [Far07], which studied stable solutions of the Lane-Emden equation,

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showing that in dimension 11 and higher there are stable, bounded, positive, radial, non-constant solutions even in the Euclidean case (this is somewhat a counterpart of Theorem B, though it leaves open the case of intermediate dimensions).

The proof of Theorem B is based on a parabolicity estimate of [DF10].

We observe (see Corollary C), that the results in Theorems A or B hold if  $M$  has dimension  $n = 2$  or  $n \leq 4$ , respectively, and it is endowed with the flat metric. In this sense, the scope of this note is complementary to the one of [FSV08b]: while [FSV08a] aims to classify the stable solutions in manifold with non-negative Ricci curvature, here we attempt a similar classification for manifold with vanishing sectional curvature.

Following are the results obtained, and their proofs.

**Theorem A.** *Suppose that  $M$  has  $\mathbb{R}^2$  with the flat metric as universal covering. Let  $u$  be a stable, bounded solution of (1) on  $M$ .*

*Then, either  $u$  is constant or any connected component of each level set of  $u$  is a geodesic.*

*Proof.* If  $p$  is the projection from  $\mathbb{R}^2$  to  $M$  and we set  $U(x) := u(p(x))$ , we have that

$$(3) \quad U \text{ is a solution of } \Delta U + f(U) \text{ on } \mathbb{R}^2,$$

because  $p$  is a local isometry and  $\nabla_g$  in normal coordinates becomes the Euclidean  $\nabla$ , since  $g$  is pulled back to the Euclidean metric.

Moreover,

$$(4) \quad U \text{ is stable, i.e., for any } \varphi \in C_0^\infty(\mathbb{R}^2) \text{ we have that} \\ \int_{\mathbb{R}^2} (|\nabla \varphi|^2 - f'(U)\varphi^2) \geq 0.$$

Indeed, the stability condition in (2) (resp., in (4)) is equivalent to the existence of a positive solution  $v \in C^2(M, (0, +\infty))$  (resp.,  $V \in C^2(\mathbb{R}^2, (0, +\infty))$ ) of the linearized equation  $\Delta_g v + f'(u)v = 0$  in  $M$  (resp.,  $\Delta V + f'(u)V = 0$  in  $\mathbb{R}^2$ ): see, e.g., [FCS80, FC85]. Hence, the stability of  $u$  in (2) gives the existence of  $v$  as above. Then, if for any  $x \in \mathbb{R}^2$  we define the lifted function  $V(x) := v(p(x))$ , we have that  $V$  is a positive solution of the linearized equation pulled back on the covering, thus proving (4).

Now we apply Theorem 1.1 of [FSV08a] to  $U$ . We remark that  $U$  satisfies the hypotheses of Theorem 1.1 of [FSV08a] thanks to (3) and (4). As a consequence,  $U$  is one-dimensional, i.e. there exist  $\omega \in S^1$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(5) \quad U(x) = h(\omega \cdot x) \text{ for any } x \in \mathbb{R}^2.$$

We claim that

$$(6) \quad \begin{array}{l} \text{either } h \text{ is constant or} \\ \text{the level sets of } h \text{ are made by points that do not accumulate.} \end{array}$$

To establish this we suppose that there exist points  $t_k$ , each different by the others, that accumulate to some  $t_o$ , for which  $h(t_k) = 0$ . By Rolle's Theorem, there exists  $\tau_k$  on the segment joining  $t_k$  to  $t_{k+1}$  such that  $h'(\tau_k) = 0$ . Of

course,  $\tau_k$  accumulates to  $t_o$  as well. We have

$$\begin{aligned} h(t_o) &= \lim_{k \rightarrow +\infty} h(t_k) = 0, \\ h'(t_o) &= \lim_{k \rightarrow +\infty} \frac{h(t_k) - h(t_o)}{t_k - t_o} = 0, \\ \text{and } h''(t_o) &= \lim_{k \rightarrow +\infty} \frac{h'(\tau_k) - h'(t_o)}{\tau_k - t_o} = 0. \end{aligned}$$

In particular,  $f(h(t_o)) = -h''(t_o) = 0$ . Thus, if we set  $g(t) = (g_1(t), g_2(t)) := (h(t), h'(t))$  and  $\psi(g_1, g_2) = (g_2, -f(g_1))$ , we have that  $g'(t) = \psi(g(t))$ . Also, the constant function  $g_o(t) := (h(t_o), 0)$  satisfies  $g'_o(t) = \psi(g_o(t))$ . Since  $g(t_o) = g_o(t_o)$ , we have that  $g(t) = g_o(t)$  for any  $t \in \mathbb{R}$ , due to the Uniqueness Theorem for ODEs, hence  $h$  is constant. This proves (6).

Then, by (5) and (6) we have that either  $U$  is constant or that all the level sets of  $U$  are straight lines with direction perpendicular to  $\omega$ . In the latter case, the level sets of  $u$  are the image by the projection  $p$  of such straight lines, and therefore they are geodesics, because all the Cristoffel symbols vanish when they are written in local coordinates, since the pull back of  $g$  is Euclidean.  $\square$

**Theorem B.** *Suppose that  $f(r) \geq 0$  for any  $r \in \mathbb{R}$  and that  $M$  has a Euclidean covering space  $\mathbb{R}^n$  with  $n \leq 4$ .*

*Let  $u$  be a stable, bounded solution of (1) on  $M$ . Then  $u$  is constant.*

*Proof.* By arguing as in (3) and (4), we see that, if  $p$  is the projection from  $\mathbb{R}^n$  to  $M$  and  $U(x) := u(p(x))$ , then  $U$  is a stable solution of  $\Delta U + f(U)$  on  $\mathbb{R}^n$ . Accordingly, by Theorem 1.1 of [DF10], we have that  $U$  is constant.  $\square$

**Corollary C.** *Suppose that the metric  $g$  of  $M$  is flat. Let  $u$  be a stable, bounded solution of (1) on  $M$ .*

*Then,*

- *if  $\dim M = 2$ , then either  $u$  is constant or any connected component of each level set of  $u$  is a geodesic;*
- *if  $\dim M \leq 4$ , and  $f(r) \geq 0$  for any  $r \in \mathbb{R}$ , then  $u$  is constant.*

*Proof.* By Theorem 3.82 on page 135 of [GHL90], we have that  $M$  has a Euclidean covering space  $\mathbb{R}^n$ .

So the results follow from Theorems A and B.  $\square$

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