On a “reversed” variational inequality

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Abstract

We are concerned with a class of penalized semilinear elliptic problems depending on a parameter. We study some multiplicity results and the limit problem obtained when the parameter goes to ∞. We obtain a “reversed” variational inequality, which is deeply investigated in low dimension.

1 Introduction

A large class of well studied equations admits, as a limit case, a variational inequality which we can call “reversed”, since the sign of the inequality is not the usual one.

A meaningful example is given by the classic *jumping* problem (see [11], [12] and the references therein), which we write in the following way

$$(J,\omega) \qquad \begin{cases} \Delta u + \alpha u - \omega(e+u)^- = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth and bounded domain of $\mathbb{R}^N$, e is a positive function and α and ω are real coefficients (in most cases $e = e_1$, the first positive eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$).

If $(\omega_n)_{n\in\mathbb{N}}$ is a sequence diverging to $+\infty$ and $(u_{\omega_n})_{n\in\mathbb{N}}$ is a sequence of solutions of (J,ω_n) which weakly converges to u in $W_0^{1,2}(\Omega)$, then u satisfies the “reversed inequality”

$$(J,\infty) \qquad \begin{cases} \int Du\cdot v - \alpha\int uv \le 0 & \forall v \text{ in } W_0^{1,2}(\Omega) \text{ such that } v\ge 0 \\ u \ge -e. \end{cases}$$

Of course we also require that u satisfies the equation in the set of x's in Ω where $u(x) > -e(x)$, which will be defined in a suitable way. And it is clear that, from this point of view, we are interested in those solutions u's which do not satisfy the equation on the whole of Ω.

We can observe that in the particular case $N = 1$, $\Omega = (a,b)$, if $u(x)$ is the trajectory depending on the time x of a material point which moves in the (unidimensional) billiard $\mathbb{R}^+$, bouncing on the boundary $\{0\}$, then u satisfies the reversed inequality (J,∞).

Note that the functionals defined on $W_0^{1,2}(\Omega)$ associated to problems (J,ω)

$$F_\omega(u) = \frac{1}{2}\int |Du|^2 - \frac{\alpha}{2}\int u^2 - \frac{\omega}{k}\int [(u+e)^-]^2$$

have an increasing lack of convexity as ω goes to $+\infty$ and tend to the functional

$$F_\infty(u) = \begin{cases} \frac{1}{2}\int |Du|^2 - \frac{\alpha}{2}\int u^2 & \text{if } u \ge -e \\ -\infty & \text{elsewhere.} \end{cases}$$

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We will consider a family of problems which, however, seems to have many links (at least for $\omega < +\infty$) with the ones above

$$(P_\omega) \qquad \begin{cases} \Delta^2 u + c\Delta u - \alpha u + \omega((u-\phi)^-)^{k-1} = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, c and α are real numbers, ω is a positive parameter and $k > 2$ and k subcritical.

The corresponding limit problem is

$$(P_\infty) \qquad \int \Delta u \Delta \psi - c \int Du \cdot D\psi - \alpha \int u\psi \leq 0 \quad \forall \psi \text{ in } H_0^1(\Omega) \cap H^2(\Omega) \text{ such that } \psi \geq 0.$$

The choice $k > 2$ allows an *a priori* bound of a suitable family of solutions. The presence of the biharmonic operator lets us show that in some cases there are limit solutions u's which do not satisfy the equation

$$\Delta^2 u + c\Delta u - \alpha u = 0$$

in the whole of Ω.

In this paper we study existence and multiplicity of solutions of (P_ω) and of (P_∞).

In particular if $N \leq 3$ it i possible to show a stronger result. In fact in this case it is possible to prove that a solution of (P_∞) satisfies the following "reversed" variational inequality

$$\int \Delta u \Delta(v-u) - c\int Du \cdot D(v-u) - \alpha \int u(v-u) \leq 0 \quad \forall v \text{ in } H_0^1(\Omega) \cap H^2(\Omega),\ v \geq \phi.$$

The name "reversed" comes from the comparison between this inequality and the "classical" variational inequalities introduced in [10]. In fact in that case, if Φ is an obstacle, that is $\Phi_{|\partial\Omega} < 0$ and Φ is positive on a set of positive measure, if one looks for

$$\min\left\{ \int |\Delta v|^2 \,\middle|\, v \in H_0^1(\Omega) \cap H^2(\Omega),\ v \geq \Phi \right\},$$

one finds that the unique solution u of this problem solves

$$\int \Delta u \Delta(v-u) \geq 0 \quad \forall v \text{ in } H_0^1(\Omega) \cap H^2(\Omega),\ v \geq \Phi.$$

At this point we also observe that problem (P_ω) can be compared to the problem introduced by Lazer and McKenna in [7] as a model to study travelling waves in suspension bridges. The problem is the following one

$$\begin{cases} \Delta^2 u + c\Delta u - bu - b(u+1)^- = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

A large number of results have been found on this context: important results are given in [6], [7], [13], [14], [15], [16], [17], [21].

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2 Setting of the problem

Let Ω be a bounded and smooth domain of $\mathbb{R}^N$, $N \geq 1$. We will make the following fundamental assumptions

$$(H) \qquad \begin{cases} \omega \in \mathbb{R},\ \omega > 0 \\ 2 < k \quad (\text{and } k < 2N/(N-4) \text{ if } N \geq 5), \\ \alpha,\ c \in \mathbb{R} \text{ and } \phi \in L^k(\Omega). \end{cases}$$

For some technical results, such as the Palais-Smale condition, we will not make other assumptions on ϕ, but in most cases we will assume $\phi \leq 0$ a.e. in Ω or $\sup\limits_{\Omega} \phi < 0$. We observe that such a requirement is related to the physical model of travelling waves in suspension bridges, where $\phi \equiv -1$ (see [6], [7], [13], [14], [15], [16], [17], [21]).

Now consider the following sequence of problems

$$(P_\omega) \qquad \begin{cases} \Delta^2 u + c\Delta u - \alpha u + \omega((u-\phi)^-)^{k-1} = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, $v^- = \max\{0, -v\}$ and $u \in H = H_0^1(\Omega) \cap H^2(\Omega)$. In H we set $\langle u, v \rangle = \int \Delta u \Delta v$, $\|u\|^2 = \int |\Delta u|^2$.

Remark 2.1. The norm just introduced is equivalent in H to the norm of $H^2(\Omega)$.

In fact $\Delta : H \to L^2(\Omega)$ is linear, injective (if $\Delta u = 0$ and $u = 0$ on $\partial\Omega$, then $u \equiv 0$), continuous if H is endowed with the $H^2(\Omega)$ norm and surjective (by regularity theorems). Then $\int |\Delta u|^2 \geq c\|u\|_{H^2(\Omega)}$ by the Open Mapping Theorem. $\triangle$

Remark 2.2. H is a closed subspace of $H^2(\Omega)$. $\triangle$

In order to study problem (P_ω), we will follow a variational approach. Consider $f_\omega : H \longrightarrow \mathbb{R}$ defined as follows

$$f_\omega(u) = \frac{1}{2}\int |\Delta u|^2 - \frac{c}{2}\int |Du|^2 - \frac{\alpha}{2}\int u^2 - \frac{\omega}{k}\int ((u-\phi)^-)^k.$$

We observe that, if $k > 1$, f_ω is of class C^1, and if $k > 2$, it is of class C^2. Moreover its critical points are solutions of (P_ω).

We will also sometimes use the following notation for the quadratic form defined on H as

$$Q_{c,\alpha}(u) = \frac{1}{2}\int |\Delta u|^2 - \frac{c}{2}\int |Du|^2 - \frac{\alpha}{2}\int u^2.$$

Remark 2.3. The following problem

$$\begin{cases} \Delta^2 u + c\Delta u = \Lambda u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has an increasing sequence of eigenvalues $\Lambda_i = \lambda_{n_i}(\lambda_{n_i} - c)$, $i \geq 1$, where the sequence $(\lambda_{n_i})_{i\in\mathbb{N}}$ is a rearrangement of the eigenvalues of $\Delta u + \lambda u = 0$, $u = 0$ on $\partial\Omega$ (if $c > \lambda_k + \lambda_j$, $j < k$, then $\lambda_k^2 - c\lambda_k < \lambda_j^2 - c\lambda_j$). The eigenfunctions of the former problem are the ones corresponding to the latter problem: $E_i = e_{n_i}$, and they are orthonormal in $L^2(\Omega)$. We recall that e_1 can be chosen strictly positive in Ω.

Observe that, since $\lambda_r \to +\infty$ as $r \to \infty$, there is possibly only a finite number of negative or null Λ_i's, and $\Lambda_i \to +\infty$ as $i \to \infty$. Note that the first eigenvalue is not simple any more, in general. Finally set $H_j = \text{Span}(E_1, \ldots, E_j)$ for any $j \geq 1$. $\triangle$

3 Palais-Smale condition

We will investigate the existence of critical points of f_ω through some variational tools that we recall in the Appendix.

Proposition 3.1. *Suppose $\alpha \neq \lambda_1^2 - c\lambda_1$ and that hypothesis (H) holds. Then f_ω satisfies $(PS)_\mathfrak{c}$ for every $\mathfrak{c}$ in $\mathbb{R}$.*

Proof. Let $(u_h)_h$ be a $(PS)_\mathfrak{c}$ sequence, that is $f_\omega(u_h) \to \mathfrak{c}$ and $f'_\omega(u_h) \to 0$. It is enough to show that $\|u_h\|$ is bounded, since $\forall\, z$ in H

$$\nabla f_\omega(z) = z + i^*(c\Delta z - \alpha z + \omega((z-\phi)^-)^{k-1}),$$

where $i^* : L^2(\Omega) \longrightarrow H$, the adjoint of the immersion $i : H \longrightarrow L^2(\Omega)$, is a compact operator. In fact, if $u_n \rightharpoonup u$ in H, then u_n converges strongly in $L^k(\Omega)$. So $((u_n-\phi)^-)^{k-1}$ converges strongly in $L^{\frac{k}{k-1}}(\Omega)$ and thus in H'. But then $u_n = \nabla f_\omega(u_n) - i^*(c\Delta u_n - \alpha u_n + \omega((u_n-\phi)^-)^{k-1})$ converges strongly, since $(\Delta u_n)_n$ and $(u_n)_n$ are bounded in $L^2(\Omega)$.

Thus suppose by contradiction that, up to a subsequence, $\|u_h\|$ diverges. Then there is v in H such that (up to a subsequence) $v_h = u_h/\|u_h\| \rightharpoonup v$ in H. Note that, dividing $f_\omega(u_h)$ by $\|u_h\|^k$ and passing to the limit, we get $\int(v^-)^k = 0$, and so $v \geq 0$.

Now observe that $\forall\, \varepsilon > 0$, $\exists\, C_\varepsilon > 0$ such that

$$\left|\int ((u_h-\phi)^-)^{k-1}\phi\right| \leq \varepsilon \int ((u_h-\phi)^-)^k + C_\varepsilon.$$

In fact

$$\left|\int ((u_h-\phi)^-)^{k-1}\phi\right| \leq \left(\int ((u_h-\phi)^-)^k\right)^{1-1/k} \|\phi\|_{L^k(\Omega)}$$

$$\leq \|\phi\|_{L^k(\Omega)} \left(\tilde{\varepsilon}\int ((u_h-\phi)^-)^k + \tilde{C}_\varepsilon\right).$$

Here we used the fact that for every $R \geq 0$, for every $0 < \alpha < p$ and for every $\epsilon > 0$

$$R^\alpha \leq \frac{\alpha}{p}\epsilon R^p + \frac{p-\alpha}{p}\left(\frac{1}{\epsilon}\right)^{\alpha/(p-\alpha)}. \tag{1}$$

In this way

$$\frac{f'_\omega(u_h)(u_h)}{\|u_h\|} = \frac{1}{\|u_h\|}\left\{\int |\Delta u_h|^2 - c\int |Du_h|^2 - \alpha\int u_h^2 + \omega\int ((u_h-\phi)^-)^{k-1}u_h\right\}$$

$$= \frac{1}{\|u_h\|}\left\{2f_\omega(u_h) + \left(\frac{2}{k}-1\right)\omega\int ((u_h-\phi)^-)^k + \omega\int ((u_h-\phi)^-)^{k-1}\phi\right\}$$

$$\leq \frac{1}{\|u_h\|}\left\{2f_\omega(u_h) + \left(\frac{2}{k}-1+\varepsilon\right)\omega\int ((u_h-\phi)^-)^k + \omega\, C_\varepsilon\right\}.$$

But $\dfrac{f'_\omega(u_h)(u_h)}{\|u_h\|} \to 0$ as $h \to 0$ and passing to the limit we get, if ε is small enough (i.e. $2/k - 1 + \varepsilon < 0$),

$$\lim_{h\to\infty} \frac{\displaystyle\int ((u_h-\phi)^-)^k}{\|u_h\|} = 0 \qquad \text{and hence} \qquad \lim_{h\to\infty} \frac{\displaystyle\int ((u_h-\phi)^-)^{k-1}\phi}{\|u_h\|} = 0.$$

Moreover, since $u_h = (u_h - \phi) + \phi$,

$$\lim_{h\to\infty} \frac{\displaystyle\int ((u_h-\phi)^-)^{k-1}u_h}{\|u_h\|} = 0.$$

But in this way

$$\frac{f'_\omega(u_h)(u_h)}{\|u_h\|^2} = 1 - c\int |Dv_h|^2 - \alpha\int v_h^2 + \frac{\omega\displaystyle\int ((u_h-\phi)^-)^{k-1}u_h}{\|u_h\|^2}$$

$$\longrightarrow 1 - c\int |Dv|^2 - \alpha\int v^2.$$

On the other hand $\dfrac{f'_\omega(u_h)(u_h)}{\|u_h\|^2} \to 0$ as $h \to \infty$. Therefore if c and α are non positive we get a contradiction. Otherwise $v \not\equiv 0$.

Now observe that $\int ((u_h-\phi)^-)^{k-1}e_1/\|u_h\| \to 0$, since

$$\frac{\displaystyle\int ((u_h-\phi)^-)^{k-1}e_1}{\|u_h\|} \le \left(\frac{\displaystyle\int e_1^k}{\|u_h\|}\right)^{1/k} \left(\frac{\displaystyle\int ((u_h-\phi)^-)^k}{\|u_h\|}\right)^{1-1/k}.$$

Therefore

$$\frac{f'_\omega(u_h)e_1}{\|u_h\|} = \frac{1}{\|u_h\|}\left\{\int \Delta u_h \Delta e_1 - c\int Du_h\cdot De_1 - \alpha\int u_h e_1 + \omega\int ((u_h-\phi)^-)^{k-1}e_1\right\}$$

$$\longrightarrow (\lambda_1^2 - c\lambda_1 - \alpha)\int ve_1.$$

But $\dfrac{f'_\omega(u_h)e_1}{\|u_h\|} \to 0$ as $h \to \infty$ and the previous limit implies $v \equiv 0$, since $\alpha \neq \lambda_1^2 - c\lambda_1$. Then a contradiction arises. $\square$

Remark 3.2. We observe that the requirement $\alpha \neq \lambda_1^2 - c\lambda_1$ is not merely a technical assumption: indeed, if $\alpha = \lambda_1^2 - c\lambda_1$ we can take the sequence $u_n = t_n e_1$, $t_n > 0$ and $t_n \to +\infty$. Such a sequence is such that $f_\omega(u_n) = 0$ and $f'_\omega(u_n) = 0$ $\forall\,\omega$ and $\forall\,n$ in $\mathbb{N}$. But, of course, it is impossible to find a converging subsequence. $\triangle$

4 Existence of one forcing solution

From now on we will also assume that $\phi \le 0$, in order to find some solutions to problem (P_ω). In particular our goal is to find some particular solutions, the ones which we call *forcing* solutions.

Definition 4.1. A function u in H is called a *forcing* solution of problem (P_ω) if it is a solution such that $(u-\phi)^- \neq 0$.

The definition just given is justified by the fact that in some cases, if a sequence of *forcing* solutions weakly converges to u, such a u is *forced* to be over ϕ and to touch ϕ somewhere.

Remark 4.2. If u is a solution of (P_ω) such that $f_\omega(u) \neq 0$, then u is a *forcing* solution. In fact, if $u \ge \phi$ a. e. in Ω, then $0 = f'_\omega(u)\psi = \int \Delta u \Delta \psi - c\int Du \cdot D\psi - \alpha \int u\psi$ for every ψ in H, and so, taking $\psi = u$ we would have $f_\omega(u) = 0$.

Conversely, if $\phi \le 0$, every *forcing* solution u is such that $f_\omega(u) > 0$. In fact

$$0 = f'_\omega(u)u = 2f_\omega(u) + \left(\frac{2}{k} - 1\right)\omega \int ((u-\phi)^-)^k + \omega \int ((u-\phi)^-)^{k-1}\phi$$

and then $f(u) > 0$. △

In this section we want to show that if $\phi \le 0$, then there exists a forcing solution u_ω of problem (P_ω) for every $\omega > 0$. Actually we show that there exists a forcing solution $\forall N \ge 1$ if $\alpha < \lambda_1^2 - c\lambda_1$ and $\forall N \ge 2$ if $\alpha > \lambda_1^2 - c\lambda_1$. In view of Proposition 3.1 and Remark 3.2 we will not take into account the case $\alpha = \lambda_1^2 - c\lambda_1$.

We finally observe that, if $\phi \le 0$ $u \equiv 0$ is always a solution of (P_ω), whatever c, α and ω are.

Lemma 4.3. *Let k be in $[1, +\infty)$ and such that $k < 2N/(N-4)$ if $N > 4$, $\phi \in L^k(\Omega)$ and $\phi \le 0$. Then*

$$\int ((u-\phi)^-)^k = O(\|u\|^k).$$

Proof. Denote by $\{u - \phi \le 0\}$ the set $\left\{x \in \Omega \,\middle|\, u(x) - \phi(x) \le 0\right\}$.
I: N > 4. Set $q = 2N/(N-4)$. Then

$$\int ((u-\phi)^-)^k = \int_{\{u-\phi\le 0\}} (-u+\phi)^k$$

and since $0 \le -u + \phi \le -u$, the last quantity is less or equal to

$$\int_{\{u-\phi\le 0\}} (-u)^k \le \left(\int_{\{u-\phi\le 0\}} (-u)^q\right)^{k/q} m(\{u-\phi \le 0\})^{1-k/q}$$

$$\le \left(\int_\Omega |u|^q\right)^{k/q} m(\{u-\phi \le 0\})^{1-k/q},$$

and by the Sobolev's Embedding Theorem it is smaller than

$$C\|u\|^k m(\{u-\phi \le 0\})^{1-k/q},$$

for a universal constant $C>0$ (here $m(A)$ stands for the Lebesgue measure of any set A). If $u\to 0$ in H, then $u\to 0$ a.e. in Ω; in this way $m(\{u-\phi\le 0\})^{1-k/q}\to 0$.
II: N = 4. Starting as in the previous step

$$\int ((u-\phi)^-)^k \le \|u\|^k_{L^s(\Omega)}\, m(\{u-\phi\le 0\})^{1-k/s}$$

for every $s>k$. As before, there exists a universal positive constant C such that the last quantity is less or equal to $C\|u\|^k m(\{u-\phi\le 0\})^{1-k/s}$.
III: N ≤ 3. In this case there exists $C>0$ such that for every u in H

$$\int ((u-\phi)^-)^k \le \|u\|^k_{L^\infty(\Omega)}\, m(\{u-\phi\le 0\}) \le C\|u\|^k\, m(\{u-\phi\le 0\}).$$

□

As already said, in the theorems involving the existence of forcing solutions, we distinguish two cases: the first one in which $\alpha<\lambda_1^2-c\lambda_1$ and the one in which $\alpha>\lambda_1^2-c\lambda_1$.

Remark 4.4. If there exists $l\ge 1$ such that $e_1\notin H_l$, then there isn't any non trivial non negative (or non positive) function in H_l. In fact $\forall v$ in H such that $v\ge 0$, $v\not\equiv 0$, we get $\langle v,e_1\rangle=\lambda_1^2\int ve_1>0$. △

Lemma 4.5. *Assume* (H) *and* $\Lambda_l\le\alpha<\Lambda_{l+1}$, $l\ge 1$. *Then*

(a)

$$\sup_{H_l} f_\omega = 0;$$

(b) *if* $\phi\le 0$, *there exists* $C_l^+>0$ *such that*

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\inf_{\substack{w\in H_l^\perp,\\ \|w\|=\rho}} f_\omega(w)\ge C_l^+;$$

(c) *if* $\alpha<\lambda_1^2-c\lambda_1$, *then*

$$\lim_{\substack{v\in H_l,\sigma\ge 0,\\ \|v-\sigma e_1\|\to\infty}} f_\omega(v-\sigma e_1)=-\infty.$$

Proof. **(a)** It is obvious, since $f_\omega(v)\le\frac{\Lambda_l-\alpha}{2}\int v^2$ $\forall v$ in H_l and $f_\omega(0)=0$.

(b) Observe that, by Lemma 4.3, for any $\varepsilon>0$ there exists $\rho>0$ such that, if $\|w\|\le\rho$, $f_\omega(w)\ge C_l^+\|w\|^2-\varepsilon\|w\|^k$, where $C_l^+=\inf\limits_{n_i\ge l+1}\dfrac{\lambda_{n_i}^2-c\lambda_{n_i}-\alpha}{\lambda_{n_i}^2}$. The thesis follows.

(c) Suppose by contradiction that there exist v_h in H_l, $\sigma_h\ge 0$ and C in $\mathbb{R}$ such that $\|v_h-\sigma_h e_1\|\to\infty$ and

$$\begin{aligned}(2)\qquad &\frac12\int|\Delta(v_h-\sigma_h e_1)|^2-\frac{c}{2}\int|D(v_h-\sigma_h e_1)|^2\\ &-\frac{\alpha}{2}\int(v_h-\sigma_h e_1)^2-\frac{\omega}{k}\int((v_h-\sigma_h e_1-\phi)^-)^k\ge C.\end{aligned}$$

Up to a subsequence we can suppose that $\frac{v_h-\sigma_h e_1}{\|v_h-\sigma_h e_1\|} \to v-\sigma e_1$ in H, in $L^k(\Omega)$ and a.e. in Ω, where $v \in H_l$ and $\sigma \geq 0$ (remember that this is a finite dimensional case). Dividing both sides of inequality (2) by $\|v_h-\sigma_h e_1\|^2$ and passing to the limit, we obtain that

$$\limsup_{h\to\infty} \|v_h-\sigma e_1\|^{k-2} \int \left(\left(\frac{v_h-\sigma_h e_1-\phi}{\|v_h-\sigma_h e_1\|} \right)^{-} \right)^{k}$$

is a real non negative number. In this way $v-\sigma e_1 \geq 0$ a.e. in Ω.

Since $\alpha < \lambda_1^2 - c\lambda_1$, $e_1 \in H_l^{\perp}$. In this way $\langle v, e_1\rangle = 0$, and so $0 \leq \langle v-\sigma e_1, e_1\rangle = -\sigma\lambda_1^2$, which is possible if and only if $\sigma = 0$. This implies $v \geq 0$ in H_l, which is impossible for Remark 4.4. □

As we will see, the proof of the following theorem is essentially based on an application of Theorem A.3, which topologically degenerates in a Mountain Pass structure if the quadratic form is positively defined.

Theorem 4.6. *Suppose (H) holds, $\phi \leq 0$ and $\alpha < \lambda_1^2 - c\lambda_1$. Then there exists a non trivial critical point u_ω of f_ω (i.e a forcing solution of problem (P_ω)). Moreover $0 < f_\omega(u_\omega)$ $\forall\,\omega$, and $\forall\,\overline{\omega} > 0$ $\sup_{\omega\geq\overline{\omega}} f_\omega(u_\omega) < +\infty$.*

Proof. There are two cases.
First case.

Suppose that $\Lambda_1 > 0$ $\forall\, i$ in $\mathbb{N}$. Then $Q_{c,\alpha}$ is positive defined and we can introduce the following scalar product in H: $(u,v) = \int \Delta u \Delta v - c\int Du\cdot Dv - \alpha\int uv$. We observe that the induced norm $|||\cdot|||$ is equivalent to the norm introduced in H. In fact there exist positive a_0 and b_0 such that $a_0\|u\|^2 \leq |||u|||^2 \leq b_0\|u\|^2$, where b_0 is obtained by the continuity of f on H, while a_0 is obtained from next Remark 4.7.

Observe that $f_\omega(0) = 0$, $\lim_{t\to+\infty} f_\omega(-te_1) = -\infty$ and that f_ω is locally coercive. In fact $Q_{c,\alpha}(u) \geq \frac{a_0}{2}\|u\|^2$. By Lemma 4.3 for every ε in $(0, a_0/2\omega)$ there exists $\rho_\omega > 0$ such that $\inf_{|||u|||=\rho_\omega} f_\omega(u) > (a-\omega\varepsilon)\rho_\omega^2$. In this way, by the Mountain Pass Theorem, there exists a critical point u_ω with positive critical value. We observe that the Palais-Smale condition also holds with this norm, as one can easily check adapting the proof of Proposition 3.1.

Moreover by Remark 4.2 we get that $(u_\omega-\phi)^- \neq 0$, since $f_\omega(u_\omega) > 0$.

Finally we find an upper bound for the critical values. In fact, if $t_\omega > 0$ is such that $f_\omega(-t_\omega e_1) \leq 0$, then

$$\left(\frac{a_0}{2} - \omega\varepsilon\right)\rho_\omega^2 \leq f_\omega(u_\omega) \leq \sup_{t\in[0,t_\omega]} f_\omega(-te_1)$$

$$\leq \sup_{t\in[0,t_\omega]} f_{\overline{\omega}}(-te_1) \leq \sup_{t\in[0,+\infty)} f_{\overline{\omega}}(-te_1) < +\infty.$$

Here we used the fact that if $\omega_1 < \omega_2$ and $u \in H$, then $f_{\omega_1}(u) \geq f_{\omega_2}(u)$.
Second case.

Now suppose that $\exists\, l \geq 1$ such that $\Lambda_l \leq \alpha < \Lambda_{l+1}$.

Lemma 4.5 implies that there exist R_ω and ρ_ω such that $R_\omega > \rho_\omega > 0$ and

$$\inf_{\substack{w\in H_l^{\perp}\\ \|w\|=\rho_\omega}} f_\omega(w) > \sup_{z\in\Sigma_{R_\omega}} f_\omega(z),$$

where $\Sigma_{R_\omega} = \{z = v - \sigma e_1 \,|\, v \in H_l, \sigma \geq 0, \|z\| = R_\omega\}$. In this way the hypotheses of the Linking Theorem are satisfied, so there exists a critical point u_ω such that

$$0 < \inf_{\substack{w \in H_l^\perp \\ \|w\|=\rho_\omega}} f_\omega(w) \leq f_\omega(u_\omega) \leq \sup_{z \in \Delta_{R_\omega}} f_\omega(z),$$

where $\Delta_{R_\omega} = \{z = v - \sigma e_1 \,|\, v \in H_l, \sigma \geq 0, \|z\| \leq R_\omega\}$. More precisely this is a forcing solution of problem (P_ω), by remark 4.2. The Linking Theorem also provides the existence of another critical point with non positive critical value, but it might be the trivial one.

We observe that also in this case we can find a uniform bound for the critical values $f_\omega(u_\omega)$. In fact from the Linking Theorem for any $\overline{\omega} > 0$

$$f_\omega(u_\omega) \leq \sup_{z \in \Delta_{R_\omega}} f_\omega(z) \leq \sup_{z \in \Delta_{R_\omega}} f_{\overline{\omega}}(z) \leq \sup_{\substack{v \in H_l, \\ \sigma \geq 0}} f_{\overline{\omega}}(v - \sigma e_1) < +\infty.$$

□

Remark 4.7. If the quadratic form $Q_{c,\alpha}$ is positive defined, there exists $a_0 > 0$ such that for every w in H

$$\int |\Delta w|^2 - c \int |Dw|^2 - \alpha \int w^2 \geq a_0 \|w\|^2.$$

In fact every w in H can be written as $w = \sum_{i=1}^{\infty} \beta_i E_i$, $\beta_i \in \mathbb{R}$, so

$$\int |\Delta w|^2 - c \int |Dw|^2 - \alpha \int w^2 = \sum \beta_i^2 \frac{(\lambda_{n_i}^2 - c\lambda_{n_i} - \alpha)}{\lambda_{n_i}^2} \lambda_{n_i}^2 \geq a_o \sum \beta_i^2 \lambda_{n_i}^2 = a_0 \|w\|^2.$$

Here a_0 is the infimum of $\frac{\lambda_{n_i}^2 - c\lambda_{n_i} - \alpha_n}{\lambda_{n_i}^2}$ for $n_i \geq 1$, and this infimum is positive, since the quotient goes to 1 as $i \to \infty$ and it is strictly positive for every finite subset of indices. △

For the case $\alpha > \lambda_1^2 - c\lambda_1$ we obtain a result which is analogous to the one of Theorem 4.6, at least if $N \geq 2$. Note that if $\alpha > \lambda_1^2 - c\lambda_1$, then there exists $l \geq 1$ such that $\Lambda_l \leq \alpha < \Lambda_{l+1}$ and $e_1 \in H_l$.

Now let e be in $H_l^\perp$ such that ess sup $e = +\infty$ (if $N \geq 4$) or $e \geq 0$ and $\exists\, x_0$ on $\partial\Omega$ such that $|Du(x_0)| = +\infty$ (if $N = 2$ or $N = 3$). In both cases $m(\{x \in \Omega \,|\, e(x) > v(x)\}) > 0$ $\forall v$ in H_l, since functions belonging to H_l are smooth. We remark that such a function e exists since the mapping

$$u \mapsto \left\{ u_{|\partial\Omega}, \frac{\partial u}{\partial \nu} \right\}$$

is linear, continuous and surjective from $W^{2,2}(\Omega)$ onto $W^{3/2,2}(\partial\Omega) \times W^{1/2,2}(\partial\Omega)$ (see [2] or [8]).

The following Lemma is the one corresponding to **(c)** of Lemma 4.5 in the case $\alpha > \lambda_1^2 - c\lambda_1$.

Lemma 4.8. *Assume (H) and $N \geq 2$. Suppose $\alpha > \lambda_1^2 - c\lambda_1$ and $\Lambda_l \leq \alpha < \Lambda_{l+1}$. Then*

$$\lim_{\substack{v \in H_l, \sigma \geq 0, \\ \|v - \sigma e\| \to \infty}} f(v - \sigma e) = -\infty.$$

Proof. Suppose by contradiction that there exist v_h in H_l, $\sigma_h \geq 0$ and C in $\mathbb{R}$ such that $\|v_h - \sigma_h e\| \to \infty$ and

$$\begin{aligned} &\frac{1}{2}\int |\Delta(v_h - \sigma_h e)|^2 - \frac{c}{2}\int |D(v_h - \sigma_h e)|^2 \\ &-\frac{\alpha}{2}\int (v_h - \sigma_h e)^2 - \frac{\omega}{k}\int ((v_h - \sigma_h e - \phi)^-)^k \geq C. \end{aligned} \tag{3}$$

Up to a subsequence we can suppose that $\frac{v_h - \sigma_h e}{\|v_h - \sigma_h e\|} \to v - \sigma e$ in H, where $v \in H_l$ and $\sigma \geq 0$. Dividing both sides of inequality (3) by $\|v_h - \sigma_h e\|^2$ and passing to the limit, we obtain $\int |\Delta(v - \sigma e)|^2 - c\int |D(v - \sigma e)|^2 - \alpha \int (v - \sigma e)^2 \geq 0$ and $v - \sigma e \geq 0$ a.e. in Ω. By the choice of e we get $\sigma = 0$, $v \geq 0$ and $\int |\Delta v|^2 - c\int |Dv|^2 - \alpha \int v^2 \geq 0$. But $v \in H_l$, so $Q_{c,\alpha}(v) = 0$ and v belongs to the subspace spanned by the eigenfunctions associated to the null eigenvalues of $Q_{c,\alpha}$. In this way $< v, e_1 >= 0$, since $\alpha > \lambda_1^2 - c\lambda_1$. But $v \not\equiv 0$ ($\|v\| = 1$), so $< v, e_1 >= \lambda_1^2 \int v e_1 > 0$ and a contradiction arises. □

Theorem 4.9. *Assume* (H), $\phi \leq 0$, $\alpha > \lambda_1^2 - c\lambda_1$ *and* $N \geq 2$. *Then* $\forall\, \omega > 0$ *there exists a non trivial critical point* u_ω *of* f_ω *(i.e. is a forcing solution of* (P_ω)*); moreover* $0 < f_\omega(u_\omega)$ $\forall\, \omega$, *and* $\forall\, \overline{\omega} > 0$ $\sup\limits_{\omega \geq \overline{\omega}} f_\omega(u_\omega) < +\infty$.

Proof. We can suppose that $\exists\, l \geq 1$ such that $\Lambda_l \leq \alpha < \Lambda_{l+1}$ and $e_1 \in H_l$. Observe that **(a)** and **(b)** of Lemma 4.5 still hold in this case, as well as Lemma 4.8.

As in the previous case, by Lemma 4.8, it is possible to apply the Linking Theorem and find a critical point u_ω for every ω such that $(u_\omega - \phi)^- \neq 0$ and $\sup\limits_{\omega \geq \overline{\omega}} f_\omega(u_\omega) < +\infty$ $\forall\, \overline{\omega} > 0$. □

Remark 4.10. In both Theorem 4.6 and Theorem 4.9, if $N \leq 3$, $\sup \phi < 0$ in Ω and ρ is small enough, the ε used in the Mountain Pass Theorem or in the Linking Theorem can be replaced by 0, since H is continuously embedded in $C^0(\Omega)$. In this way $\forall\, \omega > 0$, $\inf\limits_{\omega} f_\omega(u_\omega) > 0$ (see also Corollary 5.19). △

Definition 4.11. For any $j \geq 1$ set

$$\Lambda_j^\star = \max\left\{ \int |\Delta v|^2 - c\int |Dv|^2 \,\Big|\, v \in H_j, v \geq 0, \int v^2 = 1 \right\}.$$

Observe that in general $\Lambda_j^\star \leq \Lambda_j$. But if r is such that $e_1 = E_r$ and if $j > r$, then $\Lambda_j^\star < \Lambda_j$. In fact suppose v in H_j gives the maximum in Definition 4.11 and $v = \sum_{m=1}^{j} \beta_m E_m$. Since $E_j \neq e_1$, then $|\beta_j| < 1$ and $\beta_r > 0$, since $\forall\, v$ in H such that $v \geq 0$, $v \not\equiv 0$, $\langle v, e_1 \rangle = \lambda_1^2 \int v e_1 = \lambda_1^2 \beta_r > 0$.

Theorem 4.12. *Suppose* (H) *holds,* $\phi \leq 0$, $\alpha > \lambda_1^2 - c\lambda_1$, $l \geq 1$ *is such that* $\Lambda_l \leq \alpha < \Lambda_{l+1}$, $\alpha > \Lambda_{l+1}^\star$ *and* $N \geq 1$. *Then there exists a non trivial critical point* u_ω *of* f_ω *(which is a forcing solution of problem* (P_ω)*). Moreover* $0 < f_\omega(u_\omega)$ $\forall\, \omega$, *and* $\forall\, \overline{\omega} > 0$ $\sup\limits_{\omega \geq \overline{\omega}} f_\omega(u_\omega) < +\infty$.

Proof. The proof is very similar to the one of the previous theorem, but in this case we create a linking with E_{l+1}. In fact, proceeding as in the proof of Lemma 4.8, with e replaced by E_{l+1}, we would get that there exists v in H_l, σ in $\mathbb{R}$ such that $\|v + \sigma E_{l+1}\| = 1$ and $0 \leq Q_{c,\alpha}(v + \sigma E_1) \leq (\Lambda_{l+1}^\star - \alpha) \int (v + \sigma E_{l+1})^2$, which would imply $v + \sigma E_{l+1} = 0$, which is impossible. The rest follows in the same way. □

Remark 4.13. We remark that, however, the condition $\Lambda_l \leq \alpha < \Lambda_{l+1}$ and $\alpha > \Lambda_{l+1}^\star$ made in the previous Theorem is hardly satisfied, in general. △

5 Multiplicity of forcing solutions

In theorems related to multiplicity of forcing solutions we will consider a special case starting from the assumption that there exist $l \geq 1$ and $s \geq l+1$ such that $\Lambda_l \leq \ldots < \Lambda_s < \Lambda_{s+1}$. We will consider two cases, according to whether $\Lambda_s < \lambda_1^2 - c\lambda_1$ or $\lambda_1^2 - c\lambda_1 \leq \Lambda_l$.

It is clear, for example, that if $\overline{m}$ is such that $c \leq \lambda_1 + \lambda_{\overline{m}}$ and $\lambda_{\overline{m}} > \lambda_1$, then the eigenvalue $\Lambda_l = \lambda_{\overline{m}}^2 - c\lambda_{\overline{m}}$ satisfies $\lambda_1^2 - c\lambda_1 \leq \Lambda_l$, while if $\overline{m}$ is such that $c > \lambda_1 + \lambda_{\overline{m}}$ and $\lambda_{\overline{m}} > \lambda_1$, then $\Lambda_s = \lambda_{\overline{m}}^2 - c\lambda_{\overline{m}}$ satisfies $\Lambda_s < \lambda_1^2 - c\lambda_1$.

We now need some preliminary results and to obtain them, we consider the case $\Lambda_s < \lambda_1^2 - c\lambda_1$ and the case $\lambda_1^2 - c\lambda_1 \leq \Lambda_l$ separately.

First case: $\Lambda_s < \lambda_1^2 - c\lambda_1$.

We remember that in this case no non trivial non negative function can exist in H_s.

Lemma 5.1. *Assume* (H) *and* $\Lambda_s < \lambda_1^2 - c\lambda_1$. *Then*

$$\lim_{\substack{v \in H_s, \\ \|v\| \to \infty}} f_\omega(v) = -\infty.$$

Proof. Suppose by contradiction that there exist v_h in H_s and C in $\mathbb{R}$ such that $\|v_h\| \to \infty$ and

$$\frac{1}{2}\int |\Delta v_h|^2 - \frac{c}{2}\int |Dv_h|^2 - \frac{\alpha}{2}\int v_h^2 - \frac{\omega}{k}\int ((v_h - \phi)^-)^k \geq C. \tag{4}$$

Up to a subsequence we can suppose that $v_h/\|v_h\| \to v$ in $H_s \setminus \{0\}$. Dividing both sides of inequality (4) by $\|v_h\|^k$, we get $v \geq 0$ and this is a contradiction. $\square$

Actually each functional f_ω depends on α, too. We do not emphasize such a dependence explicitly in view of the results we will prove in the last sections, but, anyway, it should be kept in mind that $f_\omega = f_{\omega,\alpha}$. With such a convention, we can give the following

Definition 5.2. For every α in $\mathbb{R}$, $\omega > 0$ and $j \geq 1$ set

$$M_j^\omega(\alpha) = \max_{H_j} f_\omega.$$

Proposition 5.3. *Assume* (H) *and* $\Lambda_j < \lambda_1^2 - c\lambda_1$. *Then for every* $\omega > 0$

(a) $M_j^\omega(\alpha) < +\infty$;

(b) $\sup \phi < 0$ *and* $\alpha < \Lambda_j \implies M_j^\omega(\alpha) > 0$;

(c) $\alpha \geq \Lambda_j \implies M_j^\omega(\alpha) = 0$;

(d) $\lim\limits_{\alpha \to \Lambda_j} M_j^\omega(\alpha) = 0$.

Proof.

(a) Suppose by contradiction that there exists v_h in H_j such that

$$\frac{1}{2}\int |\Delta v_h|^2 - \frac{c}{2}\int |Dv_h|^2 - \frac{\alpha}{2}\int v_h^2 - \frac{\omega}{k}\int ((v_h - \phi)^-)^k \geq h. \tag{5}$$

Whatever c and α are, we get $\|v_h\| \to \infty$, and so, up to a subsequence, $v_h/\|v_h\| \to v$ in $H_j \setminus \{0\}$. Dividing both sides of inequality (5) by $\|v_h\|^k$ we get $v \geq 0$, which is impossible.

(b) Let $E \neq 0$ be an eigenfunction with eigenvalue Λ_j and such that $E - \phi \geq 0$ (this is possible since $\sup \phi < 0$ and functions with eigenvalue Λ_j are smooth). Then $f_\omega(E) = \frac{\Lambda_j - \alpha}{2} \int E^2 > 0$.
(c) This is nothing else but **(a)** of Lemma 4.5.
(d) By contradiction, suppose there exist $\alpha_h \to \Lambda_j$, v_h in H_j and $\varepsilon > 0$ such that

$$M_k^\omega(\alpha_h) = \frac{1}{2}\int |\Delta v_h|^2 - \frac{c}{2}\int |Dv_h|^2 - \frac{\alpha_h}{2}\int v_h^2 - \frac{\omega}{k}\int ((v_h - \phi)^-)^k \geq \varepsilon > 0. \tag{6}$$

If $(v_h)_h$ is bounded, then, up to a subsequence, $v_h \to v$ in H_j and

$$0 = M_j(\Lambda_j) \geq \frac{1}{2}\int |\Delta v|^2 - \frac{c}{2}\int |Dv|^2 - \frac{\Lambda_j}{2}\int v^2 - \frac{\omega}{k}\int ((v - \phi)^-)^k \geq \varepsilon > 0,$$

which is clearly absurd. Then $\|v_h\| \to \infty$ and, up to a subsequence, $v_h/\|v_h\| \to v$ in $H_j \setminus \{0\}$. Dividing both sides of inequality (6)by $\|v_h\|^k$ we get $v \geq 0$, which is impossible. □

Proposition 5.4. *Assume (H), $\phi \leq 0$ and let $\Lambda_l \leq \alpha < \Lambda_{l+1} \leq \ldots \leq \Lambda_s < \Lambda_{s+1}$, $s \geq l+1$ and $\Lambda_s < \lambda_1^2 - c\lambda_1$. Then there exist $\rho'' > \rho > \rho' \geq 0$ and $\rho_1 > 0$ such that*

$$\inf_{\substack{w \in H_l^\perp, \\ \|w\| = \rho}} f_\omega(w) > 0 \geq \sup_{u \in \mathcal{T}} f_\omega(u),$$

where $\mathcal{T} = \partial_{H_s}\mathcal{D}$ and

$$\mathcal{D} = \left\{ u = v + w \,\middle|\, v \in H_l, w \in \operatorname{Span}(E_{l+1}, \ldots, E_s), \|v\| \leq \rho_1, \rho' \leq \|w\| \leq \rho'' \right\}.$$

Proof. By **(c)** of Proposition 5.3 $M_l^\omega(\alpha) = 0$, by Lemma 5.1 and by **(b)** of Lemma 4.5 there exist $R > \rho > 0$ such that

$$\inf_{\substack{w \in H_l^\perp, \\ \|w\| = \rho}} f_\omega(w) > \max\left\{ M_l^\omega(\alpha), \sup_{\substack{v \in H_s, \\ \|v\| = R}} f_\omega(v) \right\},$$

and the thesis follows. □

Lemma 5.5. *Assume (H) and $\Lambda_s < \lambda_1^2 - c\lambda_1$. Then*

$$\lim_{\substack{v \in H_s, \sigma \geq 0 \\ \|v - \sigma e_1\| \to \infty}} f_\omega(v - \sigma e_1) = -\infty.$$

Proof. Assume by contradiction that there exist v_h in H_s, $\sigma_h \geq 0$ and C in $\mathbb{R}$ such that $\|v_h - \sigma_h e_1\| \to \infty$ and

$$\begin{aligned} &\frac{1}{2}\int |\Delta(v_h - \sigma_h e_1)|^2 - \frac{c}{2}\int |D(v_h - \sigma_h e_1)|^2 \\ &-\frac{\alpha}{2}\int (v_h - \sigma_h e_1)^2 - \frac{\omega}{k}\int ((v_h - \sigma_h e_1 - \phi)^-)^k \geq C. \end{aligned} \tag{7}$$

Up to subsequence, $\frac{v_h - \sigma_h e_1}{\|v_h - \sigma_h e_1\|} \to v - \sigma e_1$, where $v \in H_s$ and $\sigma \geq 0$. Dividing both sides of inequality (7) by $\|v_h - \sigma_h e_1\|^k$ we get $v - \sigma e_1 \geq 0$. Then $0 \leq \langle v - \sigma e_1, e_1 \rangle = -\sigma\lambda_1^2$, and so $\sigma = 0$. In this way $v \geq 0$. But $\|v\| = 1$ and this is not possible. □

Remark 5.6. Note that if $e_1 \in \mathrm{Span}(E_{l+1}, \dots, E_s)$ we can substitute e_1 with the function e chosen for the case $\alpha > \lambda_1^2 - c\lambda_1$, and then Lemma 5.5 holds $\forall N \geq 2$, and it gives a forcing solution of (P_ω) by applying the Linking Theorem. $\triangle$

Proposition 5.7. *Assume* (H), $\phi \leq 0$ *and let* $\Lambda_l < \Lambda_{l+1} = \ldots = \Lambda_s < \Lambda_{s+1}$, $s \geq l+1$ *and* $\Lambda_s < \lambda_1^2 - c\lambda_1$. *Then there exists an open neighborhood* $\mathcal{O}_s^\omega$ *of* Λ_s *such that, if* $\alpha \in \mathcal{O}_s^\omega \cap [\Lambda_l, \Lambda_s)$, *there exist* $R > \rho > 0$ *such that*

$$\inf_{\substack{w \in H_s^\perp, \\ \|w\| = \rho}} f_\omega(w) > \sup_{z \in \Sigma_R(H_s, e_1)} f_\omega(z),$$

where $\Sigma_R(H_s, e_1)$ *is the boundary in* $H_s \oplus \mathit{Span}(e_1)$ *of* $\Delta_R(H_s, e_1)$ *and*

$$\Delta_R(H_s, e_1) = \left\{ z = v - \sigma e_1 \,\middle|\, v \in H_s, \sigma \geq 0, \|z\| \leq R \right\}.$$

Proof. Define

$$\mathcal{O}_s^\omega = \left\{ \alpha \in [\Lambda_l, \Lambda_{s+1}) \,\middle|\, \exists\, \rho > 0 \text{ such that } M_s^\omega(\alpha) < \inf_{\substack{w \in H_s^\perp, \\ \|w\| = \rho}} f_\omega(w) \right\}.$$

By **(d)** of Proposition 5.3 $M_s^\omega(\alpha) \to 0$ as $\alpha \to \Lambda_s$; by **(b)** of Lemma 4.5

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \inf_{\substack{w \in H_s^\perp, \\ \|w\| = \rho}} f_\omega(w) \geq C_s^+ = \inf_{n_i \geq s+1} \frac{\lambda_{n_i}^2 - c\lambda_{n_i} - \alpha}{\lambda_{n_i}^2}.$$

But

$$C_s^+ \geq \inf_{n_i \geq s+1} \frac{\lambda_{n_i}^2 - c\lambda_{n_i} - \Lambda_s}{\lambda_{n_i}^2} > 0$$

for every $\alpha < \Lambda_s$. In this way $\mathcal{O}_s^\omega \neq \emptyset$ and it is an open neighborhood of Λ_s. Moreover by Lemma 5.5 there exists $R > \rho > 0$ such that

$$\inf_{\substack{w \in H_s^\perp, \\ \|w\| = \rho}} f_\omega(w) > \sup_{\substack{v \in H_s, \sigma \geq 0, \\ \|v - \sigma e_1\| = R}} f_\omega(v - \sigma e_1),$$

and the thesis follows. $\square$

Second case: $\lambda_1^2 - c\lambda_1 \leq \Lambda_l$.

As usual we suppose that there exists $1 \leq r \leq l$ such that $E_r = e_1$.

Proposition 5.8. *Assume* (H), $\lambda_1^2 - c\lambda_1 \leq \Lambda_l$ *and let* $j \geq r$. *Then for every* $\omega > 0$

(a) $\alpha > \Lambda_j^\star \implies M_j^\omega(\alpha) < +\infty$.

(b) $\sup \phi < 0$ *and* $\alpha < \Lambda_j \implies M_j^\omega(\alpha) > 0$.

(c) $\alpha \geq \Lambda_j \implies M_j^\omega(\alpha) = 0$.

(d) $j > r \implies \lim_{\alpha \to \Lambda_j} M_j^\omega(\alpha) = 0$.

Proof. The proof is very similar to the one of Proposition 5.3.
(a) Starting as in the proof of **(a)** of Proposition 5.3, we obtain a function $v \geq 0$ in H_j such that $\|v\| = 1$ and such that $0 \leq Q_{c,\alpha}(v) \leq \frac{\Lambda_j^\star - \alpha}{2} \int v^2$, which implies $v \equiv 0$, and this is absurd.
(b) and **(c)** are proved as in Proposition 5.3.
(d) Starting as in the proof of **(d)** of Proposition 5.3 we obtain a function v in H_j such that $\|v\| = 1$ and $v \geq 0$. But moreover $Q_{c,\alpha}(v) = 0$. So v belongs to the subspace spanned by the eigenfunctions associated to Λ_j, and this is impossible, because $j > r$. □

Lemma 5.9. *Assume (H) and $\alpha > \Lambda_j^\star$, $j \geq r$. Then*

$$\lim_{\substack{v \in H_j, \\ \|v\| \to \infty}} f_\omega(v) = -\infty.$$

Proof. Starting as in Lemma 5.1 we obtain that there is v in H_j, $v \geq 0$, $\|v\| = 1$ such that $0 \leq Q_{c,\alpha}(v) \leq \frac{\Lambda_j^\star - \alpha}{2} \int v^2$. This implies $v \equiv 0$, which is absurd. □

Proposition 5.10. *Assume (H), $\phi \leq 0$ and let $\lambda_1^2 - c\lambda_1 \leq \Lambda_l \leq \alpha < \Lambda_{l+1} \leq \ldots \leq \Lambda_s < \Lambda_{s+1}$, $s \geq l+1$. Suppose $\alpha > \Lambda_s^\star$. Then there exist $\rho'' > \rho > \rho' \geq 0$ and $\rho_1 > 0$ such that*

$$\inf_{\substack{w \in H_l^\perp, \\ \|w\| = \rho}} f_\omega(w) > 0 \geq \sup_{u \in \mathcal{T}} f_\omega(u),$$

where $\mathcal{T} = \partial_{H_s} \mathcal{D}$ and

$$\mathcal{D} = \left\{ u = v + w \,\middle|\, v \in H_l, w \in \operatorname{Span}(E_{l+1}, \ldots, E_s), \|v\| \leq \rho_1, \rho' \leq \|w\| \leq \rho'' \right\}.$$

Proof. By **(c)** of Proposition 5.8, $M_l^\omega(\alpha) = 0$; by Lemma 5.9 (applied with $j = s$) and by **(b)** of Lemma 4.5 there exist $R > \rho > 0$ such that

$$\inf_{\substack{w \in H_l^\perp, \\ \|w\| = \rho}} f_\omega(w) > \max \left\{ M_l^\omega(\alpha), \sup_{\substack{v \in H_s, \\ \|v\| = R}} f_\omega(v) \right\},$$

and the thesis follows. □

If $N \geq 2$ consider again the function e chosen before.

Lemma 5.11. *Assume (H) and let $\lambda_1^2 - c\lambda_1 \leq \Lambda_l \leq \alpha \leq \Lambda_{l+1} \leq \ldots \leq \Lambda_s < \Lambda_{s+1}$, $s \geq l+1$. If $\alpha > \Lambda_s^\star$, then*

$$\lim_{\substack{v \in H_s, \sigma \geq 0 \\ \|v - \sigma e\| \to \infty}} f_\omega(v - \sigma e) = -\infty.$$

Proof. Starting as in Lemma 5.5 we find v in H_s and $\sigma \geq 0$ such that $v - \sigma e \geq 0$, $\|v - \sigma e\| = 1$ and $0 \leq Q_{c,\alpha}(v - \sigma e)$. Then $\sigma = 0$ and $v \geq 0$, so that $0 \leq Q_{c,\alpha}(v) \leq \frac{\Lambda_k^\star - \alpha}{2} \int v^2$. This implies $v \equiv 0$, which is absurd. □

Remark 5.12. If $\Lambda_{s+1}^\star < \Lambda_s$ and $\alpha > \Lambda_{s+1}^\star$, we can substitute e with E_{s+1} and Lemma 5.11 holds $\forall N \geq 1$. But, in general, this condition is hardly satisfied (see also Remark 4.13). △

Proposition 5.13. *Assume (H), $\phi \le 0$ and let $\lambda_1^2 - c\lambda_1 \le \Lambda_l < \Lambda_{l+1} = \ldots = \Lambda_s < \Lambda_{s+1}$, $s \ge l+1$ and $N \ge 2$. Then there exists an open neighborhood $\mathcal{O}_s^\omega$ of Λ_s such that, if $\alpha \in \mathcal{O}_s^\omega \cap [\Lambda_l, \Lambda_s)$, there exist $R > \rho > 0$ such that*

$$\inf_{\substack{w\in H_s^\perp,\\ \|w\|=\rho}} f_\omega(w) > \sup_{z\in\Sigma_R(H_s,e)} f_\omega(z),$$

where $\Sigma_R(H_s, e)$ is the boundary in $H_s \oplus Span(e)$ of $\Delta_R(H_s, e)$ and

$$\Delta_R(H_s,e) = \left\{ z = v - \sigma e \,\middle|\, v \in H_s, \sigma \ge 0, \|z\| \le R \right\}.$$

Proof. Consider the set

$$\mathcal{O}_s^\omega = \left\{ \alpha \in (\Lambda_l, \Lambda_{s+1}) \,\middle|\, \alpha > \Lambda_s^\star,\ \exists\, \rho > 0 \text{ such that } M_s(\alpha) < \inf_{\substack{w\in H_s^\perp,\\ \|w\|=\rho}} f(w) \right\}.$$

Since $s \ge l+1$, $\Lambda_s^\star < \Lambda_s$; by **(b)** of Lemma 4.5 there exists $\rho > 0$ and $C_s^+ > 0$ such that $\inf_{\substack{w\in H_s^\perp,\\ \|w\|=\rho}} f_\omega(w) \ge C_s^+\rho^2$. Moreover, by **(d)** of Proposition 5.9 $M_s^\omega(\alpha) \to 0$ if $\alpha \to \Lambda_s$. We deduce that $\Lambda_s \in \mathcal{O}_s^\omega$, so $\mathcal{O}_s^\omega$ is a non empty open neighborhood of Λ_s itself. Moreover by Lemma 5.11 there exist $R > \rho > 0$ such that

$$\inf_{\substack{w\in H_s^\perp,\\ \|w\|=\rho}} f_\omega(w) > \sup_{\substack{v\in H_s,\sigma\ge 0,\\ \|v-\sigma e\|=R}} f_\omega(v - \sigma e).$$

□

Lemma 5.14. *Assume (H), $\phi \le 0$ and let $\Lambda_l < \Lambda_{l+1} \le \ldots \le \Lambda_s < \Lambda_{s+1}$, $s \ge l+1$. Then for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for every α in $[\Lambda_l + \delta, \Lambda_{s+1} - \delta]$, the unique critical point of f constrained on $H_l \oplus H_s^\perp$ in $f_\omega^{-1}([-\varepsilon_0, \varepsilon_0])$ is the trivial one.*

Proof. Suppose by contradiction that there exist $\delta > 0$, α_n in $[\Lambda_l + \delta, \Lambda_{s+1} - \delta]$, $\alpha_n \to \alpha$ and u_n in $H_l \oplus H_s^\perp \setminus \{0\}$ such that

$$f_\omega^n(u_n) := \frac{1}{2}\int |\Delta u_n|^2 - \frac{c}{2}\int |Du_n|^2 - \frac{\alpha_n}{2}\int u_n^2 + \frac{\omega}{k}\int ((u_n - \phi)^-)^k \to 0$$

and for every u in $H_l \oplus H_s^\perp$

$$\int \Delta u_n \Delta u - c\int Du_n \cdot Du - \alpha_n \int u_n u + \omega \int ((u_n - \phi)^-)^{k-1} u = 0. \tag{8}$$

Set $u_n = v_n + w_n$, where $v_n \in H_l$ and $w_n \in H_s^\perp$.
I. In (8) take $u = w_n - v_n$. Then

$$\begin{aligned} &\left(\int |\Delta w_n|^2 - c\int |Dw_n|^2 - \alpha_n \int w_n^2\right) - \left(\int |\Delta v_n|^2 - c\int |Dv_n|^2 - \alpha_n\int v_n^2\right) \\ &\qquad = -\omega \int ((u_n - \phi)^-)^{k-1}(w_n - v_n). \end{aligned} \tag{9}$$

But

$$
(10)\qquad \begin{aligned} -\omega\int((u_n-\phi)^-)^{k-1}(w_n-v_n) &\le \omega\left(\int((u_n-\phi)^-)^k\right)^{1-1/k}\|w_n-v_n\|_{L^k(\Omega)} \\ &\le C\left(\int((u_n-\phi)^-)^k\right)^{1-1/k}\|w_n-v_n\| = C\left(\int((u_n-\phi)^-)^k\right)^{1-1/k}\|u_n\|, \end{aligned}
$$

since v_n and w_n are orthogonal.

Now we want to show that there exists a positive constant a such that the l.h.s. of (9) is greater than $a\|u_n\|^2$. First of all observe that $\int|\Delta v_n|^2 - c\int|Dv_n|^2 - \alpha_n\int v_n^2 \le \max_{n_i\le l}\left(-\frac{\delta}{\lambda_{n_i}^2}\right)\|v_n\|^2$. Moreover there exists $C_s^+>0$ such that $\forall\, w$ in $H_s^\perp$

$$
\int|\Delta w_n|^2 - c\int|Dw_n|^2 - \alpha_n\int w_n^2 \ge C_s^+\|w\|^2.
$$

In this way we have proved that there exists $a>0$ such that

$$
(11)\qquad \int|\Delta u_n|^2 - c\int|Du_n|^2 - \alpha_n\int u_n^2 \ge a(\|v_n\|^2+\|w_n\|^2) = a\|u_n\|^2.
$$

Since $u_n\not\equiv 0$, (9), (10) and (11) give

$$
(12)\qquad a\|u_n\| \le C\left(\int((u_n-\phi)^-)^k\right)^{1-1/k}.
$$

II. Putting $u=u_n$ in (8) gives

$$
0 = 2f_\omega^n(u_n) + \left(\frac{2}{k}-1\right)\omega\int((u_n-\phi)^-)^k + \omega\int((u_n-\phi)^-)^{k-1}\phi
$$

and this equality implies that $\int((u_n-\phi)^-)^k$ and $\int((u_n-\phi)^-)^{k-1}\phi$ go to 0 as n goes to ∞. Then (12) implies that $\|u_n\|\to 0$. But $\left(\int((u_n-\phi)^-)^k\right)^{1-1/k} = O(\|u_n\|^{k-1})$ (see Lemma 4.3) and a contradiction arises. $\square$

Lemma 5.15. *Assume (H) and let $\Lambda_l < \Lambda_{l+1} \le \ldots \le \Lambda_s < \Lambda_{s+1}$, $s\ge l+1$, $\alpha\ne\lambda_1^2-c\lambda_1$. Denote by $P: H\longrightarrow (E_{l+1},\ldots,E_s)$ and $Q: H\longrightarrow H_l\oplus H_s^\perp$ the orthogonal projections. Suppose u_n in H is such that $f_\omega(u_n)$ is bounded, $Pu_n\to 0$ and $Q\nabla f_\omega(u_n)\to 0$. Then $(u_n)_n$ is bounded in H.*

Proof. Suppose by contradiction that there exists a sequence $(u_n)_n$ in H such that $\|u_n\|\to\infty$, $Pu_n\to 0$, $f_\omega(u_n)$ is bounded and

$$
Q(u_n + i^\star(c\Delta u_n - \alpha u_n + \omega((u_n-\phi)^-)^{k-1}) \to 0,
$$

where $i^\star: L^2(\Omega)\to H$ is the compact adjoint operator of the immersion $i: H\to L^2(\Omega)$. Up to a subsequence, we can suppose that $u_n/\|u_n\|\rightharpoonup u$ in H.

Since $f_\omega(u_n)$ is bounded, dividing it by $\|u_n\|^k$, we get $\int(u^-)^k=0$, and so $u\ge 0$.

Now observe that

$$
\langle Q\nabla f(u_n), u_n\rangle = \langle\nabla f(u_n), u_n\rangle - \langle P\nabla f(u_n), u_n\rangle = 2f_\omega(u_n)
$$

$$+\left(\frac{2}{k}-1\right)\omega\int((u_n-\phi)^-)^k+\omega\int((u_n-\phi)^-)^{k-1}\phi$$

$$-\int\Delta P(u_n+i^\star(c\Delta u_n-\alpha u_n+\omega((u_n-\phi)^-)^{k-1}))\Delta u_n=2f_\omega(u_n)$$

$$+\left(\frac{2}{k}-1\right)\omega\int((u_n-\phi)^-)^k+\omega\int((u_n-\phi)^-)^{k-1}\phi$$

$$-\int|\Delta Pu_n|^2-\int\Delta Pi^\star(c\Delta u_n-\alpha u_n+\omega((u_n-\phi)^-)^{k-1})\Delta u_n.$$

Now observe that $Pi^\star(c\Delta u_n-\alpha u_n+\omega((u_n-\phi)^-)^{k-1})\in\mathrm{Span}(E_{l+1},\dots,E_s)$, so it is a smooth function. In this way the last integral above is equal to

$$\int\Delta^2Pi^\star[(c\Delta u_n-\alpha u_n+\omega((u_n-\phi)^-)^{k-1})]u_n=\int\Delta^2Pi^\star[(c\Delta u_n-\alpha u_n$$

$$+\omega((u_n-\phi)^-)^{k-1})]Pu_n=\int\Delta^2i^\star((c\Delta u_n-\alpha u_n+\omega((u_n-\phi)^-)^{k-1}))Pu_n$$

$$=-c\int|DPu_n|^2-\alpha\int(Pu_n)^2+\omega\int((u_n-\phi)^-)^{k-1}Pu_n.$$

Moreover

$$\left|\int((u_n-\phi)^-)^{k-1}Pu_n\right|\le\left(\int((u_n-\phi)^-)^k\right)^{1-1/k}\left(\int|Pu_n|^k\right)^{1/k}$$

$$\le C\left(\int((u_n-\phi)^-)^k\right)^{1-1/k}\|Pu_n\|.$$

Finally

$$\langle Q\nabla f(u_n),u_n\rangle=2f_\omega(u_n)+\left(\frac{2}{k}-1\right)\omega\int((u_n-\phi)^-)^k+\omega\int((u_n-\phi)^-)^{k-1}\phi$$

$$-\int|\Delta Pu_n|^2+c\int|DPu_n|^2+\alpha\int(Pu_n)^2-\omega\int((u_n-\phi)^-)^{k-1}Pu_n.$$

Dividing by $\|u_n\|^{k/(k-1)}$, we get

$$0\le\frac{\left(1-\frac{2}{k}\right)\omega\int((u_n-\phi)^-)^k}{\|u_n\|^{k/(k-1)}}=-\frac{\langle Q\nabla f(u_n),u_n\rangle}{\|u_n\|^{k/(k-1)}}+\frac{2f_\omega(u_n)}{\|u_n\|^{k/(k-1)}}$$

$$+\frac{\omega\int((u_n-\phi)^-)^{k-1}\phi}{\|u_n\|^{k/(k-1)}}-\frac{\int|\Delta Pu_n|^2}{\|u_n\|^{k/(k-1)}}+\frac{c\int|DPu_n|^2}{\|u_n\|^{k/(k-1)}}$$

$$+\frac{\alpha \displaystyle\int (Pu_n)^2}{\|u_n\|^{k/(k-1)}} - \frac{\omega \displaystyle\int ((u_n-\phi)^-)^{k-1} Pu_n}{\|u_n\|^{k/(k-1)}}.$$

Throwing away the non positive terms, the last quantity is less or equal to

$$-\frac{\langle Q\nabla f_\omega(u_n), u_n\rangle}{\|u_n\|^{k/(k-1)}} + \frac{2f_\omega(u_n)}{\|u_n\|^{k/(k-1)}} + \frac{|c| \displaystyle\int |DPu_n|^2}{\|u_n\|^{k/(k-1)}} + \frac{|\alpha| \displaystyle\int (Pu_n)^2}{\|u_n\|^{k/(k-1)}}$$

$$-\frac{\omega \displaystyle\int ((u_n-\phi)^-)^{k-1} Pu_n}{\|u_n\|^{k/(k-1)}} \le o(1) + C \left(\frac{\displaystyle\int ((u_n-\phi)^-)^k}{\|u_n\|^{k/(k-1)}} \right)^{1-1/k} \frac{\|Pu_n\|}{\|u_n\|^{1/(k-1)}},$$

where $o(1) \to 0$ as $n \to \infty$. In this way $\dfrac{\int ((u_n-\phi)^-)^k}{\|u_n\|^{k/(k-1)}}$ is bounded, and, up to a subsequence, it converges to 0. Then $\dfrac{((u_n-\phi)^-)^{k-1}}{\|u_n\|}$ converges to 0 in $L^{k/(k-1)}(\Omega)$, and hence in H'.

But $Qu_n = u_n - Pu_n$, so

$$\frac{Q\nabla f(u_n)}{\|u_n\|} = \frac{u_n}{\|u_n\|} - \frac{Pu_n}{\|u_n\|} + Qi^\star \left(c\frac{\Delta u_n}{\|u_n\|} - \alpha \frac{u_n}{\|u_n\|} + \omega \frac{((u_n-\phi)^-)^{k-1}}{\|u_n\|} \right) \to 0, \tag{13}$$

and then $u_n/\|u_n\| \to u$ strongly in H and $\|u\| = 1$. But, on the other hand, from (13) we get that $u \in H_l \oplus H_s^\perp$ and it is a solution of

$$\begin{cases} \Delta^2 u + c\Delta u - \alpha u = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

in $H_l \oplus H_s^\perp$, which is also non negative from the first step. But then u is a non negative and non trivial function belonging to the subspace spanned by the eigenfunctions associated to null eigenvalues of $Q_{c,\alpha}$, and this is impossible, since e_1 doesn't belong to this space. In fact, if $u = \sum \beta_i E_i$, multiplying the equation of (14) by e_1 and integrating, we get $(\lambda_1^2 - c\lambda_1 - \alpha)\beta_1 = 0$, which would imply $\beta_1 = 0$, while $u \ge 0$, $u \not\equiv 0$, so its component along e_1 must be positive. □

Proposition 5.16. *Assume (H), $\phi \le 0$ and let $\Lambda_l < \Lambda_{l+1} \le \ldots \le \Lambda_s < \Lambda_{s+1} \le \ldots$, $s \ge l+1$ and $\alpha \ne \lambda_1^2 - c\lambda_1$. For every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for every ε', ε'' in $(0, \varepsilon_0)$ and for every α in $[\Lambda_l + \delta, \Lambda_{s+1} - \delta]$, the condition $(\nabla)(f_\omega, H_l \oplus H_s^\perp, \varepsilon', \varepsilon'')$ holds.*

Proof. Suppose by contradiction that there exists a sequence $(u_n)_n$ in H such that $\|Pu_n\| = d(u_n, H_l \oplus H_s^\perp) \to 0$, $Q\nabla f_\omega(u_n) \to 0$, $\varepsilon' \le f_\omega(u_n) \le \varepsilon''$. By Lemma 5.15 $(u_n)_n$ is bounded and, up to a subsequence, $u_n \to u$ in H with $Pu = 0$ and $Q\nabla f(u) = 0$, that is u is a critical point of f constrained on $H_l \oplus H_s^\perp$. By Lemma 5.14, $u = 0$. But then $0 = f_\omega(u) = \lim f_\omega(u_n) \ge \varepsilon'$, since f_ω is continuous, and thus we get a contradiction. □

Theorem 5.17. *Assume (H), $\phi \le 0$, $\Lambda_l < \Lambda_{l+1} = \ldots = \Lambda_s < \Lambda_{s+1} \le \ldots$ and $\lambda_1^2 - c\lambda_1 > \Lambda_s$, $s \ge l+1$. Then there exists $\tau_s^\omega > 0$ such that, if $\alpha \in (\Lambda_s - \tau_s^\omega, \Lambda_s)$, then f_ω has at least three non trivial critical points which are forcing solutions of problem (P_ω).*

Proof. The proof parallels the proof of next Theorem 5.18. $\square$

Theorem 5.18. *Assume (H), $\phi \le 0$, $N \ge 2$, $\Lambda_l < \Lambda_{l+1} = \ldots = \Lambda_s < \Lambda_{s+1} \le \ldots$, $s \ge l+1$, $\lambda_1^2 - c\lambda_1 \le \Lambda_l$. Then there exists $\tau_s^\omega > 0$ such that, if $\alpha \in (\Lambda_s - \tau_s^\omega, \Lambda_s)$, then f_ω has at least three non trivial critical points which are forcing solutions of problem (P_ω). If $\Lambda_{s+1}^\star < \Lambda_s$ the thesis is true $\forall\, N \ge 1$.*

Proof. Fix $\delta > 0$, take ε_0 as in Proposition (5.16) and define $\mathcal{U}_s^\omega = \mathcal{O}_s^\omega \cap \mathcal{A}_s^\omega(\delta)$, where

$$\mathcal{A}_s^\omega(\delta) = \left\{\alpha \in [\Lambda_l - \delta, \Lambda_{s+1} - \delta] \,\middle|\, M_s^\omega(\alpha) < \varepsilon_0\right\}.$$

By **(c)** and **(d)** of Proposition (5.8), $\mathcal{U}_s^\omega$ is not empty and it is an open neighborhood of Λ_s. Moreover Proposition (5.10) and $(\nabla)(f_\omega, H_l \oplus H_s^\perp, \varepsilon', \varepsilon'')$ hold, where $\varepsilon' < \varepsilon''$ and

$$\max\{\sup f_\omega(\mathcal{T}), 0\} < \varepsilon' < \inf_{\substack{w \in H_l^\perp,\\ \|w\|=\rho}} f_\omega(w) \qquad \text{and} \qquad M_s^\omega(\alpha) < \varepsilon'' < \varepsilon_0.$$

So the (∇)-Theorem A.5 gives 3 critical points u_ω^i such that $f_\omega(u_\omega^i) \le M_s^\omega(\alpha)$.

By Proposition (5.13) and the Linking Theorem there exist 2 critical points v_ω^1, v_ω^2 such that

$$\sup_{\Delta_R} f_\omega \ge f_\omega(v_\omega^1) \ge \inf_{\substack{w \in H_s^\perp,\\ \|w\|=\rho}} f_\omega(w) > \sup_{\Sigma_R} \ge f_\omega(v_\omega^2) \ge \inf_{\substack{w \in H_s^\perp,\\ \|w\|\le\rho}} f_\omega(w).$$

In this way $v_\omega^1 \ne u_\omega^i$ by the very definition of $\mathcal{O}_s^\omega$ $(\inf_{\substack{w \in H_s^\perp,\\ \|w\|=\rho}} f_\omega(w) > M_s^\omega(\alpha))$. $\square$

Corollary 5.19. *Under the assumptions of Theorem 5.17 or of Theorem 5.18, if $N \le 3$ and $\sup\phi < 0$, two different critical levels c_ω^1 and c_ω^2 are such that $\liminf_\omega c_\omega^1 > \limsup_\omega c_\omega^2$. Moreover there exists $\varepsilon > 0$ such that*

$$\inf_\omega f_\omega(z_\omega) \ge \varepsilon,$$

z_ω being one of the critical points of f_ω found in Theorem 5.17 or in Theorem 5.18.

Proof. Since $\sup\phi < 0$, in all the inequalities obtained for proving the existence of one solution, we can suppose that the radius of the small sphere in the Mountain Pass Theorem or the radii of the vertical spheres in the Linking Theorems $(\{w \in H_l^\perp \mid \|w\| = \rho\})$ are small enough so that the values of the functionals f_ω are not influenced by the perturbation term $\omega \int((u-\phi)^-)^k$. So there exists $\varepsilon > 0$ such that

$$f_\omega(u_\omega) = \frac{1}{2}\int |\Delta u_\omega|^2 - \frac{c}{2}\int |Du_\omega|^2 - \frac{\alpha}{2}\int u_\omega^2 \ge \varepsilon \qquad \forall\, \omega.$$

Moreover we can find $\sigma_1 > \sigma_2 > \sigma_3 > \sigma_4 > 0$ and three sequences of solutions v_ω^1 and u_ω^j, $j = 1, 2$ such that $\forall\, \omega$

$$\sigma_1 > f_\omega(v_\omega^1) = c_\omega^1 > \sigma_2 \left(= \inf_{\substack{w \in H_s^\perp,\\ \|w\|=\rho_\omega}} f_\omega(w)\right) > \sigma_3 \; (= M_s^\omega(\alpha)) \ge f_\omega(u_\omega^j) = c_\omega^2 > \sigma_4 > 0$$

and such that $v_\omega^1 \to v$ and $u_\omega^j \to u^j$, $j = 1, 2$ in H. Indeed M_s^ω is a decreasing function of ω. So in the definition of $\mathcal{O}_s^\omega$, in which we require that there exists ρ_ω such that $M_s^\omega(\alpha) < \inf_{\substack{w \in H_s^\perp,\\ \|w\|=\rho_\omega}} f_\omega(w)$, we can choose ρ_ω constant (for example equal to ρ_1) and small enough so that $(w-\phi)^- = 0$ and so $\inf\limits_{\substack{w\in H_s^\perp,\\ \|w\|=\rho_\omega}} f_\omega(w)$ is independent on ω. In this way $\mathcal{O}_s^\omega$ is independent on ω, too.

Passing to the limit we get

$$\liminf_{\omega\to+\infty} c_\omega^1 > \limsup_{\omega\to+\infty} c_\omega^2.$$

□

We finally observe that the procedure to obtain the multiplicity results above can be repeated for any functional in which the quadratic form $Q_{c,\alpha}$ is replaced by a quadratic form whose gradient has the form (linear operator)+(compact operator). For example, one can consider functionals defined on $W_0^{1,2}(\Omega)$ having the form

$$\tilde{f}(u) = \frac{1}{2}\int |Du|^2 - \frac{\alpha}{2}\int u^2 - \frac{\omega}{k}\int ((u-\phi)^-)^k.$$

See [18] for an application when $-\phi$ is replaced by e_1.

6 *A priori* estimate

From now on we will consider a sequence $(\omega_n)_n$ of real positive numbers such that $\omega_n \to +\infty$ as $n\to\infty$. For the sake of simplicity, we will write ω instead of ω_n and with $\omega\to+\infty$ we will mean that $n\to\infty$ (and then $\omega_n\to+\infty$).

Theorem 6.1. *Assume (H), $\alpha \neq \lambda_1^2 - c\lambda_1$, there exists $\tau > 0$ such that $\tau e_1 + \phi \le 0$ a.e. in Ω (for example $\sup\phi < 0$) and u_ω is a solution of (P_ω) with $\sup_\omega f_\omega(u_\omega) < +\infty$. Then*

- $\inf\limits_\omega f_\omega(u_\omega) > -\infty$;
- $\sup\limits_\omega\ \omega\int((u_\omega-\phi)^-)^k < +\infty$;
- $\inf\limits_\omega\ \omega\int((u_\omega-\phi)^-)^{k-1}\phi > -\infty$;
- *$(u_\omega)_\omega$ is bounded in H.*

Proof. Suppose by contradiction that $(u_\omega)_\omega$ is unbounded. We can suppose that, up to a subsequence, $\|u_\omega\|\to+\infty$ and that there exists v in H such that $v_\omega = u_\omega/\|u_\omega\|$ weakly converges to v in H, strongly in $L^k(\Omega)$ and a.e. in Ω. Now,

$$\begin{aligned} 0 = f'_\omega(u_\omega)u_\omega &= \int|\Delta u_\omega|^2 - c\int|Du|^2 - \alpha\int u_\omega^2 + \omega\int((u_\omega-\phi)^-)^{k-1}u_\omega \\ &= 2f_\omega(u_\omega) + \left(\frac{2}{k}-1\right)\omega\int((u_\omega-\phi)^-)^k + \omega\int((u_\omega-\phi)^-)^{k-1}\phi, \end{aligned} \tag{15}$$

and since ϕ is negative, if we divide by $\|u_\omega\|$ and pass to the limit, we obtain

$$\lim_{\omega\to+\infty}\frac{\omega\int((u_\omega-\phi)^-)^k}{\|u_\omega\|} = 0, \qquad \lim_{\omega\to+\infty}\frac{\omega\int((u_\omega-\phi)^-)^{k-1}\phi}{\|u_\omega\|} = 0$$

and also

$$\lim_{\omega\to+\infty}\frac{\omega\displaystyle\int((u_\omega-\phi)^-)^{k-1}u_\omega}{\|u_\omega\|}=0, \tag{16}$$

since $((u_\omega-\phi)^-)^{k-1}u_\omega=-((u_\omega-\phi)^-)^k+((u_\omega-\phi)^-)^{k-1}\phi$.

In this way

$$0=\frac{f'(u_\omega)(u_\omega)}{\|u_\omega\|^2}=1-c\int|Dv_\omega|^2-\alpha\int v_\omega^2+\frac{\omega\displaystyle\int((u_\omega-\phi)^-)^{k-1}u_\omega}{\|u_\omega\|^2}\to$$

$$\to 1-c\int|Dv|^2-\alpha\int v^2.$$

If α and c are ≤ 0 this is immediately absurd. Otherwise this equality implies that $v\not\equiv 0$. But

$$0=\frac{f'_\omega(u_\omega)(u_\omega)}{\omega\|u_\omega\|^k}\to-\int(v^-)^k,$$

and so $v\ge 0$. Now observe that

$$0=f'_\omega(u_\omega)\tau e_1=(\lambda_1^2-c\lambda_1-\alpha)\tau\int u_\omega e_1+\omega\int((u_\omega-\phi)^-)^{k-1}\tau e_1. \tag{17}$$

But

$$\int((u_\omega-\phi)^-)^{k-1}\tau e_1=\int((u_\omega-\phi)^-)^{k-1}(\tau e_1+\phi-u_\omega+u_\omega-\phi)$$

$$\le-\int((u_\omega-\phi)^-)^{k-1}u_\omega,$$

and by (16), we get

$$\lim_{\omega\to+\infty}\frac{\omega\displaystyle\int((u_\omega-\phi)^-)^{k-1}e_1}{\|u_\omega\|}=0.$$

Therefore from (17) we obtain

$$0=(\lambda_1^2-c\lambda_1-\alpha)\int ve_1,$$

which is possible if and only if $v\equiv 0$. A contradiction arises and so $(u_\omega)_\omega$ is bounded. From (15) the other statements of the thesis follow. □

Remark 6.2. As already remarked, no existence and bound theorem is proved in the case $\alpha=\lambda_1^2-c\lambda_1$, since it is trivial in the part of existence and it is impossible in the part of an a *priori* estimate. So the requirement $\alpha\neq\lambda_1^2-c\lambda_1$ is natural in this problem. △

Corollary 6.3. *Assume (H), $\alpha \neq \lambda_1^2 - c\lambda_1$, $\sup\phi < 0$ and u_ω is a solution of (P_ω) such that the sequence $(u_\omega)_\omega$ is bounded. Then $\sup\limits_{\omega}\ \omega\displaystyle\int((u_\omega-\phi)^-)^{k-1} < +\infty$.*

Proof. From (15) we get that there exists $M > 0$ such that $-\omega\int((u_\omega-\phi)^-)^{k-1}\phi \le M\ \forall\,\omega$, but the l.h.s of this inequality is bigger than $-\omega\sup\phi\int((u_\omega-\phi)^-)^{k-1}$, and the thesis follows. $\square$

Under the assumptions of Theorem 6.1, $(u_\omega)_\omega$ is bounded, so we can take a weakly convergent subsequence. On the other hand, if there exist a sequence of solutions $(u_\omega)_\omega$ and u in H such that $u_\omega \rightharpoonup u$ in H, then $\omega\int((u_\omega-\phi)^-)^k$ and $\omega\int((u_\omega-\phi)^-)^{k-1}\phi$ are bounded, and thus $\inf_\omega f_\omega(u_\omega) > -\infty$ and $\sup_\omega f_\omega(u_\omega) < +\infty$. In fact

$$0 = \int|\Delta u_\omega|^2 - c\int|Du_\omega|^2 - \alpha\int u_\omega^2 + \omega\int((u_\omega-\phi)^-)^{k-1}u_\omega,$$

and so the last integral is bounded. But it equals $-\int((u_\omega-\phi)^-)^k + \int((u_\omega-\phi)^-)^{k-1}\phi$ and since these integrals are both non positive, they are both bounded and the thesis follows.

7 Bounce

If u_ω is a forcing solution of (P_ω), we define $\mathcal{A}_\omega$ as (a set equivalent to)

$$\mathcal{A}_\omega = \left\{x \in \Omega \,\middle|\, u_\omega(x) < \phi(x)\right\},$$

and

$$\mathcal{A} = \left\{x \in \Omega \,\middle|\, \exists \text{ a neighborhood } U \text{ of } x \text{ and } \exists\,\omega_0 \text{ such that } \forall\,\omega \ge \omega_0\ m(U \cap \mathcal{A}_\omega) = 0\right\}.$$

We observe that $\mathcal{A}$ is an open subset of Ω, and so its complementary set $\mathfrak{B}$ is closed, where

$$\mathfrak{B} = \left\{x \in \Omega \,\middle|\, \forall \text{ neighborhood } U \text{ of } x \text{ and } \forall\,\omega_0,\ \exists\,\omega \ge \omega_0 \text{ such that } m(U \cap \mathcal{A}_\omega) > 0\right\}.$$

We also remark that such a set $\mathfrak{B}$ is, in some sense, the set of points in which u touches ϕ, or the *contact set*; actually, if $N \le 3$ and ϕ is continuous $\mathfrak{B}$ coincides with the set of points x's of Ω such that $u(x) = \phi(x)$.

Theorem 7.1. *If u_ω is a solution of (P_ω) such that $u_\omega \rightharpoonup u$ in H, then*

(i) *$u \ge \phi$ a.e. in Ω;*

(ii) *$\int \Delta u\Delta\psi - c\int Du\cdot D\psi - \alpha\int u\psi \le 0\ \ \forall\,\psi$ in H such that $\psi \ge 0$ in Ω;*

(iii) *$\int \Delta u\Delta\psi - c\int Du\cdot D\psi - \alpha\int u\psi = 0\ \ \forall\,\psi$ in H such that $\psi = 0$ on $\mathfrak{B}$.*

In other words, in the sense of distributions in Ω, we have

(ii)' *$\Delta^2 u + c\Delta u - \alpha u \le 0$ in Ω;*

(iii)' *$\Delta^2 u + c\Delta u - \alpha u = 0$ in $\Omega\setminus\mathfrak{B}$.*

In particular there exists a positive Radon measure μ such that

$$\int\Delta u\Delta\psi - c\int Du\cdot D\psi - \alpha\int u\psi + \int\psi\,d\mu = 0 \qquad \forall\,\psi \text{ in } \mathscr{D}(\Omega), \tag{18}$$

and μ is supported in $\mathfrak{B}$.

Proof. **(i)** Suppose by contradiction that $u-\phi<0$ in a set of positive measure. From the equality

$$(19)\qquad \int \Delta u_\omega \Delta\psi - c\int Du_\omega\cdot D\psi - \alpha\int u_\omega\psi = -\omega\int ((u_\omega-\phi)^-)^{k-1}\psi \qquad \forall\,\psi \text{ in } H,$$

passing to the limit, we would have $\int \Delta u\Delta\psi - c\int Du\cdot D\psi - \alpha\int u\psi = \infty$, which is clearly absurd.
(ii) Take ψ in H and $\psi\geq 0$. Passing to the limit in equation (19), we get the thesis.
(iii) Define the linear operator $L_\omega : \mathscr{D}(\Omega)\longrightarrow\mathbb{R}$ in this way: if $\psi\in\mathscr{D}(\Omega)$

$$L_\omega(\psi)=\omega\int((u_\omega-\phi)^-)^{k-1}\psi = -\int \Delta u_\omega\Delta\psi + c\int Du_\omega\cdot D\psi+\alpha\int u_\omega\psi.$$

Since $u_\omega \rightharpoonup u$ in H, L_ω converges to a linear and bounded operator L defined as $L(\psi) = -\int\Delta u\Delta\psi + c\int Du\cdot D\psi+\alpha\int u\psi$ for every ψ in $\mathscr{D}(\Omega)$. But if $\psi\geq 0$, $L_\omega(\psi)\geq 0$ and so $L(\psi)\geq 0$, that is L is a linear and positive operator defined on $\mathscr{D}(\Omega)$. By Riesz Representation Theorem (see [1] or [4]) there exists a positive Radon measure μ such that $L(\psi)=\int\psi\,d\mu\ \forall\,\psi$ in $\mathscr{D}(\Omega)$.

Let us show that $\operatorname{supp}\mu\subseteq\mathfrak{B}$. Let $x_0\in\Omega\setminus\mathfrak{B}$. Then, by definition, there exists a neighborhood U of x_0 and ω_0 such that $\forall\,\omega\geq\omega_0$ one has $m(U\cap\mathcal{A}_\omega)=0$, that is $u_\omega(x)\geq\phi(x)$ in $U\ \forall\,\omega\geq\omega_0$. Then $\forall\,\psi$ in $C^\infty_C(U)$ we have $\int\Delta u_\omega\Delta\psi - c\int Du_\omega\cdot D\psi-\alpha\int u_\omega\psi = 0\ \forall\,\omega\geq\omega_0$. Passing to the limit

$$\int\Delta u\Delta\psi - c\int Du\cdot D\psi - \alpha\int u\psi = 0 \qquad \forall\,\psi \text{ in } C^\infty_C(U),$$

that is $\Delta^2u+c\Delta u-\alpha u=0$ in U. So $x_o\notin\operatorname{supp}\mu$. ☐

Remark 7.2. We remark that the functionals $L_\omega(\psi)$ converge for every ψ in H, but in general we cannot write the limit functional as an integral, since $\mathscr{D}(\Omega)$ is not dense in H. Moreover this fact implies that the distributional versions **(ii)'** and **(iii)'** of the previous Theorem are, in some sense, weaker than the other ones. △

Remark 7.3. At this point we want to underline that the variational inequality we obtain is, in some sense, "reversed". In fact if $K_\phi=\{u\in H \mid u\geq\phi\}$ is the set of admissible functions, we are looking for a solution in K_ϕ of

$$\Delta^2u+c\Delta u-\alpha u\leq 0,$$

while the classical variational inequality is $\Delta^2u+c\Delta u-\alpha u\geq 0$ (see [5], for example). △

We now want to prove some regularity results in the case $\alpha=0$.

The fact that u is not a solution of a classical variational problem doesn't let us apply the regularization methods related to that theory. Anyway we can still get some information on the solution of the variational inequality by the following

Theorem 7.4 (Maximum Principle). *Suppose $c<\lambda_1$ and u satisfies*

$$(20)\qquad \begin{cases}\Delta^2u+c\Delta u\leq 0 & \textit{in } \Omega,\\ u=\Delta u=0 & \textit{on } \partial\Omega.\end{cases}$$

Then either $u\equiv 0$, or for every ball B contained in Ω, $\sup_B u<0$.

See [16] for a proof.

As an immediate application of this Theorem we get the following

Proposition 7.5. *Suppose $c < \lambda_1$, u is the weak limit of a sequence of solutions of (P_ω) with $\alpha = 0$. Then $\phi \le u < 0$ a.e. in Ω.*

Remark 7.6. A symmetric result holds if we consider the problem

$$\min\left\{\int_\Omega |Du|^2 \,\middle|\, u \in W_0^{1,2}(\Omega), u \ge \Phi\right\},$$

where Φ is an "obstacle" (that is $\Phi_{|\partial\Omega} < 0$ and $\Phi > 0$ in a subset with positive measure of Ω). In fact if u_0 is the unique minimal point, then $0 \le u_0$ (see [9]). We observe that the same holds for the problem

$$\min\left\{\int_\Omega |\Delta u|^2 \,\middle|\, u \in H_0^1(\Omega) \cap H^2(\Omega), u \ge \Phi\right\},$$

which gives $\Delta^2 u \ge 0$ (see [5]). $\triangle$

Proposition 7.7. *Suppose $\phi \in C^4(\overline{\Omega})$, u satisfies problem (20) and $\Delta^2\phi + c\Delta\phi > 0$ in Ω; then the "contact set" $\mathfrak{B} = \{x \in \Omega \,|\, u(x) = \phi(x)\}$ does not have interior points.*

Proof. Suppose there exists x_0 in the interior of $\mathfrak{B}$. In this case $\Delta^2 u(x_0) = \Delta^2\phi(x_0)$. But then

$$0 \ge \Delta^2 u(x_0) + c\Delta u(x_0) = \Delta^2\phi(x_0) + c\Delta\phi(x_0).$$

The last sum is strictly positive and a contradiction arises. $\square$

Suppose u satisfies $\Delta^2 u + c\Delta u \le 0$ in $\mathscr{D}(\Omega)$. We want to show that u has some finer properties of regularity. In general we cannot expect to recover the regularity results for the biharmonic operator (see [3] and [5]), but something can still be said in the case $\alpha = 0$. To do that, however, we follow the ideas of those papers to prove the following Proposition.

Proposition 7.8. *If u satisfies (20), then there exists a function W such that*

(a) *$W = \Delta u$ a.e. in Ω;*

(b) *W is lower semicontinuous;*

(c) *for every x_0 in Ω and for every sequence of balls $B_\rho(x_0)$ with center in x_0 and radius ρ, we get*

$$⨍_{B_\rho(x_0)} W \;\downarrow W(x_0) \qquad \text{as } \rho \downarrow 0.$$

Proof. Take x in Ω, $\rho > 0$ and define

$$w_\rho(x) = ⨍_{B_\rho(x)} [\Delta u(y) + cu(y)]dy.$$

First of all we observe that for every x_0 in Ω, $w_\rho(x)$ is an increasing function of ρ. In fact if u is a regular function, Green's formula gives

$$\Delta u(x_0) + cu(x_0) = ⨍_{S_\rho(x_0)} (\Delta u + cu) - \int_{B_\rho(x_0)} (\Delta^2 u + c\Delta u)G_\rho(x_0 - y)dy,$$

where G_ρ is the Green's function in the ball of radius ρ:

$$G_\rho(x-y) = \begin{cases} \gamma_3(|x-y|^{-1} - \rho^{-1} & \text{if } N=3 \\ \gamma_2 \log\left(\frac{\rho}{|x-y|}\right) & \text{if } N=2, \end{cases}$$

$\gamma_i > 0$, $i = 2, 3$.

In the same way, if $\rho' > \rho$, we get

$$\Delta u(x_0) + cu(x_0) = \fint_{S_{\rho'}(x_0)} (\Delta u + cu) - \int_{B_{\rho'}(x_0)} (\Delta^2 u + c\Delta u) G_{\rho'}(x_0 - y)dy.$$

But $0 < G_\rho \le G_{\rho'}$, so if $\Delta^2 u + c\Delta u \le 0$, we get

$$\fint_{S_\rho(x_0)} (\Delta u + cu) \ge \fint_{S_{\rho'}(x_0)} (\Delta u + cu).$$

Integrating

$$\fint_{B_\rho(x_0)} (\Delta u + cu) \ge \fint_{B_{\rho'}(x_0)} (\Delta u + cu). \tag{21}$$

If $u \in H^2(\Omega)$ and $\Delta^2 u + c\Delta u \le 0$, setting u_ε the ε-regularized functions of u, then $\Delta^2 u_\varepsilon + c\Delta u_\varepsilon \le 0$, so (21) holds with u replaced by u_ε. Letting ε going to 0, we get (21) for any u in $H^2(\Omega)$. In this way $w_\rho(x_0)$ is an increasing function of ρ and a function w is defined as

$$w_\rho(x_0) \uparrow w(x_0) \qquad \text{as } \rho \downarrow 0.$$

Every w_ρ is continuous, so w is lower semicontinuous. By Lebesgue Theorem $w_\rho \to \Delta u + cu$ a.e. in Ω. Setting

$$W(x) = w(x) - cu(x)$$

the proof is complete. $\square$

8 The Limit Problem in the case $\sup \phi < 0$ and $N \le 3$

The limit problem, as it was established in (18), holds for any ψ in $C_C^\infty(\Omega)$. Now we want to show that, if $N \le 3$ and $\sup \phi < 0$, it holds $\forall \psi$ in H. First of all define a sequence of measures $(\mu_\omega)_\omega$ in such a way that

$$\int \psi \, d\mu_\omega = \omega \int ((u_\omega - \phi)^-)^{k-1} \psi \qquad \forall \psi \text{ in } C_0^0(\Omega).$$

Moreover $H \hookrightarrow C_0^0(\Omega)$ (but not in $C_C^0(\Omega)$) and we already know that $\int \psi \, d\mu_\omega \to \int \psi \, d\mu \; \forall \psi$ in $C_C^\infty(\Omega)$. So we only need to prove that $\int \psi \, d\mu_\omega \to \int \psi \, d\mu \; \forall \psi$ in $C_0^0(\Omega)$.

To do that we remind some basic concepts of measure theory.

Definition 8.1. Let ν_ω be a sequence of measures and ν be a measure on Ω. We say that ν_ω weakly converges to ν, and we write $\nu_\omega \overset{*}{\rightharpoonup} \nu$, if the induced functionals on the dual of $C_0^0(\Omega)$ converge in the weak-*topology: $\int \psi \, d\nu_\omega \to \int \psi \, d\nu \;\; \forall \psi$ in $C_0^0(\Omega)$.

It is easy to show that, if the total variations of $(\nu)_\omega$ are bounded, that is $\sup_\omega |\nu_\omega|(\Omega) < +\infty$, this condition is equivalent to the fact that $\int \psi\, d\nu_\omega \to \int \psi\, d\nu\ \forall \psi$ in $C^\infty_C(\Omega)$.

In the problem under investigation we assume $\sup \phi < 0$, so that Corollary 6.3 implies that $\sup_\omega\ \mu_\omega(\Omega) = \sup_\omega \omega \int ((u_\omega - \phi)^-)^{k-1} < +\infty$. In this way the convergence of the induced functionals over ψ's in $C^\infty_C(\Omega)$ is equivalent to the convergence over ψ's in $C^0_0(\Omega)$. In particular $\int \psi\, d\mu \to \int \psi\, d\mu\ \forall \psi$ in H.

Lemma 8.2. *If $\nu_\omega \overset{*}{\rightharpoonup} \nu$ and $z_\omega \to z$ uniformly in $C^0_0(\Omega)$, then $\int z_\omega\, d\nu_\omega \to \int z\, d\nu$.*

Proof.

$$\left| \int z_\omega d\,\nu_\omega - \int z\, d\nu \right| \le \left| \int z_\omega\, d\nu_\omega - \int z\, d\nu_\omega \right| + \left| \int z\, d\nu_\omega - \int z\, d\nu \right|.$$

The first term of the right hand side of the previous inequality goes to 0, since it is less or equal to $\|z_\omega - z\|_\infty |\nu_\omega|(\Omega)$, while the second one goes to 0 by definition. □

An immediate consequence is the following

Theorem 8.3. *If $N \le 3$, $\phi \in C^0(\Omega)$, u_ω is a solution of (P_ω), $u_\omega \rightharpoonup u$ in H and μ is the measure defined in* (18)*, then*

$$\int \Delta u \Delta v - c \int Du \cdot Dv - \alpha \int uv = - \int v\, d\mu \qquad \forall\, v \text{ in } H. \tag{22}$$

Proof. Let $v \in H$. Then

$$\int \Delta u_\omega \Delta v - c \int Du_\omega \cdot Dv - \alpha \int u_\omega v = -\omega \int ((u_\omega - \phi)^-)^{k-1} v \qquad \forall\, \omega.$$

Passing to the limit, Lemma 8.2 implies the thesis. □

Lemma 8.4. *Suppose $N \le 3$, $\sup \phi < 0$ and u_ω is a solution of (P_ω) such that $u_\omega \rightharpoonup u$; then*

$$\lim_{\omega \to +\infty} \omega \int ((u_\omega - \phi)^-)^k = 0.$$

Proof. From Corollary 6.3 we get that the total variations of the sequence $(\mu_\omega)_\omega$ is bounded from above: $\sup_\omega \mu_\omega(\Omega) < +\infty$. So

$$\omega \int ((u_\omega - \phi)^-)^k \le \omega \int ((u_\omega - \phi)^-)^{k-1} \|(u_\omega - \phi)^-\|_{L^\infty(\Omega)}.$$

But $\|(u_\omega - \phi)^-\|_{L^\infty(\Omega)} \to 0$, and the thesis follows. □

Corollary 8.5. *Suppose $N \le 3$ and $\sup \phi < 0$. If u_ω is a solution of (P_ω) such that $u_\omega \rightharpoonup u$ in H, then $\liminf\limits_{\omega \to +\infty} \omega((u_\omega - \phi)^-)^k = 0$ a.e. in Ω.*

Proof. It follows from Lemma 8.4 and from Fatou's Lemma. □

Remark 8.6. If $u \to u$ uniformly, $f_\omega(u_\omega) \ne 0$ (i.e u_ω is a forcing solution of (P_ω)), $\phi \in C^0(\Omega)$ and $\sup \phi < 0$, then $\{x \in \Omega \,|\, u(x) = \phi(x)\} \ne \emptyset$.

In fact, since $u_\omega \to u$ uniformly, there would exist ω_0 such that $u_\omega - \phi > 0\ \forall\, \omega \ge \omega_0$. But $\int \Delta u_\omega \Delta \psi - c \int Du_\omega \cdot D\psi - \alpha \int u_\omega \psi = -\omega \int ((u_\omega - \phi)^-)^{k-1} \psi$ for every ψ in H, so $\omega \ge \omega_0$, $\int \Delta u_\omega \Delta \psi - c \int Du_\omega \cdot D\psi - \alpha \int u_\omega \psi = 0$, which implies $f_\omega(u_\omega) = 0$ (choosing $\psi = u_\omega$) and this is a contradiction. △

As a corollary of Theorem 7.1 we get the following

Theorem 8.7. *Assume $N \leq 3$, $\phi \in C^0(\Omega)$, $\sup \phi < 0$, u_ω is a solution of (P_ω) and $u_\omega \rightharpoonup u$; then:*

- *$u_\omega \to u$ uniformly and μ is supported in the contact set $\mathfrak{B} = \{x \in \Omega \mid u(x) = \phi(x)\}$, that is $\mu(\Omega \setminus \mathfrak{B}) = 0$;*
- *if $\alpha \neq \Lambda_i \ \forall i$ and if $\{x \in \Omega \mid u(x) = \phi(x)\} \neq \emptyset$ (for example if $(u_\omega - \phi)^- \neq 0 \ \forall \omega$), then $\mu(\{x \in \Omega \mid u(x) = \phi(x)\}) > 0$;*
- *if G is any neighborhood of $\partial\Omega$ such that $\overline{G} \subset \overline{\Omega} \setminus \mathfrak{B}$, then $u \in H^4_{\mathrm{loc}}(G)$, $\Delta^2 u + c\Delta u - \alpha u = 0$ a.e. in G and $\int \Delta u \Delta \psi - c \int Du \cdot D\psi - \alpha \int u\psi = 0 \ \forall \psi$ in H such that $\operatorname{supp} \psi \subset G$.*

Proof. Since $N \leq 3$, $u_\omega \to u$ uniformly. $\phi \in C^0(\Omega)$ and $\sup \phi < 0$, thus the contact set $\mathfrak{B}$ is a compact subset of Ω. Let $\psi \in C^\infty_C(\Omega \setminus \mathfrak{B})$; then $\int \Delta u_\omega \Delta \psi - c \int Du_\omega \cdot D\psi - \alpha \int u_\omega \psi + \omega \int ((u_\omega - \phi)^-)^{k-1} \psi = 0$. But if ω is big enough $(u_\omega - \phi)^- \psi = 0$ and so, passing to the limit, $\int \Delta u \Delta \psi - c \int Du \cdot D\psi - \alpha \int u\psi = 0 \ \forall \psi$ in $C^\infty_C(\Omega \setminus \mathfrak{B})$, that is μ in supported in $\mathfrak{B}$.

Now suppose $\alpha \neq \Lambda_i \ \forall i$ and $\mathfrak{B} \neq \emptyset$; assume by contradiction that $\mu = 0$. Then Theorem 8.3 gives $\int \Delta u \Delta \psi - c \int Du \cdot D\psi - \alpha \int u\psi = 0 \ \forall \psi$ in H. Since $u \not\equiv 0$, u is an eigenfunction in H of the biharmonic operator; therefore there exists i in $\mathbb{N}$ such that $\alpha = \Lambda_i$ and this is a contradiction.

Since $u_\omega \to u$ uniformly, $(u_\omega - \phi)^- = 0$ definitely in every compact subset of $\Omega \setminus \mathfrak{B}$. If ψ in H is such that $\operatorname{supp} \psi \subset G$, for ω large enough we have $\int \Delta u_\omega \Delta \psi - c \int Du_\omega \cdot D\psi - \alpha \int u_\omega \psi = 0$. Then $\int \Delta u \Delta \psi - c \int Du \cdot D\psi - \alpha \int u\psi = 0$. In particular $u \in H^4_{\mathrm{loc}}(G)$ and $\Delta^2 u + c\Delta u - \alpha u = 0$ a.e. in G. ☐

Theorem 8.8. *Suppose $N \leq 3$, $\sup \phi < 0$. If u_ω is a solution of (P_ω) and u_ω weakly converges to u in H, then $u_\omega \to u$ strongly in H.*

Proof. Putting u in (22) we get

$$\int |\Delta u|^2 - c \int |Du|^2 - \alpha \int u^2 + \int u \, d\mu = 0. \tag{23}$$

But

$$\int |\Delta u_\omega|^2 = c \int |Du_\omega|^2 + \alpha \int u_\omega^2 - \omega \int ((u_\omega - \phi)^-)^{k-1} u_\omega.$$

By Lemma 8.2, the right hand side of the last equality converges to

$$c \int |Du|^2 + \alpha \int u^2 - \int u \, d\mu.$$

By (23) we get

$$\lim_{\omega \to \infty} \int |\Delta u_\omega|^2 = \int |\Delta u|^2$$

and the thesis follows. ☐

Set $K_\phi = \{v \in H \mid v \geq \phi \text{ a.e. in } \Omega\}$, which is a convex and closed subset of H. We can now prove this

Theorem 8.9 (Reversed variational inequality). *Suppose* $N \leq 3$, $\sup \phi < 0$, *u is the limit of a sequence of forcing solutions u_ω of (P_ω). Then*

$$\int \Delta u \Delta(v-u) - c\int Du \cdot D(v-u) - \alpha \int u(v-u) \leq 0 \quad \forall v \text{ in } K_\phi. \tag{24}$$

Proof. Let v be a function of K_ϕ. Then

$$\int \Delta u_\omega \Delta(v-u_\omega) - c\int Du_\omega \cdot D(v-u_\omega) - \alpha \int u_\omega(v-u_\omega)$$

$$= -\omega \int ((u_\omega - \phi)^-)^{k-1}(v-u_\omega).$$

But $\int ((u_\omega - \phi)^-)^{k-1}(v-u_\omega) = \int ((u_\omega - \phi)^-)^{k-1}(v-\phi) - \int ((u_\omega - \phi)^-)^{k-1}(u_\omega-\phi) \geq 0$. By Theorem 8.8 and Lemma 8.4 the thesis follows. □

Remark 8.10. This is quite a surprising result, since this Theorem states the exact contrary of the statement of a classical variational inequality: if Φ is an obstacle (that is $\Phi_{|\partial\Omega} \leq 0$ and $\Phi > 0$ in a subset of positive measure in Ω) and one looks for

$$\min_{u \in K_\Phi} \int |\Delta u|^2, \tag{25}$$

then

$$\int \Delta u \Delta(v-u) \geq 0 \quad \forall v \text{ in } K_\Phi,$$

where $K_\Phi = \left\{ v \in H \,\middle|\, v \geq \Phi \text{ a.e. in } \Omega \right\}$ (see [5]). △

9 Multiplicity results for the reversed variational inequality in low dimension

As a corollary of Theorem 8.8, it is easy to prove the following

Theorem 9.1. *If $N \leq 3$, $\sup \phi < 0$, $\alpha \neq \lambda_1^2 - c\lambda_1$ and u is the weak limit of a sequence of forcing solutions u_ω of (P_ω), then there is at least one non trivial solutions of the limit problem (24).*

Proof. By Theorem 8.9 u is a solution of (24). Moreover by Corollary 5.19 there exists $\varepsilon > 0$ such that

$$f_\omega(u_\omega) = \frac{1}{2}\int |\Delta u_\omega|^2 - \frac{c}{2}\int |Du_\omega|^2 - \frac{\alpha}{2}\int u_\omega^2 \geq \varepsilon \qquad \forall \omega.$$

By Theorem 8.8 $u_\omega \to u$ strongly in H and so

$$\int |\Delta u|^2 - c\int |Du|^2 - \alpha \int u^2 \geq 2\varepsilon > 0,$$

that is $u \not\equiv 0$. □

In particular we can prove the following multiplicity result.

Theorem 9.2. *Suppose $N \leq 3$ and $\sup \phi < 0$. Under the hypotheses of Theorem 5.17 or of Theorem 5.18, there exists $\tau_s > 0$ such that $\forall\, \alpha$ in $(\Lambda_s - \tau_s, \Lambda_s)$ there exist at least two distinct non trivial solutions of equation (24).*

Proof. In Corollary 5.19 we proved that there exists $\sigma_1 > \sigma_2 > b > a > 0$ and three sequences of solutions v^1_ω and u^j_ω, $j = 1, 2$ such that $\forall\, \omega$

$$\sigma_1 > f_\omega(v^1_\omega) > \sigma_2 \left(= \inf_{\substack{w \in H_s^\perp, \\ \|w\| = \rho_\omega}} f_\omega(w) \right) > b \; (= M_s^\omega(\alpha)) \geq f_\omega(u^j_\omega) > a > 0$$

and such that $v^1_\omega \to v$ and $u^j_\omega \to u_j$, $j = 1, 2$ in H. We recall that M_s^ω is a decreasing function of ω. But since $N \leq 3$, in the definition of $\mathcal{O}_s^\omega$ (in which we require that there exists ρ_ω such that $M_s^\omega < \inf_{\substack{w \in H_s^\perp, \\ \|w\| = \rho_\omega}} f_\omega(w)$) we can choose ρ_ω constant and small enough so that $(w - \phi)^- = 0$, and so $\inf_{\substack{w \in H_s^\perp, \\ \|w\| = \rho_\omega}} f_\omega(w)$, is independent on ω.

Passing to the limit we get

$$\sigma_1 \geq \frac{1}{2} \left(\int \Delta v|^2 - c \int |Dv|^2 - \alpha \int v^2 + \int v \, d\mu \right) \geq \sigma_2 >$$

$$> b \geq \frac{1}{2} \left(\int \Delta u_j|^2 - c \int |Du_j|^2 - \alpha \int u_j^2 + \int u_j \, d\mu \right) \geq a > 0.$$

Of course we cannot distinguish u_1 and u_2 in the range $[a, b]$, so we can only establish the existence of one solution in that range. $\square$

We observe that this is quite an interesting fact. Indeed we have proved that for a "reversed" linear variational inequality there are some non trivial solutions. And such a result is not obvious at all, since the existence of one non trivial solution is not evident, either.

10 A deeper look on the case $N \leq 3$

Let us consider any solution u of (24). For simplicity consider the case $c = \alpha = 0$. Then

$$\int \Delta u \Delta v \leq \int |\Delta u|^2 \qquad \forall\, v \text{ in } K_\phi.$$

The (unique) solution of problem (25) satisfies

$$\int \Delta u \Delta(v - u) \geq 0 \qquad \forall\, v \text{ in } K_\phi,$$

as already remarked. But by Minty's Lemma (see [5]), the last inequality holds if and only if

$$\int \Delta v \Delta(v - u) \geq 0 \qquad \forall\, v \text{ in } K_\phi.$$

If we combine these two inequalities we immediately get

$$\int |\Delta u|^2 \leq \int |\Delta v|^2 \qquad \forall\, v \text{ in } K_\phi,$$

as expected.

In our case a property analogous to Minty's Lemma is not possible. In fact, if (24) was equivalent to the following "reversed" Minty's Lemma

$$\int |\Delta v|^2 \le \int \Delta u \Delta v \qquad \forall\, v \text{ in } K_\phi,$$

we would get

$$\int |\Delta u|^2 \ge \int |\Delta v|^2 \qquad \forall\, v \text{ in } K_\phi,$$

which is clearly absurd, since

$$\sup_{v \in K_\phi} \int |\Delta v|^2 = +\infty.$$

Anyway we observe that the "reversed" Minty's Lemma implies (24). In fact $\forall\, v$ in K_ϕ we have

$$0 \le \int |\Delta(v-u)|^2 = \int |\Delta v|^2 - 2\int \Delta u \Delta v + \int |\Delta u|^2$$

and by the "reversed" Minty's Lemma it is less or equal to

$$-\int \Delta u \Delta v + \int |\Delta u|^2,$$

and so (24) holds.

We also observe that the "reversed" inequality is not equivalent to a classical variational one, neither when the constraint $u \ge \phi$ is replaced by $U \le -\phi$. In fact if we consider the function $U = -u$, then U satisfies $\int \Delta U \Delta(V-U) - c\int DU \cdot D(V-U) \le 0\ \forall\, V \le -\phi$.

Finally we prove some regularity results in the case $\alpha = 0$.

Proposition 10.1. *Suppose u satisfies (24) with $\alpha = 0$, W is the function defined in Proposition 7.8 and $c < \lambda_1$. Then $W \ge 0$ in Ω.*

Proof. Fix x_o in Ω and $\rho > 0$ such that $\overline{B_\rho(x_0)} \subset \Omega$. Denote by χ_B the characteristic function of $B_\rho(x_0)$ and let z be the solution of the problem

$$\begin{cases} \Delta z + cz = -\chi_B & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

By the regularity theorem for elliptic equations, $z \in H$ and by the maximum principle $z \ge 0$ a.e. in Ω. Then $v = u + z \in K_\phi$. In this way, putting such a v in (24), we get

$$\int \Delta u \Delta z - c\int Du \cdot Dz \le 0.$$

Substituting the value of Δz and integrating by parts we get

$$\int_{B_\rho(x_0)} \Delta u = \int_{B_\rho(x_0)} W \ge 0.$$

But $W(x) = w(x) - cu(x)$ (according to the notation of Proposition 7.8) and u is continuous; dividing by the measure of $B_\rho(x_0)$ and passing to the limit, we get $W(x_0) \ge 0$. $\square$

Since $W = \Delta u$ a.e. in Ω, we get the following

Corollary 10.2. *If u satisfies problem (20) and $c < \lambda_1$, then $\Delta u \ge 0$ a.e. in Ω, that is u is subharmonic in Ω.*

Of course, thanks to the maximum principle we obtain again Proposition 7.5.

A Variational Theorems

Definition A.1. Let H be a Hilbert space, $f : H \longrightarrow \mathbb{R}$ be a C^1 function and $\mathfrak{c} \in \mathbb{R}$. We say that $(PS)_{\mathfrak{c}}$, *Palais-Smale condition at level* $\mathfrak{c}$, holds if for any u_n such that $\lim f(u_n) = \mathfrak{c}$ and $\lim \nabla f(u_n) = 0$, there exists a converging subsequence of (u_n).

Theorem A.2 (Mountain Pass). *Let B be a real Banach space and $f \in C^1(B, \mathbb{R})$ such that $f(0) = 0$ and (i) there are positive constants ρ and α such that $f_{|\partial B_\rho} \geq \alpha$, (ii) there exists e in $B \setminus B_\rho$ such that $f(e) \leq 0$.*

Suppose $(PS)_{\mathfrak{c}}$ holds $\forall\, \mathfrak{c} \geq \alpha$. Then f has a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} f(u),$$

where $\Gamma = \left\{ g \in C^0([0,1], B) \,\middle|\, g(0) = 0, g(1) = e \right\}$. Here B_ρ stands for the ball of radius ρ: $B_\rho = \left\{ u \in B \,\middle|\, \|u\| \leq \rho \right\}$.

See [19] or [20] for a proof.

Theorem A.3 (Linking Theorem). *Let H be a Hilbert space which is topological direct sum of subspaces H_1 and H_2, one of those having finite dimension. Let f be a C^1 real function defined on H and let $e \in H_1$, $e \not\equiv 0$ and ρ_1, $\rho_2 > 0$ such that*

(i) $|\rho_1 - \rho_2| < \|e\| < \rho_1 + \rho_2$;

(ii) $\sup_{\Sigma_1} f < \inf_{\Sigma_2} f$;

(iii) $-\infty < a = \inf_{B_1} f$ *and* $b = \sup_{B_2} f < +\infty$,

where B_1 is the ball in H_1 centered at 0 with radius ρ_1, Σ_1 is its boundary in H_1, B_2 is the ball in Span$(e) \oplus H_2$ centered at e with radius ρ_2 and Σ_2 is its boundary in Span$(e) \oplus H_2$. Suppose $(PS)_{\mathfrak{c}}$ holds for every $\mathfrak{c} \in [a, b]$. Then there exists two critical levels c_1 and c_2 such that

$$a \leq c_1 \leq \sup_{\Sigma_1} f < \inf_{\Sigma_2} f \leq c_1 \leq b.$$

See [11] for a proof of this Theorem.

Definition A.4. Let H be a Hilbert space, $f : H \longrightarrow \mathbb{R}$ be a C^1 function, X a closed subspace of H, a, $b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that condition $(\nabla)(f, X, a, b)$ holds if there exists $\gamma > 0$ such that $\inf \left\{ \|P_X \nabla f(u)\| \,\middle|\, a \leq f(u) \leq b, \operatorname{dist}(u, X) \leq \gamma \right\} > 0$, where $P_X : H \longrightarrow X$ is the orthogonal projection of H onto X.

Theorem A.5 ((∇)-Theorem). *Let H be a Hilbert space and H_i, $i = 1, 2, 3$ three subspaces of H such that $H = H_1 \oplus H_2 \oplus H_3$ and $\dim(H_i) < \infty$ for $i = 1, 2$. Denote by P_i the orthogonal projection of H onto H_i. Let $f : H \longrightarrow \mathbb{R}$ be a $C^{1,1}$ function. Let $\rho, \rho', \rho'', \rho_1$ be such that $\rho_1 > 0$, $0 \leq \rho' < \rho < \rho''$ and define*

$$\Delta = \left\{ u \in H_1 \oplus H_2 \,\middle|\, \rho' \leq \|P_2 u\| \leq \rho'', \|P_1 u\| \leq \rho_1 \right\} \qquad \text{and} \qquad T = \partial_{H_1 \oplus H_2} \Delta,$$

$$S_{23}(R) = \left\{ u \in H_2 \oplus H_3 \,\middle|\, \|u\| = R \right\} \qquad \text{and} \qquad B_{23} = \left\{ u \in H_2 \oplus H_3 \,\middle|\, \|u\| \leq R \right\}.$$

Assume that

$$a' = \sup f(T) < \inf f(S_{23}(\rho)) = a''.$$

Let a and b such that $a' < a < a''$ and $b > \sup f(\Delta)$. Assume $(\nabla)(f, H_1 \oplus H_3, a, b)$ holds and that $(PS)_\mathfrak{c}$ holds for every $\mathfrak{c}$ in $[a,b]$. Then f has at least two critical points in $f^{-1}([a,b]))$. Moreover, if $a_1 < \inf f(B_{23}(\rho)) > -\infty$ and $(PS)_\mathfrak{c}$ holds at any $\mathfrak{c}$ in $[a_1, b]$, then f has another critical level in $[a_1, a']$.

See [12] for the proof.

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