THE ALLEN–CAHN ACTION FUNCTIONAL IN HIGHER DIMENSIONS

LUCA MUGNAI AND MATTHIAS RÖGER

Abstract. The Allen–Cahn action functional is related to the probability of rare events in the stochastically perturbed Allen–Cahn equation. Formal calculations suggest a *reduced action functional* in the sharp interface limit. We prove the corresponding lower bound in two and three space dimensions. One difficulty is that diffuse interfaces may collapse in the limit. We therefore consider the limit of diffuse surface area measures and introduce a generalized velocity and generalized reduced action functional in a class of evolving measures.

1. Introduction

In this paper we study the (renormalized) *Allen–Cahn action functional*

$$\mathcal{S}_\varepsilon(u) := \int_0^T \int_\Omega \Big(\sqrt{\varepsilon}\partial_t u + \frac{1}{\sqrt{\varepsilon}}\big(- \varepsilon\Delta u + \frac{1}{\varepsilon} W'(u)\big)\Big)^2 \,dx\,dt. \tag{1.1}$$

This functional arises in the analysis of the stochastically perturbed Allen–Cahn equation [2, 21, 13, 30, 8, 10, 12] and is related to the probability of rare events such as switching between deterministically stable states.

Compared to the purely deterministic setting, stochastic perturbations add new features to the theory of phase separations, and the analysis of action functionals has drawn attention [8, 13, 18, 19, 26]. Kohn *et alii* [18] considered the *sharp-interface limit* $\varepsilon \to 0$ of $\mathcal{S}_\varepsilon$ and identified a *reduced action functional* that is more easily accessible for a qualitative analysis. The sharp interface limit reveals a connection between minimizers of $\mathcal{S}_\varepsilon$ and mean curvature flow.

The reduced action functional in [18] is defined for phase indicator functions $u : (0,T)\times\Omega \to \{-1,1\}$ with the additional properties that the measure of the phase $\{u(t,\cdot) = 1\}$ is continuous and the common boundary of the two phases $\{u = 1\}$ and $\{u = -1\}$ is, apart from a countable set of singular times, given as union of smoothly evolving hypersurfaces $\Sigma := \cup_{t\in(0,T)}\{t\}\times\Sigma_t$. The reduced action functional is then defined as

$$\mathcal{S}^0(u) := c_0 \int_0^T \int_{\Sigma_t} \big|v(t,x) - H(t,x)\big|^2 \,d\mathcal{H}^{n-1}(x)dt + 4\mathcal{S}^0_{nuc}(u), \tag{1.2}$$

$$\mathcal{S}^0_{nuc}(u) := 2c_0 \sum_i \mathcal{H}^{n-1}(\Sigma_i), \tag{1.3}$$

Date: July 26, 2007.

2000 *Mathematics Subject Classification.* Primary 49J45; Secondary 35R60, 60F10, 53C44.

Key words and phrases. Allen-Cahn equation, stochastic partial differential equations, large deviation theory, sharp interface limits, motion by mean curvature.

where Σ_i denotes the i^{th} component of Σ at the time of creation, where v denotes the normal velocity of the evolution $(\Sigma_t)_{t\in(0,T)}$, where $H(t,\cdot)$ denotes the mean curvature vector of Σ_t, and where the constant c_0 is determined by W,

$$c_0 := \int_{-1}^{1} \sqrt{2W(s)}\,ds. \tag{1.4}$$

(See Section 9 for a more rigourous definition of $\mathcal{S}^0$).

Several arguments suggest that $\mathcal{S}^0$ describes the Gamma-limit of $\mathcal{S}_\varepsilon$:

- The *upper bound* necessary for the Gamma-convergence was formally proved [18] by the construction of good 'recovery sequences'.
- The *lower bound* was proved in [18] for sequences $(u_\varepsilon)_{\varepsilon>0}$ such that the associated 'energy-measures' have *equipartitioned energy* and *single multiplicity* as $\varepsilon \to 0$.
- In one space-dimension Reznikoff and Tonegawa [26] proved that $\mathcal{S}_\varepsilon$ Gamma-converges to an appropriate relaxation of the one-dimensional version of $\mathcal{S}^0$.

The approach used in [18] is based on the evolution of the phases and is sensible to cancellations of phase boundaries in the sharp interface limit. Therefore in [18] a sharp lower bound is achieved only under a single-multiplicity assumption for the limit of the diffuse interfaces. As a consequence, it could not be excluded that creating multiple interfaces reduces the action.

In the present paper we prove a sharp lower-bound of the functional $\mathcal{S}_\varepsilon$ in space dimensions $n = 2, 3$ without any additional restrictions on the approximate sequences.

To circumvent problems with cancellations of interfaces we analyze the evolution of the (diffuse) *surface-area measures*, which makes information available that is lost in the limit of phase fields. With this aim we generalize the functional $\mathcal{S}^0$ to a suitable class of *evolving energy measures* and introduce a generalized formulation of velocity, similar to Brakke's generalization of Mean Curvature Flow [5].

Let us informally describe our approach and main results. Comparing the two functionals $\mathcal{S}_\varepsilon$ and $\mathcal{S}^0$ the first and second term of the sum in the integrand (1.1) describe a 'diffuse velocity' and 'diffuse mean curvature' respectively. We will make this statement precise in (6.13) and (7.1). The mean curvature is given by the first variation of the area functional, and a lower estimate for the square of the diffuse mean curvature is available in a time-independent situation [28]. The velocity of the evolution of the phase boundaries is determined by the time-derivative of the surface-area measures and the nucleation term in the functional $\mathcal{S}^0$ in fact describes a singular part of this time derivative.

Our first main result is a compactness result: the diffuse surface-area measures converge to an evolution of measures with a square integrable generalized mean curvature and a square integrable generalized velocity . In the class of such evolutions of measures we provide a generalized formulation of the reduced action functional. We prove a lower estimate that counts the propagation cost with the multiplicity of the interface. This shows that it is more expensive to move phase boundaries with higher multiplicity. Finally we prove two statements on the Gamma-convergence (with respect to $L^1(\Omega_T)$) of the action functional. The first result is for evolutions in the domain of $\mathcal{S}^0$ that have nucleations only at the initial time. This is in particular desirable since minimizers of $\mathcal{S}^0$ are supposed to be in this class. The second result proves the Gamma convergence in $L^1(\Omega_T)$ under an assumption on the structure of the set of measures arising as sharp interface limits of sequences with uniformly bounded action.

We give a precise statement of our main results in Section 4. In the remainder of this introduction we describe some background and motivation.

1.1. **Deterministic phase field models and sharp interface limits.** Most *diffuse interface models* are based on the *Van der Waals–Cahn–Hilliard* energy

$$E_\varepsilon(u) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big)\,dx. \tag{1.5}$$

The energy E_ε favors a decomposition of Ω into two regions (phases) where $u \approx -1$ and $u \approx 1$, separated by a transition layer (diffuse interface) with a thickness of order ε. Modica and Mortola [23, 22] proved that E_ε Gamma-converges (with respect to L^1-convergence) to a constant multiple of the perimeter functional $\mathcal{P}$, restricted to phase indicator functions,

$$E_\varepsilon \to c_0\mathcal{P}, \qquad \mathcal{P}(u) := \begin{cases} \frac{1}{2}\int_\Omega d|\nabla u| & \text{if } u \in BV(\Omega,\{-1,1\}),\\ \infty & \text{otherwise.}\end{cases}$$

$\mathcal{P}$ measures the surface-area of the phase boundary $\partial^*\{u = 1\} \cap \Omega$. In this sense E_ε describes a diffuse approximation of the surface-area functional.

Various tighter connections between the functionals E_ε and $\mathcal{P}$ have been proved. We mention here just two that are important for our analysis. The (accelerated) L^2-gradient flow of E_ε is given by the *Allen–Cahn equation*

$$\varepsilon\partial_t u = \varepsilon\Delta u - \frac{1}{\varepsilon}W'(u) \tag{1.6}$$

for phase fields in the time-space cylinder $(0,T)\times\Omega$. It is proved in different formulations [7, 9, 17] that (1.6) converges to the *Mean Curvature Flow*

$$H(t,\cdot) = v(t,\cdot) \tag{1.7}$$

for the evolution of phase boundaries.

Another connection between the first variations of E_ε and $\mathcal{P}$ is expressed in a (modified) conjecture of De Giorgi [6]: Considering

$$\mathcal{W}_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon}\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big)^2\,dx \tag{1.8}$$

the sum $E_\varepsilon + \mathcal{W}_\varepsilon$ Gamma-converges up to the constant factor c_0 to the sum of the Perimeter functional and the *Willmore functional* $\mathcal{W}$,

$$E_\varepsilon + \mathcal{W}_\varepsilon \to c_0\mathcal{P} + c_0\mathcal{W}, \qquad \mathcal{W}(u) = \int_\Gamma H^2\,d\mathcal{H}^{n-1}, \tag{1.9}$$

where Γ denotes the phase boundary $\partial^*\{u = 1\}\cap\Omega$. This statement was recently proved by Röger and Schätzle [28] in space dimensions $n = 2, 3$ and is one essential ingredient to obtain the lower bound for the action functional.

1.2. **Stochastic interpretation of the action functional.** Phenomena such as the nucleation of a new phase or the switching between two (local) energy minima require an energy barrier crossing and are out of the scope of deterministic models that are energy dissipative. If thermal fluctuations are taken into account such an energy barrier crossing becomes possible. In [18] 'thermally activated switching' was considered for the *stochastically perturbed Allen–Cahn equation*

$$\varepsilon\partial_t u \,=\, \varepsilon\Delta u - \frac{1}{\varepsilon}W'(u) + \sqrt{2\gamma}\eta_\lambda \tag{1.10}$$

Here $\gamma > 0$ is a parameter that represents the temperature of the system, η is a time-space white noise, and η_λ is a spatial regularization with $\eta_\lambda \to \eta$ as $\lambda \to 0$. This regularization is necessary for $n \geq 2$ since the white noise is too singular to ensure well-posedness of (1.10) in higher space-dimensions.

Large deviation theory and (extensions of) results by Wentzell and Freidlin [15, 14] yield an estimate on the probability distribution of solutions of stochastic ODEs and PDEs in the small-noise limit. This estimate is expressed in terms of a (deterministic) action functional. For instance, thermally activated switching within a time $T > 0$ is described by the set of paths

$$\mathcal{B} \,:=\, \Big\{ u(0,\cdot) \,=\, -1, \quad \|u(t,\cdot) - 1\|_{L^\infty(\Omega)} \,\leq\, \delta \text{ for some } t \leq T \Big\}, \tag{1.11}$$

where $\delta > 0$ is a fixed constant. The probability of switching for solutions of (1.10) then satisfies

$$\lim_{\gamma\to 0} \gamma \ln \mathsf{Prob}(\mathcal{B}) \,=\, -\inf_{u\in\mathcal{B}} \mathcal{S}_\varepsilon^{(\lambda)}(u). \tag{1.12}$$

Here $\mathcal{S}_\varepsilon^{(\lambda)}$ is the action functional associated to (1.10) and converges (formally) to the action functional $\mathcal{S}_\varepsilon$ as $\lambda \to 0$ [18]. Large deviation theory not only estimates the probability of rare events but also identifies the 'most-likely switching path' as the minimizer u in (1.12).

We focus here on the sharp interface limit $\varepsilon \to 0$ of the action functional $\mathcal{S}_\varepsilon$. The small parameter $\varepsilon > 0$ corresponds to a specific diffusive scaling of the time- and space domains. This choice was identified [8, 18] as particularly interesting, exhibiting a competition between *nucleation versus propagation* to achieve the optimal switching. Depending on the value of $|\Omega|^{1/d}/\sqrt{T}$ a cascade of more and more complex spatial patterns is observed [8, 18, 19]. The interest in the sharp interface limit is motivated by an interest in applications where the switching time is small compared to the deterministic time-scale, see for instance [20].

1.3. **Organization.** We fix some notation and assumptions in the next section. In Section 3 we introduce the concept of L^2-flows and generalized velocity. Our main results are stated in Section 4 and proved in the Sections 5-8. We discuss some implications for the Gamma-convergence of the action functional in Section 9. Finally, in the Appendix we collect some definitions from Geometric Measure Theory.

Acknowledgment. We wish to thank Maria Reznikoff, Yoshihiro Tonegawa, and Stephan Luckhaus for several stimulating discussions. The first author thanks the Eindhoven University of Technology for its hospitality during his stay in summer 2006.

The first author was partially supported by the *Schwerpunktprogramm DFG SPP 1095 'Multiscale Problems'* and *DFG Forschergruppe 718.*

2. Notation and Assumptions

Throughout the paper we will adopt the following notation: Ω is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary; $T > 0$ is a real number and $\Omega_T := (0,T) \times \Omega$; $x \in \Omega$ and $t \in (0,T)$ denote the space- and time-variables respectively; ∇ and Δ denote the spatial gradient and Laplacian and ∇' the full gradient in $\mathbb{R} \times \mathbb{R}^n$.

We choose W to be the standard quartic double-well potential

$$W(r) \,=\, \frac{1}{4}(1-r^2)^2.$$

For a family of measures $(\mu^t)_{t\in(0,T)}$ we denote by $\mathcal{L}^1 \otimes \mu^t$ the product measure defined by

$$\big(\mathcal{L}^1 \otimes \mu^t\big)(\eta) \,:=\, \int_0^T \mu^t(\eta(t,\cdot))\,dt$$

for any $\eta \in C^0_c(\Omega_T)$.

We next state our main assumptions.

Assumption 2.1. *Let $n = 2,3$ and let a sequence $(u_\varepsilon)_{\varepsilon>0}$ of smooth functions be given that satisfies for all $\varepsilon > 0$*

$$\mathcal{S}_\varepsilon(u_\varepsilon) \,\leq\, \Lambda_1, \tag{A1}$$

$$\int_\Omega \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big)(0,x)\,dx \,\leq\, \Lambda_2, \tag{A2}$$

where the constants Λ_1, Λ_2 are independent of $\varepsilon > 0$. Moreover we prescribe that

$$\nabla u_\varepsilon \cdot \nu_\Omega \,=\, 0 \qquad on\ [0,T] \times \partial\Omega. \tag{A3}$$

Remark 2.2. It follows from (A3) that for any $0 \leq t_0 \leq T$

$$\begin{aligned}
&\int_0^{t_0}\int_\Omega \Big(\sqrt{\varepsilon}\partial_t u_\varepsilon + \frac{1}{\sqrt{\varepsilon}}\big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\big)\Big)^2\,dxdt\\
&= \int_0^{t_0}\int_\Omega \varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}\big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\big)^2\,dxdt\\
&\quad + 2\int_\Omega \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big)(t_0,x)\,dx - 2\int_\Omega \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big)(0,x)\,dx.
\end{aligned}$$

By the uniform bounds (A1), (A2) this implies that

$$\int_{\Omega_T} \varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}\big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\big)^2\,dxdt \,\leq\, \Lambda_3, \tag{2.1}$$

$$\max_{0\leq t\leq T}\int_\Omega \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big)(t,x)\,dx \,\leq\, \Lambda_4, \tag{2.2}$$

where

$$\Lambda_3 \,:=\, \Lambda_1 + 2\Lambda_2, \qquad \Lambda_4 \,:=\, \frac{1}{2}\Lambda_1 + \Lambda_2.$$

Remark 2.3. Our arguments would also work for any boundary conditions for which $\partial_t u \nabla u \cdot \nu_\Omega$ vanishes on $\partial\Omega$, in particular for time-independent Dirichlet conditions or periodic boundary conditions.

We set

$$w_\varepsilon \;:=\; -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon) \tag{2.3}$$

and define for $\varepsilon > 0$, $t \in (0,T)$ a Radon measure μ_ε^t on $\overline{\Omega}$,

$$\mu_\varepsilon^t \;:=\; \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2(t,\cdot) + \frac{1}{\varepsilon}W(u_\varepsilon(t,\cdot))\Big)\mathcal{L}^n, \tag{2.4}$$

and for $\varepsilon > 0$ measures $\mu_\varepsilon, \alpha_\varepsilon$ on $\overline{\Omega_T}$,

$$\mu_\varepsilon \;:=\; \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^{n+1}, \tag{2.5}$$

$$\alpha_\varepsilon \;:=\; \big(\varepsilon^{1/2}\partial_t u_\varepsilon + \varepsilon^{-1/2}w_\varepsilon\big)^2\mathcal{L}^{n+1}. \tag{2.6}$$

Eventually restricting ourselves to a subsequence $\varepsilon \to 0$ we may assume that

$$\mu_\varepsilon \;\to\; \mu \qquad \text{as Radon-measures on } \overline{\Omega_T}, \tag{2.7}$$

$$\alpha_\varepsilon \;\to\; \alpha \qquad \text{as Radon-measures on } \overline{\Omega_T}, \tag{2.8}$$

for two Radon measures μ, α on $\overline{\Omega_T}$, and that

$$\alpha(\overline{\Omega_T}) \;=\; \liminf_{\varepsilon\to 0} \alpha_\varepsilon(\Omega_T). \tag{2.9}$$

3. L^2-FLOWS

We will show that the uniform bound on the action implies the existence of a square-integrable weak mean curvature and the existence of a square-integrable *generalized velocity*. The formulation of weak mean curvature is standard in Geometric Measure Theory [1, 31]. Our definition of L^2-flow and generalized velocity is similar to Brakke's formulation of mean curvature flow [5].

Definition 3.1. *Let $(\mu^t)_{t\in(0,T)}$ be any family of integer rectifiable Radon measures such that $\mu := \mathcal{L}^1 \otimes \mu^t$ defines a Radon measure on Ω_T and such that μ^t has a weak mean curvature $H(t,\cdot) \in L^2(\mu^t)$ for almost all $t \in (0,T)$.*

If there exists a positive constant C and a vector field $v \in L^2(\mu, \mathbb{R}^n)$ such that

$$v(t,x) \perp T_x\mu^t \quad \text{for } \mu\text{-almost all } (t,x) \in \Omega_T, \tag{3.1}$$

$$\Big|\int_0^T \int_\Omega \big(\partial_t\eta + \nabla\eta\cdot v\big)\, d\mu^t dt\Big| \;\le\; C\|\eta\|_{C^0(\Omega_T)} \tag{3.2}$$

for all $\eta \in C_c^1((0,T)\times\overline{\Omega})$, then we call the evolution $(\mu^t)_{t\in(0,T)}$ an L^2-flow. *A function $v \in L^2(\mu,\mathbb{R}^n)$ satisfying* (3.1), (3.2) *is called a* generalized velocity vector.

This definition is based on the observation that for a smooth evolution $(M_t)_{t\in(0,T)}$ with mean curvature $H(t,\cdot)$ and normal velocity vector $V(t,\cdot)$

$$\frac{d}{dt}\int_{M_t}\eta(t,x)\,d\mathcal{H}^{n-1}(x) - \int_{M_t}\partial_t\eta(t,x)\,d\mathcal{H}^{n-1}(x) - \int_{M_t}\nabla\eta(t,x)\cdot V(t,x)\,d\mathcal{H}^{n-1}(x)$$
$$= \int_{M_t} H(t,x)\cdot V(t,x)\eta(t,x)\,d\mathcal{H}^{n-1}(x).$$

Integrating this equality in time implies (3.2) for any evolution with square-integrable velocity and mean curvature.

Remark 3.2. Choosing $\eta(t,x) = \zeta(t)\psi(x)$ with $\zeta \in C^1_c(0,T)$, $\psi \in C^1(\overline{\Omega})$, we deduce from (3.2) that $t \mapsto \mu^t(\psi)$ belongs to $BV(0,T)$. Choosing a countable dense subset $(\psi_i)_{i\in\mathbb{N}} \subset C^0(\overline{\Omega})$ this implies that there exists a countable set $S \subset (0,T)$ of *singular times* such that any good representative of $t \mapsto \mu^t(\psi)$ is continuous in $(0,T) \setminus S$ for all $\psi \in C^1(\overline{\Omega})$.

Any generalized velocity is (in a set of good points) uniquely determined by the evolution $(\mu_t)_{t\in(0,T)}$.

Proposition 3.3. *Let $(\mu^t)_{t\in(0,T)}$ be an L^2-flow and set $\mu := \mathcal{L}^1 \otimes \mu^t$. Let $v \in L^2(\mu)$ be a generalized velocity field in the sense of Definition 3.1. Then*

$$\begin{pmatrix} 1 \\ v(t_0, x_0) \end{pmatrix} \in T_{(t_0,x_0)}\mu \tag{3.3}$$

holds in μ-almost all points $(t_0, x_0) \in \Omega_T$ where the tangential plane of μ exists. The evolution $(\mu^t)_{t\in(0,T)}$ uniquely determines v in all points $(t_0, x_0) \in \Omega_T$ where both tangential planes $T_{(t_0,x_0)}\mu$ and $T_{x_0}\mu^{t_0}$ exist.

We postpone the proof to Section 8.

In the set of points where a tangential plane of μ exists, the generalized velocity field v coincides with the normal velocity introduced in [4].

We turn now to the statement of a lower bound for sequences $(u_\varepsilon)_{\varepsilon>0}$ satisfying Assumption 2.1. As $\varepsilon \to 0$ we will obtain a phase indicator function u as the limit of the sequence $(u_\varepsilon)_{\varepsilon>0}$ and an L^2-flow $(\mu^t)_{t\in(0,T)}$ as the limit of the measures $(\mu_\varepsilon)_{\varepsilon>0}$. We will show that in $\mathcal{H}^n$-almost all points of the phase boundary $\partial^*\{u = 1\} \cap \Omega$ a tangential plane of μ exists. This implies the existence of a unique normal velocity field of the phase boundary.

4. Lower bound for the action functional

In several steps we state a lower bound for the functionals $\mathcal{S}_\varepsilon$. We postpone all proofs to Sections 5-8.

4.1. Lower estimate for the mean curvature.

We start with an application of the well-known results of Modica and Mortola [23, 22].

Proposition 4.1. *There exists $u \in BV(\Omega_T, \{-1,1\}) \cap L^\infty(0,T; BV(\Omega))$ such that for a subsequence $\varepsilon \to 0$*

$$u_\varepsilon \to u \qquad \text{in } L^1(\Omega_T), \tag{4.1}$$

$$u_\varepsilon(t,\cdot) \to u(t,\cdot) \qquad \text{in } L^1(\Omega) \text{ for almost all } t \in (0,T). \tag{4.2}$$

Moreover

$$\frac{c_0}{2}\int_{\Omega_T} d|\nabla' u| \le \Lambda_3 + T\Lambda_4, \qquad \frac{c_0}{2}\int_\Omega d|\nabla u(t,\cdot)| \le \Lambda_4 \tag{4.3}$$

holds, where c_0 was defined in (1.4).

The next proposition basically repeats the arguments in [19, Theorem 1.1].

Proposition 4.2. *There exists a countable set $S \subset (0,T)$, a subsequence $\varepsilon \to 0$ and Radon measures $\mu^t, t \in [0,T] \setminus S$, such that for all $t \in [0,T] \setminus S$*

$$\mu^t_\varepsilon \to \mu^t \text{ as Radon measures on } \overline{\Omega}, \tag{4.4}$$

such that

$$\mu = \mathcal{L}^1 \otimes \mu^t, \tag{4.5}$$

and such that for all $\psi \in C^1(\overline{\Omega})$ *the function*

$$t \mapsto \mu^t(\psi) \quad \textit{is of bounded variation in } (0,T) \tag{4.6}$$

and has no jumps in $(0,T) \setminus S$.

Exploiting the lower bound [28] for the diffuse approximation of the Willmore functional (1.8) we obtain that the measures μ^t are up to a constant integer-rectifiable with a weak mean curvature satisfying an appropriate lower estimate.

Theorem 4.3. *For almost all* $t \in (0,T)$

$$\frac{1}{c_0}\mu^t \textit{ is an integral } (n-1)\textit{-varifold,}$$
$$\mu^t \textit{ has weak mean curvature } H(t,\cdot) \in L^2(\mu^t),$$

and the estimate

$$\int_{\Omega_T} |H|^2 \, d\mu \le \liminf_{\varepsilon \to 0} \int_{\Omega_T} \frac{1}{\varepsilon} w_\varepsilon^2 \, dxdt \tag{4.7}$$

holds.

4.2. **Lower estimate for the generalized velocity.**

Theorem 4.4. *Let* $(\mu^t)_{t\in(0,T)}$ *be the limit measures obtained in Proposition 4.2. Then there exists a generalized velocity* $v \in L^2(\mu, \mathbb{R}^n)$ *of* $(\mu^t)_{t\in(0,T)}$. *Moreover the estimate*

$$\int_{\Omega_T} |v|^2 \, d\mu \le \liminf_{\varepsilon \to 0} \int_{\Omega_T} \varepsilon(\partial_t u_\varepsilon)^2 \, dxdt \tag{4.8}$$

is satisfied. In particular, $(\frac{1}{c_0}\mu^t)_{t\in(0,T)}$ *is an* L^2*-flow.*

We obtain v as a limit of suitably defined approximate velocities, see Lemma 6.2. On the phase boundary v coincides with the (standard) distributional velocity of the bulk-phase $\{u(t,\cdot) = 1\}$. However, our definition extends the velocity also to 'hidden boundaries', which seems necessary in order to prove the Gamma-convergence of the action functional; see the discussion in Section 9.

Proposition 4.5. *Define the* generalized normal velocity V *in direction of the inner normal of* $\{u = 1\}$ *by*

$$V(t,x) := v(t,x) \cdot \frac{\nabla u}{|\nabla u|}(t,x), \qquad \textit{for } (t,x) \in \partial^*\{u=1\}.$$

Then $V \in L^1(|\nabla u|)$ *holds and* $V|_{\partial^*\{u=1\}}$ *is the unique vector field that satisfies for all* $\eta \in C^1_c(\Omega_T)$

$$\int_0^T \int_\Omega V(t,x)\eta(t,x) \, d|\nabla u(t,\cdot)|(x)dt = -\int_{\Omega_T} u\partial_t \eta \, dxdt. \tag{4.9}$$

4.3. **Lower estimate of the action functional.** As our main result we obtain the following lower estimate for $\mathcal{S}_\varepsilon$.

Theorem 4.6. *Let Assumption 2.1 hold, and let μ, $(\mu^t)_{t\in[0,T]}$, and S be the measures and the countable set of singular times that we obtained in Proposition 4.2. Define the* nucleation cost $\mathcal{S}_{nuc}(\mu)$ *by*

$$\begin{aligned}\mathcal{S}_{nuc}(\mu) \,:=\, &\sum_{t_0\in S}\sup_{\psi}\Big(\lim_{t\downarrow t_0}\mu^t(\psi)-\lim_{t\uparrow t_0}\mu^t(\psi)\Big)\\ &+\sup_{\psi}\big(\lim_{t\downarrow 0}\mu^t(\psi)-\mu^0(\psi)\big)+\sup_{\psi}\big(\mu^T(\psi)-\lim_{t\uparrow T}\mu^t(\psi)\big),\end{aligned}\tag{4.10}$$

where the sup *is taken over all $\psi\in C^1(\overline{\Omega})$ with $0\le\psi\le 1$. Then*

$$\liminf_{\varepsilon\to 0}\mathcal{S}_\varepsilon(u_\varepsilon)\,\ge\,\int_{\Omega_T}|v-H|^2\,d\mu+4\mathcal{S}_{nuc}(\mu).\tag{4.11}$$

In the previous definition of nucleation cost we have tacitly chosen good representatives of $\mu^t(\psi)$ (see [3]). With this choice the jump parts in (4.10) are well-defined.

Eventually let us remark that, in view of Theorem 4.3, we can conclude that $\mathcal{S}_{nuc}$ does indeed measure only $(n-1)$-dimensional jumps.

Theorem 4.6 improves [18] in the higher-multiplicity case. We will discuss our main results in Section 9.

4.4. **Convergence of the Allen–Cahn equation to Mean curvature flow.** Let $n=2,3$ and consider solutions $(u_\varepsilon)_{\varepsilon>0}$ of the Allen–Cahn equation (1.6) satisfying (A2) and (A3). Then $S_\varepsilon(u_\varepsilon)=0$ and the results of Sections 4.1-4.3 apply: There exists a subsequence $\varepsilon\to 0$ such that the phase functions u_ε converge to a phase indicator function u, such that the energy measures μ_ε^t converge an L^2-flow $(\mu^t)_{t\in(0,T)}$, and such that μ-almost everywhere

$$H\,=\,v\tag{4.12}$$

holds, where $H(t,\cdot)$ denotes the weak mean curvature of μ^t and where v denotes the generalized velocity of $(\mu^t)_{t\in(0,T)}$ in the sense of Definition 3.1. Moreover $\mathcal{S}_{nuc}(\mu)=0$, which shows that for any nonnegative $\psi\in C^1(\overline{\Omega})$ the function $t\mapsto\mu^t(\psi)$ cannot jump upwards. From (1.6) and (5.3) below one obtains that for any $\psi\in C^1(\overline{\Omega})$ and all $\zeta\in C^1_c(0,T)$

$$-\int_0^T\partial_t\zeta\,\mu_\varepsilon^t(\psi)\,dt\,=\,-\int_{\Omega_T}\zeta(t)\Big(\frac{1}{\varepsilon}\psi(x)w_\varepsilon^2(t,x)+\nabla\psi(x)\cdot\nabla u_\varepsilon w_\varepsilon(t,x)\Big)\,dxdt.\tag{4.13}$$

We will show that suitably defined ‘diffuse mean curvatures’ converge as $\varepsilon\to 0$, see (7.1). Using this result we can pass to the limit in (4.13) and we obtain for any nonnegative functions $\psi\in C^1(\overline{\Omega})$, $\zeta\in C^1_c(0,T)$ that

$$-\int_0^T\partial_t\zeta\mu^t(\psi)\,dt\,\le\,-\int_0^T\zeta(t)\int_\Omega\Big(H^2(t,x)+\nabla\psi(x)\cdot H(t,x)\Big)\,d\mu^t(x)dt,$$

which is an time-integrated version of Brakke’s inequality.

5. Proofs of Propositions 4.1, 4.2 and Theorem 4.3

Proof of Proposition 4.1. By (2.1), (2.2) we obtain that

$$\int_{\Omega_T} \Big(\frac{\varepsilon}{2}|\nabla' u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\,dxdt \;\leq\; \Lambda_3 + T\Lambda_4.$$

This implies by [22] the existence of a subsequence $\varepsilon \to 0$ and of a function $u \in BV(\Omega_T; \{-1,1\})$ such that

$$u_\varepsilon \;\to\; u \quad \text{ in } L^1(\Omega_T)$$

and

$$\frac{c_0}{2}\int_{\Omega_T} d|\nabla' u| \;\leq\; \liminf_{\varepsilon\to 0}\int_{\Omega_T} \big(\frac{\varepsilon}{2}|\nabla' u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big)\,dxdt \;\leq\; \big(\Lambda_3 + T\Lambda_4\big).$$

After possibly taking another subsequence, for almost all $t \in (0,T)$

$$u_\varepsilon(t,\cdot) \;\to\; u(t,\cdot) \quad \text{ in } L^1(\Omega) \tag{5.1}$$

holds. Using (2.2) and applying [22] for a fixed $t \in (0,T)$ with (5.1) we get that

$$\frac{c_0}{2}\int_{\Omega} d|\nabla u|(t,\cdot) \;\leq\; \liminf_{\varepsilon\to 0}\mu_\varepsilon^t(\Omega) \;\leq\; \Lambda_4.$$

□

Before proving Proposition 4.2 we show that the time-derivative of the energy-densities μ_ε^t is controlled.

Lemma 5.1. *There exists $C = C(\Lambda_1, \Lambda_3, \Lambda_4)$ such that for all $\psi \in C^1(\overline{\Omega})$*

$$\int_0^T |\partial_t \mu_\varepsilon^t(\psi)|\,dt \;\leq\; C\|\psi\|_{C^1(\overline{\Omega})}. \tag{5.2}$$

Proof. Using (A3) we compute that

$$\begin{aligned} 2\partial_t\mu_\varepsilon^t(\psi) \;=\; &\int_\Omega \big(\sqrt{\varepsilon}\partial_t u_\varepsilon + \frac{1}{\sqrt{\varepsilon}}w_\varepsilon\big)^2(t,x)\psi(x)\,dx - \int_\Omega \big(\varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2\big)(t,x)\psi(x)\,dx \\ &- 2\int_\Omega \varepsilon\nabla\psi(x)\cdot\partial_t u_\varepsilon(t,x)\nabla u_\varepsilon(t,x)\,dx. \end{aligned} \tag{5.3}$$

By (2.1), (2.2) we estimate

$$\begin{aligned} \Big|2\int_{\Omega_T}\varepsilon\nabla\psi\cdot\partial_t u_\varepsilon\nabla u_\varepsilon\,dxdt\Big| \;&\leq\; \int_{\Omega_T}|\nabla\psi|\big(\varepsilon(\partial_t u_\varepsilon)^2 + \varepsilon|\nabla u_\varepsilon|^2\big)\,dxdt \\ &\leq (\Lambda_3 + T\Lambda_4)\|\nabla\psi\|_{C^0(\overline{\Omega})} \end{aligned} \tag{5.4}$$

and deduce from (A1), (2.1), (5.3) that

$$\int_0^T |\partial_t\mu_\varepsilon^t(\psi)|\,dt \;\leq\; (\Lambda_1 + \Lambda_3)\|\psi\|_{C^0(\overline{\Omega})} + C(\Lambda_3, T\Lambda_4)\|\nabla\psi\|_{C^0(\overline{\Omega})},$$

which proves (5.2). □

Proof of Proposition 4.2. By (2.7) $\mu_\varepsilon \to \mu$ as Radon-measures on $\overline{\Omega_T}$. Choose now a countable family $(\psi_i)_{i\in\mathbb{N}} \subset C^1(\overline{\Omega})$ which is dense in $C^0(\overline{\Omega})$. By Lemma 5.1

and a diagonal-sequence argument there exists a subsequence $\varepsilon \to 0$ and functions $m_i \in BV(0,T)$, $i \in \mathbb{N}$, such that for all $i \in \mathbb{N}$

$$\mu_\varepsilon^t(\psi_i) \to m_i(t) \quad \text{for almost-all } t \in (0,T), \tag{5.5}$$

$$\partial_t \mu_\varepsilon^t(\psi_i) \to m_i' \quad \text{as Radon measures on } (0,T). \tag{5.6}$$

Let S denote the countable set of times $t \in (0,T)$ where for some $i \in \mathbb{N}$ the measure m_i' has an atomic part in t. We claim that (5.5) holds on $(0,T) \setminus S$. To see this we choose a point $t \in (0,T) \setminus S$ and a sequence of points $(t_j)_{j\in\mathbb{N}}$ in $(0,T) \setminus S$, such that $t_j \nearrow t$ and (5.5) holds for all t_j. We then obtain

$$\lim_{j\to\infty} m_i'([t_j,t]) = 0 \quad \text{for all } i \in \mathbb{N}, \tag{5.7}$$

$$\lim_{\varepsilon\to 0} \partial_t \mu^\varepsilon(\psi_i)([t_j,t]) = m_i'([t_j,t]) \quad \text{for all } i,j \in \mathbb{N}. \tag{5.8}$$

Moreover

$$\begin{aligned} |m_i(t) - \mu_\varepsilon^t(\psi_i)| &\le |m_i(t) - m_i(t_j)| + |m_i(t_j) - \mu_\varepsilon^{t_j}(\psi_i)| + |\mu_\varepsilon^{t_j}(\psi_i) - \mu_\varepsilon^t(\psi_i)| \\ &\le |m_i'([t_j,t])| + |m_i(t_j) - \mu_\varepsilon^{t_j}(\psi_i)| + |\partial_t\mu_\varepsilon^t(\psi_i)([t_j,t])| \end{aligned}$$

Taking first $\varepsilon \to 0$ and then $t_j \nearrow t$ we deduce by (5.7), (5.8) that (5.5) holds for all $i \in \mathbb{N}$ and all $t \in (0,T) \setminus S$.

Taking now an arbitrary $t \in (0,T)$ such that (5.5) holds, by (2.2) there exists a subsequence $\varepsilon \to 0$ such that

$$\mu_\varepsilon^t \to \mu^t \quad \text{as Radon-measures on } \Omega. \tag{5.9}$$

We deduce that $\mu^t(\psi_i) = m_i(t)$ and since $(\psi_i)_{i\in\mathbb{N}}$ is dense in $C^0(\overline{\Omega})$ we can identify any limits of $(\mu_\varepsilon^t)_{\varepsilon>0}$ and obtain (5.9) for the whole sequence selected in (5.5), (5.6) and for all $t \in (0,T)$, for which (5.5) holds. Moreover for any $\psi \in C^0(\overline{\Omega})$ the map $t \mapsto \mu_\varepsilon^t(\psi)$ has no jumps in $(0,T) \setminus S$. After possibly taking another subsequence we can also ensure that as $\varepsilon \to 0$

$$\mu_\varepsilon^0 \to \mu^0, \qquad \mu_\varepsilon^T \to \mu^T$$

as Radon measures on $\overline{\Omega}$. This proves (4.4).

By the Dominated Convergence Theorem we conclude that for any $\eta \in C^0(\overline{\Omega_T})$

$$\int_{\Omega_T} \eta \, d\mu = \lim_{\varepsilon\to 0} \int_{\Omega_T} \eta \, d\mu_\varepsilon = \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \eta(t,x) \, d\mu_\varepsilon^t(x) \, dt = \int_0^T \int_\Omega \eta(t,x) \, d\mu^t(x) \, dt,$$

which implies (4.5).

By (5.2), the $L^1(0,T)$-compactness of sequences that are uniformly bounded in $BV(0,T)$, the lower-semicontinuity of the BV-norm under L^1-convergence, and (4.4) we conclude that (4.6) holds. □

Proof of Theorem 4.3. For almost all $t \in (0,T)$ we obtain from Fatou's Lemma and (2.1), (2.2) that

$$\liminf_{\varepsilon\to 0} \Big(\mu_\varepsilon^t(\Omega) + \int_\Omega \frac{1}{\varepsilon} w_\varepsilon^2(t,x) \, dx \Big) < \infty. \tag{5.10}$$

Let $S \subset (0,T)$ be as in Proposition 4.2 and fix a $t \in (0,T) \setminus S$ such that (5.10) holds. Then we deduce from [28, Theorem 4.1, Theorem 5.1] and (4.4) that

$$\frac{1}{c_0}\mu^t \text{ is an integral } (n-1)\text{-varifold},$$
$$\mu^t \geq \frac{c_0}{2}|\nabla u(t,\cdot)|,$$

and that μ^t has weak mean curvature $H(t,\cdot)$ satisfying

$$\int_\Omega |H(t,x)|^2 \, d\mu^t(x) \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega w_\varepsilon(t,x)^2 \, dx. \tag{5.11}$$

By (5.11) and Fatou's Lemma we obtain that

$$\begin{aligned}\int_0^T \int_\Omega |H(t,x)|^2 \, d\mu^t(x) \, dt &\leq \int_0^T \left(\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega w_\varepsilon(t,x)^2 \, dx\right) dt \\ &\leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\Omega_T} w_\varepsilon^2 \, dxdt,\end{aligned}$$

which proves (4.7).

For later use we also associate general varifolds to μ_ε^t and consider their convergence as $\varepsilon \to 0$. Let $\nu_\varepsilon(t,\cdot) : \Omega \to S_1^{n-1}(0)$ be an extension of $\nabla u_\varepsilon(t,\cdot)/|\nabla u_\varepsilon(t,\cdot)|$ to the set $\{\nabla u_\varepsilon(t,\cdot) = 0\}$. Define the projections $P_\varepsilon(t,x) := Id - \nu_\varepsilon(t,x)\otimes\nu_\varepsilon(t,x)$ and consider the general varifolds V_ε^t and the integer rectifiable varifold $c_0^{-1}V^t$ defined by

$$V_\varepsilon^t(f) := \int_\Omega f(x, P_\varepsilon(t,x)) \, d\mu_\varepsilon^t(x), \tag{5.12}$$

$$V^t(f) := \int_\Omega f(x, P(t,x)) \, d\mu^t(x) \tag{5.13}$$

for $f \in C_c^0(\Omega \times \mathbb{R}^{n\times n})$, where $P(t,x) \in \mathbb{R}^{n\times n}$ denotes the projection onto the tangential plane $T_x\mu^t$. Then we deduce from the proof of [28, Theorem 4.1] that

$$V_\varepsilon^t \to V^t \quad \text{as } \varepsilon \to 0 \tag{5.14}$$

in the sense of varifolds. □

6. Proof of Theorem 4.4

6.1. **Equipartition of energy.** We start with a preliminary result, showing the important *equipartition of energy*: the *discrepancy measure*

$$\xi_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^{n+1} \tag{6.1}$$

vanishes in the limit $\varepsilon \to 0$.

To prove this we combine results from [28] with a refined version of Lebesgue's dominated convergence Theorem [25], see also [27, Lemma 4.2].

Proposition 6.1. *For a subsequence $\varepsilon \to 0$ we obtain that*

$$|\xi_\varepsilon| \to 0 \quad \textit{as Radon measures on } \overline{\Omega_T}. \tag{6.2}$$

Proof. Let us define the measures

$$\xi_\varepsilon^t \,:=\, \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\Big)(t,\cdot)\mathcal{L}^n$$

on Ω. For $\varepsilon > 0$, $k \in \mathbb{N}$, we define the sets

$$\mathcal{B}_{\varepsilon,k} \,:=\, \{t \in (0,T) : \int_\Omega \frac{1}{\varepsilon} w_\varepsilon^2(t,x)\,dx \,>\, k\}. \tag{6.3}$$

We then obtain from (2.1) that

$$\Lambda_3 \,\geq\, \int_0^T \int_\Omega \frac{1}{\varepsilon} w_\varepsilon^2(t,x)\,dxdt \,\geq\, |\mathcal{B}_{\varepsilon,k}|k. \tag{6.4}$$

Next we define the (signed) Radon-measures $\xi_{\varepsilon,k}^t$ by

$$\xi_{\varepsilon,k}^t \,:=\, \begin{cases} \xi_\varepsilon^t & \text{for } t \in (0,T)\setminus \mathcal{B}_{\varepsilon,k}, \\ 0 & \text{for } t \in \mathcal{B}_{\varepsilon,k}. \end{cases} \tag{6.5}$$

By [28, Proposition 4.9], we have

$$|\xi_{\varepsilon_j}^t| \,\to\, 0 \ (j\to\infty) \text{ as Radon measures on } \Omega \tag{6.6}$$

for any subsequence $\varepsilon_j \to 0$ $(j\to\infty)$ such that

$$\limsup_{j\to\infty} \int_\Omega \frac{1}{\varepsilon_j} w_{\varepsilon_j}^2(t,x)\,dx \,<\, \infty.$$

By (2.2), (6.5) we deduce that for any $\eta \in C^0(\overline{\Omega_T},\mathbb{R}_0^+)$, $k \in \mathbb{N}$, and almost all $t \in (0,T)$

$$|\xi_{\varepsilon,k}^t|(\eta(t,\cdot)) \,\to\, 0 \quad \text{ as } \varepsilon \to 0 \tag{6.7}$$

and that

$$|\xi_{\varepsilon,k}^t|(\eta(t,\cdot)) \,=\, \big(1 - \mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t)\big)|\xi_\varepsilon^t|\big(\eta(t,\cdot)\big) \,\leq\, \Lambda_4\|\eta\|_{C^0(\overline{\Omega_T})}. \tag{6.8}$$

By the Dominated Convergence Theorem, (6.7) and (6.8) imply that

$$\int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,dt \,\to\, 0 \quad \text{ as } \varepsilon \to 0. \tag{6.9}$$

Further we obtain that

$$\begin{aligned} \int_0^T |\xi_\varepsilon^t|(\eta(t,\cdot))\,dt \,&\leq\, \int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,dt + \int_{\mathcal{B}_{\varepsilon,k}} |\xi_\varepsilon^t|(\eta(t,\cdot))\,dt \\ &\leq\, \int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,dt + \int_{\mathcal{B}_{\varepsilon,k}} \mu_\varepsilon^t(\eta(t,\cdot))\,dt. \end{aligned} \tag{6.10}$$

For $k \in \mathbb{N}$ fixed we deduce from (2.2), (6.4), (6.10) that

$$\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega |\xi_\varepsilon^t|(\eta(t,\cdot))\,dt \,\leq\, \lim_{\varepsilon\to 0} \int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,dt + \|\eta\|_{C^0(\Omega_T)}\Lambda_4\frac{\Lambda_3}{k}. \tag{6.11}$$

By (6.9) and since $k \in \mathbb{N}$ was arbitrary this proves the Proposition. □

6.2. **Convergence of approximate velocities.** In the next step in the proof of Theorem 4.4 we define approximate velocity vectors and show their convergence as $\varepsilon \to 0$.

Lemma 6.2. *Define* $v_\varepsilon : \Omega_T \to \mathbb{R}^n$ *by*

$$v_\varepsilon := \begin{cases} -\frac{\partial_t u_\varepsilon}{|\nabla u_\varepsilon|} \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} & \textit{if } |\nabla u_\varepsilon| \neq 0, \\ 0 & \textit{otherwise.} \end{cases} \tag{6.12}$$

Then there exists a function $v \in L^2(\mu, \mathbb{R}^n)$ *such that*

$$(\varepsilon|\nabla u_\varepsilon|^2 \mathcal{L}^{n+1}, v_\varepsilon) \to (\mu, v) \quad \textit{as } \varepsilon \to 0 \tag{6.13}$$

in the sense of measure function pair convergence (see the Appendix B) and such that (4.8) *is satisfied.*

Proof. We define Radon measures

$$\tilde{\mu}_\varepsilon := \varepsilon|\nabla u_\varepsilon|^2 \mathcal{L}^{n+1} = \mu_\varepsilon + \xi_\varepsilon. \tag{6.14}$$

From (2.7), (6.2) we deduce that

$$\tilde{\mu}_\varepsilon \to \mu \quad \text{as Radon measures on } \overline{\Omega_T}. \tag{6.15}$$

Next we observe that $(\tilde{\mu}_\varepsilon, v_\varepsilon)$ is a function-measure pair in the sense of [16] (see also Definition B.1 in Appendix B) and that by (2.1)

$$\int_{\Omega_T} |v_\varepsilon|^2 \, d\tilde{\mu}_\varepsilon \leq \int_{\Omega_T} \varepsilon(\partial_t u_\varepsilon)^2 \, dxdt \leq \Lambda_3. \tag{6.16}$$

By Theorem B.3 we therefore deduce that there exists a subsequence $\varepsilon \to 0$ and a function $v \in L^2(\mu, \mathbb{R}^n)$ such that (6.13) and (4.8) hold. □

Lemma 6.3. *For μ-almost all* $(t, x) \in \Omega_T$

$$v(t, x) \perp T_x\mu^t. \tag{6.17}$$

Proof. We follow [24, Proposition 3.2]. Let $\nu_\varepsilon : \Omega_T \to S_1^{n-1}(0)$ be an extension of $\nabla u_\varepsilon / |\nabla u_\varepsilon|$ to the set $\{\nabla u_\varepsilon = 0\}$ and define projection-valued maps $P_\varepsilon : \Omega_T \to \mathbb{R}^{n\times n}$,

$$P_\varepsilon := Id - \nu_\varepsilon \otimes \nu_\varepsilon.$$

Consider next the general varifolds $\tilde{V}_\varepsilon, V$ defined by

$$\tilde{V}_\varepsilon(f) := \int_{\Omega_T} f(t, x, P_\varepsilon(t, x)) \, d\tilde{\mu}_\varepsilon(t, x), \tag{6.18}$$

$$V(f) := \int_{\Omega_T} f(t, x, P(t, x)) \, d\mu^t(x) \tag{6.19}$$

for $f \in C_c^0(\Omega_T \times \mathbb{R}^{n\times n})$, where $P(t, x) \in \mathbb{R}^{n\times n}$ denotes the projection onto the tangential plane $T_x\mu^t$.

From (5.14), Proposition 6.1, and Lebesgue's Dominated Convergence Theorem we deduce that

$$\lim_{\varepsilon\to 0} \tilde{V}_\varepsilon = V \tag{6.20}$$

as Radon-measures on $\Omega_T \times \mathbb{R}^{n\times n}$.

Next we define functions $\hat{v}_\varepsilon$ on $\Omega_T \times \mathbb{R}^{n\times n}$ by

$$\hat{v}_\varepsilon(t,x,Y) \,=\, v_\varepsilon(t,x) \qquad \text{for all } (t,x) \in \Omega_T,\, Y \in \mathbb{R}^{n\times n}.$$

We then observe that

$$\int_{\Omega_T \times \mathbb{R}^{n\times n}} \hat{v}_\varepsilon^2 \, dV_\varepsilon \,=\, \int_{\Omega_T} v_\varepsilon^2 \, d\tilde{\mu}_\varepsilon \,\leq\, \Lambda_3$$

and deduce from (6.20) and Theorem B.3 the existence of $\hat{v} \in L^2(V,\mathbb{R}^n)$ such that $(V_\varepsilon, \hat{v}_\varepsilon)$ converge to $(V,\hat{v})$ as measure-function pairs on $\Omega_T \times \mathbb{R}^{n\times n}$ with values in $\mathbb{R}^n$.

We consider now $h \in C^0_c(\mathbb{R}^{n\times n})$ such that $h(Y) = 1$ for all projections Y. We deduce that for any $\eta \in C^0_c(\Omega_T,\mathbb{R}^n)$

$$\begin{aligned}\int_{\Omega_T} \eta \cdot v \, d\mu \,&=\, \lim_{\varepsilon\to 0} \int_{\Omega_T \times \mathbb{R}^{n\times n}} \eta(t,x) \cdot h(Y) \hat{v}_\varepsilon(t,x,Y) \, dV_\varepsilon(t,x,Y) \\ &= \int_{\Omega_T} \eta(t,x) \cdot \hat{v}(t,x,P(t,x)) \, d\mu(t,x),\end{aligned}$$

which shows that for μ-almost all $(t,x) \in \Omega_T$

$$\hat{v}(t,x,P(t,x)) \,=\, v(t,x). \tag{6.21}$$

Finally we observe that for h,η as above

$$\begin{aligned}&\int_{\Omega_T} \eta(t,x) \cdot P(t,x) v(t,x) \, d\mu(t,x) \\ =& \int_{\Omega_T \times \mathbb{R}^{n\times n}} \eta(t,x) h(Y) \cdot Y \hat{v}(t,x,Y) \, dV(t,x,Y) \\ =& \lim_{\varepsilon\to 0} \int_{\Omega_T \times \mathbb{R}^{n\times n}} \eta(t,x) h(Y) \cdot Y \hat{v}_\varepsilon(t,x,Y) \, dV_\varepsilon(t,x,Y) \\ =& \lim_{\varepsilon\to 0} \int_{\Omega_T} \eta(t,x) \cdot P_\varepsilon(t,x) v_\varepsilon(t,x) \, d\tilde{\mu}_\varepsilon(t,x) \,=\, 0\end{aligned}$$

since $P_\varepsilon v_\varepsilon = 0$. This shows that $P(t,x)v(t,x) = 0$ for μ-almost all $(t,x) \in \Omega_T$. □

Proof of Theorem 4.4. By (2.1) there exists a subsequence $\varepsilon \to 0$ and a Radon measure β on $\overline{\Omega}_T$ such that

$$\big(\varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2\big)\mathcal{L}^{n+1} \,\to\, \beta, \qquad \beta(\overline{\Omega_T}) \,\leq\, \Lambda_3. \tag{6.22}$$

Using (A3) we compute that for any $\eta \in C^1_c((0,T) \times \overline{\Omega})$

$$\begin{aligned}\int_{\Omega_T} \eta \, d\alpha_\varepsilon \,=& \int_{\Omega_T} \eta\big(\varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2\big) \, dxdt - 2\int_{\Omega_T} \partial_t \eta \, d\mu_\varepsilon \\ &+2\int_{\Omega_T} \varepsilon \nabla\eta \cdot \partial_t u_\varepsilon \nabla u_\varepsilon \, dxdt\,.\end{aligned} \tag{6.23}$$

As ε tends to zero the term on the left-hand side and the first two terms on the right-hand-side converge by (2.7), (2.8) and (6.22). For the third term on the right-hand side of (6.23) we obtain from (6.13) that

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\nabla\eta\cdot\varepsilon\partial_t u_\varepsilon\nabla u_\varepsilon\,dxdt \;=\; -\lim_{\varepsilon\to 0}\int_{\Omega_T}\nabla\eta\cdot v_\varepsilon\,\varepsilon|\nabla u_\varepsilon|^2\,dxdt$$
$$= -\int_{\Omega_T}\nabla\eta\cdot v\,d\mu.$$

Therefore, taking $\varepsilon\to 0$ in (6.23) we deduce that

$$\int_{\Omega_T}\eta\,d\alpha \;=\; \int_{\Omega_T}\eta\,d\beta - 2\int_{\Omega_T}\partial_t\eta\,d\mu - 2\int_{\Omega_T}\nabla\eta\cdot v\,d\mu$$

holds for all $\eta\in C^1_c((0,T)\times\overline{\Omega})$. This yields that

$$\Big|\int_{\Omega_T}\partial_t\eta+\nabla\eta\cdot v\,d\mu\Big| \;\leq\; \|\eta\|_{C^0(\overline{\Omega_T})}\frac{1}{2}\big(\alpha(\overline{\Omega_T})+\beta(\overline{\Omega_T})\big),$$

which shows together with (6.17) that v is a generalized velocity vector for $(\mu^t)_{t\in(0,T)}$ in the sense of Definition 3.1. The estimate (4.8) was already proved in Lemma 6.2. □

7. Proof of Theorem 4.6

We start with the convergence of a 'diffuse mean curvature term'.

Lemma 7.1. *Define*

$$H_\varepsilon \;:=\; \frac{1}{\varepsilon}w_\varepsilon\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2},$$

let $\tilde\mu_\varepsilon=\varepsilon|\nabla u_\varepsilon|^2\,\mathcal{L}^{n+1}$*, and let* v_ε*,* v *be as in* (6.12)*,* (6.13)*. Then*

$$(\tilde\mu_\varepsilon,H_\varepsilon)\;\to\;(\mu,H), \tag{7.1}$$
$$(\tilde\mu_\varepsilon,v_\varepsilon-H_\varepsilon)\;\to\;(\mu,v-H) \tag{7.2}$$

as $\varepsilon\to 0$ *in the sense of measure function pair convergence. In particular*

$$\int_{\Omega_T}\eta|v-H|^2\,d\mu\;\leq\;\alpha(\eta) \tag{7.3}$$

holds for all $\eta\in C^0(\overline{\Omega_T},\mathbb{R}^+_0)$.

Proof. We use similar arguments as in the proof of Proposition 6.1. For $\varepsilon>0$, $k\in\mathbb{N}$, we define sets

$$\mathcal{B}_{\varepsilon,k}\;:=\;\{t\in(0,T):\int_\Omega\frac{1}{\varepsilon}w_\varepsilon(t,x)^2\,dx\;>\;k\}. \tag{7.4}$$

We then obtain from (2.1) that

$$\Lambda_3\;\geq\;\int_{\Omega_T}\frac{1}{\varepsilon}w_\varepsilon^2\,dxdt\;\geq\;|\mathcal{B}_{\varepsilon,k}|k. \tag{7.5}$$

Next we define functionals $T^t_{\varepsilon,k}\in C^0_c(\Omega,\mathbb{R}^n)^*$ by

$$T^t_{\varepsilon,k}(\psi)\;:=\;\begin{cases}\int_\Omega\psi(x)\cdot w_\varepsilon(t,x)\nabla u_\varepsilon(t,x)\,dx & \text{for } t\in(0,T)\setminus\mathcal{B}_{\varepsilon,k},\\ \int_\Omega\psi(x)\cdot H(t,x)\,d\mu^t(x) & \text{for } t\in\mathcal{B}_{\varepsilon,k}.\end{cases} \tag{7.6}$$

Considering the general $(n-1)$-varifolds V_ε^t, V^t defined in (5.12), (5.13) we obtain from [28, Proposition 4.10] and (5.14) that

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega \psi\cdot w_{\varepsilon_j}(t,x)\nabla u_{\varepsilon_j}(t,x)\,dx \;&=\; -\lim_{j\to\infty}\delta V^t_{\varepsilon_j}(\psi)\\ &=\; -\,\delta\mu^t(\psi) \;=\; \int_\Omega \psi\cdot H(t,x)\,d\mu^t(x)\end{aligned}\tag{7.7}$$

for any subsequence $\varepsilon_j \to 0$ $(j\to\infty)$ such that

$$\limsup_{j\to\infty}\int_\Omega \frac{1}{\varepsilon_j}w_{\varepsilon_j}^2\,dxdt \;<\;\infty.$$

Therefore we deduce from (7.6), (7.7) that for all $\eta\in C_c^0(\Omega_T,\mathbb{R}^n)$, $k\in\mathbb{N}$, and almost all $t\in(0,T)$

$$T^t_{\varepsilon,k}(\eta(t,\cdot)) \;\to\; \int_\Omega \eta(t,x)\cdot H(t,x)\,d\mu^t(x) \qquad \text{as } \varepsilon\to 0 \tag{7.8}$$

and that

$$\begin{aligned}&\big|T^t_{\varepsilon,k}(\eta(t,\cdot))\big|\\ &\le \big(1-\mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t)\big)\left|\int_\Omega \eta(t,x)\cdot w_\varepsilon(t,x)\nabla u_\varepsilon(t,x)\,dx\right|\\ &\quad + \mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t)\left|\int_\Omega \eta(t,x)\cdot H(t,x)\,d\mu^t(x)\right|\\ &\le \|\eta\|_{C^0(\Omega_T)}\big(1-\mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t)\big)\Big(\int_\Omega \frac{1}{2\varepsilon}w_\varepsilon(t,x)^2\,dx\Big)^{1/2}\Big(\int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon(t,x)|^2\,dx\Big)^{1/2}\\ &\quad + \int_\Omega |\eta(t,x)||H(t,x)|\,d\mu^t(x)\\ &\le \|\eta\|_{C^0(\Omega_T)}\sqrt{\frac{k}{2}}\sqrt{\Lambda_4} + \int_\Omega |\eta(t,x)||H(t,x)|\,d\mu^t(x),\end{aligned}\tag{7.9}$$

where the right-hand side is bounded in $L^1(0,T)$, uniformly with respect to $\varepsilon>0$.

By the Dominated Convergence Theorem, (7.8) and (7.9) imply that

$$\int_0^T T^t_{\varepsilon,k}(\eta(t,\cdot))\,dt \;\to\; \int_{\Omega_T}\eta\cdot H\,d\mu \qquad \text{as } \varepsilon\to 0. \tag{7.10}$$

Further we obtain that

$$\begin{aligned}&\Big|\int_{\Omega_T}\eta\cdot w_\varepsilon\nabla u_\varepsilon\,dxdt - \int_{\Omega_T}\eta\cdot H\,d\mu\Big|\\ &\le \Big|\int_0^T T^t_{\varepsilon,k}(\eta(t,\cdot))\,dt - \int_{\Omega_T}\eta\cdot H\,d\mu\Big|\\ &\quad + \Big|\int_{\mathcal{B}_{\varepsilon,k}}\int_\Omega \eta(t,x)\cdot H(t,x)\,d\mu^t(x)dt\Big| + \Big|\int_{\mathcal{B}_{\varepsilon,k}}\int_\Omega \eta\cdot w_\varepsilon\nabla u_\varepsilon\,dx\,dt\Big|\end{aligned}\tag{7.11}$$

The last term on the right-hand side we further estimate by

$$\begin{aligned}
&\Big|\int_{\mathcal{B}_{\varepsilon,k}}\int_\Omega \eta(t,x)\cdot w_\varepsilon(t,x)\nabla u_\varepsilon(t,x)\,dx\,dt\Big| \\
&\leq \|\eta\|_{C^0(\Omega_T)}\Big(\int_{\Omega_T}\frac{1}{2\varepsilon}w_\varepsilon^2\,dxdt\Big)^{1/2}|\mathcal{B}_{\varepsilon,k}|^{1/2}\sqrt{\Lambda_4} \\
&\leq \|\eta\|_{C^0(\Omega_T)}\Lambda_3\frac{1}{\sqrt{k}}\sqrt{\Lambda_4}, \qquad (7.12)
\end{aligned}$$

where we have used (2.2) and (7.5). For the second term on the right-hand side of (7.11) we obtain

$$\begin{aligned}
\Big|\int_{\mathcal{B}_{\varepsilon,k}}\int_\Omega \eta(t,x)\cdot H(t,x)\,d\mu^t(x)dt\Big| \;&\leq\; |\mathcal{B}_{\varepsilon,k}|^{1/2}\|\eta\|_{C^0(\Omega_T)}^{1/2}\Big(\int_{\mathrm{supp}(\eta)}H^2\,d\mu\Big)^{1/2} \\
&\leq\; \frac{\sqrt{\Lambda_3}}{\sqrt{k}}\|\eta\|_{C^0(\Omega_T)}^{1/2}\sqrt{\Lambda_3}, \qquad (7.13)
\end{aligned}$$

where we have used (4.7) and (2.1). Finally, for $k\in\mathbb{N}$ fixed, by (7.10) we deduce that

$$\lim_{\varepsilon\to 0}\Big|\int_0^T T_{\varepsilon,k}^t(\eta(t,\cdot))\,dt - \int_{\Omega_T}\eta\cdot H\,d\mu\Big| \;=\; 0. \qquad (7.14)$$

Taking $\varepsilon\to 0$ in (7.11) we obtain by (7.12)-(7.14) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big|\int_{\Omega_T}\eta\cdot w_\varepsilon\nabla u_\varepsilon\,dxdt - \int_{\Omega_T}\eta\cdot H\,d\mu\Big| \\
&\leq \frac{\Lambda_3}{\sqrt{k}}\|\eta\|_{C^0(\Omega_T)}\sqrt{\Lambda_4} + \frac{1}{\sqrt{k}}\Lambda_3 \qquad (7.15)
\end{aligned}$$

for any $k\in\mathbb{N}$, which proves (7.1). Using (6.13) this implies (7.2). Finally we fix an arbitrary nonnegative $\eta\in C^0(\overline{\Omega_T})$ and deduce that the measure-function pair $(\tilde{\mu}_\varepsilon,\sqrt{\eta}(v_\varepsilon - H_\varepsilon))$ converges to $(\mu,\sqrt{\eta}(v-H))$. The estimate (7.3) then follows from Theorem B.3. □

Let $\Pi:[0,T]\times\overline{\Omega}\to[0,T]$ denote the projection onto the first component and $\Pi_\#$ the pushforward of measures by Π. For $\psi\in C^0(\overline{\Omega})$ we consider the measures

$$\alpha_\psi \;:=\; \Pi_\#\big(\psi\alpha\big),$$

on $[0,T]$, that means

$$\alpha_\psi(\zeta) \;:=\; \int_{\Omega_T}\zeta(t)\psi(x)\,d\alpha(t,x),$$

for $\zeta\in C^0([0,T])$, and set

$$\alpha_\Omega \;:=\; \Pi_\#\alpha.$$

We then can estimate the atomic part of α_Ω in terms of the nucleation cost.

Lemma 7.2. *Let $\mathcal{S}_{nuc}(\mu)$ be the nucleation cost defined in* (4.10). *Then*

$$(\alpha_\Omega)_{atomic}[0,T] \;\geq\; 4\mathcal{S}_{nuc}(\mu). \qquad (7.16)$$

Proof. Let $\eta \in C^1(\overline{\Omega_T}, \mathbb{R}_0^+)$ be nonnegative. We compute that

$$\begin{aligned}\int_{\Omega_T} \eta d\alpha_\varepsilon &= \int_{\Omega_T} \eta\Big(\varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2 + 2\partial_t u_\varepsilon w_\varepsilon\Big)\,dxdt \\ &\geq 4\int_{\Omega_T} \eta\partial_t u_\varepsilon w_\varepsilon\,dxdt \\ &= -4\int_{\Omega_T} \partial_t\eta\,d\mu_\varepsilon + 4\int_{\Omega_T} \nabla\eta\cdot\varepsilon\partial_t u_\varepsilon\nabla u_\varepsilon\,dxdt \\ &\quad + 4\mu_\varepsilon^T(\eta(T,\cdot)) - 4\mu_\varepsilon^0(\eta(0,\cdot)). \end{aligned} \tag{7.17}$$

Passing to the limit $\varepsilon \to 0$ we obtain from (2.7), (4.4), (6.13) that

$$\int_{\Omega_T} \eta d\alpha \;\geq\; -4\int_{\Omega_T} \partial_t\eta\,d\mu - 4\int_{\Omega_T} \nabla\eta\cdot v\,d\mu + 4\mu^T(\eta(T,\cdot)) - 4\mu^0(\eta(0,\cdot)). \tag{7.18}$$

We now choose $\eta(t,x) = \zeta(t)\psi(x)$ where $\zeta \in C^1([0,T],\mathbb{R}_0^+)$, $\psi \in C^1(\overline{\Omega},\mathbb{R}_0^+)$ in (7.18) and deduce that

$$\begin{aligned}\int_0^T \zeta d\alpha_\psi \;\geq\; &-4\int_0^T \partial_t\zeta\mu^t(\psi)\,dt + 4\int_0^T \zeta\int_\Omega \nabla\psi\cdot v(t,x)\,d\mu^t(x)\,dt \\ &+ 4\zeta(T)\mu^T(\psi) - 4\zeta(0)\mu^0(\psi).\end{aligned} \tag{7.19}$$

This shows that

$$\begin{aligned}\alpha_\psi \;\geq\; &4\partial_t(\mu^t(\psi)) + 4\Big(\int_\Omega \nabla\psi(x)\cdot v(t,x)\,d\mu^t(x)\Big)\mathcal{L}^1 \\ &+ 4\big(\mu^T(\psi) - \lim_{t\uparrow T}\mu^t(\psi)\big)\delta_T + 4\big(\lim_{t\downarrow 0}\mu^t(\psi) - \mu^0(\psi)\big)\delta_0.\end{aligned} \tag{7.20}$$

Evaluating the atomic parts we obtain that for any $0 < t_0 < T$

$$\alpha_\psi(\{t_0\}) \;\geq\; 4\partial_t(\mu^t(\psi))(\{t_0\}),$$

which implies that

$$\alpha_\Omega(\{t_0\}) \;\geq\; 4\sup_\psi \partial_t(\mu^t(\psi))(\{t_0\}). \tag{7.21}$$

where the supremum is taken over all $\psi \in C^1(\overline{\Omega})$ with $0 \leq \psi \leq 1$.

Moreover we deduce from (7.20)

$$\alpha_\Omega(\{0\}) \;\geq\; 4\sup_\psi\big(\lim_{t\downarrow 0}\mu^t(\psi) - \mu^0(\psi)\big), \tag{7.22}$$

$$\alpha_\Omega(\{T\}) \;\geq\; 4\sup_\psi\big(\mu^T(\psi) - \lim_{t\uparrow T}\mu^t(\psi)\big), \tag{7.23}$$

where the supremum is taken over $\psi \in C(\overline{\Omega})$ with $0 \leq \psi \leq 1$. By (7.21)-(7.23) we conclude that (7.16) holds. □

Proof of Theorem 4.6. By (7.3) we deduce that $\alpha \geq |v-H|^2\mu$. Since $\mu = \mathcal{L}^1 \otimes \mu^t$ we deduce from the Radon-Nikodym Theorem that

$$(\alpha_\Omega)_{ac}[0,T] \;\geq\; \int_{\Omega_T} |v-H|^2\,d\mu, \tag{7.24}$$

and from (7.16) that

$$(\alpha_\Omega)_{atomic}[0,T] \;\geq\; 4\mathcal{S}_{nuc}(\mu), \tag{7.25}$$

where $(\alpha_\Omega)_{ac}$ and $(\alpha_\Omega)_{atomic}$ denote the absolutely continuous and atomic part with respect to $\mathcal{L}^1$ of the measure α_Ω. Adding the two estimates and recalling (2.9) we obtain (4.11). □

8. Proofs of Proposition 3.3 and Proposition 4.5

Define for $r > 0$, $(t_0, x_0) \in \Omega_T$ the cylinders

$$Q_r(t_0, x_0) := (t_0 - r, t_0 + r) \times B^n_r(x_0).$$

Proof of Proposition 3.3. Define

$$\Sigma_n(\mu) := \big\{(t,x) \in \Omega_T : \text{ the tangential plane of } \mu \text{ in } (t,x) \text{ exists}\big\} \tag{8.1}$$

and choose $(t_0, x_0) \in \Sigma_n(\mu)$ such that

$$v \text{ is approximately continuous with respect to } \mu \text{ in } (t_0, x_0). \tag{8.2}$$

Since $v \in L^2(\mu)$ we deduce from [11, Theorem 2.9.13] that (8.2) holds μ-almost everywhere. Let

$$P_0 := T_{(t_0,x_0)}\mu, \qquad \theta_0 > 0 \tag{8.3}$$

denote the tangential plane and multiplicity at (t_0, x_0) respectively, and define for any $\varphi \in C^0_c(Q_1(0))$ the scaled functions $\varphi_\varrho \in C^0_c(Q_\varrho(t_0, x_0))$,

$$\varphi_\varrho(t,x) := \varrho^{-n}\varphi\big(\varrho^{-1}(t - t_0), \varrho^{-1}(x - x_0)\big).$$

We then obtain from (8.3) that

$$\int_{\Omega_T} \varphi_\varrho \, d\mu \to \theta_0 \int_{P_0} \varphi \, d\mathcal{H}^n \quad \text{ as } \varrho \searrow 0. \tag{8.4}$$

From (3.2), the Hahn–Banach Theorem, and the Riesz Theorem we deduce that

$$\vartheta \in C^1_c(\Omega_T)^*, \qquad \vartheta(\eta) := \int_{\Omega_T} \nabla'\eta \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} d\mu \tag{8.5}$$

can be extended to a (signed) Radon-measure on Ω_T. Since by the Radon-Nikodym Theorem $D_\mu|\vartheta|$ exists and is finite μ-almost everywhere we may assume without loss of generality that

$$D_\mu|\vartheta|(t_0, x_0) < \infty. \tag{8.6}$$

We next fix $\eta \in C^1_c(Q_1(0))$ and compute that

$$\vartheta(\varrho\eta_\varrho) = \int_{\Omega_T} \big(\nabla'\eta\big)_\varrho \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} d\mu. \tag{8.7}$$

From (8.2), (8.4) we deduce that the right-hand side converges in the limit $\varrho \to 0$,

$$\lim_{\varrho \to 0} \int_{\Omega_T} \big(\nabla'\eta\big)_\varrho \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} d\mu = \theta_0 \begin{pmatrix} 1 \\ v(t_0, x_0) \end{pmatrix} \cdot \int_{P_0} \nabla'\eta \, d\mu. \tag{8.8}$$

For the left-hand side of (8.7) we deduce that

$$\liminf_{\varrho \searrow 0} |\vartheta(\varrho\eta_\varrho)| \le \|\eta\|_{C^0_c(Q_1(0))} \liminf_{\varrho \searrow 0} \varrho^{-n+1} |\vartheta|(Q_\varrho(t_0, x_0)) \tag{8.9}$$

and observe that (8.6) implies

$$\infty > \lim_{\varrho\searrow 0} \frac{|\vartheta|(Q_\varrho(t_0,x_0))}{\mu(Q_\varrho(t_0,x_0))} \geq \liminf_{\varrho\searrow 0} \varrho^{-n}|\vartheta|(Q_\varrho(t_0,x_0))\Big(\limsup_{\varrho\searrow 0} \varrho^{-n}\mu(Q_\varrho(t_0,x_0))\Big)^{-1}$$
$$\geq c\liminf_{\varrho\searrow 0} \varrho^{-n}|\vartheta|(Q_\varrho(t_0,x_0)), \tag{8.10}$$

since by (8.4) for any $\varphi \in C^0_c(Q_2(0),\mathbb{R}^+_0)$ with $\varphi \geq 1$ on $Q_1(0)$

$$\limsup_{\varrho\searrow 0} \varrho^{-n}\mu(Q_\varrho(t_0,x_0)) \leq \limsup_{\varrho\searrow 0}\int_{\Omega_T} \varphi_\varrho\, d\mu \leq C(\varphi).$$

Therefore (8.7)-(8.10) yield

$$\theta_0 \begin{pmatrix} 1 \\ v(t_0,x_0)\end{pmatrix} \cdot \int_{P_0} \nabla'\eta\, d\mu = 0. \tag{8.11}$$

Now we observe that the integral over the projection of $\nabla'\eta$ onto P_0 vanishes. This shows that

$$\int_{P_0} \nabla'\eta\, d\mathcal{H}^n \in P_0^\perp. \tag{8.12}$$

Since η can be chosen such that the integral in (8.12) takes an arbitrary direction normal to P_0 we obtain from (8.11) that $v(t_0,x_0)$ satisfies (3.3). If $T_{x_0}\mu^{t_0}$ exists then

$$T_{(t_0,x_0)}\mu = \Big(\{0\}\times T_{x_0}\mu^{t_0}\Big) \oplus \operatorname{span}\begin{pmatrix} 1 \\ v(x_0)\end{pmatrix}$$

and we obtain that v is uniquely determined. □

To prepare the proof of Proposition 4.5 we first show that μ is absolutely continuous with respect to $\mathcal{H}^n$.

Proposition 8.1. *For any $D \subset\subset \Omega$ there exists $C(D)$ such that for all $x_0 \in D$ and almost all $t_0 \in (0,T)$*

$$\limsup_{r\searrow 0} r^{-n}\mu\big(Q_r(t_0,x_0)\big) \leq C(D)\Lambda_4 + \liminf_{\varepsilon\to 0}\int_D \frac{1}{\varepsilon}w_\varepsilon^2(t_0,x)\,dx. \tag{8.13}$$

In particular,

$$\limsup_{\rho\to 0}\frac{\mu(B_\rho(t_0,x_0))}{\rho^n} < \infty \quad \textit{for } \mu - \textit{almost every } (t_0,x_0) \tag{8.14}$$

and μ is absolutely continuous with respect to $\mathcal{H}^n$,

$$\mu << \mathcal{H}^n. \tag{8.15}$$

Proof. Let

$$r_0 := \min\Big\{1, \frac{1}{2}\operatorname{dist}(D,\partial\Omega), |t_0|, |T-t_0|\Big\}.$$

Then we obtain for all $r < r_0$, $x_0 \in D$, from (6.2) and [28, Proposition 4.5] that

$$\frac{1}{r}\int_{t_0-r}^{t_0+r} r^{1-n}\mu^t\big(B^n_r(x_0)\big)\,dt$$
$$\leq \frac{1}{r}\int_{t_0-r}^{t_0+r} r_0^{1-n}\mu^t\big(B^n_{r_0}(x_0)\big)\,dt + \frac{1}{4(n-1)^2}\frac{1}{r}\int_{t_0-r}^{t_0+r}\left(\liminf_{\varepsilon\to 0}\int_D \frac{1}{\varepsilon}w_\varepsilon^2(t,x)\,dx\right)dt. \tag{8.16}$$

By Fatou's Lemma and (2.1)

$$t \mapsto \liminf_{\varepsilon\to 0} \int_D \frac{1}{\varepsilon} w_\varepsilon^2(t,x)\,dx \quad \text{ is in } L^1(0,T) \tag{8.17}$$

and by (2.2) we deduce for almost all $t_0 \in (0,T)$ that

$$\begin{aligned} &\limsup_{r\searrow 0} \frac{1}{r}\int_{t_0-r}^{t_0+r} r^{1-n}\mu^t\big(B_r^n(x_0)\big)\,dt \\ \le\, &2r_0^{1-n}\Lambda_4 + \frac{1}{2(n-1)^2}\liminf_{\varepsilon\to 0}\int_D \frac{1}{\varepsilon} w_\varepsilon^2(t_0,x)\,dx. \end{aligned}$$

Since r_0 depends only on D,Ω the inequality (8.13) follows.

By (8.17) the right-hand side in (8.13) is finite for $\mathcal{L}^1$-almost all $t_0 \in (0,T)$ and $\theta^{*n}(\mu,(t,x))$ is bounded for almost all $t \in (0,T)$ and all $x \in \Omega$. By (2.2) we deduce that for any $I \subset (0,T)$ with $|I| = 0$

$$\mu(I\times\Omega) \,\le\, \Lambda_4|I| \,=\, 0$$

which implies (8.14).

To prove the final statement let $B \subset \Omega_T$ be given with

$$\mathcal{H}^n(B) \,=\, 0. \tag{8.18}$$

Consider the family of sets $(D_k)_{k\in\mathbb{N}}$,

$$D_k \,:=\, \{z \in \Omega_T \,:\, \theta^{*n}(\mu,z) \le k\}.$$

By (8.14), [31, Theorem 3.2], and (8.18) we obtain that for all $k \in \mathbb{N}$

$$\mu(B\cap D_k) \,\le\, 2^n k\mathcal{H}^n(B\cap D_k) \,=\, 0. \tag{8.19}$$

Moreover we have that

$$\mu(B \setminus \bigcup_{k\in\mathbb{N}} D_k) \,=\, 0 \tag{8.20}$$

by (8.14). By (8.19), (8.20) we conclude that

$$\mu(B) \,=\, 0,$$

which proves (8.15). □

To prove Proposition 4.5 we need that $\mathcal{H}^n$-almost everywhere on $\partial^*\{u=1\}$ the generalized tangent plane of μ exists. We first obtain the following relation between the measures μ and $|\nabla' u|$.

Proposition 8.2. *There exists a nonnegative function $g \in L^2(\mu,\mathbb{R}_0^+)$ such that*

$$g\,\mu \,\ge\, \frac{c_0}{2}|\nabla' u|. \tag{8.21}$$

In particular, $|\nabla' u|$ is absolutely continuous with respect to μ,

$$|\nabla' u| \,<<\, \mu. \tag{8.22}$$

Proof. Let

$$G(r) \,=\, \int_0^r \sqrt{2W(s)}\,ds. \tag{8.23}$$

On the set $\{|\nabla u_\varepsilon| \neq 0\}$ we have

$$
\begin{aligned}
|\nabla G(u_\varepsilon)| &= \frac{|\nabla G(u_\varepsilon)|}{|\nabla' G(u_\varepsilon)|} |\nabla' G(u_\varepsilon)| \\
&= \frac{|\nabla G(u_\varepsilon)|}{\sqrt{\partial_t G(u_\varepsilon)^2 + |\nabla G(u_\varepsilon)|^2}} |\nabla' G(u_\varepsilon)| \\
&= \frac{1}{\sqrt{1+|v_\varepsilon^2|}} |\nabla' G(u_\varepsilon)|. \qquad (8.24)
\end{aligned}
$$

Letting $\tilde{\mu}_\varepsilon$ as in (6.14) we get from (6.16), (2.2), and Theorem B.3 the existence of a function $g \in L^2(\mu)$ such that (up to a subsequence)

$$
\lim_{\varepsilon \to 0} (\tilde{\mu}_\varepsilon, \sqrt{1+|v_\varepsilon|^2}) \;=\; (\mu, g) \qquad (8.25)
$$

as measure-function pairs on Ω_T with values in $\mathbb{R}$.

Let $\eta \in C_c^0(\Omega_T)$. Then

$$
\begin{aligned}
&\Big| \int_{\Omega_T} \eta \sqrt{1+|v_\varepsilon|^2}\, |\nabla G(u_\varepsilon)| \, dxdt - \int \eta \sqrt{1+|v_\varepsilon|^2}\, d\tilde{\mu}_\varepsilon \Big| \\
=& \Big| \int_{\Omega_T} \eta \sqrt{1+|v_\varepsilon|^2} \Big(\sqrt{\frac{2W(u_\varepsilon)}{\varepsilon}} - \sqrt{\varepsilon} |\nabla u_\varepsilon| \Big) \sqrt{\varepsilon} |\nabla u_\varepsilon| \, dxdt \Big| \\
\leq& \Big(\int_{\Omega_T} \eta^2 (1+|v_\varepsilon|^2) \varepsilon |\nabla u_\varepsilon|^2 \, dxdt \Big)^{1/2} \Big\| \sqrt{\frac{2W(u_\varepsilon)}{\varepsilon}} - \sqrt{\varepsilon} |\nabla u_\varepsilon| \Big\|_{L^2(\Omega_T)} \\
\leq& \|\eta\|_{L^\infty} (2T\Lambda_4 + \Lambda_3)^{1/2} (2|\xi_\varepsilon|(\Omega_T))^{1/2}. \qquad (8.26)
\end{aligned}
$$

Thanks to (8.25), (8.26) and (6.2) we conclude that

$$
\lim_{\varepsilon \to 0} (|\nabla G(u_\varepsilon)| \, \mathcal{L}^{n+1}, \sqrt{1+|v_\varepsilon|^2}) = (\mu, g) \qquad (8.27)
$$

as measure-function pairs on Ω_T with values in $\mathbb{R}$.

Again by (2.1) we have

$$
\begin{aligned}
&\int_{\{0=|\nabla u_\varepsilon|<W(u_\varepsilon)\}} |\nabla' G(u_\varepsilon)| \, dxdt \\
=& \int_{\{0=|\nabla u_\varepsilon|<W(u_\varepsilon)\}} |\partial_t u_\varepsilon| \sqrt{2W(u_\varepsilon)} \, dxdt \\
\leq& \sqrt{2} \Big(\int_{\Omega_T} \varepsilon (\partial_t u_\varepsilon)^2 \, dxdt \Big)^{1/2} \Big(\int_{\{0=|\nabla u_\varepsilon|<W(u_\varepsilon)\}} \frac{W(u_\varepsilon)}{\varepsilon} \, dxdt \Big)^{1/2} \\
\leq& \sqrt{2\Lambda_3} (|\xi_\varepsilon|(\Omega_T))^{1/2},
\end{aligned}
$$

which vanishes by (6.2) as $\varepsilon \to 0$. This implies together with (8.24) and (8.27) that

$$
\begin{aligned}
\int \eta \, g \, d\mu &= \lim_{\varepsilon \to 0} \int \eta \sqrt{1+|v_\varepsilon|^2} |\nabla G(u_\varepsilon)| \, dxdt \\
&= \lim_{\varepsilon \to 0} \int_{\Omega_T} \eta |\nabla' G(u_\varepsilon)| \, dxdt \geq \frac{c_0}{2} \int_{\Omega_T} \eta \, d|\nabla' u|,
\end{aligned}
$$

where in the last line we used that

$$
\frac{c_0}{2} \int_{\Omega_T} \eta \, d|\nabla' u| \;=\; \int_{\Omega_T} \eta \, d|\nabla' G(u)| \;\leq\; \liminf_{\varepsilon \to 0} \int_{\Omega_T} \eta |\nabla' G(u_\varepsilon)| \, dxdt.
$$

Considering now a set $B \subset \partial^*\{u=1\}$ with $\mu(B) = 0$ we conclude that

$$|\nabla' u|(B) \leq \frac{2}{c_0}\int_B g\,d\mu = 0,$$

since $g \in L^2(\mu)$. □

Proposition 8.3. *In $\mathcal{H}^n$-almost-all points in $\partial^*\{u=1\}$ the tangential-plane of μ exists.*

Proof. From the Radon-Nikodym Theorem we obtain that the derivative

$$f(z) := D_{|\nabla' u|}\mu(z) := \lim_{r\searrow 0}\frac{\mu(B_r^{n+1}(z))}{|\nabla' u|(B_r^{n+1}(z))} \tag{8.28}$$

exists for $|\nabla' u|$-almost-all $z \in \Omega_T$ and that $f \in L^1(|\nabla' u|)$. By (8.15) we deduce that

$$\mu\lfloor\partial^*\{u=1\} = f|\nabla' u|. \tag{8.29}$$

Similarly we obtain that

$$\frac{1}{f(z)} = D_\mu|\nabla' u|(z)$$

is finite for μ-almost all $z \in \partial^*\{u=1\}$. By (8.22) this implies that

$$f > 0 \quad |\nabla' u|\text{-almost everywhere in } \Omega_T. \tag{8.30}$$

Since $|\nabla' u|$ is rectifiable and f measurable with respect to $|\nabla' u|$ we obtain from (8.29), (8.30) and [31, Remark 11.5] that

$$\mu\lfloor\partial^*\{u=1\} \text{ is rectifiable.} \tag{8.31}$$

Moreover $\mathcal{H}^n$-almost-all $z \in \partial^*\{u=1\}$ satisfy that

$$\lim_{r\searrow 0}\frac{\mu(B_r^{n+1}(z)\setminus\partial^*\{u=1\})}{\mu(B_r^{n+1}(z))} = 0, \tag{8.32}$$

$$\limsup_{r\searrow 0}\frac{\mu(B_r^{n+1}(z))}{\omega_n r^n} < \infty. \tag{8.33}$$

In fact, (8.32) follows from [11, Theorem 2.9.11] and (8.22), and (8.33) from Proposition 8.1 and (8.22). Let now $z_0 \in \partial^*\{u=1\}$ satisfy (8.32), (8.33). For an arbitrary $\eta \in C_c^0(B_1^{n+1}(0))$ we then deduce that

$$\begin{aligned}&\limsup_{r\to 0}\Big|\int_{\Omega_T\setminus\partial^*\{u=1\}}\eta\big(r^{-1}(z-z_0)\big)r^{-n}\,d\mu(z)\Big| \\ &\leq \|\eta\|_{C_c^0(B_1^{n+1}(0))}\limsup_{r\to 0}\frac{\mu\big(B_r^{n+1}(z_0)\setminus\partial^*\{u=1\}\big)}{\mu\big(B_r^{n+1}(z_0)\big)}\limsup_{r\to 0}\frac{\mu\big(B_r^{n+1}(z_0)\big)}{r^n} = 0\end{aligned}$$

by (8.32), (8.33). Therefore

$$\lim_{r\to 0}\int_{\Omega_T}\eta\big(r^{-1}(z-z_0)\big)r^{-n}\,d\mu(z) = \lim_{r\to 0}\int_{\partial^*\{u=1\}}\eta\big(r^{-1}(z-z_0)\big)r^{-n}\,d\mu(z)$$

if the latter limit exists. By (8.31) we therefore conclude that in $\mathcal{H}^n$-almost-all points of $\partial^*\{u=1\}$ the tangent-plane of μ exists and coincides with the tangent plane of $\mu\lfloor\partial^*\{u=1\}$. □

Proof of Proposition 4.5. Since $u \in BV(\Omega_T)$ and $u(t,\cdot) \in BV(\Omega)$ for almost all $t \in (0,T)$ we obtain that $\partial_t u, \nabla u$ are Radon measures on Ω_T and that $\nabla u(t,\cdot)$ is a Radon measure on Ω for almost all $t \in (0,T)$. Moreover we observe that $v \in L^1(|\nabla u|)$ since

$$\int_{\Omega_T} |v|\, d|\nabla u| \,\leq\, \int_{\Omega_T} |v|\, d|\nabla' u| \,\leq\, \frac{2}{c_0}\int_{\Omega_T} g|v|\, d\mu \,\leq\, \frac{2}{c_0}\|g\|_{L^2(\mu)}\|v\|_{L^2(\mu)} \,<\, \infty$$

by Theorem 4.4 and Proposition 8.2. From (3.3) and Proposition 8.3 we deduce that for any $\eta \in C^1_c(\Omega_T)$

$$-\int_{\Omega_T} \eta\, d\partial_t u \,=\, \int_{\Omega_T} \eta v\, d\nabla u \,=\, \int_{\Omega_T} \eta v \cdot \frac{\nabla u}{|\nabla u|}\, d|\nabla u| \,=\, \int_0^T \int_\Omega \eta V\, d|\nabla u(t,\cdot)|\, dt,$$

which proves (4.9). □

9. Conclusions

Theorem 4.6 suggests to define a generalized action functional $\mathcal{S}$ in the class of L^2-flows by

$$\mathcal{S}(\mu) \,:=\, \inf_v \int_{\Omega_T} |v - H|^2\, d\mu + 4\mathcal{S}_{nuc}(\mu), \tag{9.1}$$

where the infimum is taken over all generalized velocities v for the evolution $(\mu^t)_{t\in(0,T)}$. In the class of n-rectifiable L^2-flows we have

$$\mathcal{S}(\mu) \,=\, \int_{\Omega_T} |v - H|^2\, d\mu + 4\mathcal{S}_{nuc}(\mu), \tag{9.2}$$

where v is the unique normal velocity of $(\mu^t)_{t\in(0,T)}$ (see Proposition 3.3).

In the present section we compare the functional $\mathcal{S}$ with the functional $\mathcal{S}^0$ defined in [18] (see (1.2)) and discuss the implications of Theorem 4.6 on a full Gamma convergence result for the action functional. For the ease of the exposition we focus in this section on the switching scenario.

Assumption 9.1. *Let a sequence $(u_\varepsilon)_{\varepsilon>0}$ of smooth functions $u_\varepsilon : \Omega_T \to \mathbb{R}$ be given with uniformly bounded action* (A1)*, zero Neumann boundary data* (A3)*, and assume for the initial- and final states that for all $\varepsilon > 0$*

$$u_\varepsilon(0,\cdot) \,=\, -1, \quad u_\varepsilon(T,\cdot) \,=\, 1 \qquad \text{in } \Omega. \tag{9.3}$$

Following [18] we define the reduced action functional on the set $\mathcal{M} \subset BV(\Omega_T, \{-1,1\}) \cap L^\infty(0,T,BV(\Omega))$ such that

- for every $\psi \in C^0_c(\Omega)$ the function
$$t \mapsto \int_\Omega u(t,\cdot)\psi\, dx$$
is absolutely continuous on $[0,T]$;
- $(\partial^*\{u(t,\cdot) = 1\})_{t\in(0,T)}$ is up to countably many times given as a smooth evolution of hypersurfaces.

By Assumption 9.1 the functional $\mathcal{S}^0_{nuc}$ can be rewritten as

$$\mathcal{S}^0(u) := c_0 \int_0^T \int_{\Sigma_t} \big|v(t,x) - H(t,x)\big|^2 \, d\mathcal{H}^{n-1}(x)dt + 4\mathcal{S}^0_{nuc}(u), \tag{9.4}$$

$$\mathcal{S}^0_{nuc}(u) := \sum_{t_0 \in S} \sup_{\psi} \Big(\lim_{t\downarrow t_0} \frac{c_0}{2}|\nabla u(t,\cdot)|(\psi) - \lim_{t\uparrow t_0} \frac{c_0}{2}|\nabla u(t,\cdot)|(\psi)\Big) + \sup_{\psi} \lim_{t\downarrow 0} \frac{c_0}{2}|\nabla u(t,\cdot)|(\psi) \tag{9.5}$$

where the sup is taken over all $\psi \in C^1(\overline{\Omega})$ with $0 \leq \psi \leq 1$.

In [18, Proposition 2.2] a (formal) proof of the *limsup- estimate* was given for a subclass of 'nice' functions in $\mathcal{M}$. Following the ideas of that proof, using the one-dimensional construction [18, Proposition 3.1], and a density argument we expect that the limsup-estimate can be extended to the whole set $\mathcal{M}$. We do not give a rigorous proof here but rather assume the limsup-estimate in the following.

Assumption 9.2. *For all $u \in \mathcal{M}$ there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ that satisfies Assumption 9.1 such that*

$$u = \lim_{\varepsilon\to 0} u_\varepsilon, \qquad \mathcal{S}^0(u) \geq \limsup_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_\varepsilon). \tag{9.6}$$

The natural candidate for the Gamma-limit of $\mathcal{S}_\varepsilon$ with respect to $L^1(\Omega_T)$ is the $L^1(\Omega_T)$-lower semicontinuous envelope of $\mathcal{S}^0$,

$$\overline{\mathcal{S}}(u) := \inf\Big\{ \liminf_{k\to\infty} \mathcal{S}^0(u_k) \,:\, (u_k)_{k\in\mathbb{N}} \subset \mathcal{M},\, u_k \to u \text{ in } L^1(\Omega_T)\Big\}. \tag{9.7}$$

9.1. **Comparison of $\mathcal{S}$ and $\mathcal{S}^0$.** If we associate with a function $u \in \mathcal{M}$ the measure $|\nabla u|$ on Ω_T we can compare $\mathcal{S}^0(u)$ and $\mathcal{S}(\frac{c_0}{2}|\nabla u|)$.

Proposition 9.3. *Let $u \in \mathcal{M}$ and let $\mu = \mathcal{L}^1 \otimes \mu^t$ be an L^2-flow of measures. Assume that for almost all $t \in (0,T)$*

$$\mu^t \geq \frac{c_0}{2}|\nabla u(t,\cdot)| \tag{9.8}$$

and that the nucleation cost $\mathcal{S}^0_{nuc}(u)$ is not larger than the nucleation cost $\mathcal{S}_{nuc}(\mu)$. Then

$$\mathcal{S}^0(u) \leq \mathcal{S}(\mu) \tag{9.9}$$

holds. For $\mu = \frac{c_0}{2}|\nabla u|$ we obtain that

$$\mathcal{S}^0(u) = \mathcal{S}(\frac{c_0}{2}|\nabla u|). \tag{9.10}$$

Proof. The locality of the mean curvature [29] shows that the weak mean curvature of μ^t and the (classical) mean curvature coincide on $\partial\{u(t,\cdot) = 1\}$. By Proposition 4.5 any generalized velocity v and the (classical) normal velocity V are equal on the phase boundary. This shows that the integral part of $\mathcal{S}^0(u)$ is not larger than the integral part of $\mathcal{S}(\mu)$, with equality if $\mu^t = \frac{c_0}{2}|\nabla u(t,\cdot)|$ for almost all $t \in (0,T)$. This proves (9.9). For the measure $\frac{c_0}{2}|\nabla u|$ we observe that the nucleation cost $\mathcal{S}_{nuc}(\frac{c_0}{2}\mu)$ equals the nucleation cost $\mathcal{S}^0_{nuc}(u)$ and we obtain (9.10). □

If higher multiplicities occur for the measure μ, the nucleation costs of μ and u may differ and the value of $\mathcal{S}^0(u)$ might be larger than $\mathcal{S}(\mu)$ as the following example shows. Let $\Omega = (0,L)$, let $\{u = 1\}$ be the shaded regions in Figure 1, and

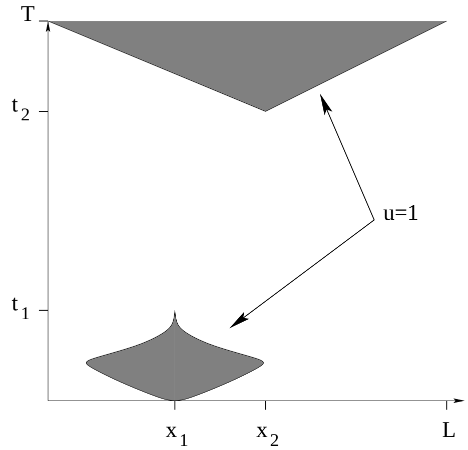

Figure 1. The phases $\{u = 1\}$

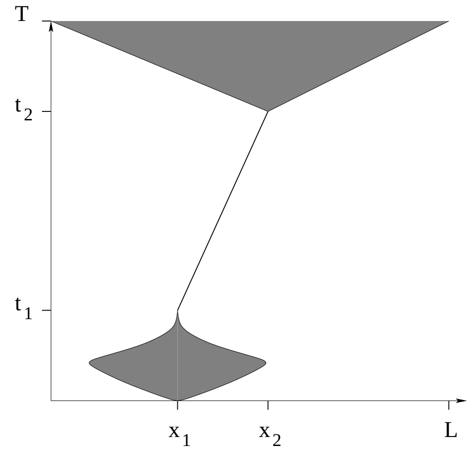

Figure 2. The measure μ

let μ be the measure supported on the phase boundary and with double density on a hidden boundary connecting the upper and lower part of the phase $\{u = 1\}$, see Figure 2. At time t_2 a new phase is nucleated but this time is not singular with respect to the evolution $(\mu^t)_{t\in(0,T)}$. On the other hand, no propagation cost occurs for the evolution $(u(t,\cdot))_{t\in(t_1,t_0)}$ whereas there is a propagation cost for $(\mu^t)_{t\in(t_1,t_2)}$. The difference in action is given by

$$\mathcal{S}^0(u) - \mathcal{S}(\mu) \;=\; 8c_0 - 2c_0\frac{(x_2 - x_1)^2}{t_2 - t_1}.$$

where x_1 is the annihilation point at time t_1 and x_2 the nucleation point at time t_2, see Figure 1. This shows that as soon as $(x_2 - x_1) < 4\sqrt{t_2 - t_1}$ we have

$$\mathcal{S}(\mu) \;<\; \mathcal{S}^0(u).$$

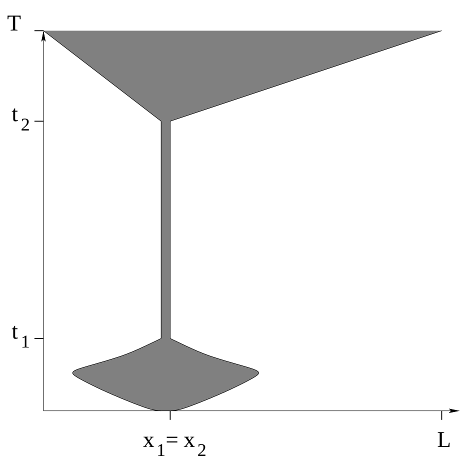

Figure 3. Phases $\{u_k = 1\}$

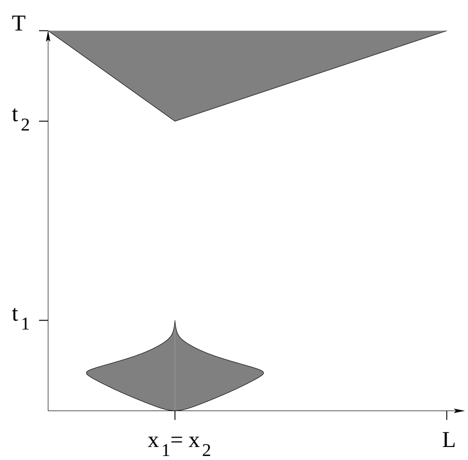

Figure 4. The limit

The same example with $x_2 = x_1$ shows that $\mathcal{S}^0$ is not lower-semicontinuous and that a relaxation is necessary in order to obtain the Gamma-limit of $\mathcal{S}_\varepsilon$. In fact consider a sequence $(u_k)_{k\in\mathbb{N}}$ with phases $\{u_k = 1\}$ given by the shaded region in Figure 3. Assume that the neck connecting the upper and lower part of the shaded region disappears with $k \to \infty$ and that u_k converges to the phase indicator function u with phase $\{u = 1\}$ indicated by the shaded regions in Figure 4. Then a nucleation cost at time t_2 appears for u. For the approximations u_k however there is no nucleation cost for $t > 0$ and the approximation can be made such that the propagation cost in (t_1, t_2) is arbitrarily small, which shows that

$$\mathcal{S}^0(u) \,>\, \liminf_{k\to\infty} \mathcal{S}^0(u_k).$$

The situation in higher space dimensions is even more involved than in the one-dimensional examples discussed above. For instance one could create a circle with double density (no new phase is created) at a time t_1 and let this double-density circle grow until a time $t_2 > t_1$ where the double-density circle splits and two circles evolve in different directions, one of them shrinking and the other one growing. In this way a new phase is created at time t_2. In this example $\mathcal{S}$ counts the creation of a double-density circle at time t_1 and the cost of propagating the double-density circle between the times t_1, t_2. In contrast $\mathcal{S}^0$ counts the nucleation cost of the new phase at time t_2, which is larger as the nucleation cost $\mathcal{S}_{nuc}$ at times t_1, but no propagation cost between the times t_1, t_2.

The analysis in [18] suggests that minimizers of the action functional exhibit nucleation and annihilation of phases only at the initial- and final time. This class is therefore particularly interesting.

Theorem 9.4. *Let $(u_\varepsilon)_{\varepsilon>0}$ satisfy Assumption 9.1 and suppose that Assumption 9.2 holds. Suppose that $u_\varepsilon \to u$ in $L^1(\Omega_T)$, $u \in \mathcal{M}$, and that u exhibits nucleation and annihilation of phases only at the final and initial time. Then*

$$\overline{\mathcal{S}}(u) \,=\, \mathcal{S}^0(u) \,\le\, \liminf_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_\varepsilon) \tag{9.11}$$

holds. In particular, $\mathcal{S}_\varepsilon$ Gamma-converges to $\mathcal{S}^0$ for those evolutions in $\mathcal{M}$ that have nucleations only at the initial time.

Proof. From the definition of the functional $\overline{\mathcal{S}}$ we deduce that

$$\overline{\mathcal{S}}(u) \,\le\, \mathcal{S}^0(u) \tag{9.12}$$

and that there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that

$$u \,=\, \lim_{k\to\infty} u_k, \qquad \overline{\mathcal{S}}(u) \,=\, \lim_{k\to\infty} \mathcal{S}^0(u_k).$$

Assumption 9.2 implies that for all $k \in \mathbb{N}$ there exists a sequence $(u_{\varepsilon,k})_{\varepsilon>0}$ such that

$$u_k \,=\, \lim_{\varepsilon\to 0} u_{\varepsilon,k}, \qquad \mathcal{S}^0(u_k) \,\ge\, \limsup_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_{\varepsilon,k}). \tag{9.13}$$

Therefore we can choose a diagonal-sequence $(u_{\varepsilon(k),k})_{k\in\mathbb{N}}$ such that

$$\overline{\mathcal{S}}(u) \,\ge\, \limsup_{k\to\infty} \mathcal{S}_{\varepsilon(k)}(u_{\varepsilon(k),k}). \tag{9.14}$$

By Proposition 4.1, 4.2 there exists a a subsequence $k \to \infty$ such that

$$u_{\varepsilon(k),k} \,\to\, u, \quad \mu_{\varepsilon(k),k} \,\to\, \mu, \quad \mu \,\ge\, \frac{c_0}{2}|\nabla u|, \tag{9.15}$$

where the last inequality follows from

$$\frac{c_0}{2}\int_\Omega \eta\, d|\nabla u(t,\cdot)| \,\leq\, \liminf_{\varepsilon\to 0}\int_\Omega \eta|\nabla G(u_\varepsilon)|\,dx \,\leq\, \liminf_{\varepsilon\to 0}\int_\Omega \eta d\mu_\varepsilon^t \,=\, \int_\Omega \eta d\mu^t,$$

with G as in (8.23). By Theorem 4.6 we further deduce that

$$\liminf_{k\to\infty}\mathcal{S}_{\varepsilon(k)}(u_{\varepsilon(k),k}) \,\geq\, \mathcal{S}(\mu).$$

This implies by (9.14) that

$$\overline{\mathcal{S}}(u) \,\geq\, \mathcal{S}(\mu). \tag{9.16}$$

Since $\mu^0 = 0$ and $\mu^t \geq \frac{c_0}{2}|\nabla u(t,\cdot)|$ the nucleation cost of μ at $t = 0$ is not lower than the nucleation cost for u. Since by assumption there are no more nucleation times we can apply Proposition 9.3 and obtain that $\mathcal{S}^0(u) \,\leq\, \mathcal{S}(\mu)$. By (9.12), (9.16) we conclude that $\mathcal{S}^0(u) = \overline{\mathcal{S}}(u) = \mathcal{S}(\mu)$.

Applying Proposition 4.1 and Theorem 4.6 to the sequence $(u_\varepsilon)_{\varepsilon>0}$ we deduce that there exists a subsequence $\varepsilon\to 0$ such that

$$\mu_\varepsilon \,\to\, \tilde{\mu}, \qquad \tilde{\mu} \,\geq\, \frac{c_0}{2}|\nabla u| \tag{9.17}$$

and such that

$$\liminf_{\varepsilon\to 0}\mathcal{S}_\varepsilon(u_\varepsilon) \,\geq\, \mathcal{S}(\tilde{\mu}).$$

Repeating the arguments above we deduce from Proposition 9.3 that $\mathcal{S}^0(u) \leq \mathcal{S}(\tilde{\mu})$ and

$$\mathcal{S}^0(u) \,\leq\, \liminf_{\varepsilon\to 0}\mathcal{S}_\varepsilon(u_\varepsilon).$$

Combining the upper bound (9.6) with (9.11) proves the Gamma convergence of $\mathcal{S}_\varepsilon$ in u. □

9.2. **Gamma convergence under an additional assumption.** Using Theorem 4.6 we can prove the Gamma convergence of $\mathcal{S}_\varepsilon$ under an additional assumption on the structure of the set of those measures that arise as limit of sequences with uniformly bounded action.

Assumption 9.5. *Consider any sequence $(u_\varepsilon)_{\varepsilon>0}$ with $u_\varepsilon \to u$ in $L^1(\Omega_T)$ that satisfies Assumption 9.1. Define the energy measures μ_ε according to* (2.5) *and let μ be any Radon measure such that for a subsequence $\varepsilon\to 0$*

$$\mu = \lim_{\varepsilon\to 0}\mu_\varepsilon. \tag{9.18}$$

Then we assume that there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that

$$u \,=\, \lim_{k\to\infty} u_k, \quad \mathcal{S}(\mu) \,\geq\, \lim_{k\to\infty}\mathcal{S}^0(u_k). \tag{9.19}$$

For any $u \in \mathcal{M}$ that exhibits nucleation and annihilation only at initial and final time the Assumption 9.5 is always satisfied: The proof of Theorem 9.4 and our results in Section 4 show that for any limit μ as in (9.18) we can apply Proposition 9.3. Therefore $\mathcal{S}^0(u) \leq \mathcal{S}(\mu)$ and the constant sequence u satisfies (9.19). However, a characterization of those $u \in \mathcal{M}$ such that Assumption 9.5 holds is open.

Theorem 9.6. *Suppose that the Assumptions 9.1, 9.2, and 9.5 hold. Then*

$$\mathcal{S}_\varepsilon \,\to\, \overline{\mathcal{S}} \qquad \text{as } \varepsilon\to 0 \tag{9.20}$$

in the sense of Gamma-convergence with respect to $L^1(\Omega_T)$.

Proof. We first prove the *limsup*-estimate for $\mathcal{S}_\varepsilon, \overline{\mathcal{S}}$. In fact, fix an arbitrary $u \in L^1(\Omega_T, \{-1,1\})$ with $\overline{\mathcal{S}}(u) < \infty$. We deduce that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ as in (9.7) such that

$$\overline{\mathcal{S}}(u) \;=\; \lim_{k\to\infty} \mathcal{S}^0(u_k). \tag{9.21}$$

By (9.6) for all $k \in \mathbb{N}$ there exists a sequence $(u_{\varepsilon,k})_{\varepsilon>0}$ such that

$$\lim_{\varepsilon\to 0} u_{\varepsilon,k} \;=\; u_k, \qquad \mathcal{S}^0(u_k) \;\geq\; \limsup_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_{\varepsilon,k}).$$

Choosing a suitable diagonal sequence $u_{\varepsilon(k),k}$ we deduce that

$$\overline{\mathcal{S}}(u) \;\geq\; \lim_{k\to\infty} \mathcal{S}_{\varepsilon(k)}(u_{\varepsilon(k),k}), \tag{9.22}$$

which proves the *limsup*-estimate.

We next prove the *liminf*-estimate. Consider an arbitrary sequence $(u_\varepsilon)_{\varepsilon>0}$ that satisfies the Assumption 9.1. By Theorem 4.6 there exists $u \in BV(\Omega_T, \{-1,1\})$ and a measure μ on Ω_T such that

$$u_\varepsilon \;\to\; u \quad \text{ in } L^1(\Omega_T), \qquad \mu_\varepsilon \;\to\; \mu \tag{9.23}$$

for a subsequence $\varepsilon \to 0$, and such that

$$\liminf_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_\varepsilon) \;\geq\; \mathcal{S}(\mu). \tag{9.24}$$

By Assumption 9.5 there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that (9.19) holds. By (9.24) and the definition of $\overline{\mathcal{S}}$ this yields that

$$\liminf_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_\varepsilon) \;\geq\; \mathcal{S}(\mu) \;\geq\; \lim_{k\to\infty} \mathcal{S}^0(u_k) \;\geq\; \overline{\mathcal{S}}(u) \tag{9.25}$$

and proves the *liminf*-estimate. □

Appendix A. Rectifiable measures and weak mean curvature

We briefly summarize some definitions from Geometric Measure Theory. We always restrict ourselves to the hypersurface case, that is 'tangential-plane' and 'rectifiability' of a measure in $\mathbb{R}^d$ means '$(d-1)$-dimensional tangential-plane' and '$(d-1)$-rectifiable'.

Definition A.1. *Let μ be a Radon-measure in $\mathbb{R}^d$, $d \in \mathbb{N}$.*

(1) *We say that μ has a* (generalized) tangent plane *in $z \in \mathbb{R}^d$ if there exist a number $\Theta > 0$ and a $(d-1)$-dimensional linear subspace $T \subset \mathbb{R}^d$ such that*

$$\lim_{r\searrow 0} r^{-d+1} \int \eta\left(\frac{y-z}{r}\right) d\mu(y) \;=\; \Theta \int_T \eta \, d\mathcal{H}^{d-1}, \qquad \text{for every } \eta \in C^0_c(\mathbb{R}^d). \tag{A.1}$$

We then set $T_z\mu := T$ and call Θ the multiplicity of μ in z.

(2) *If for μ-almost all $z \in \mathbb{R}^d$ a tangential plane exists then we call μ* rectifiable. *If in addition the multiplicity is integer-valued μ-almost everywhere we say that μ is* integer-rectifiable.

(3) *The* first variation $\delta\mu : C^1_c(\mathbb{R}^d, \mathbb{R}^d)$ *of a rectifiable Radon-measure μ on $\mathbb{R}^d$ is defined by*

$$\delta\mu(\eta) \;:=\; \int \operatorname{div}_{T_z\mu} \eta \, d\mu.$$

If there exists a function $H \in L^1_{\mathrm{loc}}(\mu)$ such that

$$\delta\mu(\eta) \;=\; -\int H \cdot \eta \, d\mu$$

we call H the weak mean-curvature vector of μ.

Appendix B. Measure-function pairs

We recall some basic facts about the notion of *measure function pairs* introduced by Hutchinson in [16].

Definition B.1. *Let $E \subset \mathbb{R}^d$ be an open subset. Let μ be a positive Radon-measure on E. Suppose $f : E \to \mathbb{R}^m$ is well defined μ-almost everywhere, and $f \in L^1(\mu, \mathbb{R}^m)$. Then we say (μ, f) is a measure-function pair over E (with values in $\mathbb{R}^m$).*

Next we define two notions of convergence for a sequence of measure-function pairs on E with values in $\mathbb{R}^m$.

Definition B.2. *Suppose $\{(\mu_k,\, f_k)\}_k$ and $(\mu,\, f)$ are measure-function pairs over E with values in $\mathbb{R}^m$. Suppose*

$$\lim_{k\to\infty} \mu_k \;=\; \mu,$$

as Radon-measures on E. Then we say $(\mu_k,\, f_k)$ converges to $(\mu,\, f)$ in the weak sense (in E) and write

$$(\mu_k,\, f_k) \to (\mu,\, f),$$

if $\mu_k \lfloor f_k \to \mu \lfloor f$ in the sense of vector-valued measures, that means

$$\lim_{k\to\infty} \int f_k \cdot \eta \, d\mu_k = \int f \cdot \eta \, d\mu,$$

for all $\eta \in C^0_c(E, \mathbb{R}^m)$.

The following result is a slightly less general version of [16, Theorem 4.4.2], however this is enough for our aims.

Theorem B.3. *Let $F : \mathbb{R}^m \to [0, +\infty)$ be a continuous, convex function with super-linear growth at infinity, that is:*

$$\lim_{|y|\to\infty} \frac{F(y)}{|y|} = +\infty.$$

Suppose $\{(\mu_k,\, f_k)\}_k$ are measure-function pairs over $E \subset \mathbb{R}^d$ with values in $\mathbb{R}^m$. Suppose μ is Radon-measure on E and $\mu_k \to \mu$ as $k \to \infty$. Then the following are true:

(1) if

$$\sup_k \int \int F(f_k) \, d\mu_k < +\infty, \tag{B.1}$$

then some subsequence of $\{(\mu_k,\, f_k)\}$ converges in the weak sense to some measure function $(\mu,\, f)$ for some f.

(2) if (B.1) *holds and $(\mu_k,\, f_k) \to (\mu,\, f)$ then*

$$\liminf_{k\to\infty} \int F(f_k) \, d\mu_k \geq \int F(f) \, d\mu. \tag{B.2}$$

References

[1] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[2] H. Allouba and J. A. Langa. Semimartingale attractors for Allen-Cahn SPDEs driven by space-time white noise. I. Existence and finite dimensional asymptotic behavior. *Stoch. Dyn.*, 4(2):223–244, 2004.

[3] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[4] Giovanni Bellettini and Luca Mugnai. Remarks on the variational nature of the heat equation and of mean curvature flow. preprint, 2007.

[5] Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[6] Ennio De Giorgi. Some remarks on Γ-convergence and least squares method. In *Composite media and homogenization theory (Trieste, 1990)*, volume 5 of *Progr. Nonlinear Differential Equations Appl.*, pages 135–142. Birkhäuser Boston, Boston, MA, 1991.

[7] Piero de Mottoni and Michelle Schatzman. Development of interfaces in $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A*, 116(3-4):207–220, 1990.

[8] Weinan E, Weiqing Ren, and Eric Vanden-Eijnden. Minimum action method for the study of rare events. *Comm. Pure Appl. Math.*, 57(5):637–656, 2004.

[9] Lawrence C. Evans, Halil Mete Soner, and Panagiotis E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45(9):1097–1123, 1992.

[10] William G. Faris and Giovanni Jona-Lasinio. Large fluctuations for a nonlinear heat equation with noise. *J. Phys. A*, 15(10):3025–3055, 1982.

[11] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[12] Jin Feng. Large deviation for diffusions and Hamilton-Jacobi equation in Hilbert spaces. *Ann. Probab.*, 34(1):321–385, 2006.

[13] Hans C. Fogedby, John Hertz, and Axel Svane. Domain wall propagation and nucleation in a metastable two-level system, 2004.

[14] Mark I. Freidlin and Alexander D. Wentzell. *Fluktuatsii v dinamicheskikh sistemakh pod deistviem malykh sluchainykh vozmushchenii*. "Nauka", Moscow, 1979. Teoriya Veroyatnostei i Matematicheskaya Statistika. [Probability Theory and Mathematical Statistics].

[15] Mark I. Freidlin and Alexander D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.

[16] John E. Hutchinson. Second fundamental form for varifolds and the existence of surfaces minimising curvature. *Indiana Univ. Math. J.*, 35(1):45–71, 1986.

[17] Tom Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.*, 38(2):417–461, 1993.

[18] Robert Kohn, Felix Otto, Maria G. Reznikoff, and Eric Vanden-Eijnden. Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation. *Comm. Pure Appl. Math.*, 60(3):393–438, 2007.

[19] Robert V. Kohn, Maria G. Reznikoff, and Yoshihiro Tonegawa. Sharp-interface limit of the Allen-Cahn action functional in one space dimension. *Calc. Var. Partial Differential Equations*, 25(4):503–534, 2006.

[20] Robert V. Kohn, Maria G. Reznikoff, and Eric Vanden-Eijnden. Magnetic elements at finite temperature and large deviation theory. *J. Nonlinear Sci.*, 15(4):223–253, 2005.

[21] Georgios T. Kossioris Markos A. Katsoulakis and Omar Lakkis. Noise regularization and computations for the 1-dimensional stochastic Allen-Cahn problem. *Interfaces Free Bound.*, 9(1):1–30, 2007.

[22] Luciano Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.*, 98(2):123–142, 1987.

[23] Luciano Modica and Stefano Mortola. Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.

[24] Roger Moser. A generalization of Rellich's theorem and regularity of varifolds minimizing curvature.

[25] Pavel I. Plotnikov and Victor N. Starovoĭtov. The Stefan problem with surface tension as a limit of the phase field model. *Differentsial′nye Uravneniya*, 29(3):461–471, 550, 1993.
[26] Maria G. Reznikoff and Yoshihiro Tonegawa. Higher multiplicity in the one-dimensional Allen-Cahn action functional. preprint, 2007.
[27] Matthias Röger. Existence of weak solutions for the Mullins-Sekerka flow. *SIAM J. Math. Anal.*, 37(1):291–301, 2005.
[28] Matthias Röger and Reiner Schätzle. On a modified conjecture of De Giorgi. *Mathematische Zeitschrift*, 254(4):675–714, 2006.
[29] Reiner Schätzle. Lower semicontinuity of the Willmore functional for currents. preprint, 2007.
[30] Tony Shardlow. Stochastic perturbations of the Allen-Cahn equation. *Electron. J. Differential Equations*, pages No. 47, 19 pp. (electronic), 2000.
[31] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University Centre for Mathematical Analysis, Canberra, 1983.

Luca Mugnai, Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig

Matthias Röger, Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig

E-mail address: mugnai@mis.mpg.de, roeger@mis.mpg.de