Shape Optimization Problems Governed by Nonlinear State Equations

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Abstract

The purpose of this paper is to give a compactness-continuity result for the solution of a non linear Dirichlet problem in terms of its domain variation. The topology in the family of domains is given by the Hausdorff metric and continuity is obtained under capacity conditions. A generalization of Sverak's result in N-dimensions is deduced as a particular case.

1 Introduction

The aim of this work is to study the dependence of the solutions of some nonlinear Dirichlet problems in terms of the domain variation and prove its continuity under weak constraints, i.e. we are interested in the continuity of the map:

$$\Omega \longrightarrow u_{\Omega,f,h} \tag{1.1}$$

where Ω is an open subset of a fixed ball B in $I\!R^N$, $n \geq 2$, $f \in H^{-1,q}(B)$, $h \in H_0^{1,p}(B)$ are fixed with $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $u_{\Omega,f,h}$ is the solution of the Dirichlet problem:

$$\begin{cases} -\Delta_p u = f & in\ \Omega \\ u - h \in H_0^{1,p}(\Omega). \end{cases} \tag{1.2}$$

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The continuity result established for the p-Laplacian is extended to a larger class of elliptic operators in divergence form, using arguments related to the Mosco convergence.

The notion of γ-convergence is therefore defined, as in the linear case, only in terms of the convergence of the solutions of the Dirichlet problems with homogeneous boundary conditions for the p-Laplacian. It is said that *a sequence $\{\Omega_n\}$ of open subsets of a fixed ball B γ_p-converges to* Ω if for any $f \in H^{-1,q}(B)$ the solutions of the Dirichlet problem:

$$-\Delta_p u_n = f \quad \text{in} \quad \Omega_n, \quad u_n \in H_0^{1,p}(\Omega_n)$$

converge strongly in $H_0^{1,p}(B)$, as $n \to +\infty$, to the solution of the corresponding problem in Ω (see [9], [10]). We see the solution on Ω_n as an element of $H_0^{1,p}(B)$ after extending it by zero. Following [12] it is sufficient to prove the continuity only for $f \equiv 1$. Using arguments related to Mosco convergence, one can extend the continuity result to equations with non-homogeneous boundary conditions.

Generally, the class of open sets is not compact with respect to the γ_p-convergence and the "limit" problem is of "relaxed type", i.e. it contains a term which depends on a Borel measure (see [6], [12]). In order to obtain a compactness result we introduce some additional constraints on the class of admissible domains which are expressed in terms of the Sobolev capacity.

In the linear case, there are two types of approaches. The first one (see [14], [16]) considers the perturbations of a domain in the so called "compact convergence" and the shape continuity result is obtained under stability condition on the limit domain. Some geometrical cases when stability occurs are given by the uniform cone condition (see [8], [15]). This approach does not give useful compactness results for shape optimization problems. The second one (see [4], [5], [17]) considers the Hausdorff complementary topology in the family of open sets, and the constraints on the moving domain are given in terms of capacity. The class of open sets, in which the continuity is established, turns out to be compact for the γ_2-convergence.

The nonlinear case is more difficult because of the failure of all standard techniques used in the linear case for the passage to the limit of the equation. On the other hand, the comparison of the limiting term with a fixed solution, previously done by subtraction via a boundary continuity lemma (see [5]), is no longer possible. The arguments for the existence and the uniqueness of the solutions come from the theory of generalized harmonic functions and monotone operators. Some preliminary results on the boundedness and on the continuous dependence of the solutions of (1.2) on the data are necessary. We refer to [12] for any further detail.

Using capacity constraints analogous to those introduced in [4] and [5] for the linear case, we prove the γ_p-compactness result for the p-Laplacian. In order to prove this statement we first study some stability results for the homogeneous equation with non homogeneous boundary data $h \in H_0^{1,p}(B)$:

$$\begin{cases} -\Delta_p u = 0 & in\ \Omega_n \\ u - h \in H_0^{1,p}(\Omega_n). \end{cases} \tag{1.3}$$

More precisely, the admissible class of domains is the following (for the definition of capacity see section 2):

$$\mathcal{W}_w(B) = \{\Omega \subseteq B : \forall x \in \partial\Omega, \forall 0 < r < R < 1;$$
$$\int_r^R \Big(\frac{C_p(\Omega^c \cap \overline{B}_{x,t}; B_{x,2t})}{C_p(\overline{B}_{x,t}; B_{x,2t})}\Big)^{\frac{1}{p-1}} \frac{dt}{t} \geq w(r, R, x)\}$$

where $B_{x,t}$ is the ball with center x and radius t, and

$$w : (0,1) \times (0,1) \times B \to I\!R^+$$

is such that:

1. $\lim_{r\to 0} w(r, R, x) = +\infty$, locally uniformly on x
2. w is lower semi continuous in the third variable.

We establish the following result.

Theorem 1.1 *Let $\Omega_n \in \mathcal{W}_w(B)$, and assume Ω_n converges in the Hausdorff complementary topology to Ω. Then $\Omega \in \mathcal{W}_w(B)$ and $\Omega_n \xrightarrow{\gamma_p} \Omega$.*

As a consequence, we prove a γ_p-compactness result in N dimensions and for $p \in]N-1, N]$ analogous to Sverak's result [17]. In this case, continuity and compactness are obtained by limiting the number of the connected components of the complementaries of admissible domains. Let us denote by

$$\mathcal{O}_l(B) = \{\Omega \subseteq B \quad | \quad \sharp\Omega^c \leq l\}$$

where by $\sharp$ one denotes the number of connected components.

Theorem 1.2 *Let $N \geq p > N-1$. Consider $\{\Omega_n\}_{n\in I\!N} \subseteq \mathcal{O}_l(B)$ and assume Ω_n converges in the Hausdorff complementary topology to Ω. Then $\Omega \subseteq \mathcal{O}_l(B)$ and Ω_n γ_p-converges to Ω.*

Let us remark firstly that one can not find an admissible function w such that $\mathcal{O}_l(B)$ is contained $\mathcal{W}_w(B)$. In theorem 1.1 the value of p is arbitrary between 1 and ∞ while in theorem 1.2, p belongs to $(N-1, N]$. In fact, for $p > N$ any sequence of open sets which converges in the Hausdorff complementary topology is γ_p-convergent. This follows quite simple from the embedding theorem of $W_0^{1,p}(B) \subseteq W^{\varepsilon,\infty}(B)$ and from the characterization of $W_0^{1,p}(\Omega)$. In this way, theorems 1.1 and 1.2 are interesting only for $p \leq N$.

These continuity results expressed for the p-Laplacian, can be easily extended to a more general class of monotone operators, by classical arguments involving the convergence in the sense of Mosco of the associated Sobolev spaces (see sections 2 and 6). In [9], [10], are given necessary and sufficient conditions for the γ_p-convergence which are expressed in terms of the local behavior in capacity. Unfortunately, these results can not be used in the study of the compactness-continuity problem we set in this paper.

2 Notation and Preliminaries.

We use standard notation for Sobolev spaces. For a fixed bounded open subset Ω of $I\!R^N$, $n \geq 2$, and two real numbers $1 < p < +\infty$, $1 < q < +\infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{D}'(\Omega)$, $H_0^{1,p}(\Omega)$, $H^{-1,q}(\Omega)$, are respectively the space of distributions , the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^{1,p}(\Omega)$, and the dual space of $H_0^{1,p}(\Omega)$.

There are many ways to define the Sobolev capacity, more or less equivalent. We shall use the local (condenser) capacity defined in the following way:

Definition 2.1 *For a compact set K contained in an arbitrary ball B,*

$$C_p(K, B) := \inf\{\int_B |D\varphi|^p dx, \quad \varphi \in C_0^\infty(B), \varphi \geq 1 \quad \text{on } K\}.$$

For the open sets contained in B the condenser capacity is defined by an interior approximating procedure by compact sets (see [13]), and for arbitrary sets by an exterior approximating procedure by open sets.

It is said that a property holds *p-quasi everywhere* (abbreviated as p-q.e.) if it holds outside a set of p- capacity zero. It is said that a property holds *almost everywhere* (abbreviated a.e.) if it holds outside a set of Lebesgue measure zero.

A function u is said *p-quasi-continuous* if for any $\epsilon > 0$ there exists an open set A_ϵ such that $C_p(A_\epsilon, B) < \epsilon$ and u is continuous in $\Omega \setminus A_\epsilon$. We recall that any function $u \in H^{1,p}(\Omega)$ has a unique (up to a set of p-capacity zero) p-quasi continuous representative.

Let us recall the following results from [2] respectively [13]:

Theorem 2.2 *If $u \in H^{1,p}(\mathbb{R}^N)$, then $u|_\Omega \in H_0^{1,p}(\Omega)$ if and only if $u = 0$ p-q.e. on Ω^c for a p-quasi-continuous representative.*

Theorem 2.3 *Let Ω be a bounded open subset of $\mathbb{R}^N$, and $u \in H^{1,p}(\Omega)$. If $u = 0$ a.e. in Ω, then $u = 0$ p-q.e. in Ω.*

For the properties of quasi-continuous representatives of Sobolev functions we refer to [18] and [13].

In order to speak about "continuity", we have to fix a topology on the space of the open subsets of B.

On the family of the open subsets of B, we define the Hausdorff complementary topology, denoted by H^c, given by the metric:

$$d_{H^c}(\Omega_1, \Omega_2) = \sup_{x \in \mathbb{R}^n} |d(x, \Omega_1^c) - d(x, \Omega_2^c)|.$$

The H^c-topology has some good properties, namely the space of the open subsets of B is compact, and if $\Omega_n \xrightarrow{H^c} \Omega$, then for any compact $K \subset\subset \Omega$ we have $K \subset\subset \Omega_n$ for n large enough. Perturbations in this topology may be very irregular, and in a general situation the continuity of the mapping $\Omega \to u_{\Omega,f,0}$ fails.

We prove in this paper the continuity results first for the p-Laplacian (γ_p-convergence), and in a second step, we prove that the γ_p-convergence implies the weak continuity for the solution of a more general Dirichlet problem associated with a monotone operator (see section 6). For this reason we introduce this larger class of operators and, in the next section, we prove for them some preliminary results necessary for the continuity, which are obviously true for the p-Laplacian.

Assume that $a : I\!R^N \times I\!R^N \to I\!R^N$, $n \geq 2$, satisfies :

$$\text{for every } \zeta \in I\!R^N \text{ the function } a(\cdot,\zeta) \text{ is measurable} \tag{2.1}$$

$$\text{for a.e. } x \in I\!R^N \text{ the function } a(x,\cdot) \text{ is continuous} \tag{2.2}$$

$$a(x,t\zeta) = |t|^{p-2}ta(x,\zeta), \quad t \in I\!R, t \neq 0. \tag{2.3}$$

For the monotonicity assumptions on $a(x,\zeta)$ we suppose as usual (see for instance [12]) that there exist two constants c_0, c_1 with $0 < c_0 \leq c_1 < \infty$ such that, for a.e. $x \in I\!R^N$ and for every $\zeta_1, \zeta_2 \in I\!R^N$ we have:

in the case $2 \leq p < +\infty$

$$(a(x,\zeta_1) - a(x,\zeta_2), \zeta_1 - \zeta_2) \geq c_0|\zeta_1 - \zeta_2|^p \tag{2.4}$$

$$|a(x,\zeta_1) - a(x,\zeta_2)| \leq c_1(|\zeta_1| + |\zeta_2|)^{p-2}|\zeta_1 - \zeta_2|, \tag{2.5}$$

in the case $1 < p \leq 2$

$$(a(x,\zeta_1) - a(x,\zeta_2), \zeta_1 - \zeta_2) \geq c_0(|\zeta_1| + |\zeta_2|)^{p-2}|\zeta_1 - \zeta_2|^2 \tag{2.6}$$

$$|a(x,\zeta_1) - a(x,\zeta_2)| \leq c_1|\zeta_1 - \zeta_2|^{p-1}. \tag{2.7}$$

In particular (2.3)-(2.7) imply that for a.e. $x \in I\!R^N$ and for any $\zeta \in I\!R^N$:

$$(a(x,\zeta),\zeta) \geq c_0|\zeta|^p \tag{2.8}$$

$$|a(x,\zeta)| \leq c_1|\zeta|^{p-1}. \tag{2.9}$$

Let us fix a ball B in $I\!R^N$. By the assumptions made on $a(x,\zeta)$ the operator $Au = -div\big(a(x,Du)\big)$ turns out to be continuous and strongly monotone from $H_0^{1,p}(B)$ into its dual $H^{-1,q}(B)$ via the pairing:

$$< Au, v >= \int_B a(x,Du) \cdot Dv dx \qquad \forall u, v \in H_0^{1,p}(B). \tag{2.10}$$

If $a(x,\zeta) = |\zeta|^{p-2}\zeta$ then A is the p-Laplace operator

$$-\Delta_p u = -div(|Du|^{p-2}Du).$$

In this case (2.4)-(2.7) are satisfied with $c_0 = 2^{2-p}$ and $c_1 = p - 1$ for $p \geq 2$, and $c_0 = 1$, $c_1 = 2^{2-p}$ for $p \leq 2$.

The scheme of the proof of the continuity is as follows. We are interested in equation (1.2)and consider a sequence of open sets $\{\Omega_n\}_{n\in I\!N}$ converging in the H^c-topology to some open set Ω.

1. The sequence of solutions of equation (1.2) on Ω_n (with a general operator) is bounded in $H_0^{1,p}(B)$ and any weak limit term of a subsequence satisfies equation (1.2) on Ω.

2. For the p-Laplacian it is sufficient to study the continuity for equation (1.3).

3. Using capacity density conditions, we prove that the limit term satisfies the boundary condition on Ω and hence it is solution on Ω.

4. The passage from the p-Laplacian to a general operator in divergence form is obtained using the Mosco convergence.

3 Some estimates for the solution and passage to the limit for the moving domain

The estimations and the passage to the limit, will be done in this section directly for a general operator. Let us consider a mapping $a(x, \zeta)$ satisfying (2.1)-(2.9). Fix a ball B in $I\!R^N$, and let A be the operator defined by (2.10). Fix also $f \in H^{-1,q}(B)$, and $h \in H_0^{1,p}(B)$. Then for any open subset Ω of B we consider the following Dirichlet problem: find u such that

$$\begin{cases} Au = f \ in \ \Omega \\ u \in H^{1,p}(\Omega), \ u - h \in H_0^{1,p}(\Omega). \end{cases} \tag{3.1}$$

In the weak sense this means

$$\int_\Omega a(x, Du) D\varphi dx =< f, \varphi > \quad \forall \varphi \in \mathcal{D}(\Omega),$$

where $<,>$ denotes the duality pairing between $H^{-1,q}(\Omega)$ and $H_0^{1,p}(\Omega)$. Following [12] one has for problem (3.1) existence and uniqueness results and some estimates for the solution in terms of the data f and h. From these estimates it follows a boundedness result for the solution of (3.1) which is uniform with respect to Ω, and a result on the continuous dependence of the solution of (3.1) on f and h.

Let us denote by $u_{\Omega,f,h}$ the solution of (3.1). Then $u_{\Omega,f,h} \in H^{1,p}(\Omega)$, and since $u_{\Omega,f,h} - h \in H_0^{1,p}(\Omega)$ we can extend the function $u_{\Omega,f,h}$ by h outside Ω to an element that belongs to $H_0^{1,p}(B)$. This extension will be denoted by $\tilde{u}_{\Omega,f,h}$.

With this definition, we have:

$$\|\tilde{u}_{\Omega,f,h}\|^p_{H_0^{1,p}(B)} = \int_\Omega |Du_{\Omega,f,h}|^p dx + \int_{B\setminus\Omega} |Dh|^p dx.$$

In [12], the "relaxed Dirichlet problem" is considered, with the presence of a measure μ. In the present paper we consider the Dirichlet problem on domains (hence with a particular measure). Therefore the estimates on the solutions from [12] can be used also for our problem (3.1).

Proposition 3.1 *For every $f \in H^{-1,q}(B)$, and $h \in H_0^{1,p}(B)$ the problem (3.1) has a unique solution $u_{\Omega,f,h}$ which satisfies the estimate:*

$$\int_\Omega |Du_{\Omega,f,h}|^p dx \leq C(\|f\|^q_{H^{-1,q}(B)} + \int_B |Dh|^p dx) \tag{3.2}$$

where C is a constant depending only on p, c_0, c_1.

Proof For the proof we refer to [12], theorem 2.1. □

Using the result above we can prove a boundedness result, uniformly with respect to Ω, for the function $\tilde{u}_{\Omega,f,h}$. In fact:

Corollary 3.2 *For every $\Omega \subseteq B$ let $u_{\Omega,f,h}$ be the solution of (3.1). Then:*

$$\|\tilde{u}_{\Omega,f,h}\|_{H_0^{1,p}(B)} \leq C$$

where C depends on f, h, c_0, c_1, p.

Proof Setting $u = u_{\Omega,f,h}$, from proposition 3.1 we have:

$$\int_\Omega |Du|^p dx \leq C(\|f\|^q_{H^{-1,q}(B)} + \int_B |Dh|^p dx).$$

Adding $\int_{B\setminus\Omega} |Dh|^p dx$ to both sides, we have :

$$\|\tilde{u}_{\Omega,f,h}\|^p_{H_0^{1,p}(B)} \leq \tilde{C}(\|f\|^q_{H^{-1,q}(B)} + \int_B |Dh|^p dx)$$

which implies the conclusion. □

The following lemma will be used to prove the continuous dependence of the solution of (3.1) on f and h . The lemma is true for any pair of functions u_1 and $u_2 \in H^{1,p}(\Omega)$. We shall use this result in the case of two solutions of (3.1) on the same Ω with respect to different pairs of data (f, h).

Lemma 3.3 *Let $u_1, u_2 \in H^{1,p}(\Omega)$. If $2 \leq p < +\infty$, then:*

$$c_0 \int_\Omega |Du_1 - Du_2|^p dx \leq \int_\Omega (a(x, Du_1) - a(x, Du_2), Du_1 - Du_2) dx. \quad (3.3)$$

If $1 < p \leq 2$, then:

$$c_0 (\int_\Omega |Du_1 - Du_2|^p dx)^{\frac{2}{p}} \leq \quad (3.4)$$

$$\leq K(u_1, u_2) \int_\Omega (a(x, Du_1) - a(x, Du_2), Du_1 - Du_2) dx$$

where $K(u_1, u_2)$ stands for :

$$K(u_1, u_2) = 2(\int_\Omega |Du_1|^p dx + \int_\Omega |Du_2|^p dx)^{\frac{2-p}{p}}.$$

Proof For the proof we refer to [12], lemma 2.2. □

Fix now $f_1, f_2 \in H^{-1,q}(B)$ and $h_1, h_2 \in H_0^{1,p}(B)$. By $u_1 = u_{\Omega, f_1, h_1}$ and $u_2 = u_{\Omega, f_2, h_2}$ we denote the solutions of (3.1) on the same Ω referring to the data (f_1, h_1) and (f_2, h_2) respectively.

The following result holds:

Lemma 3.4 *Fix f_1, f_2, h_1, h_2 as above. Let u_1, u_2 be the corresponding solutions of (3.1). If $2 \leq p < +\infty$, then:*

$$c_0 \int_\Omega |Du_1 - Du_2|^p dx \leq K(\|Dh_1 - Dh_2\|_{L^p(B)} + \|f_1 - f_2\|_{H^{-1,q}(B)}). \quad (3.5)$$

If $1 < p \leq 2$, then :

$$c_o (\int_\Omega |Du_1 - Du_2|^p dx)^{\frac{2}{p}} \leq \quad (3.6)$$

$$\leq KK(u_1, u_2)(\|Dh_1 - Dh_2\|_{L^p(B)} + \|f_1 - f_2\|_{H^{-1,q}(B)}),$$

where K is a constant depending on $\|f_1\|_{H^{-1,q}(B)}$, $\|f_2\|_{H^{-1,q}(B)}$, $\|h_1\|_{H_0^{1,p}(B)}$, $\|h_2\|_{H_0^{1,p}(B)}$, c_0, c_1, p, and $K(u_1, u_2)$ is the term in lemma 3.3.

Proof Writing (3.1) for u_1 and u_2 and the test functions $\varphi = u_1 - h_1$ and $\varphi = u_2 - h_2$, by simple subtractions we have:

$$\begin{aligned} &\int_\Omega (a(x, Du_1) - a(x, Du_2), Du_1 - Du_2)dx = \\ &= \int_\Omega (a(x, Du_1) - a(x, Du_2), Dh_1 - Dh_2)dx + \\ &+ < f_1 - f_2, u_1 - h_1 > - < f_1 - f_2, u_2 - h_2 >= I + II + III. \end{aligned} \tag{3.7}$$

Let us consider the three terms I,II, and III separately. Let us begin with I. Using (2.9) and the Holder's inequality we have:

$$\begin{aligned} &\int_\Omega (a(x, Du_1) - a(x, Du_2), Dh_1 - Dh_2)dx \leq \\ &\leq \int_\Omega |a(x, Du_1)||Dh_1 - Dh_2|dx + \int_\Omega |a(x, Du_2)||Dh_1 - Dh_2|dx \leq \\ &\leq c_1(\int_\Omega |Du_1|^{p-1}|Dh_1 - Dh_2|dx + \int_\Omega |Du_2|^{p-1}|Dh_1 - Dh_2|dx) \leq \\ &\leq c_1 \sum_{i=1}^{2} (\int_\Omega |Du_i|^p dx)^{\frac{1}{q}} (\int_\Omega |Dh_1 - Dh_2|^p dx)^{\frac{1}{p}}. \end{aligned}$$

Now, from theorem 3.1, we have for $i = 1, 2$:

$$(\int_\Omega |Du_i|^p dx)^{\frac{1}{q}} \leq c(\|f_i\|^q_{H^{-1,q}(B)} + \int_B |Dh_i|^p dx)^{\frac{1}{q}}.$$

Therefore there exists a constant α_0 such that :

$$I \leq \alpha_0 \|Dh_1 - Dh_2\|_{L^p(\Omega)} \leq \alpha_0 \|Dh_1 - Dh_2\|_{L^p(B)}, \tag{3.8}$$

where α_0 depends on $f_1, f_2, h_1, h_2, c_0, c_1, p$.

Let us consider II. From the Cauchy-Schwarz's inequality we have :

$$\begin{aligned} &II \leq \|f_1 - f_2\|_{H^{-1,q}(B)} \|u_1 - h_1\|_{H_0^{1,p}(B)} \leq \\ &\leq \|f_1 - f_2\|_{H^{-1,q}(B)} (\|\tilde{u}_1\|_{H_0^{1,p}(B)} + \|h_1\|_{H_0^{1,p}(B)}), \end{aligned}$$

and from corollary 3.2 we obtain:

$$II \leq \alpha_1 \|f_1 - f_2\|_{H^{-1,q}(B)}, \tag{3.9}$$

where α_1 depends on f_1, h_1, c_0, c_1, p.

Let us consider now III. Using the Cauchy Schwarz's inequality, as in the previous case, we find:

$$III \leq \alpha_2 \|f_1 - f_2\|_{H^{-1,q}(B)} \tag{3.10}$$

where α_2 depends on f_2, h_2, c_0, c_1, p. Now, from (3.7)-(3.10), we have:

$$\begin{aligned} &\int_\Omega (a(x, Du_1) - a(x, Du_2), Du_1 - Du_2)dx \leq \\ &\leq K(\|Dh_1 - Dh_2\|_{L^p(B)} + \|f_1 - f_2\|_{H^{-1,q}(B)}), \end{aligned} \tag{3.11}$$

where $K = \max(\alpha_0, \alpha_1, \alpha_2)$. If $p \geq 2$, from (3.3) and (3.11) we obtain :

$$c_0 \int_\Omega |Du_1 - Du_2|^p dx \leq K(\|Dh_1 - Dh_2\|_{L^p(B)} + \|f_1 - f_2\|_{H^{-1,q}(B)}),$$

while, if $1 < p \leq 2$, from (3.4) and (3.11) we obtain :

$$c_0(\int_\Omega |Du_1 - Du_2|^p dx)^{\frac{2}{p}} \leq K(u_1, u_2)K(\|Dh_1 - Dh_2\|_{L^p(B)} + \|f_1 - f_2\|_{H^{-1,q}(B)}).$$

This concludes the proof. □

Remark: Let us point out that if $h_n \to h$ in $H_0^{1,p}(B)$ and $f_n \to f$ in $H^{-1,q}(B)$, then $\alpha_0, \alpha_1, \alpha_2$ can be chosen independently of n and bounded. So the constant K in lemma 3.4 is uniformly bounded, and, using theorem 3.1, the same is true for $K(u_n, u)$.

From lemma 3.4 and the above remark we can prove the continuous dependence on f and h of the solution $u_{\Omega,f,h}$ which is uniform with respect to Ω.

Theorem 3.5 *If $h_n \to h$ strongly in $H_0^{1,p}(B)$ and $f_n \to f$ strongly in $H^{-1,q}(B)$, then:*

$$\tilde{u}_{\Omega,f_n,h_n} \to \tilde{u}_{\Omega,f,h}$$

uniformly with respect to Ω.

Proof We put $\tilde{u}_n = \tilde{u}_{\Omega,f_n,h_n}$ and $\tilde{u} = \tilde{u}_{\Omega,f,h}$. We recall that:

$$\|\tilde{u} - \tilde{u}_n\|^p_{H_0^{1,p}(B)} = \|Du - Du_n\|^p_{L^p(\Omega)} + \|Dh - Dh_n\|^p_{L^p(B\setminus\Omega)}.$$

If $p \geq 2$ from (3.5) we have:

$$\|\tilde{u}-\tilde{u}_n\|^p_{H_0^{1,p}(B)} \leq \tilde{K}(\|Dh-Dh_n\|_{L^p(B)}+\|Dh-Dh_n\|^p_{L^p(B\setminus\Omega)}+\|f-f_n\|_{H^{-1,q}(B)}),$$

and since $h_n \to h$ strongly $H_0^{1,p}(B)$ and $f_n \to f$ strongly $H^{-1,q}(B)$, we have:

$$\|\tilde{u}-\tilde{u}_n\|^p_{H_0^{1,p}(B)} \leq \varepsilon.$$

If $1 < p \leq 2$, from (3.6) we obtain:

$$\|\tilde{u}-\tilde{u}_n\|^2_{H_0^{1,p}(B)} \leq M(\|Dh-Dh_n\|_{L^p(B)}+\|Dh-Dh_n\|^2_{L^p(B\setminus\Omega)}+\|f-f_n\|_{H^{-1,q}(B)}),$$

where M is a constant which does not depend on n (see the above remark), and as in the previous case we obtain:

$$\|\tilde{u}-\tilde{u}_n\|^2_{H_0^{1,p}(B)} \leq \varepsilon.$$

$\square$

These estimations for the solutions, allow us to consider the moving domain case. Classically, if Ω_n converges to Ω in the Hausdorff complementary topology, the continuity is achieved by proving that any weak convergent subsequence tends to the solution of the equation on Ω .

A function is solution on Ω if it satisfies the equation in the distributional sense and satisfies the boundary condition. Let us consider a sequence of domains, which is convergent in the Hausdorff complementary topology. We begin with the passage to the limit in the equation.

Proposition 3.6 *Let $\{\Omega_n\}_{n\in I\!N}, \Omega \subseteq B$ and $\Omega_n \xrightarrow{H^c} \Omega$. Then $\|\tilde{u}_{\Omega_n,f,h}\|_{H_0^{1,p}(B)}$ is uniformly bounded, and if $\tilde{u}_{\Omega_n,f,h} \rightharpoonup u$, then:*

$$\int_\Omega a(x, Du)D\varphi dx =< f, \varphi > \ \forall \varphi \in \mathcal{D}(\Omega).$$

Proof Denote $u_n = u_{\Omega_n,f,h}$. In order to obtain the equality above it suffices to prove that:

$$a(x, Du_n) \overset{L^q(\Omega)}{\rightharpoonup} a(x, Du).$$

Indeed, let us consider a test function $\varphi \in \mathcal{D}(\Omega)$; then $supp\varphi = K \subset\subset \Omega$ and, since $\Omega_n \xrightarrow{H^c} \Omega$, we have that $K \subset\subset \Omega_n$, for n large enough. Then $\varphi \in \mathcal{D}(\Omega_n)$ and since u_n is a solution in Ω_n, we can write:

$$\int_{\Omega_n} a(x, Du_n) D\varphi dx =< f, \varphi > .$$

Consider the following set $K^\varepsilon = \{x \in I\!R^n : d(x, K) < \varepsilon\}$ with ε small enough such that $K^\varepsilon \subset\subset \Omega$; if n is large enough, we still have $K^\varepsilon \subset\subset \Omega_n$. Let us now prove that:

$$a(x, Du_n) \overset{L^q(K^{\frac{\varepsilon}{2}})}{\rightharpoonup} a(x, Du).$$

Fix $g \in C_0^\infty(\Omega)$ such that $g = 1$ on $K^{\frac{\varepsilon}{2}}$, and $g = 0$ on $\text{ext}(K^\varepsilon)$, and $0 \leq g \leq 1$ on K^ε, and consider the test function

$$\phi_n := g(u_n - u) \in H_0^{1,p}(K^\varepsilon).$$

If we write the equation for u_n in Ω_n with the test function ϕ_n we have

$$\int_{K^\varepsilon} a(x, Du_n) D(g(u_n - u)) dx =< f, g(u_n - u) > .$$

Performing the derivative of the test function in the right hand side we get

$$\int_{K^\varepsilon} a(x, Du_n) g D(u_n - u) dx =$$

$$=< f, g(u_n - u) > - \int_{K^\varepsilon} a(x, Du_n)(u_n - u) Dg dx \leq$$

$$\leq < f, g(u_n - u) > + (\int_B |a(x, D\tilde{u}_n)|^q dx)^{\frac{1}{q}} (\int_B |Dg(\tilde{u}_n - u)|^p dx)^{\frac{1}{p}}.$$

Remark now that the first term of the right hand side goes to zero for $n \to \infty$, at least for a subsequence still denoted with the same index, since $gu_n \rightharpoonup gu$ in $H^{1,p}(K^\varepsilon)$. The second term is the product of a uniformly bounded sequence (by the assumption on $a(x, \zeta)$) and of a vanishing term since $\tilde{u}_n \to u$ in $L^p(B)$ (by the uniformly $H_0^{1,p}(B)$ boundedness of the solution $\tilde{u}_n$). Finally we conclude with the following inequality:

$$\limsup_{n\to\infty} \int_{K^\varepsilon} a(x, Du_n) g D(u_n - u) dx \leq 0.$$

On the other hand, we get :

$$\int_{K^\varepsilon} a(x, Du) g D(u_n - u) dx \longrightarrow 0, \quad \text{as} \quad n \to \infty$$

since $Du_n \overset{L^q(K^\varepsilon)}{\rightharpoonup} Du$; therefore, by subtracting, we have

$$\limsup_{n\to\infty} \int_{K^\varepsilon} g[a(x, Du_n) - a(x, Du)] D(u_n - u) dx \leq 0.$$

By the monotonicity assumptions on $a(x, \zeta)$ and the positivity of g we find

$$g[a(x, Du_n) - a(x, Du)] D(u_n - u) \geq 0 \quad \forall x \in K^\varepsilon,$$

and hence:

$$\lim_{n\to\infty} \int_{K^\varepsilon} g[a(x, Du_n) - a(x, Du)] D(u_n - u) dx = 0.$$

By the positivity of the integrand and by the equality $g = 1$ on $K^{\frac{\varepsilon}{2}}$ we have

$$\lim_{n\to\infty} \int_{K^{\frac{\varepsilon}{2}}} [a(x, Du_n) - a(x, Du)] D(u_n - u) dx = 0.$$

From [13], lemma 3.73, we get that $a(x, Du_n) \overset{L^q(K^{\frac{\varepsilon}{2}})}{\rightharpoonup} a(x, Du)$, so that, we can write

$$\int_{K^{\frac{\varepsilon}{2}}} a(x, Du_n) D\varphi dx = < f, \varphi >,$$

where φ is the chosen test function. If we compute the limit for $n \to \infty$ we have

$$\int_{K^{\frac{\varepsilon}{2}}} a(x, Du) D\varphi dx = < f, \varphi > .$$

Since $supp\varphi = K \subset\subset K^{\frac{\varepsilon}{2}}$ the previous equality holds in Ω. As φ was arbitrarily chosen, u satisfies the equation on Ω in the sense of the distributions.

□

4 Behavior on the boundary of the limit term for the p-Laplacian

This section is devoted to the p-Laplacian, i.e. $a(x, \xi) = |\xi|^{p-2}\xi$, $1 < p < +\infty$, by $\tilde{u}_{\Omega,f,h}$ denoting the solution of (1.2). We want to prove the γ_p-convergence result, so in order to obtain the continuity of the map $\Omega \to u_{\Omega,f,0}$, from proposition 3.6, we have just to prove that $u|_\Omega \in H_0^{1,p}(\Omega)$, and hence u will be the unique solutions of the equation on Ω. This assertion is generally not true, and we refer to [12] for simple counterexamples. Remark that the hypothesis of H^c-convergence in the family of open sets is not sufficient to prove the continuity of the mapping $\Omega \to u_{\Omega,f,0}$.

In order to obtain a continuity result, we have to impose some constraints on the moving domain. For example, as in the linear case (see [14]) if Ω_n *converges compactly* to Ω (i.e. for any compact sets $K_1 \subset\subset \Omega$, and $K_2 \subset\subset ext(\Omega)$ there exists $\overline{n}$ such that for all $n \geq \overline{n}$ we have $K_1 \subset\subset \Omega_n$, $K_2 \subset\subset ext(\Omega_n)$) and Ω is *p-stable* (i.e. if for any $u \in H^{1,p}(\mathbb{R}^N)$ such that $u = 0$ a.e. on $int(\Omega^c)$, we get $u|_\Omega \in H_0^{1,p}(\Omega)$) then the shape continuity follows (the proof is the same as in the linear case, see [14]).

We notice that an arbitrary open set Ω is not generally p-stable and the approach of a Sobolev function through Lebesgue measures is not sufficiently precise. Let us think, for example, of an $H^{1,p}$ function on a domain which has a crack; even if it is of zero measure, the function has a trace on the crack, making it not negligible.

For this reason we shall consider the more general class of domains which are not p-stable. This class is defined in section 1 and is denoted by $\mathcal{W}_w(B)$.

Lemma 4.1 *Let Ω_n,Ω open set $\subseteq B$. If:*

$$\tilde{u}_{\Omega_n,0,h} \longrightarrow \tilde{u}_{\Omega,0,h} \quad \forall h \in H_0^{1,p}(B)$$

then:

$$\tilde{u}_{\Omega_n,f,0} \longrightarrow \tilde{u}_{\Omega,f,0} \quad \forall f \in H^{-1,q}(B).$$

Proof For simplicity we write $v_n = u_{\Omega_n,0,h}$, $v = u_{\Omega,0,h}$, and $u_n = u_{\Omega_n,f,0}$, $u = u_{\Omega,f,0}$. We have $u \in H_0^{1,p}(\Omega) \subseteq H_0^{1,p}(B)$.

From the boundedness of the function $\tilde{u}_n$, we have (up to passing to a subsequence):

$$\tilde{u}_n \rightharpoonup w \quad \text{in} \quad H_0^{1,p}(B).$$

We prove that $w = u$. We consider the new problem:

$$\begin{cases} -\Delta_p \varphi_n = 0 \; in \; \Omega_n \\ \quad \varphi_n = -w \; in \; B \setminus \Omega_n. \end{cases} \tag{4.1}$$

Taking in (4.1) as test function $\tilde{\varphi}_n + w - \tilde{u}_n \in H_0^{1,p}(\Omega_n)$ we have:

$$\int_B |D\tilde{\varphi}_n|^{p-2} D\tilde{\varphi}_n D(\tilde{\varphi}_n + w - \tilde{u}_n)dx = 0.$$

Now, denoting by φ the solution of (4.1) on Ω, we have that $D(\tilde{\varphi}_n + w - \tilde{u}_n) \rightharpoonup D\tilde{\varphi}$ in $L^p(B)$ and $|D\tilde{\varphi}_n|^{p-2} D\tilde{\varphi}_n \to |D\tilde{\varphi}|^{p-2} D\tilde{\varphi}$ in $L^q(B)$ (this comes from the convergence of the norms in $L^q(B)$ and the a.e. pointwise convergence which derives from $D\tilde{\varphi}_n \to D\tilde{\varphi}$ in $L^p(B)$), so passing in the limit we find:

$$\int_B |D\tilde{\varphi}|^{p-2} D\tilde{\varphi} D\tilde{\varphi} dx = 0$$

which means $\tilde{\varphi} = 0$. But $\tilde{\varphi} = -w$ in $B \setminus \Omega$, so $w = 0$ in $B \setminus \Omega$. We have just proved that the weak limit w satisfy the boundary condition, and from proposition 3.6, the equation on the limit set Ω, hence $w = u$ from the uniqueness of the solution.
We have just proved that $\tilde{u}_n \rightharpoonup \tilde{u}$. To conclude, it is sufficient to prove the convergence of the norms $\|\tilde{u}_n\|_{H_0^{1,p}(B)} \to \|\tilde{u}\|_{H_0^{1,p}(B)}$ which implies the strong convergence (see [1]). This property is related to the uniform convexity of the space $H_0^{1,p}(B)$ endowed with the L^p-norm of the gradient. Using the equation on Ω_n and on Ω we have:

$$\begin{aligned} \lim_{n\to+\infty} \|\tilde{u}_n\|^p_{H_0^{1,p}(B)} &= \lim_{n\to+\infty} < -\triangle_p \tilde{u}_n, \tilde{u}_n >= \lim_{n\to+\infty} < f, \tilde{u}_n >= \\ &=< f, \tilde{u} >=< -\triangle_p \tilde{u}, \tilde{u} >= \|\tilde{u}\|^p_{H_0^{1,p}(B)}. \end{aligned} \tag{4.2}$$

which concludes the proof. $\square$

The lemma above suggests us to study the oscillation of the solution of the problem

$$\begin{cases} -\Delta_p u = 0 & in\ \Omega \\ \quad u = h & in\ B \setminus \Omega \end{cases} \tag{4.3}$$

near the boundary, for smooth functions h. We recall first the following theorem from [13].

Theorem 4.2 *Suppose that Ω is bounded. Let $h \in H^{1,p}(\Omega) \cap C^(\overline{\Omega})$ and let u be the solution of the problem (4.3). If $y \in \partial\Omega$, then $\forall 0 < r \leq R$ it is true that*

$$osc(u, \Omega \cap B_{y,r}) \leq osc(h, \partial\Omega \cap \overline{B}_{y,2R}) +$$

$$+osc(h,\partial\Omega)\exp\Big(-c\int_r^R\Big(\frac{C_p(\Omega^c\cap\overline{B}_{y,t};B_{y,2t})}{C_p(\overline{B}_{y,t};B_{y,2t})}\Big)^{\frac{1}{p-1}}\frac{dt}{t}\Big)$$

where c is a fixed constant and

$$osc(u,A)=\sup_A u(x)-\inf_A u(x).$$

If h is a fixed function in $C_0^\infty(B)$ and $\Omega\in\mathcal{W}_w(B)$ there exists some constant M such that $\forall x,z\in\Omega\cap B_{y,r}$

$$|u(x)-u(z)|\le MR+M\exp(-cw(r,R,y)).$$

Since y is a regular point (by hypothesis $\Omega\in\mathcal{W}_w(B)$ and $\lim_{r\to0}w(r,R,y)=\infty$) we have that

$$\lim_{z\to y,z\in\Omega\cap B_{y,r}}u(z)=h(y).$$

Then we can write

$$|u(x)-h(y)|\le MR+M\exp(-cw(r,R,y)).$$

Hence

$$|u(x)-h(x)|\le|h(x)-h(y)|+MR+M\exp(-cw(r,R,y)),$$

or

$$|u(x)-h(x)|\le 2MR+M\exp(-cw(r,R,y)) \tag{4.4}$$

We are now in a position to prove theorem 1.1.

Proof of theorem 1.1. The family $\mathcal{W}_w(B)$ is compact in the Hausdorff topology. The nonlinear case can be proved as the linear one (see [4] . Therefore Ω belongs to $\mathcal{W}_w(B)$. ¿From Lemma 4.1 it is sufficient to prove that $\tilde{u}_{\Omega_n,0,h}\longrightarrow\tilde{u}_{\Omega,0,h}$, for every $h\in H_0^{1,p}(B)$.
STEP 1: Let $h\in C_0^\infty(B)$; we start by proving that $\tilde{u}_{\Omega_n,0,h}\rightharpoonup\tilde{u}_{\Omega,0,h}$.
Setting for simplicity $u_n=u_{\Omega_n,0,h}$, we have for a subsequence, still denoted with the same indices that $\tilde{u}_n\rightharpoonup u$ in $H_0^{1,p}(B)$. ¿From proposition 3.6 we know that u satisfies the equation on Ω, so it is sufficient to prove that $u=h$

p q.e. on $B\setminus\Omega$. From the Banach-Saks theorem, we can construct a sequence of convex combinations

$$\psi_n = \sum_{k=n}^{N_n} \alpha_k^n \tilde{u}_k, \quad with \quad \sum_{k=n}^{N_n} \alpha_k^n = 1 \tag{4.5}$$

such that

$$\psi_n(x) \to u(x) \ \ p\ q.e. \ \ on\ B. \tag{4.6}$$

So let us consider a point $x \in B \setminus \Omega$ where (4.6) holds. For any $\varepsilon > 0$ we shall prove that $|u(x) - h(x)| \leq \varepsilon$ which implies $u(x) = h(x)$ p q.e. on $B \setminus \Omega$. From (4.5) we can write

$$|u(x) - h(x)| \leq \sum_{k=n}^{N_n} \alpha_k^n |\tilde{u}_k(x) - h(x)|,$$

so it is sufficient to prove that there exists k_ε such that, $\forall k \geq k_\varepsilon$ we have

$$|\tilde{u}_k(x) - h(x)| \leq \varepsilon. \tag{4.7}$$

If $x \in B \setminus \Omega_k$ then (4.7) is true. In order to apply inequality (4.4) for the open set Ω_k we fix R_0 enough small such that

$$2MR_0 \leq \frac{\varepsilon}{4}.$$

Using the hypotheses on the function w (i.e. $\lim_{r\to 0} w(r,R,x) = \infty$, locally uniformly on x) we find a neighborhood of x, say U, and r_0 enough small such that $B_{x,r_0} \subseteq U$ and

$$M \exp(-cw(r,R,y)) \leq \frac{\varepsilon}{4}$$

for all $r \leq r_0$, and for all $y \in U$. Using the Hausdorff topology properties, there exists some k large enough, denoted k_0 such that for all $k \geq k_0$ we have that $B_{x,r_0} \cap \Omega_k^c \neq \emptyset$. If $x \in \Omega_k$, there exists some $z_k \in B_{x,r_0} \cap \partial\Omega_k$ and one can apply inequality (4.4) for u_k, r_0, R_0, z_k and obtain directly

$$|u_k(x) - h(x)| \leq \varepsilon \quad \forall k \geq k_0$$

which implies (4.7).

STEP 2: Let $h \in C_0^\infty(B)$. We prove that $\tilde{u}_{\Omega_n.0,h} \longrightarrow \tilde{u}_{\Omega,0,h}$. Setting for simplicity $u_n = u_{\Omega_n,0,h}$, we have $u_n \in H^{1,p}(\Omega_n)$ and :

$$\int_{\Omega_n} |Du_n|^{p-2} Du_n D\varphi dx = 0, \quad \forall \varphi \in H_0^{1,p}(\Omega_n).$$

We can choose $\varphi = u_n - h$ as a test function in the equality above, from which:

$$\int_{\Omega_n} |Du_n|^{p-2} Du_n Du_n dx = \int_{\Omega_n} |Du_n|^{p-2} Du_n Dh dx. \tag{4.8}$$

¿From the step above we have $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H_0^{1,p}(B)$, where $\tilde{u} = \tilde{u}_{\Omega,0,h}$. ¿From [3], theorem 2.1, we have (possibly passing to a subsequence):

$$D\tilde{u}_n \longrightarrow D\tilde{u}, \quad \text{a.e. in } B.$$

Let us prove the strong convergence of the solutions proving the convergence of the norms. Using (4.8) we have:

$$\|\tilde{u}_n\|^p_{H_0^{1,p}(B)} := \int_{\Omega_n} |Du_n|^p dx + \int_{B\setminus\Omega_n} |Dh|^p dx =$$

$$= \int_{\Omega_n} |Du_n|^{p-2} Du_n Du_n dx + \int_{B\setminus\Omega_n} |Dh|^p dx =$$

$$= \int_{\Omega_n} |Du_n|^{p-2} Du_n Dh dx + \int_{B\setminus\Omega_n} |Dh|^p dx =$$

$$= \int_B |D\tilde{u}_n|^{p-2} D\tilde{u}_n Dh dx + \int_{B\setminus\Omega_n} |Dh|^p dx - \int_{B\setminus\Omega_n} |D\tilde{u}_n|^{p-2} D\tilde{u}_n Dh dx =$$

$$= \int_B |D\tilde{u}_n|^{p-2} D\tilde{u}_n Dh dx.$$

Therefore passing to the limit via relation (4.8), we have:

$$\lim_{n\to+\infty} \|\tilde{u}_n\|^p_{H_0^{1,p}(B)} = \lim_{n\to+\infty} \int_B |D\tilde{u}_n|^{p-2} D\tilde{u}_n Dh dx = \int_B |D\tilde{u}|^{p-2} D\tilde{u} Dh dx =$$

$$= \int_{B\setminus\Omega} |Dh|^{p-2} Dh Dh dx + \int_\Omega |Du|^{p-2} Du Dh dx =$$

$$= \int_{B\setminus\Omega} |Dh|^p dx + \int_\Omega |Du|^{p-2} Du Du dx = \|\tilde{u}\|^p_{H_0^{1,p}(B)}.$$

In the previous relation the weak L^q convergence of $|D\tilde{u}_n|^{p-2}D\tilde{u}_n$ towards $|D\tilde{u}|^{p-2}D\tilde{u}$ is obtained from the weak L^pconvergence of $D\tilde{u}_n$ towards $D\tilde{u}$ and from the almost everywhere pointwise convergence via the Mazur's lemma.

Now, the weak convergence of $\tilde{u}_n$ in $H_0^{1,p}(B)$ and the norm-convergence give the strong convergence of the solutions (see [1]).

STEP 3: Fix now $h \in H_0^{1,p}(B)$. By the density we have the existence of a sequence $\{h_k\} \in C_0^\infty(B)$ such that $h_k \to h$ strongly in $H_0^{1,p}(B)$.

Then $\|h_k\|_{H_0^{1,p}(B)}$ is uniformly bounded and one can write:

$$\|\tilde{u}_{\Omega_n,0,h} - \tilde{u}_{\Omega,0,h}\|_{H_0^{1,p}(B)} \le \|\tilde{u}_{\Omega_n,0,h} - \tilde{u}_{\Omega_n,0,h_k}\|_{H_0^{1,p}(B)} +$$

$$+\|\tilde{u}_{\Omega_n,0,h_k} - \tilde{u}_{\Omega,0,h_k}\|_{H_0^{1,p}(B)} + \|\tilde{u}_{\Omega,0,h_k} - \tilde{u}_{\Omega,0,h}\|_{H_0^{1,p}(B)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},$$

where we use the continuity result of the step 2 in the case of smooth boundary data for the second term and from the continuity dependence of the solution $\tilde{u}_{\Omega_n,0,h}$ with respect to the variation of the boundary data which is uniform in Ω_n. □

Remark The continuity result of the above theorem is still true if we drop the hypothesis on the lower semi continuity of w in the third variable. This assumption is necessary only for the compactness of the family $\mathcal{W}_w(B)$ in the Hausdorff topology.

A particular class of domains contained in the family $\mathcal{W}_w(B)$ is the following one (see for the linear case [4]).

For $c, r > 0$ it is said that Ω satisfies the p–(r,c) capacity density condition if :

$$\forall x \in \partial\Omega, \forall \quad 0 < \delta < r \qquad \frac{C_p(\Omega^c \cap \overline{B}_{x,\delta}, B_{x,2\delta})}{C_p(\overline{B}_{x,\delta}, B_{x,2\delta})} \ge c.$$

By $\mathcal{O}_{p,c,r}(B)$ we denote the family of all open subsets of B which satisfy the p–(r,c) capacity density condition. This family is compact in the H^c-topology: for the proof in the case $p = 2$ we refer to [5], the general case being similar.

5 Extension of Sverak's result for $N-1 < p \leq N$

In this section we shall extend the continuity result of Sverak. In two dimensions and for the linear case, he obtained the continuity by fixing a uniform maximal number of the connected components of the complementary of the varying domain [17]. We extend this result for arbitrary N and for $p \in]N-1, N]$. The reason of the choice of p is that in $I\!R^N$ the curves have p positive capacity if $p > N-1$. The case $p > N$ is trivial since all the functions in $H^{1,p}(I\!R^N)$ are continuous.

To prove a Sverak type result we give some preliminary lemmas.

Lemma 5.1 *Let's consider a curve $\gamma_{[x,\xi]} \subseteq B_{x,r}$, such that $\xi \in \partial B_{x,r}$. Then:*

$$C_p(\gamma_{[x,\xi]}; B_{x,2r}) \geq C_p([x,\xi]; B_{x,2r})$$

where $[x,\xi]$ denotes the segment with extrema x and ξ.

Proof For some $\epsilon > 0$ let us consider a function $\varphi \in C_0^\infty(B_{x,2r}; I\!R^+)$ such that:

$$\int_{B_{x,2r}} |\nabla\varphi|^p dx \leq C_p(\gamma_{[x,\xi]}; B_{x,2r}) + \epsilon$$

and $\varphi \geq 1$ on a neighborhood U of $\gamma_{[x,\xi]}$. Then we denote by φ^* the Steiner symmetrization of φ with respect to the line $x\xi$. Then $\varphi^* \in H_0^{1,p}(B_{x,2r})$ and $\varphi^* \geq 1$ on $U^* \supseteq [x,\xi]$, and

$$\int_{B_{x,2r}} |\nabla\varphi^*|^p dx \leq \int_{B_{x,2r}} |\nabla\varphi|^p dx.$$

Since $C_p([x,\xi]; B_{x,2r}) \leq \int_{B_{x,2r}} |\nabla\varphi^*|^p dx$, making $\epsilon \to 0$ we conclude the proof. $\square$

Lemma 5.2 *Let $K \subset I\!R^N$ be a compact, connected set. Then for all $x \in K$ and $r < \frac{1}{2} diam K$ we have:*

$$\frac{C_p(K \cap \overline{B}_{x,r}; B_{x,2r})}{C_p(\overline{B}_{x,r}; B_{x,2r})} \geq \frac{C_p([0,1] \times \{0\}^{N-1}; B_{0,2})}{C_p(\overline{B}_{0,1}; B_{0,2})}.$$

Proof Let us consider a point $x \in K$ and set $K^\delta = \{x \in I\!R^N | d(x, K) < \delta\}$. Then $\overline{K^\delta} \subseteq K^{\delta+\epsilon}$ for all $\epsilon > 0$ and the property of the capacity on decreasing sequences of compacts together with its monotonicity gives:

$$C_p(K \cap \overline{B}_{x,r}; B_{x,2r}) = \lim_{\delta \to 0} C_p(K^\delta \cap \overline{B}_{x,r}; B_{x,2r}). \tag{5.1}$$

The set K^δ is open, contains K and so it is connected by curves. Since K^δ is not contained in $\overline{B}_{x,r}$ (since $r < \frac{1}{2}\mathrm{diam}K$), there exists a point $\xi \in \partial B_{x,r} \cap K^\delta$ and a continuous curve $\gamma_{[x,\xi]}$ which links x and ξ and lies in $\overline{B}_{x,r} \cap K^\delta$. To conclude the proof it is sufficient to use lemma 5.1 and the behavior of the capacity on homothetic sets. Indeed we get:

$$C_p(K^\delta \cap \overline{B}_{x,r}; B_{x,2r}) \geq C_p(\gamma_{[x,\xi]} \cap \overline{B}_{x,r}; B_{x,2r}) \geq C_p([x, \xi] \cap \overline{B}_{x,r}; B_{x,2r}).$$

Then using (5.1) and making $\delta \to 0$ we get:

$$C_p(K \cap \overline{B}_{x,r}; B_{x,2r}) \geq C_p([x, \xi] \cap \overline{B}_{x,r}; B_{x,2r}).$$

Since

$$\frac{C_p([x, \xi] \cap \overline{B}_{x,r}; B_{x,2r})}{C_p(\overline{B}_{x,r}; B_{x,2r})} = \frac{C_p([0, 1] \times \{0\}^{N-1}; B_{0,2})}{C_p(\overline{B}_{0,1}; B_{0,2})}$$

we conclude the proof. □

Remark that if $p > N - 1$ then (see for example [18]) every piece of a small curve has positive p-capacity.

Let's prove now theorem 1.2.

Proof of theorem 1.2 It is easy to see that $\mathcal{O}_l(B)$ is compact in the H^c-topology (see for instance [17]), so $\Omega \in \mathcal{O}_l(B)$. Following the same ideas of [4] we decompose $\overline{B} \setminus \Omega_n$ in its connected components

$$\overline{B} \setminus \Omega_n = K_1^n \cup ... \cup K_l^n.$$

Some of these components might be empty. Using the compactness of the Hausdorff topology, one can subtract a subsequence (still denoted with the same indices), such that

$$K_i^n \xrightarrow{H} K_i \quad \forall\, i = 1, .., k.$$

Then K_i are still compact, connected and

$$\Omega = B \setminus (K_1 \cup .. \cup K_l).$$

There are three types of connected components for the limit set: $K_i = \emptyset$, K_i is a point, K_i contains at least two points. Consider the following family of index which corresponds to the first two case: $I = \{i : \mathrm{diam} K_i = 0\}$ and consider the new sets

$$\Omega_n^+ = B \setminus \bigcup_{i \in \{1,..,l\} \setminus I} K_n^i.$$

Then, $\Omega_n^+ \supseteq \Omega_n$ and $\Omega_n^+ \stackrel{H^c}{\to} \Omega^+$ where $\Omega^+ = B \setminus \bigcup_{i \in \{1,..,l\} \setminus I} K^i$. For n large enough there exist some $r > 0$ such that $\mathrm{diam}(K_n^i) > r$, $\forall i \in \{1,..,l\} \setminus I$ and using lemma 5.2 and theorem 1.1 we have:

$$u_{\Omega_n^+, f, 0} \to u_{\Omega^+, f, 0}.$$

Let us fix now $f = 1$. We have:

$$u_{\Omega_n^+, 1, 0} \longrightarrow u_{\Omega^+, 1, 0} \tag{5.2}$$

and, from the boundedness of $u_{\Omega_n, 1, 0}$ (possibly passing to a subsequence):

$$u_{\Omega_n, 1, 0} \rightharpoonup u \tag{5.3}$$

from which:

$$u_{\Omega_n^+, 1, 0} - u_{\Omega_n, 1, 0} \rightharpoonup u_{\Omega^+, 1, 0} - u. \tag{5.4}$$

¿From the maximum principle we have:

$$u_{\Omega_n^+, 1, 0} \geq u_{\Omega_n, 1, 0} \geq 0. \tag{5.5}$$

¿From (5.5) and (5.3) we get $u \geq 0$, and from (5.4) and (5.5) we get $u_{\Omega^+, 1, 0} \geq u$. So:

$$0 \leq u \leq u_{\Omega^+, 1, 0} \tag{5.6}$$

from which $u \in H_0^{1,p}(\Omega^+)$. Since $C_p(\Omega^+ \setminus \Omega) = 0$ then $u \in H_0^{1,p}(\Omega)$, i.e. $u = u_{\Omega, 1, 0}$ because of proposition 3.6. We have just proved that $u_{\Omega_n, 1, 0} \rightharpoonup u_{\Omega, 1, 0}$. Now from Theorem 6.3 of [12] we obtain:

$$u_{\Omega_n, f, 0} \rightharpoonup u_{\Omega, f, 0}, \quad \forall f \in H^{-1,q}(B). \tag{5.7}$$

Now, using the equation on Ω_n and Ω, we have:

$$\lim_{n\to+\infty} \|u_{\Omega_n,f,0}\|^p_{H^{1,p}_0(\Omega_n)} = \lim_{n\to+\infty} < -\triangle_p u_n, u_n >= \lim_{n\to+\infty} < f, u_n >=$$
$$=< f, u >=< -\triangle_p u, u >= \|u_{\Omega,f,0}\|^p_{H^{1,p}_0(\Omega)}. \tag{5.8}$$

Finally, from (5.7), and (5.8) we obtain:

$$u_{\Omega_n,f,0} \longrightarrow u_{\Omega,f,0}, \quad \forall f,$$

i.e. the γ_p convergence. $\square$

6 Final remarks

The γ_p-convergence is equivalent to the convergence in the sense of Mosco of the associated Sobolev spaces. A sequence of Sobolev spaces $H^{1,p}_0(\Omega_n)$ converges in the sense of Mosco to $H^{1,p}_0(\Omega)$ if the following two conditions are satisfied:

1. $\forall u \in H^{1,p}_0(\Omega)$ $\exists u_n \in H^{1,p}_0(\Omega_n)$ such that $u_n \to u$ strongly in $H^{1,p}(I\!R^N)$;
2. $\forall u_{n_k} \in H^{1,p}_0(\Omega_{n_k})$ $u_n \to u$ weakly in $H^{1,p}(I\!R^N)$ we have $u \in H^{1,p}_0(\Omega)$.

If $\Omega_n \xrightarrow{H^c} \Omega$ then condition 1 follows from the fact that any compact set contained in Ω is also contained in Ω_n for large n. Condition 2 generally fails, but is sufficient to be proved only for a particular sequence of functions $u_{n_k} \in H^{1,p}_0(\Omega_{n_k})$, namely the solutions of equation (1.2) with $f \equiv 1$ and $h \equiv 0$ (see [12]).

As an immediate consequence, the γ_p convergence implies the stability of the solution of a more general equation (2.1)-(2.9). In this case, one can obtain only the weak continuity of the solution in $H^{1,p}_0(B)$.

Let $Au = -div\Big(a(x, Du)\Big)$, where $a(x,\zeta)$ satisfies the assumptions (2.1)-(2.9).

Theorem 6.1 *Let $\Omega_n \in \mathcal{W}_w(B)$ and assume $\Omega_n \xrightarrow{H^c} \Omega$. Then*

$$\tilde{u}_{\Omega_n,f,h} \rightharpoonup \tilde{u}_{\Omega,f,h}.$$

Proof Let $\tilde{u}_{\Omega_n,f,h}$ be the solution of the general equation (3.1). From proposition 3.6 we have $\tilde{u}_{\Omega_n,f,h} \rightharpoonup u$, where $u|_\Omega$ satisfies the equation on Ω. The capacitary constraints imply the Mosco convergence of $H_0^{1,p}(\Omega_n)$ to $H_0^{1,p}(\Omega)$ (see theorem 1.1). Using the definition of limit in the sense of Mosco we have $(u-h)|_\Omega \in H_0^{1,p}(\Omega)$ and, from the uniqueness of the solution of the equation (3.1) we obtain $u = u_{\Omega,f,h}$. □

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