A General Integral Representation Result for Continuum Limits of Discrete Energies with Superlinear Growth

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Abstract

We study the asymptotic behavior, as the mesh size ε tends to zero, of a general class of discrete energies defined on functions $u : \alpha \in \varepsilon\mathbb{Z}^N \cap \Omega \mapsto u(\alpha) \in \mathbb{R}^d$, of the form

$$F_\varepsilon(u) = \sum_{\substack{\alpha,\beta\in\varepsilon\mathbb{Z}^N \\ [\alpha,\beta]\subset\Omega}} g_\varepsilon(\alpha, \beta, u(\alpha) - u(\beta)),$$

and satisfying superlinear growth conditions. We show that all the possible variational limits are defined on $W^{1,p}(\Omega; \mathbb{R}^d)$ of the local type

$$\int_\Omega f(x, \nabla u)\, dx.$$

We show that in general f may be a quasiconvex non convex function even if very simple interactions are considered. We also treat the case of homogenization giving a general asymptotic formula that can be simplified in many situations (e.g. in the case of nearest neighbor interactions or under convexity hypotheses).

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1 Introduction

The energetic description of the asymptotic behavior of lattice systems when the mesh size tends to zero turns out to be useful both as a microscopical theoretical justification of theories in Continuum Mechanics and as a powerful means thanks to which a great number of microscopical phenomena can be read in the macroscopical setting. In this paper we describe variational limits of discrete lattice systems in

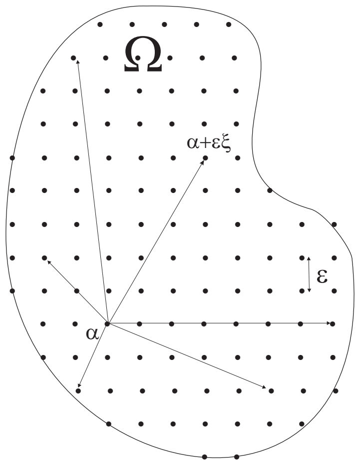

Figure 1: interactions on the lattice $\varepsilon\mathbb{Z}^N$

a vectorial and non convex setting when general "atomic" interaction energies are taken into account, that lead to continuum "elastic" theories described by bulk integral energies. We will limit our analysis to square lattices, but more general geometries, e.g. hexagonal lattices, can be easily included in this framework by a change of variables (see for instance [10] Examples 5.1 and 5.2 for details). In mathematical terms, given a fixed open set $\Omega \subset \mathbb{R}^N$ and $\varepsilon > 0$, we consider energies defined on functions $u : \alpha \in \varepsilon\mathbb{Z}^N \cap \Omega \mapsto u(\alpha) \in \mathbb{R}^d$, of the general form

$$F_\varepsilon(u) = \sum_{\substack{\alpha,\beta\in\varepsilon\mathbb{Z}^N \\ [\alpha,\beta]\subset\Omega}} g_\varepsilon(\alpha, \beta, u(\alpha) - u(\beta)),$$

In the case $N = d = 3$ we can picture the lattice $\varepsilon\mathbb{Z}^N \cap \Omega$ as the reference configuration of a set of interacting material points (see fig. 1). Here u is the field mapping the reference configuration into the deformed one, thus the total stored energy $F_\varepsilon(u)$ is obtained, according to the classical theory of cristalline structures in "hyperelastic" regime, by the superposition of the energy densities $g_\varepsilon(\alpha, \beta, u(\alpha) - u(\beta))$ weighing the pairwise interaction between points in the positions α and β in the reference configuration lattice. Note that the only assumption we make is that g_ε depends on the displacement field in α and β through the differences $u(\alpha) - u(\beta)$. This condition, expressing the invariance under translation of our energies, arises naturally in many situations as for example in frame indifferent models.

It is usually more convenient to group the energy densities as

$$F_\varepsilon(u) = \sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha\in R^\xi_\varepsilon(\Omega)} g_\varepsilon(\alpha,\alpha+\varepsilon\xi,u(\alpha+\varepsilon\xi)-u(\alpha)),$$

where $R^\xi_\varepsilon(\Omega) := \{\alpha\in\varepsilon\mathbb{Z}^N :\ [\alpha,\alpha+\varepsilon\xi]\subset\Omega\}$. Setting

$$f^\xi_\varepsilon(\alpha,\zeta) = \varepsilon^{-N} g_\varepsilon(\alpha,\alpha+\varepsilon\xi,\varepsilon|\xi|\zeta)$$

we can rewrite

$$F_\varepsilon(u) = \sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha\in R^\xi_\varepsilon(\Omega)} \varepsilon^N f^\xi_\varepsilon\left(\alpha, \frac{u(\alpha+\varepsilon\xi)-u(\alpha)}{\varepsilon|\xi|}\right), \tag{1.1}$$

thus highlighting the dependence of the energy on discrete difference quotients in the direction ξ.

The aim of this paper is to provide a characterization of all the possible variational limits, as the mesh size ε tends to zero, of a very general class of energies of the form (1.1). Upon identifying u with a function constant on each cell of the lattice $\varepsilon\mathbb{Z}^N$, we can make the asymptotic analysis precise thanks to the notions and the methods of De Giorgi's Γ-convergence (see [16], [4], [15]). On the functions $f^\xi_\varepsilon(\alpha,\cdot)$ we make assumptions of two types: a growth hypothesis of superlinear type on nearest neighbors (see 3.2) that ensures that the limit is finite only on $W^{1,p}(\Omega;\mathbb{R}^d)$, and a decay assumption as $\xi\to+\infty$ (see (3.3), (H1), (H2)) that allows to neglect very long-range interactions. Under these conditions, a compactness theorem holds asserting that, up to passing to a subsequence, the energies F_ε have a Γ-limit energy F defined on the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^d)$ and taking the form

$$F(u) = \int_\Omega f(x,Du)\ dx,$$

(see Theorem 3.1). A similar compactness result for quadratic interactions in planar networks has been observed by Vogelius [23] (see also Piatnitski and Remy [20]).

Note that the decay assumption on the density energies f^ξ_ε as $|\xi|\to+\infty$ guarantees that the non locality of our discrete functionals disappears in the limit. If this hypothesis is lifted then we may have non local Γ-limits (see [3]). On the other hand, if growth conditions are removed, the limit may be defined on sets of functions with bounded variation where a different analytical approach is needed (see [22], [5], [14], [1], [3], [8], [10]).

To perform our analysis, we develop the discrete analogue of a localization argument used, for example, in the context of homogenization theory for multiple integrals which allows us to regard our energies and their Γ-limits as functionals defined on pairs function-set and then to prove that all the hypotheses of an integral representation theorem are fulfilled. In order to treat minimum problems with boundary data, we also derive a compactness theorem in the case that our functionals are subject to Dirichlet boundary conditions (see (3.30) and Theorem 3.10).

An interesting special case is when the arrangement of the "material points" presents a periodic feature; *i.e.,* in terms of f_ε, we have

$$f^\xi_\varepsilon(\cdot, z) = f^\xi\left(\frac{\cdot}{\varepsilon}, z\right) \qquad f^\xi(\cdot, z) \quad Q_k\text{-periodic}$$

where $Q_k = (0,k)^N$. By adapting the integral homogenization arguments to our discrete setting, we prove that the whole family F_ε Γ-converges to a limit energy of the form

$$F(u) = \int_\Omega f_{hom}(Du)\, dx.$$

Note that in this setting we also include, when $k = 1$, the situation when $f^\xi(\alpha, z)$ is independent of α. If not only nearest neighbor interactions are present, the formula for f_{hom} highlights a multiple-scale effect also in this case (see [4]). An interesting example showing the effect of nonlinearities of "geometrical" origin is contained in a work by Friesecke and Theil [18], where an interpretation in terms of the Cauchy-Born rule is given.

Here f_{hom} is given by the following homogenization formula

$$f_{hom}(M) = \lim_{h\to+\infty} \frac{1}{h^N} \min\left\{\mathcal{F}_h(u),\ u|_{\partial Q_h} = M\alpha\right\} \tag{1.2}$$

where

$$\mathcal{F}_h(u) = \sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha\in R^\xi(Q_h)} f^\xi\left(\alpha, \frac{u(\alpha+\xi) - u(\alpha)}{|\xi|}\right)$$

and $u|_{\partial Q_h} = M\alpha$ means that "near" the boundary of Q_h the function u is the discrete interpolation of the affine function Mx (for more precise definitions see (3.29) and Theorem 4.1). This formula generalizes that obtained in [11] in a one dimensional-scalar setting.

In general (1.2) cannot be simplified to a cell problem formula and gives rise to a quasiconvex not convex function even for simple interactions. Indeed, in Section 7 we provide an example of quasiconvex not convex f_{hom} drawing inspiration from Šverák's construction of a quasiconvex function which is not polyconvex (see [21]).

In Sections 5 and 6 we study some important cases when the formula for f_{hom} can be simplified. For convex interactions a periodicity cell problem formula holds: if f^ξ is a convex function in the second variable for all $\xi \in \mathbb{Z}^N$, then (1.2) can be written as

$$f_{hom}(M) = \frac{1}{k^N} \min\left\{\mathcal{F}(u),\ u\, Q_k\text{-periodic}\right\}$$

where

$$\mathcal{F}(u) = \sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha\in\{0,1,\dots,k-1\}^N} f^\xi \left(\alpha, \frac{u(\alpha+\xi)-u(\alpha)}{|\xi|} + M\cdot\frac{\xi}{|\xi|}\right),$$

(see Theorem 5.1). An analogous result for discrete quadratic forms has been obtained by Piatnitski and Remy [20]. Our result has been used by Braides and Francfort [7] as a step for the derivation of optimal bounds for composite conducting networks in the particular case of quadratic interactions (see Remarks 3.2 and 5.2).

If we consider only interactions along independent directions a reduction to the 1-dimensional case occurs: if $k = 1$, that is f^ξ does not depend on α, and

$$f^\xi \equiv 0 \quad \forall\xi \in \mathbb{Z}^N : \xi \neq je_i,\ i \in \{1, 2, \dots, N\},\ j \in \mathbb{N}, \tag{1.3}$$

where $\{e_1, e_2, \dots, e_N\}$ is the standard orthonormal base in $\mathbb{R}^N$, then

$$f_{hom}(M) = \sum_{i=1}^{N} (\tilde{f}_i)(M^i)$$

being $(\tilde{f}_i)$ convex functions defined by a 1-dimensional homogenization formula and M^i the i-th column of M (see Theorem 6.3). Note that here a superposition principle holds, in the sense that the limit energy is obtained by relaxing the energies due to the interactions in every coordinate direction independently and then summing over them.

From the results obtained in the 1-dimensional setting in [11] (see Theorems 6.1, 6.2), we deduce that the limit energy density f_{hom} can be rewritten by a non-asymptotic formula if only nearest and next-to-nearest neighbor interactions along the coordinate directions are considered (see Remark 6.5). In particular, in the case of only nearest neighbor interactions, the only effect of the passage from the discrete setting to the continuum is a separate convexification process in the coordinate directions.

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2 Notation and Preliminaries

We denote by $\{e_1, e_2, \dots, e_N\}$ the standard basis in $\mathbb{R}^N$, by $|\cdot|$ the usual euclidean norm and by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^N$. We denote by $\mathcal{M}^{d\times N}$ and $\mathcal{M}^{d\times d}_{sym}$ the space of $d\times N$ matrices and symmetric $d\times d$ matrices, respectively. For $P \in \mathcal{M}^{d\times N}$, $Q \in \mathcal{M}^{N\times l}$, $P\cdot Q$ denotes the standard row by column product. For $x,\ y \in \mathbb{R}^N$,

$[x, y]$ denotes the segment between x and y. If Ω is a bounded open subset of $\mathbb{R}^N$, $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω while $\mathcal{A}_0(\Omega)$ denotes the family of all open subsets of Ω whose closure is a compact subset of Ω. If $B \subset \mathbb{R}^N$ is a Borel set, we will denote by $|B|$ its Lebesgue measure. We use standard notation for L^p and Sobolev spaces.

We also recall the standard notation for slicing arguments (see [4]). Let $\xi \in S^{N-1}$, and let $\Pi_\xi = \{y \in \mathbb{R}^N : \langle y, \xi\rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \Pi_\xi$ and $E \subset \mathbb{R}^N$ we define $E^\xi = \{y \ s.t.\ \exists t \in \mathbb{R} : y + t\xi \in E\}$ and $E^\xi_y = \{t \in \mathbb{R} : y + t\xi \in E\}$. Moreover, if $u : E \to \mathbb{R}$ we set $u_{\xi,y} : E^\xi_y \to \mathbb{R}$ by $u_{\xi,y}(t) = u(y + t\xi)$.

We also introduce a useful notation for difference quotient along any direction. Fix $\xi \in \mathbb{R}^N$; for $\varepsilon > 0$ and for every $u : \mathbb{R}^N \to \mathbb{R}^d$ we define

$$D^\xi_\varepsilon u(x) := \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon|\xi|}.$$

2.1 Γ- convergence

We recall the notion of Γ-convergence in $L^p(\Omega; \mathbb{R}^d)$ (see [16],[15],[4]). A sequence of functionals $F_j : L^p(\Omega; \mathbb{R}^d) \to [0, +\infty]$ is said to Γ-converge to a functional $F : L^p(\Omega; \mathbb{R}^d) \to [0, +\infty]$ at $u \in L^p(\Omega; \mathbb{R}^d)$ as $j \to +\infty$, and we write $F(u) = \Gamma\text{-}\lim_j F_j(u)$, if the following two conditions hold:

(i) (lower semicontinuity inequality) for all sequences (u_j) converging to u in $L^p(\Omega; \mathbb{R}^d)$ we have that $F(u) \le \liminf_j F_j(u_j)$;

(ii) (existence of a recovery sequence) there exists a sequence (u_j) converging to u in $L^p(\Omega; \mathbb{R}^d)$ such that $F(u) = \lim_j F_j(u_j)$.

We say that F_j Γ-converges to F if $F(u) = \Gamma\text{-}\lim_j F_j(u)$ at all points $u \in L^p(\Omega; \mathbb{R}^d)$ and that F is the Γ-limit of F_j. The main reason for the introduction of this convergence is the following fundamental theorem.

Theorem 2.1 *Let $F = \Gamma\text{-}\lim_j F_j$, and let a compact set $K \subset L^p(\Omega; \mathbb{R}^d)$ exist such that $\inf_{L^p(\Omega;\mathbb{R}^d)} F_j = \inf_K F_j$ for all j. Then*

$$\exists \min_{L^p(\Omega;\mathbb{R}^d)} F = \lim_j \inf_{L^p(\Omega;\mathbb{R}^d)} F_j.$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_j(u_j) = \lim_j \inf_{L^p(\Omega;\mathbb{R}^d)} F_j$ then its limit is a minimum point for F.

If (F_ε) is a family of functionals indexed by $\varepsilon > 0$ then we say that F_ε Γ-converges to F as $\varepsilon \to 0^+$ if $F = \Gamma\text{-}\lim_j F_{\varepsilon_j}$ for all (ε_j) converging to 0. If we define the *lower and upper Γ-limits* by

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$
$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

respectively, then F_ε Γ-converges to F as $\varepsilon \to 0^+$ if and only if $F'(u) = F''(u) = F(u)$. Note that the functions F' and F'' are lower semicontinuous (see [15] Proposition 6.8).

2.2 Integral representation on Sobolev spaces

In this section we recall an integral representation result on Sobolev spaces for functionals defined on pairs function-sets (see [13]).

Theorem 2.2 *Let $1 \le p < \infty$ and let $F : W^{1,p}(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be a functional satisfying the following conditions:*

(i) (locality) *F is local, i.e. $F(u,A) = F(v,A)$ if $u = v$ a.e. on $A \in \mathcal{A}(\Omega)$;*

(ii) (measure property) *for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ the set function $F(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$;*

(iii) (growth condition) *there exists $c > 0$ and $a \in L^1(\Omega)$ such that*

$$F(u,A) \le c \int_A (a(x) + |Du|^p)\, dx$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$;

(iv) (translation invariance in u) *$F(u+z,A) = F(u,A)$ for all $z \in \mathbb{R}^d$, $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$;*

(v) (lower semicontinuity) *for all $A \in \mathcal{A}(\Omega)$ $F(\cdot,A)$ is sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}(\Omega;\mathbb{R}^d)$.*

Then there exists a Carathéodory function $f : \Omega \times \mathbb{M}^{d\times N} \to [0,+\infty)$ satisfying the growth condition

$$0 \le f(x,M) \le c(a(x) + |M|^p)$$

for all $x \in \Omega$ and $M \in M^{d\times N}$, such that

$$F(u,A) = \int_A f(x,Du(x))\, dx$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$.
If in addition it holds

(vi) (translation invariance in x)

$$F(Mx, B(y,\varrho)) = F(Mx, B(z,\varrho))$$

for all $M \in M^{d\times N}$, $y,z \in \Omega$, and $\varrho > 0$ such that $B(y,\varrho) \cup B(z,\varrho) \subset \Omega$, then f does not depend on x.

3 Compactness and integral representation

In this section we define the class of discrete energies we are going to consider in the rest of the paper and we prove a general compactness theorem, asserting that any sequence of energies in this class has a subsequence whose Γ-limit F is an integral functional.

In what follows Ω will denote a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary. We consider the family of functionals $F_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined as

$$F_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha\in R^\xi_\varepsilon(\Omega)} \varepsilon^N f^\xi_\varepsilon\left(\alpha, D^\xi_\varepsilon u(\alpha)\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.1}$$

where for any $\xi \in \mathbb{Z}^N$ and $\varepsilon > 0$

$$R^\xi_\varepsilon(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^N : \ [\alpha, \alpha + \varepsilon\xi] \subset \Omega\},$$

$$\mathcal{A}_\varepsilon(\Omega) := \{u : \mathbb{R}^N \to \mathbb{R}^d \ : \ u \text{ constant on } \alpha + [0,\varepsilon)^N \text{ for any } \alpha \in \varepsilon\mathbb{Z}^N \cap \Omega\},$$

and $f^\xi_\varepsilon : (\varepsilon\mathbb{Z}^N \cap \Omega) \times \mathbb{R}^d \to [0,+\infty)$ is a given function. On f^ξ_ε we make the following assumptions:

$$f^{e_i}_\varepsilon(\alpha, z) \geq c_1(|z|^p - 1) \quad \forall(\alpha, z) \in (\varepsilon\mathbb{Z}^N \cap \Omega) \times \mathbb{R}^d,\ i \in \{1, \dots, N\} \tag{3.2}$$

$$f^\xi_\varepsilon(\alpha, z) \leq C^\xi_\varepsilon(|z|^p + 1) \quad \forall(\alpha, z) \in (\varepsilon\mathbb{Z}^N \cap \Omega) \times \mathbb{R}^d,\ \xi \in \mathbb{Z}^N, \tag{3.3}$$

where $c_1 > 0$ and $\{C^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfies:

$$\limsup_{\varepsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^N} C^\xi_\varepsilon < +\infty; \tag{H1}$$

$$\forall\delta > 0 \quad \exists\, M_\delta > 0 : \quad \limsup_{\varepsilon\to 0^+} \sum_{|\xi|>M_\delta} C^\xi_\varepsilon < \delta. \tag{H2}$$

The main result of this section is stated in the following theorem.

Theorem 3.1 **(compactness)** *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy (3.2), (3.3) and let* (H1)-(H2) *hold. Then for every sequence (ε_j) of positive real numbers converging to 0, there exists a subsequence (ε_{j_k}) and a Carathéodory function quasiconvex in the second variable $f : \Omega \times \mathbb{R}^{d\times N}$ satisfying*

$$c(|M|^p - 1) \leq f(x, M) \leq C(|M|^p + 1),$$

with $0 < c < C$, such that $(F_{\varepsilon_{j_k}}(\cdot))$ Γ-converges with respect to the $L^p(\Omega;\mathbb{R}^d)$-topology to the functional $F : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined as

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.4}$$

Remark 3.2 (**quadratic forms**) Under the hypotheses of Theorem 3.1, if in addition for any $\xi \in \mathbb{Z}^N$ and $\varepsilon > 0$ $f_\varepsilon^\xi(\alpha, \cdot)$ is a positive quadratic form on $\mathbb{R}^d$, that is

$$f_\varepsilon^\xi(\alpha, z) = \langle A_\varepsilon^\xi(\alpha) z, z\rangle \quad A_\varepsilon^\xi(\alpha) \in \mathcal{M}^{d\times d}_{sym}$$

then, by the properties of Γ-convergence (see [15]), the limit energy density $f(x, \cdot)$ is a quadratic form on $\mathcal{M}^{d\times N}$, that is

$$f(x, M) = A(x)\,(M, M)\,, \quad A(x) \in T_2\mathcal{M}^{d\times N} \tag{3.5}$$

where $T_2\mathcal{M}^{d\times N}$ is the vectorial space of all two times covariant tensors on $\mathcal{M}^{d\times N}$.

To prove Theorem 3.1 we use a localization technique, which is a standard argument dealing with limits of integral functionals (see for example [6] in the context of homogenization theory). We stress the fact that here this analysis becomes more difficult to perform because of the non locality of our discrete energies.

The first step is to define a "localized" version of our energies: given an open set A we isolate the contributions due to interactions within A as follows. For $u \in \mathcal{A}_\varepsilon(\Omega)$, $A \in \mathcal{A}(\Omega)$ and $\xi \in \mathbb{Z}^N$, set

$$\mathcal{F}_\varepsilon^\xi(u, A) := \sum_{\alpha \in R_\varepsilon^\xi(A)} \varepsilon^N f_\varepsilon^\xi\left(\alpha, D_\varepsilon^\xi u(\alpha)\right), \tag{3.6}$$

where

$$R_\varepsilon^\xi(A) := \{\alpha \in \varepsilon\mathbb{Z}^N : \ [\alpha, \alpha + \varepsilon\xi] \subset A\}.$$

The function $\mathcal{F}_\varepsilon^\xi$ represents the energy due to the interactions within A along the direction ξ. Then the local version of the functional in (3.1) is given by

$$F_\varepsilon(u, A) = \begin{cases} \sum_{\xi \in \mathbb{Z}^N} \mathcal{F}_\varepsilon^\xi(u, A) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

We will prove also the following result.

Theorem 3.3 (**local compactness**) *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (3.2), (3.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to* 0*, let (ε_{j_k}) and f be as in Theorem 3.1. Then for any $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ there holds*

$$\Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) = \int_A f(x, \nabla u)\, dx.$$

We will derive the proof of Theorems 3.1 and 3.3 as a direct consequence of some propositions and lemmas which are fundamental steps to show that our limit functionals satisfy all the hypotheses of the Representation Theorem 2.2.

In the next two propositions we show that, thanks to hypotheses (3.2) and (3.3), the Γ-lim inf and the Γ-lim sup of F_ε are finite only on $W^{1,p}(\Omega; \mathbb{R}^d)$ and satisfy standard p-growth conditions.

Proposition 3.4 *Let $\{f_\varepsilon^{e_i}\}_{\varepsilon,i}$ satisfy (3.2). If $u \in L^p(\Omega;\mathbb{R}^d)$ is such that $F'(u,A) < +\infty$, then $u \in W^{1,p}(A;\mathbb{R}^d)$ and*

$$F'(u,A) \geq c\ \left(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N})} - |A|\right) \tag{3.8}$$

for some positive constant c independent on u and A.

Proof. Let $\varepsilon_n \to 0^+$ and let u_n converge to u in $L^p(\Omega;\mathbb{R}^d)$ and be such that $\liminf_n F_{\varepsilon_n}(u_n,A) < +\infty$. By the growth condition (3.2) we get

$$F_{\varepsilon_n}(u_n,A) \geq c_1 \sum_{i=1}^N \sum_{\alpha \in R_{\varepsilon_n}^{e_i}(A)} \varepsilon_n^N |D_{\varepsilon_n}^{e_i} u_n(\alpha)|^p - c_1 N|A|.$$

For any $i \in \{1,\ldots,N\}$, consider the sequence of piecewise-affine functions (v_n^i) defined as follows

$$v_n^i(x) := u_n(\alpha) + D_{\varepsilon_n}^{e_i} u_n(\alpha)(x_i - \alpha_i) \quad x \in \left(\alpha + [0,\varepsilon_n)^N\right) \cap \Omega,\ \alpha \in R_{\varepsilon_n}^{e_i}(A).$$

Note that v_n^i is a function of bounded variation and we will denote by $\frac{\partial v_n^i}{\partial x_i}$ the density of the absolutely continuous part of $D_{x_i} v_n^i$ with respect to the Lebesgue measure. Moreover for $\mathcal{H}^{N-1}$-a.e. $y \in (A)^{e_i}$ the slices $(v_n^i)_{e_i,y} \in W^{1,p}\left((A)_y^{e_i};\mathbb{R}^d\right)$. For any $\eta > 0$, set

$$A_\eta := \{x \in A : dist(x,A^c) > \eta\}.$$

Then, fixed $\eta > 0$, it is easy to check that $v_n^i \to u$ in $L^p(A_\eta;\mathbb{R}^d)$ for every $i \in \{1,\ldots,N\}$; moreover, since $\frac{\partial v_n^i}{\partial x_i}(x) = D_{\varepsilon_n}^{e_i} u_n(\alpha)$ for $x \in \alpha + [0,\varepsilon_n)^N$, we get

$$F_{\varepsilon_n}(u_n,A) \quad \geq \quad c_1 \sum_{i=1}^N \int_{A_\eta} \left|\frac{\partial v_n^i}{\partial x_i}(x)\right|^p\, dx - c_1 N|A|. \tag{3.9}$$

We apply now a standard slicing argument. By Fubini's Theorem and Fatou's Lemma for any i we get

$$\liminf_n \int_{(A_\eta)} \left|\frac{\partial v_n^i}{\partial x_i}(x)\right|^p \geq \int_{(A_\eta)^{e_i}} \liminf_n \int_{(A_\eta)_y^{e_i}} \left|(v_n^i)'_{e_i,y}(t)\right|^p\, dt\, d\mathcal{H}^{N-1}(y).$$

Since, up to passing to a subsequence, we may assume that, for $\mathcal{H}^{N-1}$-a.e. $y \in (A_\eta)^{e_i}$ $(v_n^i)_{e_i,y} \to u_{e_i,y}$ in $L^p\left((A_\eta)_y^{e_i};\mathbb{R}^d\right)$, we deduce that $u_{e_i,y} \in W^{1,p}\left((A_\eta)_y^{e_i};\mathbb{R}^d\right)$ for $\mathcal{H}^{N-1}$-a.e. $y \in (A_\eta)^{e_i}$ and

$$\liminf_n \int_{(A_\eta)} \left|\frac{\partial v_n^i}{\partial x_i}(x)\right|^p \geq \int_{(A_\eta)^{e_i}} \int_{(A_\eta)_y^{e_i}} \left|u'_{e_i,y}(t)\right|^p\, dt\, d\mathcal{H}^{N-1}(y).$$

Then, by (3.9), we have

$$\liminf_n F_{\varepsilon_n}(u_n, A) \geq c_1 \sum_{i=1}^N \int_{(A_\eta)^{e_i}} \int_{(A_\eta)_y^{e_i}} \left|u'_{e_i,y}(t)\right|^p \, dt \, d\mathcal{H}^{N-1}(y) - c_1 N|A|.$$

Since, in particular, the previous inequality implies that

$$\sum_{i=1}^N \int_{(A_\eta)^{e_i}} \int_{(A_\eta)_y^{e_i}} \left|u'_{e_i,y}(t)\right|^p \, dt \, d\mathcal{H}^{N-1}(y) < +\infty,$$

thanks to the characterization of $W^{1,p}$ by slicing, we obtain that $u \in W^{1,p}\left(A_\eta; \mathbb{R}^d\right)$ and

$$\begin{aligned} \liminf_n F_{\varepsilon_n}(u_n, A) &\geq c_1 \sum_{i=1}^N \int_{A_\eta} \left|\frac{\partial u}{\partial x_i}(x)\right|^p dx - c_1 N|A| \\ &\geq c\left(\int_{A_\eta} \|\nabla u(x)\|^p \, dx - |A|\right). \end{aligned}$$

Letting $\eta \to 0^+$, we get the conclusion. $\square$

Proposition 3.5 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (3.3) and let* (H1) *hold. Then for every $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ there holds*

$$F''(u, A) \leq C \left(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N})} + |A|\right) \tag{3.10}$$

for some positive constant C independent on u and A.

Proof. We first show that inequality (3.5) holds for u smooth and then we recover the proof for any $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ by using a density argument.

Let $u \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^d)$ and consider the family $(u_\varepsilon) \subset \mathcal{A}_\varepsilon(\Omega)$ defined as

$$u_\varepsilon(\alpha) := u(\alpha), \quad \alpha \in \varepsilon\mathbb{Z}^N.$$

Then $u_\varepsilon \to u$ in $L^p(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0^+$. Moreover, for any $\alpha \in \varepsilon\mathbb{Z}^N$, we have

$$D_\varepsilon^\xi u_\varepsilon(\alpha) = \frac{1}{|\xi|} \int_0^1 \nabla u(\alpha + \varepsilon\xi s)\xi \, ds,$$

so that, by Jensen's inequality, we get

$$\begin{aligned} |D_\varepsilon^\xi u_\varepsilon(\alpha)|^p &= \frac{1}{|\xi|^p} \left|\int_0^1 \nabla u(\alpha + \varepsilon\xi s)\xi \, ds\right|^p \\ &\leq \frac{1}{|\xi|^p} \int_0^1 |\nabla u(\alpha + \varepsilon\xi s)\xi|^p \, ds \leq \int_0^1 |\nabla u(\alpha + \varepsilon\xi s)|^p \, ds. \end{aligned}$$

By the regularity hypothesis on u and by Fubini's Theorem, we easily obtain the following inequalities

$$
\begin{aligned}
&\varepsilon^N \int_0^1 |\nabla u(\alpha + \varepsilon\xi s)|^p \, ds = \int_{\alpha+[0,\varepsilon)^N} \int_0^1 |\nabla u(\alpha + \varepsilon\xi s)|^p \, ds \, dx \\
&\leq \int_{\alpha+[0,\varepsilon)^N} \int_0^1 |\nabla u(x + \varepsilon\xi s)|^p \, ds \, dx + c(u) \int_{\alpha+[0,\varepsilon)^N} \int_0^1 |x-\alpha|^p \, ds \, dx \\
&\leq \int_0^1 \int_{\alpha+s\varepsilon\xi+[0,\varepsilon)^N} |\nabla u(x)|^p \, dx \, ds + c(u)\varepsilon^p\varepsilon^N,
\end{aligned}
$$

where by $c(u)$ we denote a constant depending only on u. By (3.3) and the last inequality, we then have

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon, A) \quad &\leq \quad \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(A)} \int_0^1 \int_{\alpha+s\varepsilon\xi+[0,\varepsilon)^N} |\nabla u(x)|^p \, dx \, ds \\
&\quad + (1 + c(u)\varepsilon^p) \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(A)} \varepsilon^N \\
&\leq \quad \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi \left(\int_{A^\varepsilon} |\nabla u(x)|^p \, dx + (1 + c(u)\varepsilon^p) \, |A^\varepsilon| \right),
\end{aligned}
$$

where

$$A^\varepsilon := A + [0,\varepsilon)^N.$$

Eventually, letting $\varepsilon \to 0^+$, by (H1) we get

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) \leq C \left(\int_A |\nabla u(x)|^p \, dx + |A| \right)$$

and the conclusion follows by the definition of F''. Now let $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and let $(u_n) \subset C_c^\infty(\mathbb{R}^N;\mathbb{R}^d)$ converge to u in the $W^{1,p}(\Omega;\mathbb{R}^d)$-topology. Then, by the lower-semicontinuity of F'', we obtain

$$
\begin{aligned}
F''(u, A) \quad &\leq \quad \liminf_n F''(u_n, A) \leq \lim_n C \left(\|\nabla u_n\|^p_{L^p(A;\mathbb{R}^{d\times N})} + |A| \right) \\
&= \quad C \left(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N})} + |A| \right).
\end{aligned}
$$

□

The next technical lemma asserts that finite difference quotients along any direction can be controlled by finite difference quotients along the coordinate directions.

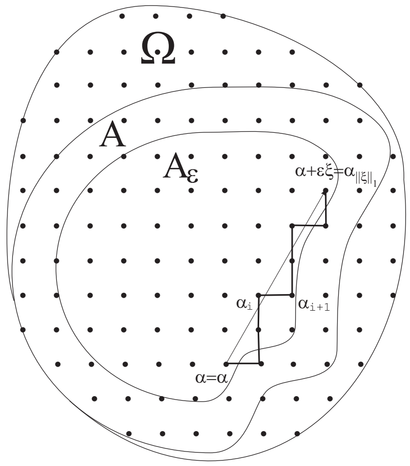

Figure 2

Lemma 3.6 *Let $A \in \mathcal{A}(\Omega)$ and set $A_\varepsilon := \{x \in A : \ dist(x, \partial A) > 2\sqrt{N}\varepsilon\}$. Then for any $\xi \in \mathbb{Z}^N$ and $u \in \mathcal{A}_\varepsilon(\Omega)$ there holds*

$$\sum_{\alpha \in R^\xi_\varepsilon(A_\varepsilon)} |D^\xi_\varepsilon u(\alpha)|^p \leq C \sum_{i=1}^N \sum_{\alpha \in R^{e_i}_\varepsilon(A)} |D^{e_i}_\varepsilon u(\alpha)|^p. \tag{3.11}$$

Proof. Let us fix some notations : for $\xi \in \mathbb{Z}^N$ and $\alpha \in \varepsilon\mathbb{Z}^N$, set

$$I^\xi_\varepsilon(\alpha) := \left\{\beta \in \varepsilon\mathbb{Z}^N : \ \left(\beta + [-\varepsilon, \varepsilon]^N\right) \cap [\alpha, \alpha + \varepsilon\xi] \neq \emptyset\right\};$$

moreover we will denote by $\|\cdot\|_1$ the norm on $\mathbb{R}^N$ defined as

$$\|\xi\|_1 := \sum_{i=1}^N |\xi_i|, \quad \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N.$$

Let $\alpha \in R^\xi_\varepsilon(A_\varepsilon)$ and consider $\{\alpha_h\}_{h=1}^{\|\xi\|_1} \subset I^\xi_\varepsilon(\alpha)$ such that

$$\alpha_{\|\xi\|_1} = \alpha + \varepsilon\xi, \quad \alpha_1 = \alpha, \quad \alpha_h = \alpha_{h-1} + \varepsilon e_{i(h)},$$

for some $i(h) \in \{1, \ldots, N\}$ (see fig. 2). Then, since

$$D^\xi_\varepsilon u(\alpha) = \frac{1}{|\xi|} \sum_{h=1}^{\|\xi\|_1} D^{e_{i(h)}}_\varepsilon u(\alpha_h)$$

by Jensen's inequality, we get

$$
\begin{aligned}
|D_\varepsilon^\xi u(\alpha)|^p &= \left(\frac{\|\xi\|_1}{|\xi|}\right)^p \left| \frac{1}{\|\xi\|_1} \sum_{h=1}^{\|\xi\|_1} D_\varepsilon^{e_{i(h)}} u(\alpha_h) \right|^p \\
&\leq \left(\frac{\|\xi\|_1}{|\xi|}\right)^p \frac{1}{\|\xi\|_1} \sum_{h=1}^{\|\xi\|_1} |D_\varepsilon^{e_{i(h)}} u(\alpha_h)|^p.
\end{aligned}
$$

Since for any $h = 1, \dots, N$, $\alpha_h \in R_\varepsilon^{e_{i(h)}}(A)$ and all the norms are equivalent in a finite dimensional space, we infer that

$$
\sum_{\alpha \in R_\varepsilon^\xi(A_\varepsilon)} |D_\varepsilon^\xi u(\alpha)|^p \leq C \sum_{i=1}^N \sum_{\beta \in R_\varepsilon^{e_i}(A)} \frac{\gamma_\varepsilon^\xi(\beta)}{\|\xi\|_1} |D_\varepsilon^{e_i} u(\beta)|^p,
$$

where

$$
\gamma_\varepsilon^\xi(\beta) := \#\{\alpha \in R_\varepsilon^\xi(A_\varepsilon) : \ \beta \in I_\varepsilon^\xi(\alpha)\}.
$$

Hence, the proof is complete if we show that $\gamma_\varepsilon^\xi(\beta) \leq C|\xi|$. To this aim, notice that

$$
\{\alpha \in R_\varepsilon^\xi(A_\varepsilon) : \ \beta \in I_\varepsilon^\xi(\alpha)\} \subseteq \varepsilon\mathbb{Z}^N \cap Q_\varepsilon^\xi(\beta),
$$

where

$$
Q_\varepsilon^\xi(\beta) := \{x \in \mathbb{R}^N : x = y + t\xi, \ y \in \beta + [-\varepsilon, \varepsilon]^n, \ t \in [-\varepsilon, \varepsilon]^N\}.
$$

Thus, we infer that

$$
\gamma_\varepsilon^\xi(\beta) \leq C \frac{\left|Q_\varepsilon^\xi(\beta)\right|}{\varepsilon^N}.
$$

Now we use a slicing argument to provide an estimate of $\left|Q_\varepsilon^\xi(\beta)\right|$. By Fubini's Theorem, we get

$$
\begin{aligned}
|Q_\varepsilon^\xi(\beta)| &= \int_{\left(Q_\varepsilon^\xi(\beta)\right)^\xi} \mathcal{H}^1 \left(Q_\varepsilon^\xi(\beta)\right)_y^\xi \, d\mathcal{H}^{N-1}(y) \\
&\leq \mathcal{H}^{N-1}\left((Q_\varepsilon^\xi(\beta))^\xi\right) 2(\sqrt{N} + |\xi|)\varepsilon \leq c(N)|\xi|\varepsilon^N,
\end{aligned}
$$

where the last inequality holds, since for any $\xi \in \mathbb{Z}^N$

$$
\mathcal{H}^{N-1}\left((Q_\varepsilon^\xi(\beta))^\xi\right) \leq c(N)\varepsilon^{N-1}.
$$

□

In the next two propositions we establish the subadditivity and the inner regularity of the set function $F''(u, \cdot)$. To this end we use a careful modification of De Giorgi's cut-off functions argument, which appears frequently in the proof of the integral representation of Γ-limits of integral functionals (see [6],[15]). We underline that the non locality of our energies requires a deeper analysis in which a key role is played by hypothesis (H2), which allows us to show that very long range interactions do not lead to non local terms in the limit.

Proposition 3.7 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (3.2),(3.3) and let* (H1)-(H2) *hold. Let $A, B \in \mathcal{A}(\Omega)$ and let $A', B' \in \mathcal{A}(\Omega)$ be such that $A' \subset\subset A$ and $B' \subset\subset B$. Then for any $u \in W^{1,p}(\Omega;\mathbb{R}^d)$,*

$$F''(u, A' \cup B') \le F''(u, A) + F''(u, B).$$

Proof. Without loss of generality, we may suppose $F''(u, A)$ and $F''(u, B)$ finite. Let $u_\varepsilon, v_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ both converge to u in $L^p(\Omega;\mathbb{R}^d)$ and be such that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u, A), \quad \limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, B) = F''(u, B).$$

By (3.2) and Lemma 3.6, we infer that

$$\sup_{\xi\in\mathbb{Z}^N} \sup_{\varepsilon>0} \sum_{\alpha\in R_\varepsilon^\xi(A_\varepsilon)} \varepsilon^N |D_\varepsilon^\xi u_\varepsilon(\alpha)|^p < +\infty, \tag{3.12}$$

$$\sup_{\xi\in\mathbb{Z}^N} \sup_{\varepsilon>0} \sum_{\alpha\in R_\varepsilon^\xi(B_\varepsilon)} \varepsilon^N |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p < +\infty, \tag{3.13}$$

where A_ε and B_ε are defined as in Lemma 3.6. Moreover, since (u_ε) and (v_ε) converge to u in the $L^p(\Omega;\mathbb{R}^d)$-topology, we have

$$\sum_{\alpha\in\varepsilon\mathbb{Z}^N\cap\Omega'} \varepsilon^N \left(|u_\varepsilon(\alpha)|^p + |v_\varepsilon(\alpha)|^p\right) \le \|u_\varepsilon\|^p_{L^p(\Omega;\mathbb{R}^d)} + \|v_\varepsilon\|^p_{L^p(\Omega;\mathbb{R}^d)} \le C < +\infty, \tag{3.14}$$

$$\sum_{\alpha\in\varepsilon\mathbb{Z}^N\cap\Omega'} \varepsilon^N \left(|u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p\right) \le \|u_\varepsilon - v_\varepsilon\|^p_{L^p(\Omega;\mathbb{R}^d)} \to 0^+ \tag{3.15}$$

for any $\Omega' \subset\subset \Omega$. Set

$$d := \operatorname{dist}(A', A^c)$$

and for any $i \in \{1, \dots, N\}$ define

$$A_i := \{x \in A : \ \operatorname{dist}(x, A') < i\frac{d}{N}\}.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \le 2\frac{N}{d}$. Then for any $i \in \{1, \dots, N\}$ consider the family of functions $w_\varepsilon^i \in \mathcal{A}_\varepsilon(\Omega)$ still converging to u in $L^p(\Omega;\mathbb{R}^d)$ defined as

$$w_\varepsilon^i(\alpha) := \varphi_i(\alpha)u_\varepsilon(\alpha) + (1 - \varphi_i(\alpha))\, v_\varepsilon(\alpha).$$

Note that, for any $\xi \in \mathbb{Z}^N$, we have

$$\begin{aligned} D_\varepsilon^\xi w_\varepsilon^i(\alpha) = \varphi_i(\alpha+\varepsilon\xi) D_\varepsilon^\xi u_\varepsilon(\alpha) \quad &+ \quad (1 - \varphi_i(\alpha+\varepsilon\xi))\, D_\varepsilon^\xi v_\varepsilon(\alpha) \\ &+ (u_\varepsilon(\alpha) - v_\varepsilon(\alpha))\, D_\varepsilon^\xi \varphi(\alpha). \end{aligned} \tag{3.16}$$

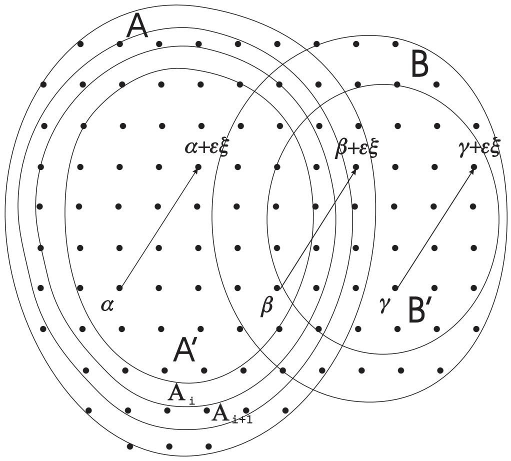

Figure 3 : $\alpha \in R^{\xi}_{\varepsilon}(A_i), \beta \in R^{\xi}_{\varepsilon}(S^{\varepsilon,\xi}_i), \gamma \in R^{\xi}_{\varepsilon}(B' \setminus A^c_{i+1})$

Fix $i \in \{1, 2, \dots, N-3\}$. Given $\xi \in \mathbb{Z}^N$ and $\alpha \in R^{\xi}_{\varepsilon}(A' \cup B')$, then either $\alpha \in R^{\xi}_{\varepsilon}(A_i)$, or $\alpha \in R^{\xi}_{\varepsilon}(\overline{A}^c_{i+1} \cap B')$, or

$$[\alpha, \alpha + \varepsilon\xi] \cap \big(\overline{A}_{i+1} \setminus A_i\big) \cap B' \neq \emptyset.$$

Then, if we set

$$\big(\overline{A}_{i+1} \setminus A_i\big)^{\varepsilon,\xi} := \{x = y + t\xi,\ |t| \leq \varepsilon,\ y \in \overline{A}_{i+1} \setminus A_i\},$$

$$S^{\varepsilon,\xi}_i := \big(\overline{A}_{i+1} \setminus A_i\big)^{\varepsilon,\xi} \cap (A' \cup B'),$$

we get

$$R^{\xi}_{\varepsilon}(A' \cup B') \subseteq R^{\xi}_{\varepsilon}(A_i) \cup R^{\xi}_{\varepsilon}(B' \setminus \overline{A}_{i+1}) \cup R^{\xi}_{\varepsilon}\left(S^{\varepsilon,\xi}_i\right)$$

(see fig. 3). Thus, since $D^{\xi}_{\varepsilon} w^i_{\varepsilon}(\alpha) = D^{\xi}_{\varepsilon} u_{\varepsilon}(\alpha)$ if $\alpha \in R^{\xi}_{\varepsilon}(A_i)$ and $D^{\xi}_{\varepsilon} w^i_{\varepsilon}(\alpha) = D^{\xi}_{\varepsilon} v_{\varepsilon}(\alpha)$ if $\alpha \in R^{\xi}_{\varepsilon}(\overline{A}^c_{i+1} \cap B')$, we get by (3.3) and (3.16)

$$\begin{aligned} \mathcal{F}^{\xi}_{\varepsilon}(w^i_{\varepsilon}, A' \cup B') &\leq \mathcal{F}^{\xi}_{\varepsilon}(u_{\varepsilon}, A) + \mathcal{F}^{\xi}_{\varepsilon}(v_{\varepsilon}, B) \\ &+ \ C\ C^{\xi}_{\varepsilon} \sum_{\alpha \in R^{\xi}_{\varepsilon}(S^{\varepsilon,\xi}_i)} \varepsilon^N \left(|D^{\xi}_{\varepsilon} u_{\varepsilon}(\alpha)|^p + |D^{\xi}_{\varepsilon} v_{\varepsilon}(\alpha)|^p + N^p\, |u_{\varepsilon}(\alpha) - v_{\varepsilon}(\alpha)|^p + 1\right). \end{aligned} \tag{3.17}$$

If $\varepsilon|\xi| \leq \frac{d}{2N}$, then

$$S^{\varepsilon,\xi}_i \subseteq \big(A_{N-1} \setminus \overline{A'}\big) \cap B' =: S_N \subset\subset A \cap B. \tag{3.18}$$

If $\varepsilon|\xi| \geq \frac{d}{2N}$, then

$$\frac{1}{\varepsilon^p|\xi|^p} \leq \frac{2^p N^p}{d^p},$$

and so

$$|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p \leq CN^p \left(|u_\varepsilon(\alpha)|^p + |u_\varepsilon(\alpha+\varepsilon\xi)|^p\right),$$

and the same inequality holds for v_ε. Thus, in this case we get by (3.17)

$$\begin{aligned}
&\mathcal{F}_\varepsilon^\xi\left(w_\varepsilon^i, A'\cup B'\right) \leq \mathcal{F}_\varepsilon^\xi(u_\varepsilon, A) + \mathcal{F}_\varepsilon^\xi(v_\varepsilon, B) \\
&+\ CN^p\ C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(A'\cup B')} \varepsilon^N \left(|u_\varepsilon(\alpha)|^p + |u_\varepsilon(\alpha+\varepsilon\xi)|^p + |v_\varepsilon(\alpha)|^p + |v_\varepsilon(\alpha+\varepsilon\xi)|^p + 1\right).
\end{aligned} \tag{3.19}$$

Let $M_\delta > 0$ be such that $\limsup_{\varepsilon\to 0^+} \sum_{|\xi|>M_\delta} C_\varepsilon^\xi < \delta$. Then, by (3.17), (3.18) and (3.19), summing over $\xi \in \mathbb{Z}^N$,for ε small enough we get

$$\begin{aligned}
&F_\varepsilon\left(w_\varepsilon^i, A'\cup B'\right) \leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\
&\quad +C \sum_{|\xi|\leq M_\delta} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(S_i^{\varepsilon,\xi})} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\,|u_\varepsilon(\alpha)-v_\varepsilon(\alpha)|^p + 1\right) \\
&\quad +C \sum_{M_\delta<|\xi|\leq\frac{d}{2N\varepsilon}} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(S_N)} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\,|u_\varepsilon(\alpha)-v_\varepsilon(\alpha)|^p + 1\right) \\
&\quad +CN^p \sum_{|\xi|>\frac{d}{2N\varepsilon}} C_\varepsilon^\xi \sum_{\alpha\in\varepsilon\mathbb{Z}^N\cap A'\cup B'} \varepsilon^N \left(|u_\varepsilon(\alpha)|^p + |v_\varepsilon(\alpha)|^p + 1\right).
\end{aligned}$$

Note that, for ε small enough and $|\xi| \leq M_\delta$ we have that $R_\varepsilon^\xi(S_i^{\varepsilon,\xi}) \cap R_\varepsilon^\xi(S_j^{\varepsilon,\xi}) \neq \emptyset$ if and only if $|i-j| = 1$, and $\bigcup_{i=1}^{N-3} R_\varepsilon^\xi(S_i^{\varepsilon,\xi}) \subseteq R_\varepsilon^\xi(A_\varepsilon \cap B_\varepsilon)$. Thus, summing over $i \in \{1, 2, \ldots, N-3\}$, averaging, and taking into account (3.12), (3.13), (3.14) and (3.15), we get

$$\begin{aligned}
\frac{1}{N-3}\sum_{i=1}^{N-3} F_\varepsilon(w_\varepsilon^i, A'\cup B') \leq\ & F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\
& +\frac{C}{N-3}\left(1 + N^p O(\varepsilon)\right) + C(\delta + O(\varepsilon))(1 + N^p O(\varepsilon)) \\
& +C\left(\delta + O(\varepsilon)\right)(N^p)
\end{aligned} \tag{3.20}$$

For any $\varepsilon > 0$ there exists $i(\varepsilon) \in \{1, \ldots, N-3\}$ such that

$$F_\varepsilon(w_\varepsilon^{i(\varepsilon)}, A'\cup B') \leq \frac{1}{N-3}\sum_{i=1}^{N-3} F_\varepsilon(w_\varepsilon^i, A'\cup B'). \tag{3.21}$$

Then, since $w_\varepsilon^{i(\varepsilon)}$ still converges to u in $L^p(\Omega;\mathbb{R}^d)$, by (3.20) and (3.21), letting $\varepsilon \to 0^+$, we get

$$F''(u, A' \cup B') \leq F''(u, A) + F''(u, B) + \frac{C}{N-3} + C\delta\,(1 + N^p)\,.$$

Eventually, letting first $\delta \to 0^+$ and then $N \to +\infty$, we obtain the thesis. □

Proposition 3.8 *Let* $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ *satisfy (3.2),(3.3) and let* (H1)-(H2) *hold. Then for any* $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ *and for any* $A \in \mathcal{A}(\Omega)$*, there holds*

$$\sup_{A' \subset\subset A} F''(u, A') = F''(u, A).$$

Proof. Since $F''(u,\cdot)$ is an increasing set function, it suffices to prove that

$$\sup_{A' \subset\subset A} F''(u, A') \geq F''(u, A).$$

To do this, we apply the same argument of the proof of Proposition 3.7. Given $\delta > 0$, there exists $A'' \subset\subset A$ such that

$$|A \setminus \overline{A''}| + \|\nabla u\|^p_{L^p(A \setminus \overline{A''})} \leq \delta.$$

Let $\tilde{\Omega} \supset\supset \Omega$ and let $\tilde{u} \in W^{1,p}(\tilde{\Omega};\mathbb{R}^d)$ an extension of u. By reasoning as in the proof of Proposition 3.5, we may find $v_\varepsilon \in \mathcal{A}_\varepsilon(\tilde{\Omega})$ such that v_ε converges to $\tilde{u}$ in $L^p(\tilde{\Omega};\mathbb{R}^d)$ and

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(v_\varepsilon, A \setminus \overline{A''}) \leq C\left(|A \setminus \overline{A''}| + \|\nabla u\|^p_{L^p(A \setminus \overline{A''})}\right) \leq C\delta. \tag{3.22}$$

We remark that this extension on $\tilde{\Omega}$ is just a technical tool to exploit an analogue of inequality (3.14) and obtain a control of the interactions near the boundary of Ω. Let $A' \in \mathcal{A}(\Omega)$ be such that $A'' \subset\subset A' \subset\subset A$ and let $u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ converge to u in $L^p(\Omega;\mathbb{R}^d)$, with

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, A') = F''(u, A').$$

Set

$$d := \operatorname{dist}(A'', A'^c)$$

and for any $i \in \{1,\dots,N\}$ define

$$A_i := \{x \in A : \ \operatorname{dist}(x, A') < i\frac{d}{N}\}.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \leq 2\frac{N}{d}$. Then for any $i \in \{1,\dots,N\}$ consider the family of functions $w_\varepsilon^i \in \mathcal{A}_\varepsilon(\Omega)$ still converging to u in $L^p(\Omega;\mathbb{R}^d)$ defined as

$$w_\varepsilon^i(\alpha) := \varphi_i(\alpha)u_\varepsilon(\alpha) + (1 - \varphi_i(\alpha))\,v_\varepsilon(\alpha).$$

Now we can set
$$S_i^{\varepsilon,\xi} := \left(\overline{A}_{i+1} \setminus A_i\right)^{\varepsilon,\xi} \cap A,$$
so that
$$R_\varepsilon^\xi(A) \subseteq R_\varepsilon^\xi(A_i) \cup R_\varepsilon^\xi(A \setminus \overline{A}_{i+1}) \cup R_\varepsilon^\xi\left(S_i^{\varepsilon,\xi}\right).$$
Let $\delta > 0$ and let $M_\delta > 0$ be such that $\limsup_{\varepsilon\to 0^+} \sum_{|\xi|>M_\delta} C_\varepsilon^\xi < \delta$. Then, by reasoning as in the proof of Proposition 3.7,for ε small enough, we get

$$\begin{aligned}
&F_\varepsilon(w_\varepsilon^i, A) \le F_\varepsilon(u_\varepsilon, A') + F_\varepsilon(v_\varepsilon, A \setminus \overline{A''})\\
&\quad + C \sum_{|\xi|\le M_\delta} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(S_i^{\varepsilon,\xi})} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\, |u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p + 1\right)\\
&\quad + C \sum_{M_\delta<|\xi|\le \frac{d}{2N\varepsilon}} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(S_N)} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\, |u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p + 1\right)\\
&\quad + CN^p \sum_{|\xi|>\frac{d}{2N\varepsilon}} C_\varepsilon^\xi \left(\|u_\varepsilon\|^p_{L^p(\Omega;\mathbb{R}^d)} + \|v_\varepsilon\|^p_{L^p(\tilde{\Omega};\mathbb{R}^d)} + 1\right).
\end{aligned}$$

Since u_ε and v_ε satisfy (3.12), (3.13), (3.14) and (3.15) with A_ε replaced by A'_ε and B_ε by $(A \setminus \overline{A''_\varepsilon})$ then we can choose $i(\varepsilon) \in \{1, \dots, N-3\}$ such that

$$\begin{aligned}
&F_\varepsilon(w_\varepsilon^{i(\varepsilon)}, A) \le \frac{1}{N-3} \sum_{i=1}^{N-3} F_\varepsilon(w_\varepsilon^i, A) \le \\
&F_\varepsilon(u_\varepsilon, A') + C\delta + \frac{C}{N-3}\left(1 + N^p O(\varepsilon)\right) + \\
&C(\delta + O(\varepsilon))\left(1 + N^p O(\varepsilon)\right) + CN^p(\delta + O(\varepsilon)).
\end{aligned} \tag{3.23}$$

Then, since $w_\varepsilon^{i(\varepsilon)}$ still converges to u in $L^p(\Omega; \mathbb{R}^d)$, by (3.23), letting $\varepsilon \to 0^+$, we get
$$F''(u, A) \le \sup_{A' \subset\subset A} F''(u, A') + C\left(\frac{1}{N-3} + \delta + \delta N^p\right).$$
Eventually, letting first $\delta \to 0^+$ and then $N \to +\infty$, we obtain the thesis. □

The following proposition asserts that $F''(\cdot,\cdot)$ satisfies hypothesis *(i)* of Theorem 2.2. The argument we use for the proof is still the same exploited in the last two propositions.

Proposition 3.9 *Let* $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ *satisfy (3.2),(3.3) and let* (H1)-(H2) *hold. Then for any* $A \in \mathcal{A}(\Omega)$ *and for any* $u,\ v \in W^{1,p}(\Omega; \mathbb{R}^d)$*, such that* $u = v$ *a.e. there holds*
$$F''(u, A) = F''(v, A).$$

Proof. Thanks to Proposition 3.8, we may assume that $A \in \mathcal{A}_0(\Omega)$. We first prove

$$F''(u, A) \geq F''(v, A). \tag{3.24}$$

Once more we apply the argument used in the previous proposition. Given $\delta > 0$, there exists $A_\delta \subset\subset A$ such that

$$|A \setminus \overline{A_\delta}| + \|\nabla u\|^p_{L^p(A \setminus \overline{A_\delta})} \leq \delta.$$

Let $v_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ and $u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ be such that

$$v_\varepsilon \to v \text{ in } L^p(\Omega; \mathbb{R}^d) \tag{3.25}$$

$$u_\varepsilon \to u \text{ in } L^p(\Omega; \mathbb{R}^d) \tag{3.26}$$

and

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u, A),$$

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(v_\varepsilon, A \setminus \overline{A_\delta}) = F''(v, A \setminus \overline{A_\delta}) \leq C\left(|A \setminus \overline{A_\delta}| + \|\nabla u\|^p_{L^p(A \setminus \overline{A_\delta})}\right) \leq C\delta \tag{3.27}$$

Set

$$d := \operatorname{dist}(A_\delta, A^c)$$

and for any $i \in \{1, \ldots, N\}$ define

$$A_i := \{x \in A : \ \operatorname{dist}(x, A_\delta) < i\frac{d}{N}\}.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla \varphi_i\|_\infty \leq 2\frac{N}{d}$. Then for any $i \in \{1, \ldots, N\}$ consider the family of functions $w^i_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ converging to v in $L^p(\Omega; \mathbb{R}^d)$ defined as

$$w^i_\varepsilon(\alpha) := \varphi_i(\alpha) u_\varepsilon(\alpha) + (1 - \varphi_i(\alpha))\, v_\varepsilon(\alpha).$$

Then, following the same steps as in the proof of Propositions 3.7 and 3.8, we can choose $i(\varepsilon) \in \{1, \ldots, N-3\}$ such that

$$\begin{aligned} F_\varepsilon(w^{i(\varepsilon)}_\varepsilon, A) &\leq \frac{1}{N-3}\sum_{i=1}^{N-3} F_\varepsilon(w^i_\varepsilon, A) \\ &\leq F_\varepsilon(u_\varepsilon, A) + C\delta + \frac{C}{N-3}\,(1 + N^p O(\varepsilon)) \\ &+ C(\delta + O(\varepsilon))\,(1 + N^p O(\varepsilon)) + C(\delta + O(\varepsilon))N^p. \end{aligned} \tag{3.28}$$

Then, since $w^{i(\varepsilon)}_\varepsilon$ still converges to v in $L^p(\Omega; \mathbb{R}^d)$, by (3.28), letting $\varepsilon \to 0^+$, we get

$$F''(v, A) \leq F''(u, A) + C\left(\frac{1}{N-3} + \delta + \delta N^p\right).$$

Eventually, letting first $\delta \to 0^+$ and then $N \to +\infty$, we obtain 3.24. Reversing the roles of u and v we obtain the thesis. $\square$

Proof of Theorems 3.1 and 3.3. By the compactness property of the Γ-convergence and by Proposition 3.8, there exists a subsequence (ε_{j_k}) such that, for any $(u, A) \in W^{1,p}(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)$, there holds

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) := F(u, A)$$

(see [6] Theorem 10.3). Moreover, by Proposition 3.4,

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u) = +\infty$$

for $u \in L^p(\Omega; \mathbb{R}^d) \setminus W^{1,p}(\Omega; \mathbb{R}^d)$. So far, it suffices to check that, for every $(u, A) \in W^{1,p}(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)$, $F(u, A)$ satisfies all the hypotheses of Theorem 2.2. In fact, it can be easily seen that the superadditivity property of $F_\varepsilon(u, \cdot)$ is conserved in the limit. Thus, as an easy consequence of Propositions 3.5, 3.7, 3.8, 3.9 and thanks to De Giorgi - Letta Criterion (see [6]), hypotheses (i), (ii), (iii) hold true. Moreover, as $F_\varepsilon(u, A)$ depends on u only through its difference quotients, hypothesis (iv) is satisfied and finally, by the lower semicontinuity property of Γ-limit, also hypothesis (v) is fulfilled. $\square$

3.1 Convergence of minimum problems

In order to treat minimum problems with boundary data, we also derive a compactness theorem in the case that our functionals are subject to Dirichlet boundary conditions.

Given $\varphi \in Lip\ (\mathbb{R}^N)$ and $l \in \mathbb{N}$, set, for any $\varepsilon > 0$ and $A \in \mathcal{A}(\Omega)$

$$\mathcal{A}^l_{\varepsilon,\varphi}(A) := \{u \in \mathcal{A}_\varepsilon(\mathbb{R}^N) \,:\, u(\alpha) = \varphi(\alpha) \text{ if } \big(\alpha + [-l\varepsilon, l\varepsilon]^N\big) \cap A^c \neq \emptyset\big)\}. \quad (3.29)$$

Then define $F^{\varphi,l}_\varepsilon : L^p(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty]$ as

$$F^{\varphi,l}_\varepsilon(u, A) = \begin{cases} F_\varepsilon(u, A) & \text{if } u \in \mathcal{A}^l_{\varepsilon,\varphi}(A) \\ +\infty & \text{otherwise.} \end{cases} \quad (3.30)$$

By simplicity of notation we set $\mathcal{A}_{\varepsilon,\varphi}(A) := \mathcal{A}^1_{\varepsilon,\varphi}(A)$ and $F^\varphi_\varepsilon := F^{\varphi,1}_\varepsilon$.

Theorem 3.10 *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy (3.2), (3.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to 0, let (ε_{j_k}) and f be as in Theorem 3.1. For any $\varphi \in Lip\ (\mathbb{R}^N)$, let $F^\varphi : L^p(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be defined as*

$$F^\varphi(u, A) = \begin{cases} \displaystyle\int_A f(x, \nabla u)\, dx & \textit{if } u - \varphi \in W^{1,p}_0(A; \mathbb{R}^d) \\ +\infty & \textit{otherwise.} \end{cases}$$

Then, for any $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary and $l \in \mathbb{N}$, $(F^{\varphi,l}_{\varepsilon_{j_k}}(\cdot, A))$ Γ-converges with respect to the $L^p(\Omega;\mathbb{R}^d)$-topology to the functional $F^\varphi(\cdot, A)$.

Proof. For the sake of simplicity we prove the Theorem with $l = 1$, the proof being the same in the other cases. Let us first prove the Γ-liminf inequality. Let (u_k) be a sequence of functions belonging to $\mathcal{A}_{\varepsilon_{j_k},\varphi}(A)$ converging to u in the L^p topology such that

$$\liminf_k F^\varphi_{\varepsilon_{j_k}}(u_k, A) = \lim_k F^\varphi_{\varepsilon_{j_k}}(u_k, A) < +\infty.$$

Then, from (3.2), we get in particular that

$$\sup_k \sum_{i=1}^N \sum_{\alpha \in R^{e_i}_{\varepsilon_{j_k}}(A)} \varepsilon^N_{j_k} |D^{e_i}_{\varepsilon_{j_k}} u_n(\alpha)|^p < +\infty. \tag{3.31}$$

Thanks to the boundary conditions on u_k it is easy to deduce that

$$\sup_k \sum_{i=1}^N \sum_{\alpha \in R^{e_i}_{\varepsilon_{j_k}}(\Omega)} \varepsilon^N |D^{e_i}_{\varepsilon_{j_k}} u_n(\alpha)|^p < +\infty.$$

Then, by reasoning as in the proof of Proposition 3.4, we can prove that $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and, since (u_k) converge to φ in $L^p(\Omega \setminus A;\mathbb{R}^d)$, we get that $u - \varphi \in W^{1,p}_0(A;\mathbb{R}^d)$. By Theorem 3.3 one has

$$\liminf_k F^\varphi_{\varepsilon_{j_k}}(u_k, A) = \liminf_k F_{\varepsilon_{j_k}}(u_k, A) \geq F^\varphi(u, A).$$

To prove the Γ-limsup inequality, let us first consider $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that supp $(u - \varphi) \subset\subset A$. Let $u_k \in \mathcal{A}_{\varepsilon_{j_k}}(\Omega)$, be such that (u_k) converges to u in $L^p(\Omega;\mathbb{R}^d)$ and

$$\limsup_k F_{\varepsilon_{j_k}}(u_k, A) = F^\varphi(u, A).$$

Then, by reasoning as in the proof of Proposition 3.8, given $\delta > 0$, we can find suitable cut-off functions ϕ_k with supp $(u - \varphi) \subset\subset$ supp $\phi_k \subset\subset A$ such that, set

$$v_k(\alpha) := \phi_k(\alpha)u_k(\alpha) + (1 - \phi_k(\alpha))\varphi(\alpha),$$

then (v_k) still converges to u in $L^p(\Omega;\mathbb{R}^d)$, $v_k \in \mathcal{A}_{\varepsilon_{j_k},\varphi}(\Omega)$ for k large enough and

$$\limsup_k F_{\varepsilon_{j_k}}(v_k, A) \leq \limsup F_{\varepsilon_{j_k}}(u_k, A) + \delta.$$

Thus, thanks to the definition of Γ-limsup we have

$$\Gamma\text{-limsup} F^\varphi_{\varepsilon_{j_k}}(u, A) \leq F^\varphi(u, A) + \delta.$$

By the arbitrariness of δ, we obtain the required inequality. In the general case the thesis follows by a density argument, thanks to the lower semicontinuity of Γ-limsup and to the continuity of F with respect to the strong convergence in $W^{1,p}(\Omega;\mathbb{R}^d)$. $\square$

As a consequence of the previous theorem we derive the following result about the convergence of minimum problems with boundary data.

Corollary 3.11 *Under the hypotheses of Theorem 3.10 we get that, for any $\varphi \in Lip\ (\mathbb{R}^N)$, $l \in \mathbb{N}$ and $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary*

$$\lim_k \inf\{F_{\varepsilon_{j_k}}(u,A):\ u \in \mathcal{A}^l_{\varepsilon_{j_k},\varphi}\} = \min\{F(u,A):\ u-\varphi \in W^{1,p}_0(A;\mathbb{R}^d)\}.$$

Moreover, if (u_k) is a converging sequence such that

$$\lim_k F_{\varepsilon_{j_k}}(u_k,A) = \lim_k \inf\{F_{\varepsilon_{j_k}}(u,A):\ u \in \mathcal{A}^l_{\varepsilon_{j_k},\varphi}\},$$

then its limit is a minimizer for $\min\{F(u,A):\ u-\varphi \in W^{1,p}_0(A;\mathbb{R}^d)\}$.

Proof. Let (u_k) be a sequence such that $F_{\varepsilon_{j_k}}(u_k,A) < +\infty$. Then, by (3.2) and by the boundary conditions on u_k, it is easy to show that

$$\sup_n \sum_{i=1}^N \sum_{\alpha\in\varepsilon_n\mathbb{Z}^N\cap K} \varepsilon^N |D^{e_i}_{\varepsilon_{j_k}} u_k(\alpha)|^p < +\infty,$$

for any compact set K of $\mathbb{R}^N$. By virtue of this property, up to passing to a continuous extension of u_k vanishing outside a bounded open set containing Ω, we get

$$\lim_{|h|\to 0} \sup_k \|\tau_h u_k - u_k\|_{L^p(\mathbb{R}^N;\mathbb{R}^d)} = 0,$$

where we have set

$$(\tau_h u)(x) := u(x+h),\ x \in \mathbb{R}^N,\ h \in \mathbb{R}^N.$$

Then, by Frechét-Kolmogorov Theorem, there exists a subsequence (u_{k_n}) converging in $L^p(\Omega;\mathbb{R}^d)$ to a function $u \in L^p(\Omega;\mathbb{R}^d)$. Arguing as in the previous proof it is easy to show that $u-\varphi \in W^{1,p}_0(\Omega)$. The thesis follows thanks to Theorem 3.10 and Theorem 2.1. $\square$

We can also derive the analogue of Theorem 3.10 and Corollary 3.11 about the convergence of minimum problems with periodic conditions.

Let $\mathcal{Q}(\Omega)$ be the family of all open N-cubes contained in Ω. For any $\varepsilon > 0$, $r > 0$, $Q = (x_0, x_0 + r)^N \in \mathcal{Q}(\Omega)$ and $\varphi \in Lip\ (\mathbb{R}^N)$, set

$$r_\varepsilon = \varepsilon\left(\left[\frac{r}{\varepsilon}\right] - 2\right),$$

$$\mathcal{A}^{\#}_{\varepsilon,\varphi}(Q) = \left\{u \in \mathcal{A}_\varepsilon(\mathbb{R}^N) : \ u - \hat{\varphi} \ \ r_\varepsilon - \text{periodic}\right\},$$

where $\hat{\varphi} \in \mathcal{A}_\varepsilon(\mathbb{R}^N)$, $\hat{\varphi}(\alpha) = \varphi(\alpha)$ for any $\alpha \in \varepsilon\mathbb{Z}^N$. Then define $F^{\varphi,\#}_\varepsilon : L^p(\Omega;\mathbb{R}^d)\times \mathcal{Q}(\Omega) \to [0,+\infty]$ as

$$F^{\varphi,\#}_\varepsilon(u,Q) = \begin{cases} F_\varepsilon(u,Q) & \text{if } u \in \mathcal{A}^{\#}_{\varepsilon,\varphi}(Q) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{3.32}$$

Theorem 3.12 *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy (3.2), (3.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to* 0*, let (ε_{j_k}) and f be as in Theorem 3.1. Then, for any $\varphi \in Lip\ (\mathbb{R}^N)$, let $F^{\#} : L^p(\Omega;\mathbb{R}^d)\times\mathcal{Q}(\Omega) \to [0,+\infty]$ be defined as*

$$F^{\varphi,\#}(u,Q) = \begin{cases} \displaystyle\int_Q f(x,\nabla u)\,dx & \textit{if } u \in W^{1,p}_{\#}(Q;\mathbb{R}^d) \\ \\ +\infty & \textit{otherwise.} \end{cases}$$

Then, for any $Q \in \mathcal{Q}(\Omega)$, $(F^{\varphi,\#}_{\varepsilon_{j_k}}(\cdot,Q))$ Γ-converges with respect to the $L^p(\Omega;\mathbb{R}^d)$-topology to the functional $F^{\varphi,\#}(u,Q)$.

Proof. To prove the Γ-liminf inequality, let (u_k) be a sequence of functions belonging to $\mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}(Q)$ converging to u in the L^p topology such that

$$\liminf_k F^{\varphi,\#}_{\varepsilon_{j_k}}(u_k,Q) = \lim_k F^{\varphi,\#}_{\varepsilon_{j_k}}(u_k,Q) < +\infty.$$

Then, arguing as in the proof of Theorem 3.10 and observing that $r_\varepsilon \to r$, we can conclude that $u - \varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^d)$ and

$$\liminf_k F^{\varphi,\#}_{\varepsilon_{j_k}}(u_k,Q) \geq F^{\varphi,\#}(u,Q).$$

By a density argument it suffices to prove the Γ-limsup inequality for u such that $u-\varphi \in W^{1,\infty}_{\#}(Q';\mathbb{R}^d)$ for any open N-cube Q' such that $(x_0+\delta, x_0+r-\delta) \subseteq Q' \subseteq Q$ for some $\delta > 0$. Note that, for such a u, $\mathcal{A}_{\varepsilon_{j_k},u} \subseteq \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}$ for k large enough. Then the existence of a recovery sequence is assured by Theorem 3.10. □

As a consequence of the previous theorem, by reasoning as in the proof of Corollary 3.11 one can prove the following result.

Corollary 3.13 *Under the hypotheses of Theorem 3.12 we get that, for any $\varphi \in Lip\ (\mathbb{R}^N)$ and $Q \in \mathcal{Q}(\Omega)$*

$$\liminf_k \{F_{\varepsilon_{j_k}}(u,Q) : \ u \in \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}(Q)\} = \min\{F(u,Q) : \ u - \varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^d)\}.$$

Moreover, if (u_k) is a converging sequence such that

$$\lim_k F_{\varepsilon_{j_k}}(u_k,Q) = \liminf_k \{F_{\varepsilon_{j_k}}(u,Q) : \ u \in \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}\},$$

then its limit is a minimizer for $\min\{F(u,Q) : \ u - \varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^d)\}$.

4 Homogenization

In this section we will show that if the functions f_ε^ξ are obtained by rescaling by ε functions f^ξ periodic in the space variable, then a Γ-convergence result holds true. This models the case when the arrangement of the "material points" presents a periodic feature (see fig. 4).

Let $\mathbf{k} = (k_1, \dots, k_N) \in \mathbf{Z}^N$ be given and set

$$\mathcal{R}_{\mathbf{k}} := (0, k_1) \times \cdots \times (0, k_N).$$

For any $\xi \in \mathbf{Z}^N$, let $f^\xi : \mathbf{Z}^N \times \mathbf{R}^d \to [0, +\infty)$ be such that $f^\xi(\cdot, z)$ is $\mathcal{R}_{\mathbf{k}}$-periodic for any $z \in \mathbf{R}^d$. Then we consider f_ε^ξ of the following form

$$f_\varepsilon^\xi(\alpha, z) := f^\xi\left(\frac{\alpha}{\varepsilon}, z\right). \tag{4.1}$$

In this case, the growth conditions (3.2) and (3.3) and hypotheses (H1) and (H2) can be rewritten as follows:

$$f^{e_i}(\alpha, z) \geq c_1(|z|^p - 1), \quad \forall i \in \{1, \dots, N\}, \tag{4.2}$$

$$f^\xi(\alpha, z) \leq C^\xi(|z|^p + 1), \tag{4.3}$$

where

$$\sum_{\xi \in \mathbf{Z}^N} C^\xi < +\infty. \tag{$H3$}$$

In the sequel we will use the following notation: for any $x = (x_1, \dots, x_N) \in \mathbf{R}^N$ define

$$[x]_{\mathbf{k}} := \left(\left[\frac{x_1}{k_1}\right] k_1, \dots, \left[\frac{x_N}{k_N}\right] k_N\right).$$

Moreover, for any $A \in \mathcal{A}(\Omega)$, $\varepsilon > 0$, $l \in \mathbf{N}$ and $M \in \mathcal{M}^{d\times N}$ we denote by $\mathcal{A}^l_{\varepsilon,M}(A)$ the set defined in formula (3.29) with $\varphi(x) = Mx$. By simplicity of notation, we set $\mathcal{A}^1_{\varepsilon,M}(A) := \mathcal{A}_{\varepsilon,M}(A)$. Finally for every $r > 0$ we set $Q_r := (0, r)^N$.

The following theorem is the main result of this section and its proof is obtained by adapting a homogenization argument to the discrete setting. We remark that a central role is played by Theorems 3.1 and 3.3 and by the convergence of minimum problems with boundary data stated in Corollary 3.11. Moreover, we recall that the following result has been already proven in [11] in the one-dimensional case, where a more straightforward proof is possible.

Theorem 4.1 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (4.1)-(4.3) and let* (H3) *hold. Then,* (F_ε) *Γ-converges with respect to the $L^p(\Omega; \mathbf{R}^d)$-topology to the functional $F : L^p(\Omega; \mathbf{R}^d) \to [0, +\infty]$ defined as*

$$F(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\, dx & \text{if } u \in W^{1,p}(\Omega; \mathbf{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.4}$$

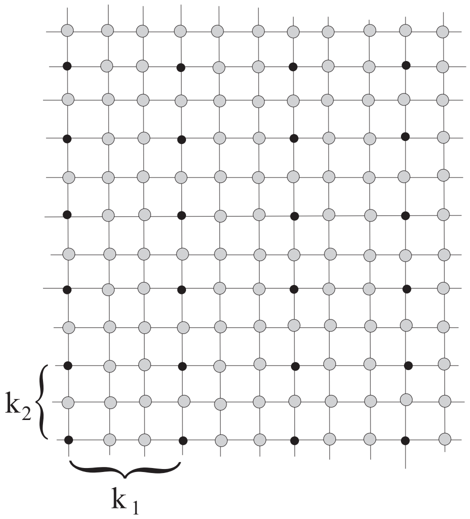

Figure 4: example of periodic structure

where $f_{hom} : \mathcal{M}^{d\times N} \to [0, +\infty)$ *is given by the following homogenization formula*

$$f_{hom}(M) := \lim_{h\to+\infty} \frac{1}{h^N} \min\Big\{ \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi(\beta, D_1^\xi v(\beta)), \quad v \in \mathcal{A}_{1,M}(Q_h) \Big\} \tag{4.5}$$

Proof. Let (ε_n) be a sequence of positive numbers converging to 0. Then, by Theorems 3.1 and 3.3, we can extract a subsequence (not relabelled) such that (F_{ε_n}) Γ-converges to a functional F defined as in (3.4) and such that, for any $u \in W^{1,p}(\Omega; \mathbf{R}^d)$, $A \in \mathcal{A}(\Omega)$

$$\Gamma\text{-}\lim_n F_{\varepsilon_n}(u, A) = \int_A f(x, \nabla u)\, dx.$$

The theorem is proved if we show that f does not depend on the space variable x and $f \equiv f_{hom}$. To prove the first claim, by Theorem 2.2, it suffices to show that, set

$$F(u, A) = \int_A f(x, \nabla u)\, dx,$$

then

$$F(Mx, B(y, \rho)) = F(Mx, B(z, \rho))$$

for all $M \in \mathcal{M}^{d\times N}$, $y, z \in \Omega$ and $\rho > 0$ such that $B(y, \rho) \cup B(z, \rho) \subset \Omega$. We will prove that

$$F(Mx, B(y, \rho)) \leq F(Mx, B(z, \rho)),$$

the proof of the opposite inequality being analogous. By the inner regularity of $F(Mx,\cdot)$, given by Proposition 3.8, it suffices to show that for any $\rho' < \rho$ we get

$$F(Mx, B(y,\rho')) \leq F(Mx, B(z,\rho)). \tag{4.6}$$

Let, then, $v_n \in \mathcal{A}_{\varepsilon_n}(\Omega)$ be such that (v_n) converges to Mx in $L^p(\Omega;\mathbf{R}^d)$ and

$$\lim_n F_{\varepsilon_n}(v_n, B(z,\rho)) = F(Mx, B(z,\rho)). \tag{4.7}$$

For $n \in \mathbf{N}$, define $u_n \in \mathcal{A}_{\varepsilon_n}(\Omega)$ as

$$u_n(\alpha) := \begin{cases} v_n\left(\alpha - \varepsilon_n \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}}\right) + \varepsilon_n M \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}} & \text{if } \alpha \in \varepsilon_n \mathbf{Z}^N \cap B(y,\rho') \\ M\alpha & \text{otherwise.} \end{cases}$$

Then it is easy to verify that (u_n) converges to Mx in $L^p(\Omega;\mathbf{R}^d)$. Moreover for n large enough

$$R^{\xi}_{\varepsilon_n}(B(y,\rho')) - \varepsilon_n \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}} \subseteq R^{\xi}_{\varepsilon_n}(B(z,\rho)).$$

Thus, since, by the periodicity hypothesis, $f^{\xi}\left(\alpha - \varepsilon_n \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}}, z\right) = f^{\xi}(\alpha, z)$ and $D^{\xi}_{\varepsilon} u_n(\alpha) = D^{\xi}_{\varepsilon} v_n\left(\alpha - \varepsilon_n \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}}\right)$, we get for n large enough

$$F_{\varepsilon_n}(u_n, B(y,\rho')) \leq F_{\varepsilon_n}(v_n, B(z,\rho)).$$

Eventually, by (4.7), we obtain

$$\begin{aligned} F(Mx, B(y,\rho')) &\leq \liminf_{n\to+\infty} F_{\varepsilon_n}(u_n, B(y,\rho')) \\ &\leq \lim_{n\to+\infty} F_{\varepsilon_n}(v_n, B(z,\rho)) = F(Mx, B(z,\rho)). \end{aligned}$$

In order to prove that $f \equiv f_{hom}$, first note that, by the lower semicontinuity of F in $W^{1,p}(\Omega;\mathbf{R}^d)$, f is quasiconvex, so that, by the p-growth properties of f, for any $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary and for any $M \in \mathcal{M}^{d\times N}$ there holds

$$\begin{aligned} f(M) &= \frac{1}{|A|} \min\left\{\int_A f(\nabla u)\,dx : \quad u - Mx \in W^{1,p}_0(A;\mathbf{R}^d)\right\} \\ &= \frac{1}{|A|} \min\left\{F(u,A) : \quad u - Mx \in W^{1,p}_0(A;\mathbf{R}^d)\right\} \\ &= \frac{1}{|A|} \liminf_n \inf\left\{F_{\varepsilon_n}(u,A) : \quad u \in \mathcal{A}_{\varepsilon_n,M}(A)\right\}, \end{aligned}$$

where the last equality follows by Corollary 3.11. In particular, if $x_0 \in \Omega$ and $r > 0$ are such that $Q_r(x_0) := (x_0, x_0 + r)^N \subseteq \Omega$, then

$$f(M) = \lim_n \frac{1}{r^N} \inf\{F_{\varepsilon_n}(u, Q_r(x_0)) : \ u \in \mathcal{A}_{\varepsilon_n,M}(Q_r(x_0))\}.$$

Without loss of generality, we may suppose $x_0 = 0$. If we set

$$T_n := \left[\frac{r}{\varepsilon_n}\right] + 1,$$

then it is easy to show that $\mathcal{A}_{\varepsilon_n,M}(Q_r) = \mathcal{A}_{\varepsilon_n,M}(Q_{\varepsilon_n T_n})$ and that for $\xi \in \mathbf{Z}^N$ $R^{\xi}_{\varepsilon_n}(Q_r) = R^{\xi}_{\varepsilon_n}(Q_{\varepsilon_n T_n})$. Thus

$$f(M) = \lim_n \frac{1}{r^N} \inf\{F_{\varepsilon_n}(u, Q(0, \varepsilon_n T_n)) : \ u \in \mathcal{A}_{\varepsilon_n,M}(Q_{\varepsilon_n T_n})\}.$$

Eventually, through the change of variable

$$\beta = \frac{\alpha}{\varepsilon}, \quad v(\beta) = \frac{1}{\varepsilon} u(\varepsilon\beta), \tag{4.8}$$

we get

$$\begin{aligned} f(M) &= \lim_n \left(\frac{\varepsilon_n}{r}\right)^N \inf\Big\{ \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta \in R^{\xi}_1(Q_{T_n})} f^{\xi}(\beta, D^{\xi}_1 v(\beta)), \quad v \in \mathcal{A}_{1,M}(Q_{T_n}) \Big\} \\ &= \lim_n \frac{1}{T_n^N} \inf\Big\{ \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta \in R^{\xi}_1(Q_{T_n})} f^{\xi}(\beta, D^{\xi}_1 v(\beta)), \quad v \in \mathcal{A}_{1,M}(Q_{T_n}) \Big\}, \end{aligned}$$

where the last equality holds since

$$\lim_n T_n \frac{\varepsilon_n}{r} = 1.$$

Then the thesis will follow by the next proposition.

Proposition 4.2 *Let f^{ξ} satisfy (4.2),(4.3) and (H3) for any $\xi \in \mathbf{Z}^N$. Then the limit*

$$\lim_{h \to +\infty} \frac{1}{h^N} \inf\Big\{ \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta \in R^{\xi}_1(Q_h)} f^{\xi}(\beta, D^{\xi}_1 v(\beta)), \quad v \in \mathcal{A}_{1,M}(Q_h) \Big\}$$

exists for all $M \in \mathcal{M}^{d \times N}$.

Proof. Let $M \in \mathcal{M}^{d \times N}$ be fixed and set

$$F_1(v, A) := \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta \in R^{\xi}_1(A)} f^{\xi}(\beta, D^{\xi}_1 v(\beta)),$$

$$f_h(M) := \frac{1}{h^N} \inf\{F_1(v, Q_h), \quad v \in \mathcal{A}_{1,M}(Q_h)\}.$$

Moreover for any $R > 0$, set

$$F^R_1(v, A) := \sum_{|\xi| \leq R} \sum_{\beta \in R^{\xi}_1(A)} f^{\xi}(\beta, D^{\xi}_1 v(\beta)),$$

$$f_h^R(M) := \frac{1}{h^N} \inf \left\{ F_1^R(v, Q_h), \quad v \in \mathcal{A}_{1,M}\left(Q_h\right) \right\}.$$

We prove that

$$\lim_{R \to +\infty} \sup_h |f_h^R(M) - f_h(M)| = 0. \tag{4.9}$$

To this end, since $f_h^R(M) \le f_h(M)$ for any $h \in \mathbf{N}$ and $R > 0$, it suffices to prove that for any $\delta > 0$, there exist $R_\delta > 0$ such that

$$f_h(M) \le f_h^R(M) + \delta, \quad \forall R > R_\delta \ , h \in \mathbf{N}$$

Fix $\delta > 0$ and let $v_h^R \in \mathcal{A}_{1,M}(Q_h)$ be such that

$$\frac{1}{h^N} F_h^R(v_h^R, Q_h) \le f_h^R(M) + \frac{1}{R}. \tag{4.10}$$

By testing the minimum problem defining $f_h^R(M)$ with $v(\alpha) = M\alpha$, we get, by (4.3) and (H3), that

$$f_h^R(M) \le \frac{1}{h^N} F_h^R(M\alpha, Q_h) \le C|M|^p.$$

Thus, by (4.10) and (4.2), we obtain that

$$\sup_{h,R} \frac{1}{h^N} \sum_{i=1}^{N} \sum_{\beta \in R_1^{e_i}(Q_h)} |D_1^{e_i} v_h^R(\beta)|^p < +\infty.$$

Then, by arguing as in the proof of Lemma 3.6 and thanks to the particular geometry of the sets Q_h, we deduce that

$$\sup_{h,R} \frac{1}{h^N} \sup_{\xi \in \mathbf{Z}^N} \sum_{\beta \in R_1^{\xi}(Q_h)} |D_1^{\xi} v_h^R(\beta)|^p < +\infty.$$

Eventually, we have

$$\begin{aligned} f_h(M) \quad &\le \quad \frac{1}{h^N} F_h(v_h^R, Q_h) \le \frac{1}{h^N} F_h^R(v_h^R, Q_h) + \frac{1}{h^N} \sum_{|\xi| > R} C^\xi \sum_{\beta \in R_1^\xi(Q_h)} |D_1^\xi v_h^R(\beta)|^p \\ &\le f_h^R(M) + \frac{1}{R} + C \sum_{|\xi| > R} C^\xi. \end{aligned}$$

Thus, it suffices to choose $R_\delta > 0$ such that for $R > R_\delta$

$$\frac{1}{R} + C \sum_{|\xi| > R} C^\xi \le \delta.$$

So far, in order to prove the thesis, it suffices to show that for any $R > 0$ there exists the limit

$$\lim_h f_h^R(M).$$

Set

$$f_h^{R,R}(M) := \frac{1}{h^N} \inf \Big\{ F_1^R(v, Q_h), \quad v \in \mathcal{A}_{1,M}^{[R]}(Q_h) \Big\}.$$

Using backward the scaling argument exploited in the proof of the previous proposition and thanks to Theorem 3.10 and Corollary 3.11, one can show that, for any subsequence $(h_n) \subset \mathbf{N}$ it is possible to extract a further subsequence (not relabelled) such that

$$\lim_n f_{h_n}^R(M) = \lim_n f_{h_n}^{R,R}(M). \tag{4.11}$$

Thus, to complete the proof, it is sufficient to prove that there exists the limit

$$\lim_h f_h^{R,R}(M).$$

Let $h \in \mathbf{N}$ and let $v_h \in \mathcal{A}_{1,M}^{[R]}(Q_h)$ be such that

$$\frac{1}{h^N} F_1^R(v_h, Q_h) \le f_h^{R,R}(M) + \frac{1}{h}.$$

For any $k > h$ define a function $u_k \in \mathcal{A}_{1,M}^{[R]}(Q_k)$ as follows

$$u_k(\alpha) = \begin{cases} v_h(\alpha - h\mathbf{i}) + hM\mathbf{i} & \text{if } \alpha \in h\mathbf{i} + Q_h, \ \mathbf{i} \in \big\{0, \dots, \big[\frac{k}{h}\big] - 1\big\}^N \\ \\ M\alpha & \text{otherwise} \end{cases}$$

Note that for any $\xi \in \mathbf{Z}^N$, $|\xi| \le R$ we have

$$\begin{aligned} R_1^\xi(Q_k) \quad \subseteq \quad & \Big(\bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} R_1^\xi(h\mathbf{i} + Q_h) \Big) \cup R_1^\xi \Big(Q_k \setminus \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} (h\mathbf{i} + Q_h) \Big) \\ & \cup \Big(\bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} \big(h\mathbf{i} + \big(\{0, \dots, h + R\}^N \setminus \{0, \dots, h - R\}^N\big)\big) \Big). \end{aligned}$$

Moreover $D_1^\xi u_k(\alpha) = M\frac{\xi}{|\xi|}$ if $\alpha \in R_1^\xi \Big(Q_k \setminus \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} (h\mathbf{i} + Q_h) \Big)$ or $\alpha \in \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} \big(h\mathbf{i} + \big(\{0, \dots, h + R\}^N \setminus \{0, \dots, h - R\}^N\big)\big)$, and

$$\#\Big(R_1^\xi \Big(Q_k \setminus \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} (h\mathbf{i} + Q_h) \Big) \Big) \le k^N - \left[\frac{k}{h}\right]^N h^N,$$

$$\#\left(\{0,\dots,h+R\}^N \setminus \{0,\dots,h-R\}^N\right) \le (h+R)^N - (h-R)^N.$$

Then, by (4.3) and (H3), we get

$$\begin{aligned}
f_k^{R,R}(M) \quad &\le \quad \frac{1}{k^N} F_1^R(u_k, Q_k) \le \left[\frac{k}{h}\right]^N \frac{1}{k^N} F_1^R(v_h, Q_h) \\
&\quad + C|M|^P \frac{1}{k^N}\left(k^N - \left[\frac{k}{h}\right]^N h^N + \left[\frac{k}{h}\right]^N \left((h+R)^N - (h-R)^N\right)\right) \\
&\le \left[\frac{k}{h}\right]^N \frac{h^N}{k^N}\left(f_h^{R,R}(M) + \frac{1}{h}\right) \\
&\quad + C|M|^P \frac{1}{k^N}\left(k^N - \left[\frac{k}{h}\right]^N h^N + \left[\frac{k}{h}\right]^N \left((h+R)^N - (h-R)^N\right)\right).
\end{aligned}$$

By letting k tend to $+\infty$, we then get

$$\limsup_k f_k^{R,R}(M) \le f_h^{R,R}(M) + \frac{1}{h} + C|M|^P \frac{1}{h^N}\left((h+R)^N - (h-R)^N\right)$$

Eventually, letting h tendo to $+\infty$, we obtain

$$\limsup_k f_k^{R,R}(M) \le \liminf_h f_h^{R,R}(M),$$

that is the conclusion. □

Remark 4.3 In formula (4.5) we can replace $\mathcal{A}_{1,M}(Q_h)$ by $\mathcal{A}^l_{1,M}(Q_h)$ for any fixed $l \in \mathbf{N}$, the proof being exactly the same.

Remark 4.4 The function f_{hom} in Theorem 4.1 also satisfies

$$f_{hom}(M) = \lim_{h\to+\infty} \frac{1}{h^N} \inf\Bigg\{ \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}(Q_{h-2})\Bigg\}, \tag{4.12}$$

where, for every $k \in \mathbf{R}$,

$$\mathcal{A}_{1,\#}(Q_k) := \{v \in \mathcal{A}_1(\mathbf{R}^N) : \ v \ k\text{-periodic}\}.$$

This characterization can be proved by arguing as in the proof of Theorem 4.1 and Proposition 4.2 taking into account Corollary 3.13 and recalling that, since f_{hom} is quasiconvex, there holds

$$\begin{aligned}
f_{hom}(M) \quad &= \quad \frac{1}{r^N} \min\left\{ \int_{Q_r} f_{hom}(M + \nabla\psi)\, dx : \psi \in W^{1,p}_{\#}(Q_r; \mathbf{R}^d)\right\} \\
&= \quad \frac{1}{r^N} \min\left\{ F(M\alpha + \psi, Q_r) : \psi \in W^{1,p}_{\#}(Q_r; \mathbf{R}^d)\right\}.
\end{aligned}$$

As a consequence of Theorem 3.10, Corollary 3.11 and Theorem 4.1 we immediately derive the following result about Γ-convergence and convergence of minimum problems for homogeneous functionals subject to Dirichlet boundary conditions.

Theorem 4.5 *For any $\varphi \in Lip\ (\mathbf{R}^N)$ and $l \in \mathbf{N}$ let $F_\varepsilon^{\varphi,l}$ be defined by (3.30) and let $F^\varphi : L^p(\Omega;\mathbf{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be defined as*

$$F^\varphi(u,A) = \begin{cases} \displaystyle\int_A f_{hom}(\nabla u)\, dx & \text{if } u-\varphi \in W_0^{1,p}(A;\mathbf{R}^d) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.13}$$

Under the hypotheses of Theorem 4.1, $F_\varepsilon^\varphi(\cdot,A)$ Γ-converges with respect to the $L^p(\Omega;\mathbf{R}^d)$-topology to $F^\varphi(\cdot,A)$ for any $A \in \mathcal{A}$.

Corollary 4.6 *Under the hypotheses of Theorem 4.5, for any $\varphi \in Lip\ (\mathbf{R}^N)$, $l \in \mathbf{N}$ and $A \in \mathcal{A}(\Omega)$*

$$\lim_{\varepsilon\to 0} \inf\{F_\varepsilon(u,A) :\ u \in \mathcal{A}^l_{\varepsilon,\varphi}\} = \min\{F(u,A) :\ u-\varphi \in W_0^{1,p}(A;\mathbf{R}^d)\}.$$

Moreover, for any (ε_j) converging to zero as j tends to infinity, if (u_j) is a converging sequence such that

$$\lim_j F_{\varepsilon_j}(u_j,A) = \lim_j \inf\{F_{\varepsilon_j}(u,A) :\ u \in \mathcal{A}^l_{\varepsilon_j,\varphi}\},$$

then its limit is a minimizer for $\min\{F(u,A) :\ u-\varphi \in W_0^{1,p}(A;\mathbf{R}^d)\}$.

An analogous result about the convergence of minimum problems with periodic conditions follows by Theorem 3.12 and Corollary 3.13.

5 The convex case: a cell problem formula

In this section we will see that in the convex case the function f_{hom} can be rewritten by a single periodic minimization problem on the periodic cell $\mathcal{R}_\mathbf{k}$. Set

$$\hat{k} := \prod_{i=1}^N k_i,$$

$$I_\mathbf{k} := \prod_{i=1}^N \{0,\dots,k_i-1\}$$

and

$$\mathcal{A}_{1,\#}(\mathcal{R}_\mathbf{k}) := \{u \in \mathcal{A}_1(\mathbf{R}^N)\ :\ u \text{ is } \mathcal{R}_\mathbf{k}-\text{periodic}\}$$

Theorem 5.1 *Let $(f_\varepsilon^\xi)_{\varepsilon,\xi}$ satisfies all the assumptions of Theorem 4.1 and in addition let $f_\varepsilon^\xi(\alpha,\cdot)$ be convex for all $\alpha \in \varepsilon\mathbf{Z}^N$, $\varepsilon > 0$ and $\xi \in \mathbf{Z}^N$. Then the conclusion of Theorem 4.1 holds with f_{hom} satisfying*

$$f_{hom}(M) = \frac{1}{\hat{k}} \inf\Big\{ \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}\left(\mathcal{R}_{\mathbf{k}}\right)\Big\},$$

for all $M \in \mathcal{M}^{d\times N}$.

Proof. Set

$$\overline{f}(M) := \frac{1}{\hat{k}} \inf\Big\{ \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}\left(\mathcal{R}_{\mathbf{k}}\right)\Big\}.$$

We first prove that

$$f_{hom}(M) \le \overline{f}(M). \tag{5.1}$$

With fixed $\delta > 0$, let $v \in \mathcal{A}_{1,\#}\left(\mathcal{R}_k\right)$ be such that

$$\frac{1}{\hat{k}} \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \le \overline{f}(M) + \delta.$$

For $n \in \mathbf{N}$, since in particular $v \in \mathcal{A}_{1,\#}(Q_{n\hat{k}})$, we get

$$\begin{aligned} f^{\#}_{n\hat{k}+2}(M) &\le \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in R_1^\xi(Q_{n\hat{k}})} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \\ &\le n^N \hat{k}^{N-1} \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \end{aligned}$$

where the last inequality follows by the periodicity of v, $(f(\cdot, z))$ and by the fact that $Q_{n\hat{k}}$ is union of $n^N\hat{k}^{N-1}$ periodicity cells. Eventually, by Remark 4.12, we get

$$\begin{aligned} f_{hom}(M) &\le \limsup_n \frac{1}{(n\hat{k}+2)}^N f^{\#}_{n\hat{k}+2}(M) \\ &\le \frac{1}{\hat{k}} \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \le \overline{f}(M) + \delta, \end{aligned}$$

and inequality (5.1) follows, by letting δ tend to 0. Let us prove that

$$f_{hom}(M) \ge \overline{f}(M).$$

For any $R > 0$, set

$$f^R_{hom}(M) := \lim_{h\to+\infty} \frac{1}{h^N} \inf\Big\{ \sum_{|\xi|\le R} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi(\beta, D_1^\xi v(\beta)), \quad v \in \mathcal{A}^{[R]}_{1,M}\left(Q_h\right)\Big\},$$

$$\overline{f}^R(M) := \frac{1}{\hat{k}} \inf\Big\{ \sum_{|\xi|\leq R} \sum_{\beta\in I_{\mathbf{k}}} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}\left(\mathcal{R}_{\mathbf{k}}\right)\Big\}.$$

By (4.9) and (4.11)we easily derive that

$$\lim_{R\to+\infty} f^R_{hom}(M) = f_{hom}(M).$$

Analogously one can prove that

$$\lim_{R\to+\infty} \overline{f}^R(M) = \overline{f}(M).$$

Thus it suffices to prove that for any $R > 0$

$$f^R_{hom}(M) \geq \overline{f}^R(M). \tag{5.2}$$

For $n \in \mathbf{N}$, let $u \in \mathcal{A}^{[R]}_{1,M}(Q_{n\hat{k}})$ and let $v \in \mathcal{A}_{1,\#}(Q_{n\hat{k}})$ be such that

$$v(\alpha) = u(\alpha) - M\alpha, \quad \forall \alpha \in Q_{n\hat{k}}.$$

Moreover set

$$I^n_{\mathbf{k}} := \prod_{i=1}^N \{0, \dots, n\prod_{j\neq i} k_j - 1\}.$$

Then, we get

$$\begin{aligned}
&\frac{1}{(n\hat{k})^N} \sum_{|\xi|\leq R} \sum_{\beta\in R_1^\xi(Q_{n\hat{k}})} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \\
&\quad= \frac{1}{(n\hat{k})^N} \sum_{|\xi|\leq R} \sum_{\beta\in\{0,\dots,n\hat{k}\}^N} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) - O(\frac{1}{n}) \\
&\quad= \frac{1}{\hat{k}} \sum_{|\xi|\leq R} \sum_{\beta\in I_k} \frac{1}{\hat{k}^{N-1}n^N} \sum_{\gamma\in I^n_{\mathbf{k}}} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta + \sum_{i=1}^N \gamma_i k_i e_i)\right) - O(\frac{1}{n}) \\
&\quad\geq \frac{1}{\hat{k}} \sum_{|\xi|\leq R} \sum_{\beta\in I_k} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + \frac{1}{\hat{k}^{N-1}n^N} \sum_{\gamma\in I^n_{\mathbf{k}}} D_1^\xi v(\beta + \sum_{i=1}^N \gamma_i k_i e_i)\right) - O(\frac{1}{n}),
\end{aligned}$$

where in the last inequality we have used the convexity hypothesis on f^ξ. Eventually, set

$$v_n(\beta) := \frac{1}{\hat{k}^{N-1}n^N} \sum_{\gamma\in I^n_{\mathbf{k}}} v(\beta + \sum_{i=1}^N \gamma_i k_i e_i).$$

It is easy to show that $v_n \in \mathcal{A}_{1,\#}(\mathcal{R}_k)$ and so, by the previous inequality, we get

$$\begin{aligned}
&\frac{1}{(n\hat{k})^N}\sum_{|\xi|\le R}\sum_{\beta\in R_1^\xi(Q_{n\hat{k}})} f^\xi\left(\beta, M\frac{\xi}{|\xi|}+D_1^\xi v(\beta)\right)\\
&\quad\ge \frac{1}{\hat{k}}\sum_{|\xi|\le R}\sum_{\beta\in I_k} f^\xi\left(\beta, M\frac{\xi}{|\xi|}+D_1^\xi v_n(\beta)\right) - O(\frac{1}{n})\\
&\quad\ge \overline{f}^R(M) - O(\frac{1}{n}).
\end{aligned}$$

Passing to the inf with respect to $u \in \mathcal{A}^R_{1,\#}(Q_{n\hat{k}})$, we get

$$f^{R,R}_{n\hat{k}+2}(M) \ge \overline{f}^R(M) - O(\frac{1}{n}),$$

and then, letting n tend to $+\infty$, we obtain (5.2). $\square$

Remark 5.2 (**quadratic forms**) Under the hypotheses of Theorem 5.1, if in addition for any $\xi \in \mathbf{Z}^N$ $f^\xi(\alpha,\cdot)$ is a positive quadratic form on $\mathbf{R}^d$, that is

$$f^\xi(\alpha,z) = \langle A^\xi(\alpha)z, z\rangle \quad A^\xi(\alpha) \in \mathcal{M}^{d\times d}_{sym},$$

then, thanks to Remark 3.2, the limit energy density $f_{hom}(\cdot)$ is a homogeneous quadratic form on $\mathcal{M}^{d\times N}$ and formula (3.5) becomes

$$\begin{aligned}
f_{hom}(M) &= A_{hom}\,(M,M)\\
&= \frac{1}{\hat{k}}\inf\Big\{\sum_{\xi\in\mathbf{Z}^N}\sum_{\beta\in I_{\mathbf{k}}}\langle A^\xi(\beta)\cdot\left(M\frac{\xi}{|\xi|}+D_1^\xi v(\beta)\right), \left(M\frac{\xi}{|\xi|}+D_1^\xi v(\beta)\right)\rangle,\\
&\qquad v \in \mathcal{A}_{1,\#}(\mathcal{R}_{\mathbf{k}})\Big\}
\end{aligned}$$

with $A_{hom} \in T_2(\mathcal{M}^{d\times N})$.
If $N = d = 1$ and only nearest-neighbor interactions are taken into account, that is

$$f^\xi \equiv 0 \text{ if } \xi \ne e_1, \qquad f^{e_1}(\alpha,z) = a(\alpha)z^2,$$

with $a : \mathbf{Z}^N \to (0,+\infty)$ k-periodic, the previous minimum problem can be easily solved (see [9])giving the analogue in the discrete setting of a well known homogenization result for integral functionals (see [6]). In fact, in this case

$$A_{hom} = \frac{1}{k}\left(\sum_{\beta=0}^{k-1}\frac{1}{a(\beta)}\right)^{-1}$$

is the harmonic mean of $a(\cdot)$.

Remark 5.3 Note that if $\mathcal{R}_{\mathbf{k}} = (0,1)^N$, that is f^ξ does not depend on the space variable α, in Theorem 5.1 we obtain

$$f_{hom}(M) = \sum_{\xi \in \mathbf{Z}^N} f^\xi \left(M \frac{\xi}{|\xi|} \right).$$

6 Interactions along independent directions and reduction to the 1-dimensional case

In this section we first recall some results proven in the 1-dimensional setting in [11], where a non-asymptotic formula defining the limit energy density f_{hom} is provided when only nearest and next-to-nearest neighbor interactions are considered.

Then in Theorem 6.3 we will show that, if only interactions along the coordinate directions are taken into account, the N-dimensional problem can be reduced to a 1-dimensional one.

The following two theorems have been proven in [11] in the case $d = 1$. Their proof in the case $d > 1$ is the same.

Theorem 6.1 (**nearest-neighbor interactions**) *Let $\Omega = (0,l) \subset \mathbf{R}$ and let $F_\varepsilon : L^p(\Omega;\mathbf{R}^d) \to [0,+\infty)$ be defined as*

$$F_\varepsilon(u) := \begin{cases} \displaystyle\sum_{i=1}^{l-2} \varepsilon f\left(\frac{u(\varepsilon(i+1)) - u(\varepsilon i)}{\varepsilon} \right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

with $f : \mathbf{R}^d \to [0,+\infty)$ satisfying $f(z) \geq C(|z|^p - 1)$. Then the conclusions of Theorem 4.1 hold with

$$f_{hom}(z) = f^{**}(z).$$

Theorem 6.2 (**next-to-nearest neighbor interactions**) *Let $\Omega = (0,l) \subset \mathbf{R}$ and let $F_\varepsilon : L^p(\Omega;\mathbf{R}^d) \to [0,+\infty)$ be defined as*

$$F_\varepsilon(u) := \begin{cases} \displaystyle\sum_{i=1}^{l-2} \varepsilon f^1\left(\frac{u(\varepsilon(i+1)) - u(\varepsilon i)}{\varepsilon} \right) + \sum_{i=1}^{l-3} \varepsilon f^2\left(\frac{u(\varepsilon(i+2)) - u(\varepsilon i)}{2\varepsilon} \right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in \mathcal{A}_\varepsilon(\Omega) & \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} & \end{cases}$$

with $f^1, f^2 : \mathbf{R}^d \to [0,+\infty)$ satisfying $f^1(z) \geq C(|z|^p - 1)$. Then the conclusions of Theorem 4.1 hold with

$$f_{hom}(z) = \tilde{f}^{**}(z),$$

where $\tilde{f}(z) = f^2(z) + \frac{1}{2}\inf\{f^1(z_1) + f^1(z_2),\ z_1 + z_2 = 2z\}$.

Back to the general N-dimensional setting, we consider now energies of the form

$$F_\varepsilon(u) = \begin{cases} \sum_{i=1}^{N} \mathcal{F}^i_\varepsilon(u,\Omega) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{6.1}$$

where, for any $i \in \{1,\dots,N\}$, $\mathcal{F}^i_\varepsilon : \mathcal{A}_\varepsilon(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ is defined as

$$\mathcal{F}^i_\varepsilon(u,A) := \sum_{k=1}^{+\infty} \sum_{\alpha\in R^{ke_i}_\varepsilon(A)} \varepsilon^N f^k_i(D^{ke_i}_\varepsilon u(\alpha)) \tag{6.2}$$

with $f^k_i : \mathbf{R}^d \to [0,+\infty)$ satisfying

$$f^1_i(z) \geq c(|z|^p - 1), \qquad f^k_i(z) \leq C^k_i(|z|^p + 1)$$

and

$$\sum_{i=1}^{N}\sum_{k=1}^{+\infty} C^k_i < +\infty.$$

This is a particular case of the model considered in Section 4, with $f^\xi \equiv 0$ if $\xi \neq ke_i$, $f^{ke_i}(0,z) = f^k_i(z)$, $i \in \{1,\dots,N\}$, $k\in\mathbf{N}$, and $\mathcal{R}_{\mathbf{k}} = (0,1)^N$.

The following theorem shows that, in this case, the homogenization formula defining f_{hom} can be rewritten as a sum of N one-dimensional homogenization formulas.

Theorem 6.3 *Let F_ε be defined by (6.2). Then the Γ-convergence result stated in Theorem 4.1 holds with f_{hom} satisfying*

$$f_{hom}(M) = \sum_{i=1}^{N} \tilde{f}_i(M^i) \tag{6.3}$$

for any $M = (M^1,\dots,M^N) \in \mathcal{M}^{d\times N}$, where $\tilde{f}_i : \mathbf{R}^d \to \mathbf{R}$, $i\in\{1,\dots,N\}$, is defined by the following 1-dimensional homogenization formula

$$\tilde{f}_i(z) := \lim_{h\to+\infty} \frac{1}{h} \inf\left\{ \sum_{k=1}^{+\infty}\sum_{j=1}^{h-k-1} f^k_i\left(\frac{v(j+k)-v(j)}{k}\right),\ v\in\mathcal{A}_{1,z}((0,h)) \right\}$$

Proof. We first prove that

$$f_{hom}(M) \geq \sum_{i=1}^{N} \tilde{f}_i(M^i).$$

To do this, by the definition of $f_{hom}(M)$, it suffices to show that for any $i \in \{1,\dots,N\}$, $u \in \mathcal{A}_{1,M}(Q_h)$ we have

$$\frac{1}{h^N}\mathcal{F}_1^i(u,Q_h) \geq \tilde{f}_i(M^i) + O(h). \tag{6.4}$$

We use a slicing argument. For $i \in \{1,\dots,N\}$, set

$$m_h^i(z) := \frac{1}{h}\inf\left\{\sum_{k=1}^{+\infty}\sum_{j=1}^{h-k-1} f_i^k\left(\frac{v(j+k)-v(j)}{k}\right),\ v \in \mathcal{A}_{1,z}((0,h))\right\}.$$

By simplicity of notation, we prove (6.4) for $i=1$. Given $u \in \mathcal{A}_{1,M}(Q_h)$, we may write

$$\mathcal{F}_1^1(u,Q_h) = \sum_{\beta\in\{1,\dots,h-1\}^{N-1}}\sum_{k=1}^{+\infty}\sum_{j=1}^{h-k-1} f_1^k\left(\frac{u(j+k,\beta)-u(j,\beta)}{k}\right) \tag{6.5}$$

Since for any $\beta \in \{1,\dots,h-1\}^{N-1}$ the function $v(j) := u(j,\beta) - \tilde{M}\beta$ belongs to $\mathcal{A}_{1,M^1}(0,h)$, where $\tilde{M} := \left(M^2,\dots,M^N\right)$, from (6.5) we get

$$\frac{1}{h^N}\mathcal{F}_1^1(u,Q_h) \geq \frac{1}{h^{N-1}}\#\left(\{1,\dots,h-1\}^{N-1}\right)m_h^1(M^1) \geq m_h^1(M^1).$$

We then easily infer inequality (6.4).

We now prove that

$$f_{hom}(M) \leq \sum_{i=1}^N \tilde{f}_i(M^i). \tag{6.6}$$

With fixed $\eta > 0$, for any $i \in \{1,\dots N\}$ let $v_h^i \in \mathcal{A}^2_{1,M^i}(0,h)$ be such that

$$\frac{1}{h}\sum_{k=1}^{+\infty}\sum_{j=1}^{h-k-1} f_i^k\left(\frac{v_h^i(j+k)-v_h^i(j)}{k}\right) \leq m_h^i(M^i) + \eta, \tag{6.7}$$

and set

$$u_h(\alpha) := \sum_{i=1}^N v_h^i(\alpha_i), \quad \alpha = (\alpha_1,\dots,\alpha_N).$$

Note that $u_h \in M\alpha + \mathcal{A}_{1,\#}(Q_{h-2})$. Moreover by the analogue of (6.5) applied to $\mathcal{F}_1^i(u,Q_h)$ for any $i \in \{1,\dots,N\}$ and by (6.7), we easily deduce that

$$\frac{1}{h^N}\sum_{i=1}^N \mathcal{F}_1(u_h,Q_h) \leq \sum_{i=1}^N m_h^i(M^i) + N\eta.$$

Eventually, by the characterization of f_{hom} given by formula (4.12), letting first h tend to $+\infty$ and then η tend to 0, we get (6.6). □

Remark 6.4 Note that formula (6.3) highlights that a superposition principle holds, in the sense that the limit energy is obtained by relaxing the energies due to the interactions in every coordinate direction independently and then summing over them.

Remark 6.5 (a)(*nearest-neighbors*) By Theorem 6.1, if $f_i^k = 0$ for all $k \neq 1$, then formula (6.3) can be rewritten as

$$f_{hom}(M) = \sum_{i=1}^{N} (f_i^1)^{**}(M^i);$$

(b)(*next-to-nearest neighbors*) By Theorem 6.2, if $f_i^k = 0$ for all $k \neq 1, 2$, then formula (6.3) can be rewritten as

$$f_{hom}(M) = \sum_{i=1}^{N} (\tilde{f}_i)^{**}(M^i),$$

with

$$\tilde{f}_i(z) = f_i^2(z) + \frac{1}{2}\inf\{f_i^1(z_1) + f_i^1(z_2),\ z_1 + z_2 = 2z\}.$$

7 An example of quasiconvex not convex limit energy density

In the following we provide an example of vector-valued discrete interaction energies defined in the plane whose continuous counterpart has an energy density which is a quasiconvex (not polyconvex) function. Our example draws inspiration from Šverák's construction of a quasiconvex function which is not polyconvex (see [21]). Let $N = d = 2$, $p > 1$ and define $f_i : \mathbf{R}^2 \to [0, +\infty)$, $i = 1, 2, 3$, as

$$f_i(z) = \begin{cases} 1 + |z|^p & \text{if } z \neq \pm\frac{\xi_i}{|\xi_i|} \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi_1 = e_1$, $\xi_2 = e_2$, $\xi_3 = e_1 + e_2$. Let F_ε be defined as

$$F_\varepsilon(u) = \sum_{i=1}^{3} \sum_{\alpha \in R_\varepsilon^{\xi_i}} \varepsilon^2 f_i\left(D_\varepsilon^{\xi_i} u(\alpha)\right),$$

then the conclusions of Theorems 4.1, 4.5 and Corollary 4.6 hold with f_{hom} given by

$$f_{hom}(M) = \lim_{h \to +\infty} \frac{1}{h^N} \min\Big\{\sum_{i=1}^{3} \sum_{\beta \in R_1^{\xi_i}(Q_h)} f_i(D_1^{\xi_i} v(\beta)), \quad v \in \mathcal{A}_{1,M}(Q_h)\Big\}$$

Theorem 7.1 *f_{hom} is not convex.*

Proof. By testing the minimum problem defining f_{hom} with the identity function and its opposite, we immediately obtain that

$$f_{hom}(I) = f_{hom}(-I) = 0$$

where I is the identity matrix in $M^{2\times 2}$. The claim is proven if we show that $f_{hom}(0) > 0$. We argue by contradiction. Without loss of generality we may assume that Theorem 4.5 hold with $A = Q_1$. Were $f_{hom}(0)$ zero, there should exist a sequence $u_n \in \mathcal{A}_{\varepsilon_n,0}(Q_1)$ such that $u_n \to 0$ in $L^p(Q_1;\mathbf{R}^2)$ and

$$\lim_n F_{\varepsilon_n}(u_n) = 0. \tag{7.1}$$

Set

$$\begin{aligned} T^+ &:= \{(x_1,x_2) \in \mathbf{R}^2 : \ 0 \le x_1 \le 1, \quad x_1 \le x_2 \le 1\} \\ T^- &:= \{(x_1,x_2) \in \mathbf{R}^2 : \ 0 \le x_1 \le 1, \quad 0 \le x_2 \le x_1\}, \end{aligned}$$

we consider the family of piecewise affine functions $v_n : Q_1 \to \mathbf{R}^2$ defined as follows

$$v_n(x) = \begin{cases} u_n(\alpha) + D^{e_1}_{\varepsilon_n} u_n(\alpha)(x_1-\alpha_1) & \\ +D^{e_2}_{\varepsilon_n} u_n(\alpha+\varepsilon_n e_1)(x_2-\alpha_2) & \text{if } x \in \alpha + \varepsilon_n T^- \\ & \\ u_n(\alpha) + D^{e_1}_{\varepsilon_n} u_n(\alpha+\varepsilon_n e_2)(x_1-\alpha_1) & \\ +D^{e_2}_{\varepsilon_n} u_n(\alpha)(x_2-\alpha_2) & \text{if } x \in \alpha + \varepsilon_n T^+. \end{cases}$$

Note that $v_n|_{\partial Q_1} = 0$. Moreover it is easy to check that

$$F_{\varepsilon_n}(u_n) = \int_{Q_1} \tilde{f}(\nabla v_n)\, dx. \tag{7.2}$$

where $\tilde{f} : M^{2\times 2} \to [0,+\infty)$ is defined as

$$\tilde{f}(\zeta) := f_1(\zeta_1) + f_2(\zeta_2) + f_3\left(\frac{\zeta_1+\zeta_2}{\sqrt{2}}\right) \qquad \zeta = (\zeta_1,\zeta_2) \in M^{2\times 2}.$$

In particular, by (7.1)

$$\lim_n \int_{Q_1} \tilde{f}(\nabla v_n)\, dx = 0 \tag{7.3}$$

Since we have

$$\tilde{f}(\zeta) \ge c(|\zeta_{11} - \zeta_{22}|^p + |\zeta_{12} + \zeta_{21}|^p),$$

by (7.1) and (7.2) we obtain

$$\lim_n \int_{Q_1} (|\nabla_1 v_n^1 - \nabla_2 v_n^2|^p + |\nabla_1 v_n^2 + \nabla_2 v_n^1|^p)\, dx = 0. \tag{7.4}$$

Since

$$\Delta v_n^1 = \mathrm{div}(\nabla_1 v_n^1 - \nabla_2 v_n^2, \nabla_1 v_n^2 + \nabla_2 v_n^1)$$

$$\Delta v_n^2 = \mathrm{div}(\nabla_1 v_n^2 + \nabla_2 v_n^1, -\nabla_1 v_n^1 + \nabla_2 v_n^2),$$

using the L^p estimates for the Laplace operator (see [19]) we obtain that

$$\begin{aligned} \|\nabla v_n^i\|^p_{L^p(Q_1;\mathbf{R}^2)} &\leq \|\Delta v_n^i\|^p_{W^{-1,p}(Q_1;\mathbf{R}^2)} \\ &\leq \int_{Q_1} (|\nabla_1 v_n^1 - \nabla_2 v_n^2|^p + |\nabla_1 v_n^2 + \nabla_2 v_n^1|^p)\ dx \end{aligned}$$

for $i = 1, 2$. Then, by (7.4) and the previous estimates, ∇v_n converges to 0 strongly in $L^p(Q_1; M^{2\times 2})$, so that

$$\lim_n \int_{Q_1} \tilde{f}(\nabla v_n)\ dx = \tilde{f}(0)\ |Q_1| > 0.$$

Hence we reach a contradiction. □

Remark 7.2 In the particular case $1 < p < 2$, thanks to the growth hypotheses on f_i, f_{hom} is a quasiconvex not polyconvex function (see [6], Remark 6.9).

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