p-Poincaré inequality vs. ∞-Poincaré inequality; some counter-examples

Estibalitz Durand-Cartagena, Nageswari Shanmugalingam, and Alex Williams

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Abstract

We point out some of the differences between the consequences of p-Poincaré inequality and that of ∞-Poincaré inequality in the setting of doubling metric measure spaces. Based on the geometric characterization of ∞-Poincaré inequality given in [DJS], we give a geometric implication of p-Poincaré inequality and show throughout examples that the characterization in the p finite case is not possible. The examples we give are metric measure spaces which are doubling and support an ∞-Poincaré inequality, but support no finite p-Poincaré inequality. In particular, these examples show that one cannot expect a self-improving property for ∞-Poincaré inequality in the spirit of Keith-Zhong [KZ]. We also show that the persistence of Poincaré inequality under measured Gromov-Hausdorff limits fails for ∞-Poincaré inequality.

1 Introduction

Some of recent research in analysis on metric measure space has focused on the geometric properties of Poincaré inequalities. Given $1 \leq p < \infty$, when analyzing the behavior of minimizers of energy functionals, a p-Poincaré inequality allows us to control the variance of a Sobolev function on a ball in terms of the average value on the (perhaps on a concentric larger) ball of the p-th power of the gradient, thus ensuring good behavior of the functions. Papers such as [HeKo], [Ke], [KZ], [Se1], [Se], [Ko], and [Ch] have studied some geometric properties of p-Poincaré inequalities. However, some of the geometric and analytic consequences of Poincaré inequalities seem at the surface to be independent of the index p in p-Poincaré inequality, as in the papers [Ke]

(Lip − lip-condition), [Se] (quasiconvexity), and [Ch] (measurable differentiable structure and persistence of Poincaré inequality under pointed measured Gromov-Hausdorff limits of metric spaces).

Because of Hölder's inequality, the larger the value of p the easier it would be for a metric measure space to support a p-Poincaré inequality. Hence the weakest Poincaré inequality, the ∞-Poincaré inequality, is satisfied by the most general class of metric measure spaces considered in the theory of analysis in metric spaces. Such an ∞-Poincaré inequality was studied in [DJS]. In the light of the above discussion, it is natural to ask whether the results obtained for p-Poincaré inequality with finite p also holds for the case $p = \infty$. Such is the case with quasiconvexity, as shown in [DJS]. However, not all results translate well from the $p < \infty$ situation to the $p = \infty$ case. The goal of this current note is to point out some of the differences between the consequences of p-Poincaré inequality and that of ∞-Poincaré inequality. These differences appear to be due to the fact that unlike the L^p-norm for finite p, the L^∞-norm is not sensitive to small local perturbations.

We begin this paper by introducing the notations needed in the paper in Section 2, and giving further details of the properties under consideration. In [DJS] it is shown that a complete connected doubling metric measure space supports an ∞-Poincaré inequality if and only if it is thick quasiconvex. In Section 3, we study a concept analogous to thick quasiconvexity associated with p-Poincaré inequality for finite $p \geq 1$, and give an example which illustrates that this analogous geometric property does not imply the validity of a p-Poincaré inequality. The metric measure space given in this example is doubling and supports an ∞-Poincaré inequality, but supports no finite p-Poincaré inequality. So this example shows in addition that one cannot expect a self-improving property for ∞-Poincaré inequalities; the fact that a doubling metric measure space supporting a p-Poincaré inequality for some $1 < p < \infty$ also supports a q-Poincaré inequality for some $1 \leq q < p$ was shown by Keith and Zhong [KZ]. In Section 3 we also discuss the persistence of ∞-Poincaré inequalities under Gromov-Hausdorff convergence. The discussion in Chapter 9 of [Ch] demonstrates that if $\{X_n, d_n, \mu_n\}_n$ is a sequence of metric measure spaces with μ_n doubling measures supporting a p-Poincaré inequality, and in addition the constants associated with the doubling property and Poincaré inequality are uniformly bounded, and furthermore, this sequence of metric measure spaces converges in the measured Gromov-Hausdorff sense to a metric measure space (X, d, μ), then this limit space also is doubling and supports a p-Poincaré inequality. We will provide an example which demonstrates that this persistence of Poincaré inequality under measured Gromov-Hausdorff limits fails for ∞-Poincaré inequality.

Section 4 of this paper concentrates on doubling weights in $\mathbb{R}^n$. Positive doubling weights w in $\mathbb{R}^n$ support an ∞-Poincaré inequality. This is seen by the fact that a weighted measure μ, given by $d\mu = w\, d\mathscr{L}^n$, has the same class of null sets as $\mathscr{L}^n$ when the weight w is positive $\mathscr{L}^n$-almost everywhere, and the property of thick quasiconvexity—which is equivalent to ∞-Poincaré inequality—therefore holds for the measure μ just as it does for $\mathscr{L}^n$. The question is then whether a positive doubling weight in $\mathbb{R}^n$ necessarily supports a p-Poincaré inequality for some finite $p \geq 1$. In Section 4 we construct a positive doubling weight that does not support p-Poincaré inequality for any finite p, that is, an Euclidean ∞-admissible weight that is *not* an $\mathcal{A}_\infty$ weight. This weight provides us with a second example of a metric measure space that supports a weak ∞-Poincaré inequality but no finite p-Poincaré inequality. Note that there are doubling weights in $\mathbb{R}^n$ that are not Lebesgue measure-a.e. positive, see [St, page 40, 8.8(b)]. Finally, in Section 5 we discuss some further open problems related to consequences of ∞-Poincaré inequalities.

2 Notation and Preliminaries

We assume throughout the paper that (X, d, μ) is a metric measure space, that is, a metric space equipped with a metric d and a Borel measure μ such that $0 < \mu(B) < \infty$ for each open ball $B \subset X$.

A measure μ is *doubling* if there is a constant $C_\mu > 0$ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C_\mu \, \mu(B(x, r)).$$

In the above definition of doubling measure, we can equivalently replace open balls $B(z, R)$ with closed balls $\overline{B}(z, R) = \{y \in X \, : \, d(y, z) \leq R\}$.

By a *curve* γ we will mean a continuous mapping $\gamma : [a, b] \to X$. Recall that the *length* of a continuous curve $\gamma : [a, b] \to X$ is the number

$$\ell(\gamma) = \sup \Big\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \Big\}$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. We will say that a curve γ is *rectifiable* if $\ell(\gamma) < \infty$. The integral of a Borel function g over a rectifiable path γ is usually defined via the arc-length parametrization γ_0 of γ in the following way:

$$\int_\gamma \rho ds = \int_0^{\ell(\gamma)} g \circ \gamma_0(t) dt.$$

Recall here that every rectifiable curve γ admits a parametrization by the arc-length; that is, with $\gamma_0 : [a, b] \to X$, for all $t_1, t_2 \in [a, b]$ with $t_1 \leq t_2$, we have $\ell(\gamma_0|_{[t_1, t_2]}) = t_2 - t_1$. Hence from now on we only consider curves that are arc-length parametrized. The book [V, Chapter 1] has an elegant discussion about paths and path integrals; while the discussion in this reference is given in the setting of Euclidean domains, the discussion and proofs there are so general as to be valid in the setting of any metric space.

We recall the definition of *p-modulus,* an outer measure on the collection of all paths in X.

Definition 2.1. (Modulus of a family of curves) Let Γ be a family of non-constant rectifiable curves in X. For $1 \le p \le \infty$ we define the *p-modulus of* Γ by

$$\operatorname{Mod}_p(\Gamma) = \begin{cases} \inf_\rho \int_X \rho^p \, d\mu & \text{if } 1 \le p < \infty, \\ \inf_\rho \|\rho\|_{L^\infty(X)} & \text{if } p = \infty, \end{cases}$$

where the infimum is taken over all non-negative Borel functions $\rho : X \to [0, \infty]$ such that $\int_\gamma \rho \, ds \ge 1$ for all $\gamma \in \Gamma$. If some property holds for all rectifiable curves in X except for a family Γ with $\operatorname{Mod}_p \Gamma = 0$, then we say that the property holds for p-a.e. curve.

Example 2.2. (Cusp domains) Fixing $m \in \mathbb{N}$, let $X \subset \mathbb{R}^2$ be the region

$$X := \{(x, y) \in \mathbb{R}^2 \, : \, x \ge 0 \text{ and } 0 \le y \le x^m\}$$

be endowed with the Euclidean distance and the 2-dimensional Lebesgue measure $\mathscr{L}^2$.

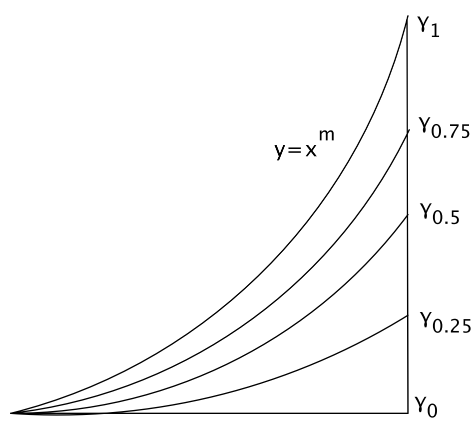

Figure 1: Fibrating the cusp domain

One can prove that the p-modulus of curves in X passing through the origin is positive if and only if $p > m + 1$. To see this, let ρ be admissible for computing the p-modulus of the family of curves connecting the origin to the vertical line segment $\{1\} \times \mathbb{R}$ inside this domain. For each $0 \le a \le 1$ let γ_a be the curve given by $\gamma_a(t) = (t, at^m)$, where t ranges between 0 and 1. Then γ_a is a curve in X connecting the origin to the vertical line segment. Furthermore, the family $\{\gamma_a\}_{0\le a\le 1}$ fibrates $X\cap([0,1]\times[0,1])$, and since ρ is admissible, we have that

$$\int_{\gamma_a}\rho\,ds \;=\; \int_0^1 \rho\circ\gamma_a(t)\,\sqrt{1+m^2a^2t^{2m-2}}\,dt \;\ge\; 1.$$

Thus by Hölder's inequality, with q denoting the Hölder conjugate of p,

$$\begin{aligned}1 &\le \int_0^1 \rho\circ\gamma_a(t)t^{m/p}\,t^{-m/p}\,\sqrt{1+m^2a^2t^{2m-2}}\,dt\\ &\le \left(\int_0^1 \rho\circ\gamma_a(t)^p t^m\,dt\right)^{1/p}\left(\int_0^1 t^{-mq/p}\left(1+m^2a^2t^{2m-2}\right)^{q/2}\,dt\right)^{1/q}.\end{aligned}$$

Since $1\le\sqrt{1+m^2a^2t^{2m-2}}\le\sqrt{1+m^2}$ for $0\le t\le 1$, the integral

$$C_1 := \int_0^1 t^{-mq/p}\left(1+m^2a^2t^{2m-2}\right)^{q/2}\,dt$$

is finite if and only if $\int_0^1 t^{-mq/p}\,dt$ is finite, and this happens precisely when $p>m+1$. When $p>m+1$, from the above we see that

$$\int_0^1 \rho\circ\gamma_a(t)^p t^m\,dt \ge C_1^{1-p} > 0.$$

It follows that (by setting $\rho(x,y)=0$ for $x>1$ without loss of generality),

$$\int_X \rho^p\,d\mathscr{L}^2 = \int_0^1\int_0^1 \rho\circ\gamma_a(t)^p\,t^m\,dt\,da \ge C_1^{1-p} > 0,$$

and so the p-modulus of the family of all curves passing through the origin, which contains γ_a, $0\le a\le 1$ as a sub-family, is at least $C_1^{1-p}>0$.

For $1\le p<m+1$, with the aid of the admissible function $\rho(x,y)=1/x$, $(x,y)\in X$, we see that the p-modulus of the family of all curves in X passing through the origin is zero. A more careful analysis using the function $\rho(x,y)=(\ln(R/r))^{-1}\,x^{-1}$ for $(x,y)\in X$ which is admissible for computing the p-moduli of curves connecting $\{r\}\times\mathbb{R}$ to $\{R\}\times\mathbb{R}$ in X for $0<r<R$, and then letting $r\to 0$ also shows that when $p=m+1$ the p-modulus is zero. Observe that the measure $\mathscr{L}^2|_X$ on X is doubling with doubling constant 2^m.

A useful generalization of Sobolev spaces to general metric spaces is Newtonian Spaces $N^{1,p}(X)$ introduced in [Sh, Sh1]. The case $p=\infty$ was studied in [DJ]. The definition is based on the notion of upper gradients of Heinonen and Koskela [HeKo], and weak upper gradients of Koskela and MacManus [KoMc].

Definition 2.3. A non-negative Borel function g on X is a *p-weak upper gradient* of an extended real-valued function f on X if

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g$$

for p-a.e. rectifiable curve γ in X.

Let $\widetilde{N}^{1,p}(X,d,\mu)$, where $1 \leq p \leq \infty$, be the class of all p-integrable functions on X for which there exists a p-weak upper gradient in $L^p(X)$. For $f \in \widetilde{N}^{1,p}(X,d,\mu)$ we define

$$\|f\|_{\widetilde{N}^{1,p}} := \|f\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p-weak upper gradients g of f. Now, we define in $\widetilde{N}^{1,p}(X,d,\mu)$ an equivalence relation by $f_1 \sim f_2$ if and only if $\|f_1 - f_2\|_{\widetilde{N}^{1,p}} = 0$. Then the space $N^{1,p}(X,d,\mu) = N^{1,p}(X)$ is defined as the quotient $\widetilde{N}^{1,p}(X,d,\mu)/\sim$ and it is equipped with the norm $\|f\|_{N^{1,p}(X)} = \|f\|_{\widetilde{N}^{1,p}}$.

The following Poincaré inequality is now standard in literature on analysis on metric spaces.

Definition 2.4. Let $1 \leq p \leq \infty$. We say that (X,d,μ) supports a *weak p-Poincaré inequality* if there are constants $\lambda_p, C_p > 0$ such that when $f : X \to \mathbb{R} \cup \{-\infty,\infty\}$ is a measurable function and $g : X \to [0,\infty]$ is an every upper gradient of f, and $B(x,r)$ is a ball in X,

$$\fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \leq C_p\, r \Big(\fint_{B(x,\lambda_p r)} g^p d\mu \Big)^{1/p}$$

if $1 \leq p < \infty$, and

$$\fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \leq C_\infty\, r \|g\|_{L^\infty(B(x,\lambda_\infty r))}$$

if $p = \infty$. Note that it is necessary to have $\lambda_p \geq 1$. The word *weak* refers to the possibility that λ_p may be larger than 1. Here for arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$f_A = \fint_A f := \frac{1}{\mu(A)} \int_A f\, d\mu.$$

In most situations the constant C_p in the p-Poincaré inequality depends strongly on p (the dependence is crucial as we make p smaller), but the constant λ_p depends usually on the shapes of holes in X and rarely on p. Hence, in the remainder of this paper we will drop the subscript in λ_p and merely denote it λ.

Next we discuss the notion of *pointed measured Gromov-Hausdorff convergence* of a sequence of pointed metric measure spaces, $\{(X_n, d_n, \mu_n, p_n)\}_n$ with $p_n \in X_n$, to a pointed metric measure space (X, d, μ, p) with $p \in X$. We first recall the notion of Hausdorff convergence of compact metric spaces. See [G] or [BBI, Chapter 7.4] for further details.

A metric space is said to be proper if closed and bounded subsets of that space are compact. Given a proper metric space (Z, d_Z), and two compact sets K_1, K_2 of Z, the *Hausdorff distance* $d_H(K_1, K_2)$ is the number

$$d_H(K_1, K_2) := \inf\left\{\varepsilon > 0 \,:\, K_2 \subset \bigcup_{z\in K_1} B(z,\varepsilon) \text{ and } K_1 \subset \bigcup_{z\in K_2} B(z,\varepsilon)\right\}.$$

A sequence of compact sets $\{K_n\}_n$ in Z is said to *converge in the Hausdorff topology* to a compact set $K \subset Z$ if $d_H(K_n, K) \to 0$ as $n \to \infty$. Given a sequence of proper metric subspaces $\{A_n\}_n$ of Z, $p \in A_n$ for all n, and a proper metric subspace $A \subset Z$ and a point $p \in A$, we say that the pointed sequence $\{(A_n, p)\}_n$ converges in the Hausdorff topology to (A, p) if for all $r > 0$ the sequence of compact sets $\{\overline{B}(p, r) \cap A_n\}_n$ converges in the Hausdorff topology to $\overline{B}(p, r) \cap A$.

We next recall the *pointed measured Gromov-Hausdorff convergence* . The notion of *pointed measured Gromov-Hausdorff convergence* was introduced by Fukaya in [Fu]. See also [Ke3], [Ch] and references therein.

Definition 2.5. A sequence of proper pointed metric measure spaces $\{(X_n, d_n, \mu_n, p_n)\}$ is said to converge to another complete pointed metric measure space (X, d, μ, p) if there exists a proper pointed metric metric space (Z, ρ, q) and isometric embeddings $i : X \to Z$ and $i_n : X_n \to Z$ for each $n \in \mathbb{N}$ such that $i_n(p_n) = q = i(p)$, $(i_n(X_n), q)$ converges to $(i(X), q)$ in the above-mentioned sense of Hausdorff topology on Z, and such that $(i_n) * \mu_n$ converges to $i * \mu$ in the weak* sense.

In the above definition, $i_n * \mu_n$ is the push-forward of the measure μ_n under the isometry i_n; for sets $A \subset Z$, we have $i_n * \mu_n(A) = \mu_n(i_n^{-1}(A))$. We say that a sequence of Borel measures ν_n on Z *converges in the weak* sense* to a Borel measure ν if for all compactly supported continuous functions φ on Z,

$$\lim_{n\to\infty} \int_Z \varphi \, d\nu_n \;=\; \int_Z \varphi \, d\nu.$$

If for all Borel sets $A \subset Z$ we have $\nu_n(A) \to \nu(A)$ as $n \to \infty$, then ν_n converges in the weak* sense to ν, but the converse is not always true, as shown by the measures μ_n given by $d\mu_n = [1-(1-n^{-1})^2]^{-1}\, \chi_{B((0,0),1)\setminus B((0,0),1-n^{-1})}\, d\mathscr{L}^2$ and $\mu = (2\pi)^{-1}\, \mathcal{H}^1 | S^1((0,0),1)$ on $\mathbb{R}^2$.

If a sequence of compact sets $\{K_n\}_n$ of Z converges in the Hausdorff topology to a compact set $K \subset Z$, then this sequence converges in the Gromov-Hausdorff sense to

K, but again the converse need not hold, as demonstrated by the example $Z = \mathbb{R}^2$, $K_n = \overline{B}((n,0),1)$, and $K = \overline{B}((0,0),1)$. The notion of Gromov-Hausdorff convergence is therefore more flexible and depends more on the shapes of the sequence of metric spaces approximating the shape of the limit space.

3 p-thick quasiconvexity

It is known that if a complete doubling metric measure space supports a weak p-Poincaré inequality, then the space is *quasiconvex* , that is, there exists a constant such that every pair of points can be connected with a curve whose length is at most the constant times the distance between the points (see [Se1] or [HeKo]). See [Ko] for further improvements of the quasiconvexity condition. In what follows, we consider a stronger geometric property. We will prove that every pair of sets of positive measure which are a positive distance apart can be connected by a "thick" family of quasiconvex curves in the sense that the modulus of this family of curves is positive. The following definition makes this idea more precise.

Definition 3.1. A metric measure space (X,d,μ) is said to be a *p-thick quasiconvex* space (where $1 \le p \le \infty$) if there is a constant $C \ge 1$ such that for all $x, y \in X$, all $0 < \varepsilon < \frac{1}{4}d(x,y)$, and all measurable sets $E \subset B(x,\varepsilon)$ and $F \subset B(y,\varepsilon)$ satisfying $\mu(E)\mu(F) > 0$, we have that

$$\mathrm{Mod}_p(\Gamma(E,F,C)) > 0.$$

Here $\Gamma(E,F,C)$ denotes the set of all curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \le Cd(p,q)$. Following [DJS], we say that X is *thick quasiconvex* if it is ∞-thick quasiconvex.

Remark 3.2. Note that every complete p-thick quasiconvex space X supporting a doubling measure is quasiconvex. The converse is not true in general. The Sierpinski carpet is a quasiconvex space which is not ∞-thick quasiconvex ([DJS, Corollary 4.15]), and so it is not p-thick quasiconvex either for any $1 \le p \le \infty$.

Lemma 3.3. *Whenever $1 \le p \le \infty$ the Euclidean space $\mathbb{R}^n$ is p-thick quasiconvex with quasiconvexity constant $C = 1$.*

Proof. An easy modification of Lemma 3.8 tells us that it suffices to prove that $\mathbb{R}^n$ is p-thick quasiconvex for $p = 1$.

Let $x, y \in \mathbb{R}^n$ be two distinct points, and $0 < \varepsilon < |x-y|/10$. Let $E \subset B(x,\varepsilon)$ and $F \subset B(y,\varepsilon)$ be two measurable sets of positive measure, and $\Gamma(E,F,1)$ be the collection of all straight line segments connecting points in E to points in F. We wish to show that $\operatorname{Mod}_1(\Gamma(E,F,1)) > 0$. To do this, let $z \in E$ be a point of density 1 of E, and $w \in F$ be a point of density 1 of F; since both E and F have positive measure, by Lebesgue differentiation theorem such points exist. Let L be the line passing through z and w, P_1 be the $(n-1)$-dimensional hyperplane perpendicular to L, and P_2 the $(n-1)$-dimensional hyperplane parallel to P_1, such that the balls $B(z, 2\varepsilon)$ and $B(w, 2\varepsilon)$ lie between these two hyperplanes.

Let E_1 be the orthogonal projection of E to P_1 and F_1 the orthogonal projection of F to P_2. By Fubini's theorem, we know that $\mathcal{H}^{n-1}(E_1) > 0$ and $\mathcal{H}^{n-1}(F_1) > 0$, and that the projection z_1 of z to P_1 is a point of $\mathcal{H}^{n-1}$-density 1 for E_1 and the projection w_1 of w to P_2 is a point of $\mathcal{H}^{n-1}$-density 1 for F_1. Let Γ be the collection of all line segments *parallel to* L and connecting points in E_1 to points in F_1. We now show that $\operatorname{Mod}_1(\Gamma) > 0$. Suppose $\operatorname{Mod}_1(\Gamma) = 0$. Then lines parallel to L and passing through $\mathcal{H}^{n-1}$-almost every point in E_1 does not intersect F_1, and lines parallel to L and passing through $\mathcal{H}^{n-1}$-almost every point in F_1 does not intersect E_1 (see the discussion of [V, Chapter 1, Section 7.2]). Let $\nu_1 = \chi_{E_1}\mathcal{H}^{n-1}$, and $\nu_2 = \chi_{F_1}\mathcal{H}^{n-1}$. It follows that the projection ν_2' of the measure ν_2 to the hyperplane P_1 is mutually singular with ν_1. However, because w_1 is a $\mathcal{H}^{n-1}$-point of density 1 for F_1, and the projection of w_1 to P_1 is the same as z_1, it follows that z_1 is a point of density 1 for both ν_1 and the projection ν_2' of ν_2 to P_1; this is not possible since by the mutual singularity of the two measures, and by the definition of the two measures, for $r > 0$,

$$\nu_2'(B(z_1, r)) + \nu_1(B(z_1, r)) \leq \mathcal{H}^{n-1}|_{P_1}(B(z_1, r)).$$

It follows that $\operatorname{Mod}_1(\Gamma) > 0$. Since every curve in Γ has a subcurve in $\Gamma(E, F, 1)$, it follows that $\operatorname{Mod}_1(\Gamma(E, F, 1) \geq \operatorname{Mod}_1(\Gamma) > 0$, which concludes the proof. $\square$

Remark 3.4. The proof of the above lemma also tells us that in the Euclidean setting, given two parallel $(n-1)$-dimensional hyperplanes of $\mathbb{R}^n$ and two sets of $\mathcal{H}^{n-1}$-measure positive, one lying in one of the hyperplanes and the other lying in the other hyperplane, the set of all geodesic line segments connecting points in one set to points in the second set has positive 1-modulus.

The following result was proven in [DJS, Theorem 4.6].

Theorem 3.5. ([DJS, Theorem 4.6]) *Suppose that X is a connected complete metric space supporting a doubling Borel measure μ which is non-trivial and finite on balls. Then the following conditions are equivalent:*

(a) *X supports a weak ∞-Poincaré inequality.*

(b) *X is thick quasiconvex.*

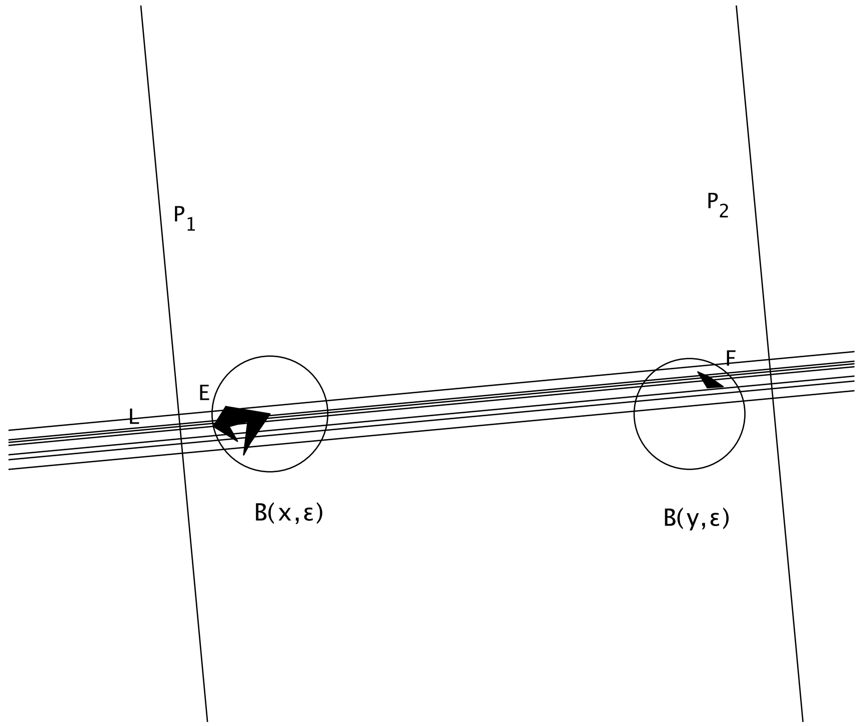

Figure 2: Thick quasiconvex family

(c) $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy semi-norms.*

(d) X *supports a weak ∞-Poincaré inequality for functions in* $N^{1,\infty}(X)$.

The proof of the above theorem given in [DJS] shows that the constants C_∞, λ of ∞-Poincaré inequality, the constant C of thick quasiconvexity, and the constant of comparison of energy semi-norms of the two function spaces mentioned in the above theorem depend only on each other and on the doubling constant of the measure on X. In this paper we use the equivalence of Condition (a) and Condition (b).

The following proposition gives an analog of Condition (a) implying Condition (b) for the case of finite p; the converse is not true in general, as will be shown below.

Proposition 3.6. *Let (X, d, μ) be a metric measure space with μ a doubling measure. If X supports a weak p-Poincaré inequality for functions in $N^{1,p}(X)$ with upper gradients in $L^p(X)$, then X is p-thick quasiconvex.*

Proof. Let $x, y \in X$ such that $x \neq y$, and let $0 < \varepsilon < d(x, y)/4$. Fix $n \in \mathbb{N}$ and let $\Gamma_n = \Gamma(B(x, \varepsilon), B(y, \varepsilon), n)$ be the collection of all rectifiable curves connecting $B(x, \varepsilon)$ to $B(y, \varepsilon)$ such that $\ell(\gamma) \leq n\, d(x, y)$. By the choice of ε, if p, q are the end points of γ, then $d(p, q)/4 \leq d(x, y) \leq 4d(p, q)$.

Suppose that $\mathrm{Mod}_p(\Gamma_n) = 0$. Then, there exists a non-negative Borel measurable function $g \in L^p(X)$ such that $\|g\|_{L^p(X)} = 1$ and for all $\gamma \in \Gamma_n$, the path integral $\int_\gamma g\, ds = \infty$.

Next, for each $k \geq 1$ consider the family of functions $g_k = \frac{1}{k}\, g$. It is clear that $\|g_k\|_{L^p(X)} = 1/k$. By the Maximal Function Theorem (see [He, 2.2]),

$$\mu(\{z \in X:\, M(g_k^p)(z) > 1\}) \leq \frac{C}{1}\int_X g_k^p = \; C\|g_k\|^p_{L^p(X)} < C\frac{1}{k^p}. \tag{1}$$

Recall here that $M(g)(z) = \sup_{r>0} \fint_{B(z,r)} |g| d\mu$.

Let

$$S_k = \{z \in X \,:\, M(g_k^p)(z) \leq 1\} = \{z \in X \,:\, M(g^p)(z) \leq k^p\}.$$

Observe that if $k_1 \leq k_2$ then $S_{k_1} \subset S_{k_2}$, and the set $G = X \setminus \bigcup_{k\geq 1} S_k$ has measure zero by inequality (1). Let

$$u_k(z) = \inf_{\gamma \text{ connecting } z \text{ to } B(x,\varepsilon)} \int_\gamma (1+g_k)\, ds.$$

Note that $u_k = 0$ on $B(x,\varepsilon)$ for each k. If $z \in B(y,\varepsilon)$ and γ is a rectifiable curve connecting z to $B(x,\varepsilon)$, then either $\gamma \in \Gamma_n$ in which case $\int_\gamma (1+g_k)\, ds \geq \int_\gamma g_k\, ds = \infty$, or else $\gamma \notin \Gamma_n$, in which case $\ell(\gamma) > n\, d(x,y)$ and so $\int_\gamma (1+g_k)\, ds \geq \int_\gamma 1\, ds > n\, d(x,y)$, and hence $u_k(z) \geq n\, d(x,y)$. It follows that the function $v_k = \min\{u_k, n\, d(x,y)\}$ has the properties that

1. $v_k = 0$ on $B(x,\varepsilon)$,
2. $v_k = n\, d(x,y)$ on $B(y,\varepsilon)$,
3. $1 + g_k$ is an upper gradient of v_k on X,
4. $v_k \in N^{1,p}(X)$.

Since $\mu(G) = 0$ we can find points $x_0 \in B(x,\varepsilon/4) \setminus G$ and $y_0 \in B(y,\varepsilon/4) \setminus G$. Let $k \in \mathbb{N}$ such that $x_0, y_0 \in S_k$. By using the chain of balls $B_i = B(x_0, 2^{1-i}d(x,y))$ if $i \geq 0$ and $B_i = B(y_0, 2^{1+i}d(x,y))$ if $i \leq -1$, and by the weak p-Poincaré inequality,

$$\begin{aligned}
n\, d(x,y) = v_k(y_0) = |v_k(x_0) - v_k(y_0)| &\leq \sum_{i\in\mathbb{Z}} |v_{k B_i} - v_{k B_{i+1}}| \\
&\leq C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |v_k - v_{k B_i}|\, d\mu \\
&\leq C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \Big(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (1+g_k)^p\, d\mu\Big)^{1/p} \\
&\leq C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \Big(1 + \Big(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (g_k)^p\, d\mu\Big)^{1/p}\Big) \\
&\leq C d(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|}(1+1) \leq C d(x,y).
\end{aligned}$$

Observe that x_0, y_0 are Lebesgue points of v_k since $v_k = 0$ on the open set $B(x, \varepsilon) \ni x_0$ and $v_k = n\,d(x, y)$ on $B(y, \varepsilon) \ni y_0$. Thus we must have $n \le C$, with C depending solely on the doubling constant and the constant of the Poincaré inequality. Hence if $n > C$ then the curve family $\Gamma_n = \Gamma(B(x, \varepsilon), B(y, \varepsilon), n)$ must have positive p-Modulus, completing the proof in the simple case that $E = B(x, \varepsilon)$ and $F = B(y, \varepsilon)$. The proof for more general E, F is very similar, where we modify the definition of u_k by looking only at curves that connect z to E, and then observing that almost every point in E and almost every point in F are Lebesgue points for the modified function v_k, with $v_k = 0$ on E and $v_k = nd(x, y)$ on F. This completes the proof of the proposition. $\square$

If $p = \infty$, the converse of Proposition 3.6 is true ([DJS, Theorem 4.5]). The following example shows that for finite p, the converse of Proposition 3.6 is not true in general. The metric measure space in this example, being thick quasiconvex and hence supporting a weak ∞-Poincaré inequality, also demonstrates that there is no self-improvement of ∞-Poincaré inequality in the spirit of [KZ].

Example 3.7. Let $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ be the unit square. Let Q_1 be the set obtained by dividing Q into nine equal squares of side-length $1/3$ and removing the central *open* square. The set Q_1 is the union of 8 squares of side-length $1/3$. Repeating this procedure on each of the 8 squares making up Q_1 we obtain the set Q_2, a union of 8^2 squares, each of side-length $1/3^2$. Repeating this process we get a sequence of sets Q_j consisting of 8^j squares of side-length $1/3^j$. Notice that each Q_j has positive area, so we can define a probability measure μ_j concentrated on Q_j obtained by renormalizing the Lebesgue measure (restricted to Q_j) to have measure one. The metric measure space under consideration is

$$X = Q_1 \cup (Q_2 + (1, 0)) \cup (Q_3 + (2, 0)) \cup \cdots (Q_j + (j - 1, 0)) \cup \cdots$$

endowed with the measure

$$\mu = \sum_i \chi_{Q_j + (j-1,0)} \cdot \mu_j,$$

and with the Euclidean metric restricted to X. Here, $Q_j + (j - 1, 0)$ is the set obtained by translating Q_j in the direction parallel to the x-axis by $j - 1$ units;

$$Q_j + (j - 1, 0) := \{(x + j - 1, y) \in \mathbb{R}^2 \,:\, (x, y) \in Q_j\},$$

and μ_j is the measure given by

$$\mu_j = (9/8)^j \,\mathscr{L}^2|_{Q_j + (j-1,0)}.$$

It can be directly verified that the measure μ is doubling on X.

Suppose that (X, d, μ) supports a weak p-Poincaré inequality for some finite p with constants C_p and λ. By [BS, Theorem 4.4], uniform domains in (X, d, μ) also support a

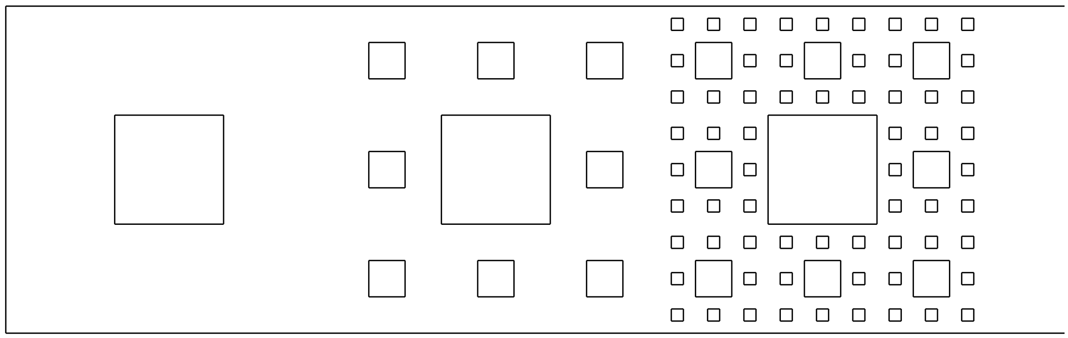

Figure 3: Sierpiński strip

weak p-Poincaré inequality with constants C'_p and λ', where C'_p and λ' depend solely on C_p, λ, and the uniformity constant of the uniform domain. Recall here that a domain $\Omega \subset X$ is C*-uniform,* $C \geq 1$ if for every pair of points $x, y \in \Omega$ there is a C-quasiconvex curve γ in Ω connecting x and y and for all $z \in \Omega$,

$$\min\{\ell(\gamma_{xz}), \ell(\gamma_{yz})\} \geq C\, d(x, X \setminus \Omega).$$

Here γ_{xz} and γ_{yz} are subcurves connecting z to x and y, respectively. For each j, the domains $Q_j+(j-1)$ are uniform domains in X, with the same uniformity constant. To see this, note that the unit square $Q = [0,1] \times [0,1]$ is a C_0-uniform domain in $\mathbb{R}^2$. Let $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in Q_j + (j-1, 0)$; then $(x_1 - j + 1, y_1), (x_2 - j + 1, y_2) \in Q$, and so there is a C_0-uniform curve β in Q connecting these two points. The curve $\beta_j := \beta + (j-1, 0)$ may not lie in $Q_j + (j-1, 0)$, but if it does not, then it intersects the translations of squares removed from Q in order to obtain Q_j; these removed squares are of Whitney type in Q (because the distance from the removed square to the boundary of Q is at least $1/3$ the side-length of the removed square), and so modifying the path β in the following manner yields a curve γ connecting P_1 to P_2 in $Q_j + (j-1, 0)$ which is $\sqrt{2}\, C_0$-uniform. If β_j intersects the translation of one of the removed open squares, then with P'_1, P'_2 denoting the points of intersection of β_j with the boundary of the removed square, we may replace the sub-curve of β_j inside this removed square with the shortest of the two components obtained by removing P'_1, P'_2 from the boundary of the removed square. This sub-curve replacing the original sub-curve of β_j has length no more than $\sqrt{2}$ times the length of the sub-curve being replaced; furthermore, its distance from the boundary (with respect to X) of $Q_j + (j-1, 0)$ is comparable to the corresponding quantity of the original sub-curve, with comparison constant $\sqrt{2}$. Hence the curve γ obtained by modifying β_j as above results in a uniform curve in $Q_j + (j-1, 0)$ with uniformity constant $\sqrt{2}\, C_0$, which is independent of j. One should keep in mind here that the boundary of $Q_j + (j-1, 0)$ in X is the union of the two vertical line segments $\{j-1\} \times [0,1]$ and $\{j\} \times [0,1]$, whose translation by $(-j+1, 0)$ is a subset of the boundary of Q in $\mathbb{R}^2$.

As explained above, the domains $Q_j + (j-1)$ are uniform domains in X, with the same uniformity constant. Therefore, for each $j \geq 1$ the space $(Q_j + (j-1,0), d, \mu_j)$ supports a weak p-Poincaré inequality for some finite p with constants C_p' and λ' and so it is clear that the space (Q_j, d_j, μ_j), where d_j is the Euclidean distance restricted to each Q_j, also supports a weak p-Poincaré inequality with the same constants C_p' and λ'.

The sequence of pointed spaces $\{(Q_j + (j-1,0), d_j, \mu_j, (j-1,0))\}$ converges in the measured Gromov-Hausdorff topology to the space (S, d, μ), where μ is the weak* limit of the probability measures μ_j and $S = \bigcap Q_j$ is the *Sierpinski carpet.* By the construction of the carpet, it is easy to see that the sequence of compact subsets $\{Q_j\}_j$ of $\mathbb{R}^2$ converges in the Hausdorff topology to the Sierpinski carpet equipped with the Euclidean metric; hence the convergence of $\{(Q_j+(j-1,0), d_j, (j-1,0))\}$ holds in the Gromov-Hausdorff sense as well. The sequence of measures μ_j converges to an Ahlfors s-regular measure, where the number s is given by $3^s = 8$. In particular, μ coincides with the Hausdorff measure on (S, d) of dimension s, see [Fal, Page 130, Theorem 9.3] together with [M, Theorem 1.23] or [Se, Section 4.1]. Furthermore, μ is a doubling measure, and in fact is an Ahlfors regular measure on the carpet S.

Since p-Poincaré inequality (with uniformly bounded constants) persists to the limit of a sequence of converging pointed metric measure spaces (see [Ch, Theorem 9.6] or [Ke3, Theorem 3]), the limit space (S, d, μ) would support a weak p-Poincaré inequality, which is known to be not true, see for example [BoP, Proposition 4.5] or [Se].

However, it is clear that (X, d, μ) is p-thick quasiconvex. We can use a simple modification of the proof of Lemma 3.3 to families of curves, obtained as a union of line segments parallel to the two coordinate axes. These curves are at most 2-quasiconvex. The idea is that for any pair of points $x, y \in X$, one can find a narrow 2-quasiconvex tube of curves connecting balls centered at x and y of positive p-modulus. If the largest index j for which one of x, y lies in $Q_j + (j-1,0)$ is large, then this tube of curves is correspondingly narrow and has a small p-modulus. Thus the p-modulus of curves connecting the balls has no quantitative lower bound, and that is the reason why the space does not support a weak p-Poincaré inequality for any finite p.

As we have seen before, p-thick quasiconvexity does not guarantee a weak p-Poincaré inequality for finite p. However, the next lemma shows that it is a sufficient condition to obtain a weak ∞-Poincaré inequality.

Lemma 3.8. *If (X, d, μ) is a p-thick quasiconvex for some $p < \infty$ then (X, d, μ) supports a weak ∞-Poincaré inequality.*

Proof. Since (X, d, μ) is a p-thick quasiconvex space, we know that there exists $C \geq 1$ such that for all $x, y \in X$, $0 < \varepsilon < \frac{1}{4} d(x,y)$, and all measurable sets $E \subset B(x, \varepsilon)$, $F \subset$

$B(y,\varepsilon)$ satisfying $\mu(E)\mu(F)>0$ we have that $\mathrm{Mod}_p(\Gamma(E,F,C))>0$, where $\Gamma(E,F,C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p\in E$ and $q\in F$ with $\ell(\gamma_{p,q})\le Cd(p,q)$. Let g be a non-negative Borel measurable function on X such that for all $\gamma\in\Gamma(E,F,C)$ we have $\int_\gamma g\,ds\ge 1$. Since the curves $\gamma\in\Gamma(E,F,C)$ are of length at most $4C\,d(x,y)$ and hence lie in the ball $B:=B(x,8C\,d(x,y))$, and so we may assume, without loss of generality, that the support of g lies in B. Because $0<\mu(B)<\infty$ we obtain by Hölder's inequality that

$$\|g\|_{L^p(X)}\le \mu(B)^{\frac{1}{p}-\frac{1}{s}}\|g\|_{L^s(X)}\quad \text{for all } s\in(p,\infty).$$

Letting $s\to\infty$ we get that $\|g\|_{L^p(X)}\le\mu(B)^{\frac{1}{p}}\|g\|_{L^\infty(X)}$, and so

$$\big(\mathrm{Mod}_p(\Gamma(E,F,C))\big)^{\frac{1}{p}}\le\mu(B)^{\frac{1}{p}}\,\mathrm{Mod}_\infty(\Gamma(E,F,C)).$$

The last inequality says that if $\mathrm{Mod}_p(\Gamma(E,F,C))>0$, then $\mathrm{Mod}_\infty(\Gamma(E,F,C))>0$ and so (X,d,μ) is an ∞-thick quasiconvex space. By the geometric characterization in [DJS, Theorem 4.6], we conclude that (X,d,μ) supports a weak ∞-Poincaré inequality. □

Remark 3.9. By the aid of Lemma 3.8, the space (X,d,μ) in Example 3.7 is a doubling metric measure space which supports a weak ∞-Poincaré inequality but does not support a weak p-Poincaré inequality for any finite p. Observe that in [DJS] the authors construct a space which supports a weak ∞-Poincaré inequality but does not support a weak p-Poincaré inequality for any finite p. However, the measure considered in that example was not doubling.

Finally in this section, we give an example which shows that the ∞-Poincaré inequality does not persist under Gromov-Hausdorff convergence.

Example 3.10. We consider the sets Q_j constructed in Example 3.7 above, and the corresponding Hausdorff limit of $\{Q_j\}_j$, which is the Sierpinski carpet $S=\bigcap Q_j$. The sequence of metric measure spaces under consideration is $\{(Q_j,d_j,\mu_j)\}$, where d_j is the Euclidean distance restricted to each Q_j and μ_j is a probability measure concentrated on Q_j. As mentioned in Example 3.7, the sequence of pointed spaces $\{(Q_j,d_j,\mu_j,(0,0))\}$ converges to the space $(S,d,\mu,(0,0))$, where μ is the Hausdorff measure corresponding to the Hausdorff dimension of S. The metric measure spaces in the sequence $\{(Q_j,d_j,\mu_j)\}$ are ∞-thick quasiconvex (and therefore support an ∞-Poincaré inequality). Keep in mind that in this example the constant that appears in the ∞-thick quasiconvexity property for each Q_j depends only on the constant of the quasiconvexity of the space. Therefore the constants are uniformly bounded by the quasiconvexity constant. However, the limit space (S,d,μ) does not support a weak ∞-Poincaré inequality (see [DJS, Corollary 4.14]).

4 ∞-admissible weights

The following definition of admissible weights is from [HKM].

Definition 4.1. A non-negative locally integrable function w in $\mathbb{R}^n$ is a *p-admissible weight* with $p \geq 1$ if $0 \leq w < \infty$ $\mathscr{L}^n$-a.e., the measure μ given by $d\mu = w d\mathscr{L}^n$ is doubling, and $(\mathbb{R}^n, |\cdot|, \mu)$ admits a weak p-Poincaré inequality.

Definition 4.2. A non-negative function w on $\mathbb{R}^n$ is a *Muckenhoupt A_p weight* with $p \geq 1$, if for some $C > 0$ and all balls $B \subset \mathbb{R}^n$,

$$\frac{1}{\mathscr{L}^n(B)} \int_B w dx < \begin{cases} C \left(\frac{1}{\mathscr{L}^n(B)} \int_B w^{1/(1-p)} dx \right)^{1-p} & \text{for } p > 1, \\ C \operatorname{ess\,inf}_B w & \text{for } p = 1. \end{cases}$$

The $\mathcal{A}_\infty$ class of weights is defined by

$$\mathcal{A}_\infty = \bigcup_{p>1} A_p.$$

It is well-known that A_p weights are p-admissible (see [HKM, Theorem 15.21]). In [BBK] it is shown that if $n = 1$, all p-admissible weights have to be of class A_p as well. However, when $n \geq 2$, not all p-admissible weights are of class A_p. For example, when $n \geq 2$, by [HKM, Corollary 15.35]), the measures $d\mu = |x|^\alpha d\mathscr{L}^n$ with $\alpha > 0$ are p-admissible in $\mathbb{R}^n$ for all $p > 1$, but belong to A_p if and only if $p > 1 + n\alpha$.

It is clear that $\mathcal{A}_\infty$ weights are ∞-admissible. Indeed, if w is an $\mathcal{A}_\infty$ weight, then it is an A_p weight for some finite p and so it supports a weak p-Poincaré inequality for some finite p. By Hölder's inequality we obtain that it also supports a weak ∞-Poincaré inequality.

Our goal of this section is to show that not all ∞-admissible weights in $\mathbb{R}^n$ are $\mathcal{A}_\infty$ weights. For the case $n = 1$ we will construct a weighted measure $d\mu = w d\mathscr{L}^n$ with $w > 0$ $\mathscr{L}^1$-a.e. such that $(\mathbb{R}^n, \mu)$ does not support p-Poincaré inequality for any $p < \infty$. This weight also is an example of a metric measure space that supports a weak ∞-Poincaré inequality but no finite weak p-Poincaré inequality.

There exist doubling measures on $\mathbb{R}^n$ that are totally singular, i.e., measures that have no absolutely continuous part with respect to the Lebesgue measure. In $\mathbb{R}$, the *Riesz product*

$$d\nu(x) = \prod_{k=1}^{\infty} (1 + a \cos(3^k \cdot 2\pi x)) d\mathscr{L}^1(x) \quad \text{for } |a| < 1$$

is a doubling measure (see [St, page 40, Section 8.8(a)]) such that ν and the Lebesgue measure $\mathscr{L}^1$ are mutually singular with the Lebesgue measure, see [Z], [St]. The sequence of measures ν_k given by $d\nu_k(x) := \prod_{j=1}^{k}(1 + a\cos(3^j \cdot 2\pi x))d\mathscr{L}^1(x)$ converges weakly to the measure ν. However, a stronger statement holds. By the definition of Fourier-Stieltjes integral given in [Z, page 10] and by [Z, page 209, (7.4)], we know that $\nu_k \to \nu$ in measure, that is, whenever $A \subset \mathbb{R}$ is a Borel set, we have $\lim_k \nu_k(A) = \nu(A)$.

The first step is to construct a sequence of weights w_k, $k \geq 1$, in $\mathbb{R}$ such that $w_k d\mathscr{L}^1$ approximates $d\nu$ better as k goes to ∞. We do this via convolution. We could of course take $w_k = \prod_{j=1}^{k}(1 + a\cos(3^k \cdot 2\pi x))$, but this choice does not give us the estimates for the weight w needed subsequently.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the characteristic function $\psi = 2^{-1}\chi_{[-1,1]}$, and for $k \in \mathbb{N}$ let $\psi_k(x) = k\,\psi(kx)$. Observe that $\int_{\mathbb{R}} \psi_k\, d\mathscr{L}^1 = 1$ and that $\psi_k = 2^{-1}k\,\chi_{[-1/k,1/k]}$. For $k \in \mathbb{N}$, we set

$$w_k(x) = \int_{\mathbb{R}} \psi_k(y - x)\, d\nu(y) \;=\; \frac{k}{2}\,\nu([x - 1/k, x + 1/k]).$$

Then w_k is lower semi-continuous and positive everywhere on $\mathbb{R}$. The measure μ_k given by this weight, $d\mu_k = w_k\, d\mathscr{L}^1$, is doubling, with the doubling constant independent of the integer k. To see this, note that for $z \in \mathbb{R}$ and $r > 0$,

$$\mu_k(\overline{B}(z, r)) = \mu_k([z - r, z + r]) = \int_{z-r}^{z+r} w_k(x)\, dx = \frac{k}{2}\int_{z-r}^{z+r} \nu([x - 1/k, x + 1/k])\, dx.$$

If $r \leq 10/k$, then because ν is a doubling measure we can find a constant $C > 0$, that depends only on the doubling constant of ν, such that for all $x_1, x_2 \in [z - r, z + r]$ we have

$$\nu([x_1 - 1/k, x_1 + 1/k]) \leq C\,\nu([x_2 - 1/k, x_2 + 1/k]).$$

In this case, from the above, we have

$$(2)\quad \frac{k}{C}\, r\, \nu([z-1/k, z+1/k]) \leq \mu_k(\overline{B}(z, r)) \leq Ck\, r\, \nu([z-1/k, z+1/k]) \text{ when } r \leq 10/k.$$

If $r \geq 1/(4k)$, then we can find intervals $I_1, \ldots, I_n$, with $n \approx kr$, such that

- $[z - r, z + r] = \bigcup_{j=1}^{n} I_j$,
- $I_j = [a_j, b_j]$ satisfies $a_1 = z - r$, $b_n = z + r$, and $b_j = a_{j+1}$ for $j = 1, \ldots, n - 1$,
- $(8k)^{-1} \leq b_j - a_j \leq (4k)^{-1}$.

Then for all $x \in I_j$ we have by the doubling property of ν, a constant C which depends only on the doubling constant of ν (and in particular, independent of k) such that for all $x \in I_j$,

$$\nu([x - 1/k, x + 1/k]) \leq C\,\nu([a_j - 1/k, a_j + 1/k]),$$

and so when $r \geq 1/(4k)$,

$$\begin{aligned}\mu_k([z-r,z+r]) &= \frac{k}{2}\sum_{j=1}^{n}\int_{I_j}\nu([x-1/k,x+1/k])\,dx\\ &\approx \frac{k}{2}\sum_{j=1}^{n}\frac{1}{k}\,\nu([a_j-1/k,a_j+1/k])\\ &\approx \sum_{j=1}^{n}\nu([a_j-1/k,a_j+1/k]) \approx \nu([z-r,z+r]).\end{aligned}\tag{3}$$

Here the constant of comparison in the above is dependent solely on the doubling constant of ν, and in particular is independent of k, z, r. If $r \leq 5/k$, then $2r \leq 10/k$ and so by (2),

$$\mu_k(\overline{B}(z,2r)) \approx k\,2r\,\nu(\overline{B}(z,1/k)) \approx 2\,\mu_k(\overline{B}(z,r)),$$

that is, $\mu_k(\overline{B}(z,2r)) \leq C\,\mu_k(\overline{B}(z,r))$ when $r \leq 5/k$. When $r \geq 5/k$, then we certainly have $r \geq 1/(4k)$ and so by (3) and the doubling property of ν,

$$\mu_k(\overline{B}(z,2r)) \approx \nu(\overline{B}(z,2r)) \leq C\,\nu(\overline{B}(z,r)) \leq C\,\mu_k(\overline{B}(z,r)).$$

Thus μ_k is doubling. Let $[a,b] \subset \mathbb{R}$. By the doubling property of ν, for $k \in \mathbb{N}$ we have

$$\begin{aligned}\mu_k([a,b]) &= \int_a^b \nu([x-1/k,x+1/k])\,dx\\ &\approx \int_a^b \nu([x-1/(k+1),x+1/(k+1)])\,dx = \mu_{k+1}([a,b]),\end{aligned}\tag{4}$$

with the constant of comparison independent of k, a, b.

Now let

$$w(x) = w_1(x)\chi_{(-\infty,2]}(x) + \sum_{k=2}^{\infty} w_k(x-k)\,\chi_{[k,k+1]}(x),$$

and we consider the corresponding weighted measure μ given by $d\mu = w\,d\mathscr{L}^1$. We now prove that μ is a doubling measure on $\mathbb{R}$.

Let $z \in \mathbb{R}$ and $r > 0$. Then $\mu(\overline{B}(z,r)) = \int_{z-r}^{z+r} w(x)\,dx$. If $[z-r,z+r] \subset [k,k+1]$ for some $2 \leq k \in \mathbb{N}$ or if $[z-r,z+r] \subset (-\infty,1]$, then by (4),

$$\mu(\overline{B}(z,2r)) \approx \mu_k(\overline{B}(z-k,2r)) \leq C\,\mu_k(\overline{B}(z-k,r)) = C\,\mu(\overline{B}(z,r)).$$

Here, if $[z-r,z+r] \subset (-\infty,1]$ then $k = 1$. So without loss of generality, we may assume that there is an integer $k \geq 2$ such that $k \in (z-r,z+r)$. We consider two cases. In the first case, $r \le 1/4$. Then $[z-2r, z+2r] \subset [k-1, k+1]$, and so by (4), the doubling property of μ_k, μ_{k-1} established above, and by (4) again,

$$\begin{aligned}
\mu(\overline{B}(z,2r)) =& \mu_{k-1}([z-2r-k+1,1]) + \mu_k([0,z+2r-k]) \\
=& \mu_{k-1}([z-2r-k,0]) + \mu_k([0,z+2r-k]) \approx \mu_k([z-2r-k,z+2r-k]) \\
&\le C\,\mu_k([z-r-k,z+r-k]) = C\,\mu_k([z-r-k,0]) + C\,\mu_k([0,z+r-k]) \\
&= C\,\mu_k([z-r-k+1,1]) + C\,\mu_k([0,z+r-k]) \\
&\le C\,(\mu_{k-1}([z-r-k+1,1]) + \mu_k([0,z+r-k]) = C\,\mu(\overline{B}(z,r)).
\end{aligned}$$

Here we also used the fact that by the construction of ν we have

$$\nu([a,b]) = \nu([a+1,b+1]) \tag{5}$$

and hence for all $k \in \mathbb{N}$ we have $w_k(x) = w_k(x+1)$ and so $\mu_k([a,b]) = \mu_k([a+1,b+1])$. Thus if we have $r \le 1/4$, the doubling property of μ holds for balls of radii r. Next, if $r \ge 1/4$, then for $k \ge 2$ we have $r \ge 1/(4(k-1))$, and so by combining (3) with the above-mentioned invariance of ν, μ_k under translation of intervals by integers (that is, periodicity 1), and the doubling property of ν,

$$\mu(\overline{B}(z,2r)) \approx \nu(\overline{B}(z-k,2r)) \le C\,\nu(\overline{B}(z-k,r)) \approx \mu(\overline{B}(z,r)).$$

Hence μ is doubling on $\mathbb{R}$.

By [M, Theorem 1.26] we know that the sequence of doubling measures $w_k d\mathscr{L}^1$ converges weakly to ν as $k \to \infty$, that is,

$$\lim_{k\infty} \int_{\mathbb{R}} \phi(x) w_k(x) d\mathscr{L}^1(x) = \int_{\mathbb{R}} \phi(x) d\nu(x),$$

for all $\phi \in C_0(\mathbb{R})$. The statement of [M, Theorem 1.26] is for continuous mollifiers ψ, but the proof given there is valid for all compactly supported non-negative Borel functions ψ satisfying $\int_{\mathbb{R}} \psi\, dx = 1$ as it is in our case.

The metric measure space $(\mathbb{R}, |\cdot|, \mu)$ with the doubling weighted measure $d\mu = w d\mathscr{L}^1$ does not support a p-Poincaré inequality for any finite p. To see this we argue as follows. Since ν is singular with respect to $\mathscr{L}^1$ there exists a set Borel $E_0 \subset \mathbb{R}$ such that $\nu(E_0) = 0$ and $\mathscr{L}^1(E_0) > 0$. Since $\mathscr{L}^1$ is an inner measure, we can in addition take E_0 to be a compact subset of $\mathbb{R}$. Without loss of generality we may assume that $E_0 \subset [0,1]$. Let $E = \bigcup_{k \in \mathbb{N}} E_0 + k$. Then by the periodicity 1 invariance (5) of ν and $\mathscr{L}^1$, for all $k \in \mathbb{N}$ we have $\nu(E \cap [k,k+1]) = 0$ and $\mathscr{L}^1(E \cap [k,k+1]) = \mathscr{L}^1(E_0) > 0$. Let $g = \chi_E$, and we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \int_0^x g(y)\, dy.$$

It is clear that g is an upper gradient of f and that $f(k) = k\,\mathscr{L}^1(E_0) > 0$ when k is a positive integer. Furthermore, for $k \in \mathbb{N}$ we get

$$f(x) = f(k+1) + f(x-k-1) \quad \text{for all } x \geq k+1.$$

In the interval $[k, k+1]$ we have $f(x) = f(x+1) - (f(k+1) - f(k))$. Since $w_k \approx w_{k+1}$ in $[0,1]$ with the constant of comparison dependent solely on the doubling constant of ν, we have whenever $k \in \mathbb{N}$,

$$\fint_{[k,k+2]} |f - f(k+1)| d\mu \geq C > 0,$$

with the constant C dependent solely on the doubling constant of ν and $\mathscr{L}^1(E_0)$. Note that as μ_k converges in the weak* topology, we have $\liminf_k \mu_k([0,2]) \geq \nu([0,2]) > 0$. It follows that for sufficiently large $k \in \mathbb{N}$,

$$\fint_{[k,k+2]} g^p \, d\mu \leq \frac{2}{\nu([0,2])}\, \mu([k,k+2]) = 2\, \frac{\mu_k(E_0) + \mu_{k+1}(E_0)}{\nu([0,2])}.$$

Since $\nu(E_0) = 0$ and E_0 is compact, it follows that

$$0 \leq \limsup_{k\to\infty} \mu_k(E_0) \leq \nu(E_0) = 0,$$

and thus the function-upper gradient pair (f, g) cannot sustain a weak p-Poincaré inequality.

On the other hand, $(\mathbb{R}, |\cdot|, \mu)$ is a p-thick quasiconvex space. First of all observe that the only quasiconvex simple curves in the space are the segments joining two points in the real line. Fix two points $a, b \in \mathbb{R}$ and choose g an admissible function for computing the p-modulus of the simple quasiconvex curve connecting a to b, that is, a measurable function g such that

$$\int_a^b g \, d\mathscr{L}^1 \geq 1.$$

By Hölder's inequality, $\int_a^b g^p d\mathscr{L}^1 \geq C(a,b,p) > 0$ for each $p \geq 1$. As mentioned before, for all $k \in \mathbb{N}$ we have $w_k(x) = k/2\, \nu([x-1/k, x+1/k])$; hence by the doubling property of ν there is a constant $c = c(a,b) > 0$ such that $w \geq c$ on $[a,b]$. Therefore,

$$\int_{\mathbb{R}} g^p d\mu = \int_{\mathbb{R}} g^p w d\mathscr{L}^1 \geq c \int_a^b g^p d\mathscr{L}^1 \geq c\, C(a,b,p) > 0,$$

which gives us a strictly positive p-modulus of the segments. It follows from Lemma 3.8 in the previous section that this weight w is ∞-thick quasiconvex and hence is ∞-admissible.

Remark 4.3. Observe that the weight given in the previous example is defined in the real line. We can make an analogous construction in higher dimensions by using the measure

$$\omega(x_1, \dots, x_n) := \prod_{j=1}^{n} w(x_j).$$

In this case, the function that violates the Poincaré inequality is obtained as a function of the single variable x_1, that is, $F(x_1, \dots, x_n) = f(x_1)$ with f the function defined as in the previous example.

5 Open problems.

We conclude the discussion in this paper by listing some problems in this section, for which we have so far neither a counterexample nor a proof.

Cheeger proved in [Ch] that doubling p-Poincaré spaces admit a differentiable structure for which Lipschitz functions are differentiable μ-a.e. A remarkable fact is that although the exponent p is present in the hypothesis of this result, it has no role in the conclusions. Keith, in [Ke2] (see also [Ke]) weakened the hypotheses so as not depend on p. He defined the $\mathrm{Lip} - \mathrm{lip}$ condition as follows: A metric measure space X is said to satisfy a $\mathrm{Lip} - \mathrm{lip}$ condition if there exists a constant $K \geq 1$ such that

$$\mathrm{Lip}\, f(x) \leq K \,\mathrm{lip}\, f(x)$$

for all Lipschitz functions $f : X \longrightarrow \mathbb{R}$, for μ-a.e. $x \in X$ (the exceptional set of measure zero is of course allowed to depend on f). As a consequence of [Ch, Theorem 6.1] and the fact that $\mathrm{lip}\, f$ is also a weak upper gradient of any Lipschitz function f, we know that complete doubling metric measure spaces which admit a weak p-Poincaré inequality satisfy the $\mathrm{Lip} - \mathrm{lip}$ condition as well. The thesis [Ke, Section 1.4] conjectures that this generalization can be understood as a version of Cheeger's theorem for $p = \infty$. The following example shows that these two conditions are not equivalent.

Example 5.1. Let $X \subset \mathbb{R}^2$ be the set obtained by removing certain thin rectangles from $[0,1] \times [0,1]$ as follows:

$$X = [0,1] \times [0,1] \setminus \bigcup_{2 \leq n \in \mathbb{N}} \left(\frac{1}{n} - \frac{1}{n^4},\ \frac{1}{n} \right) \times \left(\frac{1}{2},\ 1 \right).$$

We consider the complete space (X, d, μ) where d is the Euclidean distance and μ is the 2-dimensional Lebesgue measure $\mathscr{L}^2$ restricted to X. The space is not quasiconvex and so it cannot support any weak p-Poincaré inequality for $1 \leq p \leq \infty$. However, since it is an open set of $\mathbb{R}^2$ (except for the boundary, which has zero $\mathscr{L}^2$-dimensional measure), it satisfies the $\mathrm{Lip}-\mathrm{lip}$ condition. Furthermore, it can be checked that since the rectangles $(n^{-1}-n^{-4}, n^{-1}) \times (2^{-1}, 1)$ removed from $[0,1]\times[0,1]$ are sufficiently thin, the measure on X is doubling.

Some of the classical theorems in analysis in the Euclidean setting can be extended to doubling metric measure spaces. The Lebesgue differentiation Theorem is such an example: if f is a locally integrable function on a doubling metric space X, then

$$f(x) = \lim_{r\to 0}\Big(\fint_{B(x,r)} f^p d\mu\Big)^{1/p},$$

for $\mu-$a.e. point in X. In other words, almost every point in X is a *Lebesgue point* for f, see for example [He, Theorem 1.8]. One of the difficulties when working with the L^∞-norm is that the Lebesgue differentiation Theorem is no longer true. That is, there are examples for which

$$f(x) \neq \lim_{r\to 0}\|f\|_{L^\infty(B(x,r))},$$

in a set of positive measure. This fact makes proving that a weak ∞-Poincaré inequality implies a $\mathrm{Lip}-\mathrm{lip}$ condition a difficult task.

Question 1: Is it true that when a complete doubling metric measure space supports a weak ∞-Poincaré inequality, it must necessarily satisfy a $\mathrm{Lip}-\mathrm{lip}$ condition? Even if such a space does not satisfy a $\mathrm{Lip}-\mathrm{lip}$ condition, does it support a non-trivial (that is, there is a Lipschitz function whose derivative is non-vanishing on a set of positive measure) measurable differentiable structure in the sense of [Ch, Ke]? We point out here that by the results in [Ke], the $\mathrm{Lip}-\mathrm{lip}$ condition together with the doubling measure by itself guarantees a measurable differentiable structure, but this structure may not be natural in the sense that there may be a Lipschitz function whose derivative vanishes on an open connected set without the function itself being constant on that open connected set. If the metric space satisfies a weak ∞-Poincaré inequality in addition to doubling and a $\mathrm{Lip}-\mathrm{lip}$ condition, then the Poincaré inequality forces the function, whose derivative vanishes on a connected open set, to be itself constant on that connected open set.

The second problem is that for a finite p, we can approximate functions from above in $L^p(X)$ by lower semi-continuous functions (it follows from Vitali-Caratheodory theorem [F, pp. 209–213]). This is highly used when proving that the p-Modulus of the collection of all curves that connect x_0 itself to $X\setminus B(x_0,R)$ is positive (see [HeKo]). Unfortunately such an approximation by lower semi-continuous functions in the L^∞-norm does not hold true, and so we cannot conclude that the ∞-modulus of the collection of all curves connecting x_0 to $X\setminus B(x_0,R)$ is positive if X is only known to support a weak $\infty-$Poincaré inequality.

Question 2: Is it true that if X supports a weak ∞-Poincaré inequality, then the ∞-modulus of all curves connecting x_0 to $X\setminus B(x_0,R)$ is positive?

Question 3: There exist metric measure spaces that are ∞-thick quasiconvex but are not p-thick quasiconvex for any finite $p \geq 1$ (see for example [DJS]); however, the examples we know of, are not doubing measure spaces. We have seen in this paper that there are doubling metric measure spaces that are ∞-thick quasiconvex and hence support an ∞-Poincré inequality but fail to support a weak p-Poincaré inequality for any finite p; however, these example spaces given in this paper are p-thick quasiconvex for some finite p. Are there doubling ∞-thick quasiconvex spaces which are not p-thick quasiconvex?

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Address:

E.D.: Departamento de Análisis Matemático, Facultad de CC. Matemáticas, Universidad Complutense de Madrid, 28040-Madrid, Spain.
E-mail: `estibalitzdurand@mat.ucm.es`

N.S.: Department of Mathematical Sciences, P.O.Box 210025, University of Cincinnati, Cincinnati, OH 45221–0025, U.S.A.
E-mail: `nages@math.uc.edu`

A.W.: Department of Mathematics, Texas Tech University, Lubbock, TX 79409-1042, U.S.A.
E-mail: `alejandros_son@yahoo.com`