HAUSDORFF DIMENSIONS OF SELF-SIMILAR AND SELF-AFFINE FRACTALS IN THE HEISENBERG GROUP

ZOLTÁN M. BALOGH AND
JEREMY T. TYSON

ABSTRACT. We study the Hausdorff dimensions of invariant sets for self-similar and self-affine iterated function systems in the Heisenberg group.

Date: March 1, 2004.

1991 *Mathematics Subject Classification.* Primary 22E30, 28A78; Secondary 26A18, 28A78.

Key words and phrases. Heisenberg group, Hausdorff dimension, iterated function system.

Z. M. B. supported by a grant from the Swiss NSF.

J. T. T. supported by NSF grant DMS 0228807. The research for this paper was done while J. T. T. was a visitor at the University of Berne during 2003. He wishes to thank the department for its hospitality.

Contents

1. Introduction

Analysis on the Heisenberg group is motivated by its appearance in several complex variables and quantum mechanics. In addition, as the simplest non-abelian example, the Heisenberg group serves as a testing ground for questions and conjectures on more general Carnot groups and sub-Riemannian spaces. Geometric measure theory and rectifiability play an important role in these settings in connection with sub-elliptic PDE's and control theory. For recent results in the subject we refer to [3], [5], [12], [13], [15], [18].

This paper is part of a larger program [5], [4] for studying properties of fractal sets in the sub-Riemannian metric setting of the Heisenberg group. The results presented here concern the Hausdorff dimensions of invariant sets associated to self-similar and self-affine iterated function systems.

Let us recall that the (first) *Heisenberg group* $\mathbb{H} = \mathbb{H}^1$ is the unique non-abelian Carnot group of rank two and dimension three. Explicitly, $\mathbb{H} = \mathbb{R}^3$ with the group law

$$(x,t) * (x',t') = (x + x', t + t' + 2\langle x, Jx' \rangle) \tag{1.1}$$

where $J : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the map

$$J(x_1, x_2) = (-x_2, x_1)$$

and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^2$.

The sub-Riemannian nature of $\mathbb{H}$ is reflected in the so-called *horizontal distribution* $H\mathbb{H}$, which is the distinguished subbundle of the full tangent bundle $T\mathbb{H}$ defined by

$$H_p\mathbb{H} := \text{span}\{X_p, Y_p\}.$$

Here X and Y denote the left-invariant vector fields in $\mathbb{H}$ whose values at a point $p = (x_1, x_2, t)$ are

$$X_p = \partial_{x_1} + 2x_2\partial_t, \qquad Y_p = \partial_{x_2} - 2x_1\partial_t.$$

Equivalently, $H_p\mathbb{H}$ can be characterized as the kernel of the canonical contact form $d\tau = dt + 2x_1dx_2 - 2x_2dx_1$ on $\mathbb{H}$ at the point p.

The Heisenberg group is equipped with a non-Euclidean metric structure via the so-called *Heisenberg metric*. This is the left-invariant metric on $\mathbb{H}$ defined as follows:

$$d_H(p,q) = |p^{-1} * q|_H, \qquad p,q \in \mathbb{H}, \tag{1.2}$$

where $*$ denotes the group law from (1.1) and $|\cdot|_H$ denotes the Heisenberg norm given by

$$|(x,t)|_H = (|x|^4 + t^2)^{1/4}. \tag{1.3}$$

Before passing to the main results of this paper, let us begin by describing an application which served as motivation for our studies.

The relationship between the Heisenberg and Euclidean geometry on $\mathbb{H} = \mathbb{R}^3$ is rather intricate. For example, the Hausdorff dimension of $(\mathbb{H}, d_H)$ is equal to 4; in fact balls in the metric d_H have measure proportional to the fourth power of their radius. This implies, for instance, that the Heisenberg metric d_H cannot be locally bi-Lipschitz equivalent with any Riemannian metric.

The following problem asks for an understanding of the relationship between the Hausdorff measures $\mathcal{H}_E^\alpha$ and $\mathcal{H}_H^\beta$ on $\mathbb{H} = \mathbb{R}^3$ associated with the Heisenberg and Euclidean metrics d_H and d_E. A version of this question was posed by Gromov [15, 0.6.C].

Problem 1.4. For fixed $\alpha \in [0,3]$, what are the possible values of $\beta = \dim_H S$ when S ranges over all subsets of $\mathbb{H}$ with $\dim_E S = \alpha$?

Here we denote by $\mathcal{H}_H^s$, resp. $\mathcal{H}_E^s$, the s-dimensional Hausdorff measure associated with the relevant metric d_H, resp. d_E, and by $\dim_H$, resp. $\dim_E$ the corresponding Hausdorff dimensions.

Problem 1.4 is a fundamental question for understanding properties of Hausdorff measures with respect to the Heisenberg metric. It asks which subsets of $\mathbb{H}$ are "most nearly Euclidean" (β is smallest for fixed α) and which are "most nearly non-Euclidean" (β is largest for fixed α). Recently, a nearly complete answer to Problem 1.4 was obtained by Balogh–Rickly–Serra-Cassano [5]. We formulate a slightly different version of the original statement in [5].

Theorem 1.5 (Balogh–Rickly–Serra-Cassano [5], Theorems 1.1 and 1.2)**.** *Let $S \subset \mathbb{H}$ with $\dim_E S = \alpha \in [0,3]$ and $\dim_H S = \beta \in [0,4]$. Then*

$$(1.6)\qquad \max\{\alpha, 2\alpha-2\} =: \beta_-(\alpha) \le \beta \le \beta_+(\alpha) := \min\{2\alpha, \alpha+1\}.$$

Moreover,

(i) *for each $\alpha \in [0,3]$ there exists a set $S^\alpha \subset \mathbb{H}$ with $\mathcal{H}^\alpha_E(S^\alpha) < \infty$ and $\mathcal{H}^{\beta_+(\alpha)}_H(S^\alpha) > 0$,*
(ii) *for each $\alpha \in [0,2) \cup \{3\}$ there is a set $S_\alpha \subset \mathbb{H}$ with $\mathcal{H}^\alpha_E(S_\alpha) > 0$ and $\mathcal{H}^{\beta_-(\alpha)}_H(S_\alpha) = \mathcal{H}^\alpha_H(S_\alpha) < \infty$, and*
(ii') *for each $\alpha \in [2,3)$ and each $\delta \in (0,1)$ there is a set $S_{\alpha,\delta} \subset \mathbb{H}$ with $\mathcal{H}^{\alpha-\delta}_E(S_{\alpha,\delta}) > 0$ and $\mathcal{H}^{\beta_-(\alpha)}_H(S_{\alpha,\delta}) = \mathcal{H}^{2\alpha-2}_H(S_{\alpha,\delta}) < \infty$.*

The techniques in [5] did not suffice to obtain examples showing the precise sharpness in the lower bound in (1.6) in the case $2 \le \alpha < 3$. In particular there were no examples of sets S with the property that

$$\dim_E S = \dim_H S = 2.$$

As a consequence of our main results (which we describe shortly) we are able to find such examples and complete the solution to Gromov's problem 1.4. More precisely, we may record the following theorem.

Theorem 1.7. *For each $\alpha \in [0,3]$ there exists $S_\alpha \subset \mathbb{H}$ with $\mathcal{H}^\alpha_E(S_\alpha) > 0$ and $\mathcal{H}^{\beta_-(\alpha)}_H(S_\alpha) < \infty$, where $\beta_-(\alpha) = \max\{\alpha, 2\alpha-2\}$.*

A case of special interest in the above statement is $\alpha = \beta_-(\alpha) = 2$ The example which figures in this case in Theorem 1.7 is a self-similar set $Q_H \subset \mathbb{H}$ which we term as the *Heisenberg square*. It is obtained as invariant set for a certain self-similar iterated function system. Such systems are the main objects of study in this paper. We describe this example in more detail later on in this introduction; here we give a few relevant facts which indicate how the proof of Theorem 1.7 goes. There is a 1-Lipschitz projection mapping $\pi : \mathbb{H} \to \mathbb{R}^2$ given by $\pi(x,t) = x$ which maps Q_H onto the closed unit square $Q = [0,1]^2$. Thus $\mathcal{H}^2_E(Q_H) \ge \mathcal{H}^2_E(Q) = 1 > 0$. On the other hand, the self-similar construction of Q_H gives rise to natural coverings by families of self-similar copies of Q_H, and using these covers to estimate the Heisenberg Hausdorff measure yields $\mathcal{H}^2_H(Q_H) < \infty$.

The case $\alpha = 2$ is the key to establishing Theorem 1.7 in full generality. The examples for $2 < \alpha < 3$ are constructed as certain "product-type" sets using the Heisenberg square Q_H together with vertical Cantor sets.

With this motivation in mind we turn to the principal objects of study in this paper, namely, invariant sets for iterated function systems in $(\mathbb{H}, d_H)$. Recall that an *iterated function system* (for short, an *IFS*) on a complete metric space (X, d) is a finite collection

$$\mathcal{F} = \{f_1, \dots, f_M\}$$

of contraction maps of (X, d) (i.e., Lipschitz maps with Lipschitz constant strictly less than one). The *invariant set* for $\mathcal{F}$ is the unique nonempty compact set in X which is invariant under the action of the elements of $\mathcal{F}$.

In [4], we studied regularity and connectivity questions for invariant sets of Heisenberg iterated function systems. The present work is devoted to the study of the dimensions of such invariant sets.

Throughout this paper, we restrict our attention to the case of *affine iterated function systems* (*AIFS*). That is, we assume that each IFS consists entirely of affine maps. Moreover, we are interested in affine contractions of $\mathbb{H}$ that arise as lifts of affine mappings of $\mathbb{R}^2$ as follows.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$. A map $F : \mathbb{H} \to \mathbb{H}$ is called a *lift* of f if

$$\pi \circ F = f \circ \pi,$$

where $\pi : \mathbb{H} \to \mathbb{R}^2$ is the projection map defined above. It is an important observation that each affine map of $\mathbb{H}$ which is Lipschitz with respect to the Heisenberg metric d_H arises as a lift of an affine map of $\mathbb{R}^2$. Conversely, each affine map of $\mathbb{R}^2$ lifts to affine Lipschitz maps of $\mathbb{H}$. See Proposition 2.2 of [4] and section 2 of this paper. Such lifts are not unique, but any two lifts of a given map of $\mathbb{R}^2$ differ only by the addition of a vertical constant.

Each AIFS $\{F_1, \dots, F_M\}$ on $\mathbb{H}$ therefore arises as a lift of an AIFS $\{f_1, \dots, f_M\}$ on $\mathbb{R}^2$, and conversely, each AIFS on $\mathbb{R}^2$ can be lifted to AIFS's on $\mathbb{H}$. The space of all AIFS's $\mathcal{F}_H$ on $\mathbb{H}$ which arise as lifts of a given AIFS $\mathcal{F}$ on $\mathbb{R}^2$ is naturally parameterized by an M-dimensional Euclidean space, corresponding to the ambiguity in the vertical constants mentioned above.

We call the invariant sets for Heisenberg AIFS's *(self-affine) horizontal fractals*. This terminology comes from the fact that these objects are in some sense tangent to the horizontal distribution $H\mathbb{H}$. In this paper, we study the Hausdorff dimensions of horizontal fractals with respect to the metrics d_H and d_E on $\mathbb{H} = \mathbb{R}^3$.

To give a more concrete feeling for what is going on we describe in detail our basic example, the so-called *Heisenberg square* Q_H. By this name we denote the invariant set for any horizontal lift of the standard planar IFS

$$\mathcal{F} = \{f_0, f_1, f_2, f_3\}, \tag{1.8}$$

where

$$f_0(x) = \frac{1}{2}x, \quad f_1(x) = \frac{1}{2}(x + e_1),$$
$$f_2(x) = \frac{1}{2}(x + e_2), \quad f_3(x) = \frac{1}{2}(x + e_1 + e_2).$$

Here $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the standard basis vectors in $\mathbb{R}^2$. Figure 1.1 shows several versions of the Heisenberg square, corresponding to different lifts $\mathcal{F}_H$ of the IFS $\mathcal{F}$ from (1.8).

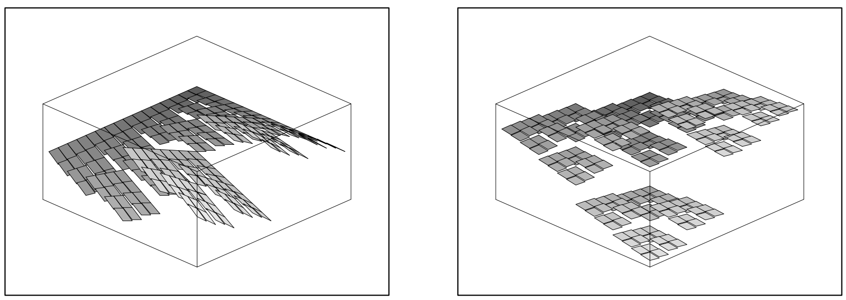

FIGURE 1.1. Heisenberg squares: horizontal lifts of $Q = [0, 1]^2$.

As indicated above, our first result gives the dimensions of Heisenberg squares.

Theorem 1.9. *Let $\mathcal{F}$ be the IFS in (1.8) and let $\mathcal{F}_H$ be any horizontal lift of $\mathcal{F}$. Denote by $Q = [0, 1]^2$, resp. Q_H, the invariant set for $\mathcal{F}$, resp. $\mathcal{F}_H$. Then*

$$\dim_H Q_H = \dim_E Q_H = \dim_E Q = 2.$$

In fact we have

$$0 < 1 = \mathcal{H}^2_E(Q) \le \mathcal{H}^2_E(Q_H) \quad \text{and} \quad \mathcal{H}^2_H(Q_H) < \infty.$$

The Heisenberg squares have been considered previously. Strichartz [24] used Q_H (and versions of this object in more general Carnot groups) to construct "dyadic-type" Carnot tilings. See also [25]. Part of Theorem 1.9, namely the equality $\dim_H Q_H = 2$ can be found in [24]. However, Strichartz obtained Q_H in a different way as the graph of an L^∞ function and not as a horizontal lift. Due to our different approach we obtain a more complete statement and a much simpler proof of Theorem 1.9. In fact, we describe a significantly more general result from which Theorem 1.9 arises as an easy corollary.

Let us mention that the Heisenberg square Q_H is also interesting for another reason. In [4], we prove the following result: there exists a horizontal lift $\mathcal{F}_H$ of the IFS $\mathcal{F}$ from (1.8), so that each selection $\beta : Q \to \mathbb{H}$, $\beta(x) = (x, g(x))$, of the set-valued map $\alpha(x) = \pi^{-1}(x) \cap Q_H$, is a function of bounded variation. Combining this result and Theorem 1.9, we see that there exists a surface $S = g(Q^o)$ in $\mathbb{H}$ with

$$0 < \mathcal{H}_H^2(S) < \infty \tag{1.10}$$

and g a function of bounded variation. This contrasts with a recent result of Ambrosio and Kirchheim [1]. According to Theorem 7.2 of [1], there are no surfaces $S = g(\Omega)$ in $\mathbb{H}$, where Ω is a domain in $\mathbb{R}^2$ and $\beta = (\mathrm{id}, g)$ is a Lipschitz map from Ω to $(\mathbb{H}, d_H)$, which satisfy (1.10).

As mentioned above, Theorem 1.9 is a special case of more general results concerning the dimensions of self-similar and self-affine horizontal fractals in $\mathbb{H}$. The results in question are Heisenberg analogs of theorems of Falconer [10] and Solomyak [23] on the generic dimensions of invariant sets. To set the stage we recall in brief some results from [10] and [23]. A more detailed description will appear in section 5.

To a finite collection $\mathcal{A}$ of contractive linear maps of $\mathbb{R}^n$, Falconer [10] associates a *critical exponent* $s_E(\mathcal{A})$. In the case when each element of $\mathcal{A}$ is in the conformal group $CO(n) = \mathbb{R}_+ \times O(n)$, the critical exponent of $\mathcal{A}$ is equal to the *similarity dimension* of $\mathcal{A}$, i.e., the unique value s satisfying the equation

$$\sum_{A \in \mathcal{A}} ||A||^s = 1, \tag{1.11}$$

where $|| \cdot ||$ denotes the operator norm. (It is not required that the elements of $\mathcal{A}$ be distinct.)

In the case of when a self-similar IFS satisfies the open set condition (cf. section 2) we have the following remarkable equality of dimensions:

Theorem 1.12. *Let $\mathcal{F}$ be a self-similar iterated function system in the plane which satisfies the open set condition and let $\mathcal{F}_H$ be a lift of $\mathcal{F}$ to the Heisenberg group. Then*

$$\dim_E K = \dim_E K_H = \dim_H K_H = s,$$

where s denotes the similarity dimension for the associated family of conformal matrices. Moreover,

$$0 < \mathcal{H}_E^s(K) \le \mathcal{H}_E^s(K_H) \quad \textit{and} \quad \mathcal{H}_H^s(K_H) < \infty.$$

Since the IFS $\mathcal{F}$ from (1.8) which generates the square in $\mathbb{R}^2$ satisfies the open set condition, Theorem 1.9 follows from Theorem 1.12.

The major question adressed in this paper is what happens in the absence of the open set condition in the more general setting of affine maps. The definition of the critical exponent $s_E(\mathcal{A})$ from [10] is more complicated and will be recalled in section 5. By results of Falconer and Solomyak in the Euclidean case one still has a dimension formula which holds in a generic sense. To recall this statement fix a collection $\mathcal{A} = \{A_1, \dots, A_M\}$ as above. For each $b = (b_1, \dots, b_M)$ in $\mathbb{R}^{nM}$, define an AIFS $\mathcal{F}(b) = \{f_1, \dots, f_M\}$ on $\mathbb{R}^n$, where $f_i(x) = A_i x + b_i$, $i = 1, \dots, M$. Let $K(b)$ be the invariant set for $\mathcal{F}(b)$.

Theorem 1.13 (Falconer, Solomyak). *Let $\mathcal{A}$ and $K(b)$ be as above. Then*

(i) $\dim_E K(b) \le s_E(\mathcal{A})$ *for all* $b \in \mathbb{R}^{nM}$*; and*
(ii) *if* $||A_i|| < 1/2$ *for each* i*, then* $\dim_E K(b) = \min\{n, s_E(\mathcal{A})\}$ *for a.e.* $b \in \mathbb{R}^{nM}$.

Falconer proved this result first with 1/2 replaced by 1/3 [10, Proposition 5.1 and Theorem 5.3]. Solomyak [23, Proposition 3.1] observed that the hypotheses could be weakened as indicated. The constant 1/2 is sharp for generic statements of this type, as was observed by Edgar in [8]. See also the proof of Proposition 3.1 in [23].

Each lift of an affine map $f(x) = Ax + b$ of $\mathbb{R}^2$ to the Heisenberg group is an affine map $F(x,t) = \tilde{A}_b(x,t) + \tilde{b}$, where $\tilde{A}_b$ is a certain block-lower triangular matrix defined in terms of A and b and $\tilde{b} = (b, \tau)$, τ an arbitrary real parameter. See (2.1). For a given $b \in \mathbb{R}^{2M}$ and an AIFS $\mathcal{F}(b)$ on $\mathbb{R}^2$, denote by $\mathcal{F}_H(b,\tau)$ the lifted IFS on $\mathbb{H}$ corresponding to a specific choice of $\tau \in \mathbb{R}^M$ and by $K_H(b,\tau)$ its invariant set. Also, denote by $\tilde{s}_E(b;\mathcal{A})$ the critical exponent for the family $\{\tilde{A}_{1,b_1}, \dots, \tilde{A}_{M,b_M}\}$.

From Theorem 1.13 we immediately deduce that

$$\dim_E K_H(b,\tau) \le \tilde{s}_E(b;\mathcal{A}) \tag{1.14}$$

for all b and τ. However, the upper bound in (1.14) is not the correct value for $\dim_E K_H(b,\tau)$. In fact, we will prove the following result.

Theorem 1.15. *Let $\mathcal{F}(b)$, $b \in \mathbb{R}^{2M}$, be an IFS of contractive, affine maps in $\mathbb{R}^2$ and let $\mathcal{F}_H(b,\tau)$, $\tau \in \mathbb{R}^M$, be any horizontal lift to $\mathbb{H}$ as above. Then*

(i) $\dim_E K_H(b,\tau) \le \tilde{s}_E(\mathcal{A}) := \tilde{s}_E(0;\mathcal{A})$ *for all* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$*; and*
(ii) *if* $||A_i|| < 1/2$ *for all* i*, then* $\dim_E K_H(b,\tau) = \min\{3, \tilde{s}_E(\mathcal{A})\}$ *for a.e.* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$.

Observe that there is no contradiction between the almost sure results of Theorems 1.15(ii) and 1.13(ii) since the matrices $\tilde{A}_{i,b_i}$ depend on the auxiliary parameter $b \in \mathbb{R}^{2M}$. Thus it cannot be guaranteed that the almost sure conclusion in Theorem 1.13 is applicable for any particular choice of the lift $K_H(b,\tau)$ in Theorem 1.15(ii).

To study the Heisenberg dimensions of $K_H(b,\tau)$ we introduce a *Heisenberg critical exponent* $\tilde{s}_H(\mathcal{A})$ (see section 5) associated with a family $\mathcal{A}$ of contractive linear maps of $\mathbb{R}^2$. This quantity differs substantially from its Euclidean counterpart and represents a major conceptual novelty of this paper. We then have the following

Theorem 1.16. *Let $\mathcal{F}(b)$, $b \in \mathbb{R}^{2M}$, be an IFS of contractive, affine maps in $\mathbb{R}^2$ and let $\mathcal{F}_H(b,\tau)$, $\tau \in \mathbb{R}^M$, be any horizontal lift to $\mathbb{H}$ as above. Then*

(i) $\dim_H K_H(b,\tau) \le \tilde{s}_H(\mathcal{A})$ *for all* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$*; and*
(ii) *if* $||A_i|| < 1/2$ *for each* i*, then* $\dim_H K_H(b,\tau) = \min\{4, \tilde{s}_H(\mathcal{A})\}$ *for a.e.* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$.

From the definitions of $s_E, \tilde{s}_E, \tilde{s}_H$ (cf. section 5) it is straightforward to verify that

$$\min\{2, s_E(\mathcal{A})\} \le \min\{3, \tilde{s}_E(\mathcal{A})\} \le \min\{4, \tilde{s}_H(\mathcal{A})\}.$$

Furthermore, if $0 \le s_E(\mathcal{A}) \le 1$ then $s_E(\mathcal{A}) = \tilde{s}_E(\mathcal{A}) = \tilde{s}_H(\mathcal{A})$ and if $1 \le s_E(\mathcal{A}) \le 2$ then $s_E(\mathcal{A}) = \tilde{s}_E(\mathcal{A})$.

In the self-similar case, the critical exponents $s_E(\mathcal{A})$, $\tilde{s}_E(\mathcal{A})$ and $\tilde{s}_H(\mathcal{A})$ all agree and are equal to the similarity dimension of $\mathcal{A}$. Denoting this common value by s, we have

$$\dim_E(K(b)) = \min\{2, s\},$$

$$\dim_E(K_H(b,\tau)) = \min\{3, s\},$$

and

$$\dim_H(K_H(b,\tau)) = \min\{4, s\}$$

for almost every b and τ. In particular, if $s \le 2$ then

$$\dim_E K(b) = \dim_E K_H(b,\tau) = \dim_H K_H(b,\tau) = s \tag{1.17}$$

for a.e. b,τ.

Overview. The structure of this paper is as follows. In section 2 we collect some additional definitions and recall background material. We also fix notation which will be in force for the rest of the paper.

Section 3 is devoted to the self-similar case. We prove Theorems 1.15 and 1.16 in this special setting first, in preparation for the general case. We also discuss the open set condition for horizontal lifts, and give the proof of Theorem 1.12.

In section 4 we discuss Gromov's question on the relationship between $\dim_E$ and $\dim_H$.

The various critical exponents for a general affine family and its horizontal lifts are defined and discussed in section 5. Section 6 is devoted to the proofs of Theorems 1.15 and 1.16 in full generality.

In an appendix (section 7), we sketch the proof of an interesting fact from linear algebra which arises in connection with inequalities between the various critical exponents associated with a family of contractive linear maps.

Acknowledgements. We are grateful to Regula Hofer-Isenegger for the Maple code which produced the image of the Heisenberg square in Figure 1.1.

2. Definitions, notation and preliminary results

Affine maps on $\mathbb{H}$. We start by recalling the following relation between Lipschitz affine maps of $\mathbb{H}$ and horizontal lifts of affine maps of $\mathbb{R}^2$. See Proposition 2.2 of [4]. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be an affine map of the form

$$F(x,t) = (Ax + t\cdot a + b, \langle d, x\rangle + ct + \tau),$$

where A is a real 2×2 matrix, $a, b, d \in \mathbb{R}^2$ and $c, \tau \in \mathbb{R}$. Then F is Lipschitz with respect to the metric d_H if and only if the relations

$$a = 0, \quad d = -2A^T Jb, \quad c = \det A$$

hold. Thus every Lipschitz affine map $F\mathbb{H} \to \mathbb{H}$ arises as horizontal lift of the affine map $f(x) = Ax + b$ to $\mathbb{H}$ and is given by

$$F(x,t) = \tilde{A}_b \begin{pmatrix} x \\ t \end{pmatrix} + \tilde{b},$$

where

$$\tilde{A}_b = \begin{pmatrix} A & 0 \\ -2(Jb)^T A & \det A \end{pmatrix}, \qquad \tilde{b} = \begin{pmatrix} b \\ \tau \end{pmatrix}, \tag{2.1}$$

and τ is a real constant. The Lipschitz constant of F as a self-map of $(\mathbb{H}, d_H)$ is equal to the Lipschitz constant of f as a self-map of $(\mathbb{R}^2, d_E)$. Furthermore, F is a similarity with respect to d_H if and only if the above relations hold and $A \in CO(2)$ is a conformal matrix; in this case the Lipschitz constant agrees with the operator norm of the linear part of f.

For example, choose $A = rI$, $r > 0$ (where I denotes the 2×2 identity matrix) and $b = 0$. The lift of $f(x) = rx$ corresponding to $\tau = 0$ is the Heisenberg *dilation* $F(x,t) = (rx, r^2 t)$. Similarly, choose $A = I$ and

$b \in \mathbb{R}^2$ arbitrary. Then the lift of $f(x) = x + b$ corresponding to $\tau \in \mathbb{R}$ is the *left translation* by (b, τ):

$$F(x,t) = (b,\tau) * (x,t) = (x + b, t + \tau - 2\langle Jb, x\rangle).$$

Affine iterated function systems. Let X be either $\mathbb{R}^n, n = 2, 3$ or $\mathbb{H}$. Recall that an *affine iterated function system (AIFS)* is a finite collection $\mathcal{F}$ of contracting affine maps of X. The *invariant set* for $\mathcal{F}$ is the unique nonempty compact set $K \subset X$ which is fully invariant under the action of $\mathcal{F}$:

$$K = \bigcup_{f\in\mathcal{F}} f(K).$$

The existence of invariant sets for iterated function systems follows from the completeness of the space of compact subsets of X with the Hausdorff metric. See, for example, [19, 4.13] or [17, Theorem 1.1.4].

It follows from the previous paragraph that, to each AIFS

$$\mathcal{F}(b) = \{f_1, \dots, f_M\}$$

in $\mathbb{R}^2$, where $b = (b_1, \dots, b_M) \in \mathbb{R}^{2M}$ and $f_i(x) = A_i x + b_i$, there correspond horizontally lifted IFS's on $\mathbb{H}$ given by

$$\mathcal{F}_H(b,\tau) = \{F_1, \dots, F_M\}, \qquad \tau = (\tau_1, \dots, \tau_M) \in \mathbb{R}^M,$$

where

$$F_i(x,t) = \tilde{A}_{i,b_i} \begin{pmatrix} x \\ t \end{pmatrix} + \tilde{b}_i, \tag{2.2}$$

and $\tilde{A}_{i,b_i}$ and $\tilde{b}_i$ are given by (2.1). The space of all lifts $\mathcal{F}_H(b,\tau)$ corresponding to a fixed AIFS $\mathcal{F}(b)$ depends on the M real parameters $\tau_1, \dots, \tau_M$.

Recall that the similarity dimension of $\mathcal{A} = \{A_1, \dots, A_M\}$ is the unique positive solution s to the equation (1.11). It follows from remarks made in the previous paragraph that the Heisenberg similarity dimension of the family $\{\tilde{A}_{1,b_1}, \dots, \tilde{A}_{M,b_M}\}$ is equal to the same value s, regardless of the choice of $b_1, \dots, b_M$.

Symbolic dynamics. The dynamical attributes of an iterated function system are encoded via its representation as a quotient of the standard sequence space. Let A be an alphabet consisting of the letters $1, \dots, M$. Let $W_m = A^m$, $m \geq 1$, (resp. $\Sigma = A^{\mathbb{N}}$) denote the space of words of length m (resp. words of infinite length) with letters drawn from A. We denote elements of these spaces by concatenation of letters, i.e., $w = w_1 w_2 \cdots w_m \in W_m$ or $w = w_1 w_2 \cdots \in \Sigma$, where $w_j \in A$ for each j. Let $W = \cup_{m\geq 1} W_m$ be the collection of all words of finite length. For $w \in W$ we write Σ_w for the set of words in Σ which begin with w; Σ_w is called the *cylinder set* with label w.

Assume now that $\mathcal{F} = \{f_i\}_{i \in A}$ is an IFS in a complete metric space (X,d) with invariant set K. For each finite word $w = w_1 \cdots w_m$ let $f_w = f_{w_1} \circ \cdots \circ f_{w_m}$ and $K_w = f_w(K)$. Then $K = \cup_{w \in W_m} K_w$ for each m and $\max_{w \in W_m} \operatorname{diam} K_w \to 0$ as $m \to \infty$. We also define K_w for infinite words $w = w_1 w_2 \cdots$ by setting $K_w = \cap_m K_{w_1 \cdots w_m}$. In this case K_w consists of a single point in K.

We consider on Σ the product topology induced by the discrete topology on A and we define a map $p = p_{\mathcal{F}} : \Sigma \to K$ by setting $p(w)$ equal to the unique point in K_w. Then p is a continuous surjection between compact sets [17, Theorem 1.2.3]. Observe that

$$p(w) = \lim_{m \to \infty} f_{w_1 \cdots w_m}(x_0), \qquad w = w_1 w_2 \cdots \in \Sigma, \tag{2.3}$$

where x_0 is an arbitrarily chosen point in X.

Hausdorff measure and dimension. Let $X = (X,d)$ be a metric space. For $\alpha \geq 0$ we denote by $\mathcal{H}_d^\alpha$ the *α-dimensional Hausdorff measure* on X, defined as

$$\mathcal{H}_d^\alpha(A) := \lim_{\delta \searrow 0} \inf \sum_n \operatorname{diam}(A_n)^\alpha,$$

where the infimum is taken over all countable covers of A by sets $A_1, A_2, \ldots$ satisfying $\operatorname{diam} A_n < \delta$. Then the *Hausdorff dimension* of $A \subset X$ is

$$\dim_d(A) = \inf\{\alpha : \mathcal{H}_d^\alpha(A) = 0\} = \sup\{\alpha : \mathcal{H}_d^\alpha(A) = \infty\}.$$

We will use these concepts only in the cases $(X,d) = (\mathbb{R}^2, d_E)$, $(X,d) = (\mathbb{R}^3, d_E)$, and $(X,d) = (\mathbb{H}, d_H)$. We write $\mathcal{H}_E^\alpha$ and $\dim_E$ for the Hausdorff measures and dimension in $\mathbb{R}^2$ and $\mathbb{R}^3$ and $\mathcal{H}_H^\alpha$ and $\dim_H$ for the corresponding objects in $\mathbb{H}$.

Since d_E is locally bounded by d_H on $\mathbb{H} = \mathbb{R}^3$ [5, Lemma 2.1], we have the absolute continuity relation

$$\mathcal{H}_E^\alpha \ll \mathcal{H}_H^\alpha$$

for the α-Hausdorff measures on $\mathbb{H}$ for any $\alpha \geq 0$ [5, Proposition 3.2(i)]. Thus

$$\dim_E A \leq \dim_H A \tag{2.4}$$

for any set $A \subset \mathbb{H}$.

The open set condition. An iterated function system $\mathcal{F}$ on a complete metric space X is said to satisfy the *open set condition* if there exists a bounded open set $O \subset X$ so that $f(O) \subset O$ for all $f \in \mathcal{F}$ and $f(O) \cap g(O) = \emptyset$ for all $f, g \in \mathcal{F}$, $f \neq g$.

The relevance of this condition for the computation of the Hausdorff dimensions of self-similar sets derives from the following result, which was proved by Moran [20] in 1946 and rediscovered by Hutchinson [16] in the 1980's.

Proposition 2.5. *Let $\mathcal{F}$ be a self-similar IFS in $\mathbb{R}^n$ which satisfies the open set condition. Let K be the invariant set of $\mathcal{F}$. Let $\mathcal{A}$ denote the collection of conformal matrices which arise as the linear parts of elements of $\mathcal{F}$ (counted with multiplicity).*

Then the Hausdorff dimension of K is equal to the similarity dimension s of $\mathcal{A}$. Moreover,

$$0 < \mathcal{H}^s_E(K) < \infty.$$

Schief [22], building on ideas of Bandt–Graf [7], proved the following (somewhat suprising) converse to Proposition 2.5:

Proposition 2.6. *Let $\mathcal{F}$ be a self-similar IFS in $\mathbb{R}^n$ whose invariant set K satisfies $\mathcal{H}^s_E(K) > 0$, where $\mathcal{A}$ is defined as in Proposition 2.5. Then $\mathcal{F}$ satisfies the open set condition.*

3. The self-similar case

In this section, we discuss relations between the Euclidean dimension of a planar self-similar invariant set and the Heisenberg dimensions of its horizontal lifts. In particular, we prove Theorem 1.12 on the equality of dimensions in the presence of the open set condition. The principal theorem of this section (Theorem 3.9) states that the Heisenberg and Euclidean dimensions agree generically. It is a special case of Theorems 1.15 and 1.16.

Throughout this section, we assume that $\mathcal{F}$ is an IFS consisting of similarity maps of the plane, and that $\mathcal{F}_H$ is a horizontal lift of $\mathcal{F}$ to $\mathbb{H}$. We denote by K, resp. K_H, the invariant sets of $\mathcal{F}$, resp. $\mathcal{F}_H$.

Since π is a 1-Lipschitz map from $(\mathbb{R}^3, d_E)$ to $(\mathbb{R}^2, d_E)$ and $\pi(K_H) = K$, we have the following *a priori* inequality:

$$\dim_E K \le \dim_E K_H \le \dim_H K_H. \tag{3.1}$$

Observe that the second inequality follows from (2.4).

The following example shows that we need not always have equality throughout (3.1). In this example it is the first equality which is strict.

Example 3.2. Fix $r \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$ and let $f_1(x) = rx$ and $f_2(x) = e_1 + r(x - e_1)$, where $e_1 = (1, 0)$. The invariant set for $\mathcal{F} = \{f_1, f_2\}$ is $[0, 1]$. The formula in (2.2) gives the horizontal lifts F_i, $i = 1, 2$, as

$$\begin{aligned} F_1(x,t) &= (rx, r^2 t + \tau_1), \\ F_2(x,t) &= (e_1 + r(x - e_1), r^2 t - 2r(1-r)x_2 + \tau_2), \end{aligned}$$

where $x = (x_1, x_2)$ and $\tau_1, \tau_2 \in \mathbb{R}$. Choose $\tau_1 = 0$ and

$$\tau_2 > \frac{2r(1-r)}{1-2r^2}.$$

It is straightforward to show that the open set $U = B(0,1) \times (0, 2\tau_2)$ verifies the open set condition for $\mathcal{F}_H = \{F_1, F_2\}$. The maps F_1 and F_2 are similarities of $\mathbb{H}$ with contraction ratio r. By Proposition 3.3 below,

$$\dim_H K_H = \frac{\log 2}{\log 1/r} > 1 = \dim_E K.$$

In fact, the Euclidean dimension of K_H is also equal to $\log 2 / \log 1/r$. The proof of this latter fact requires Falconer's theory of dimensions of self-affine fractals which will be recalled in section 5.

In the above example we made use of the following proposition, which extends the Moran–Hutchinson result to the Heisenberg setting.

Proposition 3.3. *Let $\mathcal{F}_H$ be a self-similar iterated function system in $\mathbb{H}$ which satisfies the open set condition. Assume that $\mathcal{F}_H$ is a lift of $\mathcal{F}$, and define $\mathcal{A}$ as in Proposition 2.5. Then the Heisenberg dimension of K_H is equal to the similarity dimension of $\mathcal{A}$.*

Kigami [17, Proposition 1.5.8] gave a new proof of the theorem of Moran–Hutchinson. His proof extends to the Heisenberg setting, as we now demonstrate.

Kigami's proof uses the following more general result, which is Theorem 1.5.7 of [17].

Theorem 3.4. *Let $\mathcal{F} = \{f_1, \ldots, f_M\}$ be an iterated function system in a complete metric space X. Let K be the invariant set of $\mathcal{F}$. Assume that there exist $r_1, \ldots, r_M \in (0,1)$ and positive constants C_1, C_2, M and r_0 so that the following two conditions hold:*

(i) $\operatorname{diam} f_w(K) \le C_1 r_w$ *for each* $w \in W$, *and*
(ii) *for any* $p \in K$ *and any* $0 < r \le r_0$, *the number of words* $w = w_1 \cdots w_m \in W$ *satisfying the conditions*

$$r_{w_1} \cdots r_{w_{m-1}} > r \ge r_{w_1} \cdots r_{w_m} \tag{3.5}$$

and

$$\operatorname{dist}(p, f_w(K)) \le C_2 r \tag{3.6}$$

is at most M.

Then the Hausdorff dimension of K *is given by the unique positive solution* s *to the equation*

$$\sum_{i=1}^{M} r_i^s = 1. \tag{3.7}$$

Moreover, $0 < \mathcal{H}^s(K) < \infty$.

Proof of Proposition 3.3. We verify the assumptions of Theorem 3.4 with r_i equal to the Lipschitz constant of $F_i \in \mathcal{F}_H$.

Let U be a bounded open set in $\mathbb{H}$ which verifies the OSC for $\mathcal{F}_H$. Without loss of generality $\operatorname{diam}_H \overline{U} = 1$; since $K_H \subset \overline{U}$ by Exercise 1.2 of [17], we conclude that $\operatorname{diam}_H K_H \le 1$.

By the choice of r_i, we have $\operatorname{diam}_H F_w(K_H) \le r_w$ for all words w. This establishes Theorem 3.4(i) with $C_1 = 1$.

Next fix $p = (x,t) \in K_H$ and $0 < r \le 1$, and consider a word w satisfying (3.5) and (3.6) with $C_2 = 1$. Then $F_w(U) \subset B_H(p, 2r)$, where $B_H(p,r)$ denotes the ball in the Heisenberg metric about p of radius r. Since the sets $F_w(U)$ are pairwise disjoint for such words w,

$$\sum_w |F_w(U)| \le |B_H(p,2r)| = 16r^4 |B_H(0,1)|,$$

where the sum is taken over all words w satisfying (3.5) and (3.6). Here $|U|$ denotes the three-dimensional Lebesgue measure of a set $U \subset \mathbb{H}$. From (3.5) we see that $|F_w(U)| \ge r_{\min}^4 r^4 |U|$ and so the number of words w is bounded by

$$M := \frac{16|B_H(0,1)|}{r_{\min}^4 |U|}.$$

□

Since the open set condition passes to horizontal lifts (see Proposition 3.14 of [4]), we may record the following corollary stated as Theorem 1.12 to Proposition 3.3.

Corollary 3.8. *Let* $\mathcal{F}$ *be a self-similar iterated function system in the plane which satisfies the open set condition and let* $\mathcal{F}_H$ *be a lift of* $\mathcal{F}$ *to the Heisenberg group. Then*

$$\dim_E K = \dim_E K_H = \dim_H K_H = s,$$

where s denotes the similarity dimension for the associated family of conformal matrices. Moreover,

$$0 < \mathcal{H}^s_E(K) \le \mathcal{H}^s_E(K_H) \quad \textit{and} \quad \mathcal{H}^s_H(K_H) < \infty.$$

Generic equality of dimensions for self-similar fractals. In this subsection, we show that equality holds throughout (3.1) in a generic sense even in the absence of the open set condition.

Consider a family $\mathcal{A} = \{A_1, \dots, A_M\}$ of 2×2 conformal matrices. For each $b = (b_1, \dots, b_M) \in \mathbb{R}^{2M}$, consider the AIFS $\mathcal{F}(b) = \{f_1, \dots, f_M\}$, where $f_i(x) = A_i x + b_i$, $i = 1, \dots, M$. We view the matrices $A_1, \dots, A_M$ as fixed and $b_i, \dots, b_M$ as varying.

The following theorem gives an upper bound for the Hausdorff dimensions of self-similar lifts. In conjunction with Theorem 1.13, it implies the generic equality of Heisenberg and Euclidean dimensions.

Theorem 3.9. *Let $\mathcal{F}(b)$ be a self-similar IFS in $\mathbb{R}^2$ as above and let $\mathcal{F}_H(b,\tau)$, $\tau \in \mathbb{R}^M$, be any horizontal lift to $\mathbb{H}$. Then*

$$\mathcal{H}^s_H(K_H(b,\tau)) < \infty,$$

where s is the similarity dimension of $\mathcal{A}$. In particular,

$$\dim_H K_H(b,\tau) \le s.$$

Corollary 3.10. *If $||A_i|| < \frac{1}{2}$ for each i and $s \le 2$, then*

$$\dim_E K(b) = \dim_E K_H(b,\tau) = \dim_H K_H(b,\tau) = s$$

for a.e. $b \in \mathbb{R}^{2M}$ and all $\tau \in \mathbb{R}^M$.

Proof of Theorem 3.9. Without loss of generality assume that the Heisenberg diameter of $K_H(b)$ is one.

Since $\mathcal{F}(b)$ consists of similarities of $\mathbb{R}^2$, $\mathcal{F}_H(b,\tau)$ consists of similarities of $\mathbb{H}$. The value $r_i := ||A_i||$ is the common contraction ratio for $f_i \in \mathcal{F}(b)$ and its lift $F_i \in \mathcal{F}_H(b,\tau)$.

Given $\delta > 0$, choose m so that $r^m_{\max} < \delta$. The sets $A_w := F_w(K_H(b,\tau))$, $w \in W_m$, cover $K_H(b,\tau)$ and $\operatorname{diam}_H A_w = r_w < \delta$. Thus

$$\mathcal{H}^s_{H,\delta}(K_H(b,\tau)) \le \sum_{w \in W_m} (\operatorname{diam}_H A_w)^s = \sum_{w \in W_m} r^s_w = \left(\sum_{i=1}^M r^s_i \right)^m = 1.$$

Hence $\mathcal{H}^s_H(K_H(b,\tau)) \le 1$ and $\dim_H K_H(b,\tau) \le s$. ☐

Remark 3.11. The theory developed by Falconer in [10] and recalled in section 5 applies to self-affine systems in arbitrary Euclidean spaces $\mathbb{R}^n$. Each self-similar IFS in $\mathbb{R}^2$ lifts to self-similar IFS's in $(\mathbb{H}, d_H)$ which are self-affine when viewed as IFS's on $(\mathbb{R}^3, d_E)$. It is an interesting exercise to use Theorem 1.13 to verify that the Euclidean dimension of the lifted fractal agrees with the similarity dimension.

4. Comparison of Euclidean and Heisenberg dimensions

In this section we discuss the application of Theorem 1.9 to the problem of Gromov. In particular, we will prove Theorem 1.7, whose statement we now recall:

Theorem 4.1. *For each $\alpha \in [0,3]$ there exists $S_\alpha \subset \mathbb{H}$ with*

$$\mathcal{H}_E^\alpha(S_\alpha) > 0 \text{ and } \mathcal{H}_H^{\max\{\alpha,2\alpha-2\}}(S_\alpha) < \infty.$$

Let us also recall that relevant examples for the cases $0 \le \alpha < 2$ and $\alpha = 3$ of Theorem 4.1 were previously given by Balogh–Rickly–Serra-Cassano [5]; see Theorem 1.5.

Proof of Theorem 4.1. By Theorem 1.9, each horizontal lift Q_H of the unit square serves as the desired example S_2 in Theorem 4.1 in the case $\alpha = 2$. Indeed $\mathcal{H}_H^2(S_2) < \infty$ while $\mathcal{H}_E^2(S_2) \ge \mathcal{H}_E^2(Q) = 1$.

To treat the case $2 < \alpha < 3$, we construct certain product-type sets over Q_H. Let $p = \alpha - 2$ and consider a Cantor set C_p in the t-axis with $0 < \mathcal{H}_E^p(C_p) < \infty$ and $0 < \mathcal{H}_H^{2p}(C_p) < \infty$. The construction of such a set is standard, see e.g. [3, p. 300] or [5, §4]. To wit, choosing $s < 1$ so that

$$2s^p = 1,$$

we view C_p as the invariant set associated with the system $\mathcal{G}_H = \{G_1, G_2\}$, where G_1 and G_2 are the $\sqrt{s}$-Lipschitz maps of $(\mathbb{H}, d_H)$ defined by $G_1(x,t) = (\sqrt{s}x, st)$ and $G_2(x,t) = (\sqrt{s}x, 1 + s(t-1))$.

The set S_α is defined as the following product of Q_H with C_p:

$$S_\alpha := \{(x, t+t') : (x,t) \in Q_H, (0,t') \in C_p\}.$$

The estimate $\mathcal{H}_E^\alpha(S_\alpha) = \mathcal{H}_E^{2+p}(S_\alpha) > 0$ is a consequence of the Euclidean product structure of S_α, as follows. For $x \in Q$ define

$$t_x = \max\{t : (x,t) \in Q_H\}$$

and $\Phi : Q \times C_p \to S_\alpha$,

$$\Phi(x,(0,t)) = (x, t_x + t).$$

It is easy to see that Φ is a bi-Lipschitz embedding of $Q \times C_p$ into S_α. Thus it suffices to show that

$$\mathcal{H}^\alpha_E(Q \times C_p) = \mathcal{H}^{2+p}_E(Q \times C_p) > 0.$$

This follows from [19, Theorem 8.10], since $\mathcal{H}^2_E(Q) = 1$ and $\mathcal{H}^p_E(C_p) > 0$.

To show the estimate $\mathcal{H}^{2\alpha-2}_H(S_\alpha) = \mathcal{H}^{2+2p}_H(S_\alpha) < \infty$ we use the obvious covering of S_α by similarity images of Q and C_p. Fix $\delta > 0$ and choose

$$m > \frac{1}{4} + \frac{1}{2p} + \frac{\log 1/\delta}{\log 2}.$$

Set $n = [2pm]$, where $[x]$ denotes the greatest integer less than or equal to x, and consider the covering of S_α with the sets

$$S_{vw} := \{(x, t+t') : (x,t) \in F_w(Q_H), (0,t') \in G_v(C_p)\},$$

where w and v range over the sets $W_m = \{1,2,3,4\}^m$ and $V_n = \{1,2\}^n$ respectively.

To estimate $\operatorname{diam}_H(S_{vw})$, choose $(x, t+t')$ and $(\tilde{x}, \tilde{t}+\tilde{t}')$ in S_{vw} with

$$\operatorname{diam}_H(S_{vw}) = d_H((x,t+t'),(\tilde{x},\tilde{t}+\tilde{t}'))$$

and compute

$$\begin{aligned}\operatorname{diam}_H(S_{vw})^4 &= |\tilde{x}-x|^4 + (\tilde{t}-t+\tilde{t}'-t'-2\langle x, J\tilde{x}\rangle)^2 \\ &\le 2\left(|\tilde{x}-x|^4 + (\tilde{t}-t-2\langle x, J\tilde{x}\rangle)^2 + (\tilde{t}'-t')^2\right) \\ &\le 2\left((\frac{1}{2})^{4m} + \alpha^{2n}\right) < \delta^4.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{H}^{2+2p}_{H,\delta}(S_\alpha) &\le \sum_{w\in W_m}\sum_{v\in V_n} \operatorname{diam}_H(S_{vw})^{2+2p} \\ &\le 2^{1/4}\cdot 4^m \cdot 2^n \cdot \left((\frac{1}{2})^{4m} + \alpha^{2n}\right)^{(1+p)/2} \\ &\le C(p) 2^{2m(1+p)}\left((\frac{1}{2})^{2m(1+p)} + \alpha^{2m(1+p)p}\right) = 2C(p) < \infty\end{aligned}$$

as desired. □

5. The self-affine case I

In this section, we collect some preliminary material on the Euclidean and Heisenberg critical exponents for a family of linear maps, and Hausdorff-type measures on sequence space defined using these quantities. We also give an example of an AIFS in $\mathbb{H}$ whose dimension can be estimated using our theorems.

Singular value functions and critical exponents. Let $n \geq 2$ be an integer. For a contracting linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ denote by $1 > \alpha_1 \geq \cdots \geq \alpha_n > 0$ the *singular values* of A, defined as the lengths of the principal semi-axes of the ellipsoid $A(B^n(0,1))$, or equivalently as the positive square roots of the eigenvalues of $A^T A$. The *singular value function* $\varphi^s(A)$ is defined for $s \geq 0$ as

$$\varphi^s(A) = \alpha_1 \alpha_2 \cdots \alpha_{m-1} \alpha_m^{s-m+1}, \qquad 0 < s \leq n, \tag{5.1}$$

where m is the integer such that $m - 1 < s \leq m$,

$$\varphi^0(A) = 1,$$

and

$$\varphi^s(A) = (\alpha_1 \cdots \alpha_n)^{s/n}, \quad s > n.$$

Given a collection $\mathcal{A} = \{A_1, \ldots, A_M\}$ of linear maps in $\mathbb{R}^n$, define the *critical exponent* $s_E(\mathcal{A})$ as the unique nonnegative solution s to the equation

$$\lim_{m \to \infty} \left(\sum_{w \in W_m} \varphi^s(A_w) \right)^{1/m} = 1, \tag{5.2}$$

where $A_w = A_{w_1} \cdots A_{w_m}$ and $w = w_1 \cdots w_m \in W_m := \{1, \ldots, M\}^m$. This critical exponent is the value which appears in the theorem of Falconer and Solomyak from the introduction. If each element of $\mathcal{A}$ is conformal, $s_E(\mathcal{A})$ is equal to the similarity dimension of $\mathcal{A}$.

Recall that each lift of an affine map $f(x) = Ax + b$ of $\mathbb{R}^2$ to $\mathbb{H}$ is an affine map $F(x,t) = \tilde{A}_b(x,t) + \tilde{b}$, where $\tilde{A}_b$ and $\tilde{b} = (b, \tau)$, $\tau \in \mathbb{R}$, are defined as in (2.1). For fixed $b \in \mathbb{R}^{2M}$ and an AIFS $\mathcal{F}(b)$ on $\mathbb{R}^2$, we denote by $\mathcal{F}_H(b,\tau)$ the lifted IFS on $\mathbb{H}$ corresponding to a specific choice of $\tau \in \mathbb{R}^M$ and by $K_H(b,\tau)$ its invariant set. Also, denote by $\tilde{s}_E(b;\mathcal{A})$ the critical exponent for the family $\{\tilde{A}_{1,b_1}, \ldots, \tilde{A}_{M,b_M}\}$, as defined above, and abbreviate $\tilde{s}_E(\mathcal{A}) := \tilde{s}_E(0;\mathcal{A})$.

We now recall the statement of Theorem 1.15 from the introduction.

Theorem 5.3. *Let $\mathcal{F}(b)$, $b \in \mathbb{R}^{2M}$, be an IFS of affine maps in $\mathbb{R}^2$ and let $\mathcal{F}_H(b,\tau)$, $\tau \in \mathbb{R}^M$, be any horizontal lift to $\mathbb{H}$ as above. Then*

(i) $\dim_E K_H(b,\tau) \leq \tilde{s}_E(\mathcal{A})$ *for all* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$*; and*
(ii) *if* $||A_i|| < 1/2$ *for all* i*, then* $\dim_E K_H(b,\tau) = \min\{3, \tilde{s}_E(\mathcal{A})\}$ *for a.e.* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$.

If α_1, α_2 are the singular values of A then the singular values of the 3×3 matrix $\tilde{A}_0$ are α_1, α_2 and $\alpha_1\alpha_2$, as can easily be seen from (2.1). It follows that $\tilde{s}_E(\mathcal{A})$ is the unique nonnegative solution s to the equation

$$\lim_{m\to\infty}\left(\sum_{w\in W_m}\tilde{\varphi}^s(A_w)\right)^{1/m}=1, \tag{5.4}$$

where $\tilde{\varphi}^s$ is the modified singular value function

$$\tilde{\varphi}^s(A)=\begin{cases}\alpha_1^s, & 0<s\le 1,\\ \alpha_1\alpha_2^{s-1}, & 1<s\le 2,\\ \alpha_1^{s-1}\alpha_2^{s-1}, & 2<s\le 3,\\ \alpha_1^{2s/3}\alpha_2^{2s/3}, & 3<s,\end{cases} \tag{5.5}$$

and $\tilde{\varphi}^0(A)=1$. Note that $\varphi^s=\tilde{\varphi}^s$ for $0\le s\le 2$.

For $\tilde{s}_E(\mathcal{A})\le 3$, the estimate

$$\tilde{s}_E(\mathcal{A})\le\tilde{s}_E(b;\mathcal{A}), \tag{5.6}$$

clearly follows from (1.14) and Theorem 5.3(ii) for a.e. $b\in\mathbb{R}^{2M}$. In fact, (5.6) holds without restriction. This is a purely linear algebraic fact, which can be proved using a minor adaptation of a theorem of Golub [14] on singular values of rank one perturbations of diagonal matrices. For further details, see the appendix.

Next, for a contracting linear map A of $\mathbb{R}^2$ as above with singular values $1>\alpha_1\ge\alpha_2>0$, define the *Heisenberg singular value function* $\psi^s(A)$, $0\le s\le 4$, as

$$\psi^s(A)=\begin{cases}\alpha_1^s, & 0<s\le 1,\\ \alpha_1^{(s+1)/2}\alpha_2^{(s-1)/2}, & 1<s\le 3,\\ \alpha_1^2\alpha_2^{s-2}, & 3<s\le 4,\end{cases} \tag{5.7}$$

and $\psi^0(A)=1$. Note that $\varphi^s=\psi^s$ for $0\le s\le 1$ and $\tilde{\varphi}^s\le\psi^s$ for all $0\le s\le 3$.

Given a family of linear maps $\mathcal{A}=\{A_1,\dots,A_M\}$ on $\mathbb{R}^2$, we define the *Heisenberg critical exponent* $\tilde{s}_H(\mathcal{A})$ as the unique nonnegative solution s to the equation

$$\lim_{m\to\infty}\left(\sum_{w\in W_m}\psi^s(A_w)\right)^{1/m}=1. \tag{5.8}$$

We now restate Theorem 1.16 from the introduction.

Theorem 5.9. *Let $\mathcal{F}(b)$, $b\in\mathbb{R}^{2M}$, be an IFS of affine maps in $\mathbb{R}^2$ and let $\mathcal{F}_H(b,\tau)$, $\tau\in\mathbb{R}^M$, be any horizontal lift to $\mathbb{H}$ as above. Then*

(i) $\dim_H K_H(b,\tau) \le \tilde{s}_H(\mathcal{A})$ *for all* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$*; and*
(ii) *if* $||A_i|| < 1/2$ *for each* i*, then* $\dim_H K_H(b,\tau) = \min\{4, \tilde{s}_H(\mathcal{A})\}$ *for a.e.* $b \in \mathbb{R}^{2M}$ *and* $\tau \in \mathbb{R}^M$*.*

The singular value functions defined in (5.1) and (5.7) should be interpreted as follows. In the Euclidean case, the image of a cube Q of side length 1 in $\mathbb{R}^n$ under A is a rectilinear parallelpiped with sides of length $\alpha_1, \dots, \alpha_n$. The singular value function

$$\varphi^s(A) = \frac{\alpha_1}{\alpha_m} \cdots \frac{\alpha_{m-1}}{\alpha_m} \cdot \alpha_m^s$$

has the following interpretation: the term $(\alpha_1/\alpha_m)\cdots(\alpha_{m-1}/\alpha_m)$ counts (roughly) the number of cubes Q' of side length α_m needed to cover $A(Q)$ and the term α_m^s represents the sth power of the diameter of such a cube Q'.

In the Heisenberg case, the image of $Q \subset \mathbb{R}^3 = \mathbb{H}$ under a lift $\tilde{A}$ of A is a (skewed) parallelpiped, whose base is a rectangle with sides of length α_1 and α_2 and which has Euclidean height $\alpha_1\alpha_2$ and Heisenberg height $\sqrt{\alpha_1\alpha_2}$. In this case, the singular value function

$$\psi^s(A) = \begin{cases} 1 \cdot \alpha_1^s, & 0 < s \le 1, \\ \frac{\alpha_1}{\sqrt{\alpha_1\alpha_2}}(\sqrt{\alpha_1\alpha_2})^s, & 1 < s \le 3, \\ (\frac{\alpha_1}{\alpha_2})^2 \alpha_2^s, & 3 < s \le 4, \end{cases}$$

has a similar interpretation: the terms 1, $\alpha_1/\sqrt{\alpha_1\alpha_2}$ and $(\alpha_1/\alpha_2)^2$ count the number of Heisenberg cubes Q' of the appropriate size needed to cover $\tilde{A}(Q)$ and the final term α_1^s, $(\sqrt{\alpha_1\alpha_2})^s$, or α_2^s represents the sth power of the (Heisenberg) diameter of such a cube Q'.

Measures of Hausdorff type on Σ. Fix $s \ge 0$. Following Falconer [10, Section 4], we define certain measures of Hausdorff type on symbolic space Σ. A collection Λ of finite words is called a *partition* of Σ if Σ is the disjoint union of the cylinder sets Σ_w, $w \in \Lambda$.

Let $\mathcal{A}$ be a finite collection of linear maps in $\mathbb{R}^n$, $n = 2, 3$. For $m \in \mathbb{N}$ and $S \subset \Sigma$ let

$$\mathcal{M}^s_{E,m}(S) := \inf_{\Lambda} \sum_{\substack{w \in \Lambda \\ S \cap \Sigma_w \neq \emptyset}} \varphi^s(A_w),$$

where the infimum is taken over all partitions Λ of Σ with words of length at least m. Next, let

$$\mathcal{M}^s_E(S) = \lim_{m \to \infty} \mathcal{M}^s_{E,m}(S).$$

Then $\mathcal{M}^s_E$ is an outer measure on Σ. The Borel subsets of Σ are $\mathcal{M}^s_E$-measurable, so $\mathcal{M}^s_E$ restricts to a Borel measure on Σ. The technical term for $\mathcal{M}_E^s$ is the *Method II measure* constructed from the premeasure $\tau(\Sigma_w) = \varphi^s(A_w)$ on the net $\{\Sigma_w : w \in W\}$. See Rogers [21] for the relevant definitions and vocabulary.

In a similar manner, we define $\widetilde{\mathcal{M}}_{H,m}^s$ and $\widetilde{\mathcal{M}}_H^s$ by replacing φ^s in the above equation with ψ^s. Then $\widetilde{\mathcal{M}}_H^s$ is again a Method II Borel net measure on Σ.

By Proposition 4.1 of [10], the Euclidean critical exponent $s_E(\mathcal{A})$ defined via (5.2) is also equal to

$$\inf\{s : \mathcal{M}_E^s(\Sigma) = 0\} = \sup\{s : \mathcal{M}_E^s(\Sigma) = \infty\}.$$

In a similar manner, we show the following result.

Proposition 5.10. *The Heisenberg critical exponent $\tilde{s}_H(\mathcal{A})$ defined via (5.8) is equal to*

$$\inf\{s : \widetilde{\mathcal{M}}_H^s(\Sigma) = 0\} = \sup\{s : \widetilde{\mathcal{M}}_H^s(\Sigma) = \infty\}.$$

The proof is completely analogous to the proof of [10, Proposition 4.1] and will be omitted. The relevant features of the singular value function ψ^s which are necessary for the proof are:

(i) $\psi^s(A_w)$ is submultiplicative in w: $\psi^s(A_{ww'}) \le \psi^s(A_w)\psi^s(A_{w'})$,
(ii) $\psi^s(A_w)$ is decreasing in s.

These properties are easily proved using the definition of ψ^s.

The following technical result on Method II net measures will be used in the proof of Theorem 1.16. The case $\mu = \mathcal{M}_E^s$, $\tau(\Sigma_w) = \varphi^s(A_w)$ is Lemma 4.2 in [10], but the result holds for any Method II net measure μ on Σ satisfying the assumptions. In particular, it holds for $\mu = \widetilde{\mathcal{M}}_H^s$, $\tau(\Sigma_w) = \psi^s(A_w)$. Compare Theorem 54 of [21].

Lemma 5.11. *Let $\mu = \sup_{\delta>0} \mu_\delta$ be a nonatomic Method II net measure on Σ of infinite total mass, defined from a finite premeasure τ on the cylinder sets $\{\Sigma_w : w \in W\}$. Assume that $\mu_\delta(C_j) \to 0$ as $j \to \infty$ for every $\delta > 0$ and every sequence $C_1 \supset C_2 \supset \cdots$ of compact subsets of Σ with $\mu(\cap_j C_j) = 0$.*

Then there exists a compact subset $C_0 \subset \Sigma$ so that $0 < \mu(C_0) < \infty$ and there exists a constant $C < \infty$ so that

$$\mu(C_0 \cap \Sigma_w) \le C\tau(\Sigma_w) \tag{5.12}$$

for all $w \in W$.

The following example shows that the second inequality could be strict in (3.1) for self-affine fractals.

Example 5.13. Fix integers $n \geq p \geq 2$ and consider the planar AIFS $\mathcal{F} = \{f_{11}, \dots, f_{np}\}$, where $f_{ij}(x_1, x_2) = ((x_1 + i)/n, (x_2 + j)/p)$. The invariant set for $\mathcal{F}$ is the unit square $Q = [0,1]^2$, viewed as the self-affine set obtained by gluing together np rectangles with sides of length $1/n$ and $1/p$. In this case $A_{ij} = \begin{pmatrix} 1/n & 0 \\ 0 & 1/p \end{pmatrix}$ and $b_{ij} = \begin{pmatrix} i/n \\ j/p \end{pmatrix}$. For $w \in W_m = \{1, \dots, np\}^m$, the singular values of A_w are p^{-m} and n^{-m}. Then

$$\lim_{m\to\infty} \left(\sum_{w \in W_m} \psi^s(A_w) \right)^{1/m} = \begin{cases} np^{1-s}, & 0 \leq s \leq 1, \\ n^{(3-s)/2} p^{(1-s)/2}, & 1 \leq s \leq 3, \\ n^{3-s} p^{-1}, & 3 \leq s \leq 4. \end{cases}$$

Thus

$$\dim_H K_H(\tau) \leq \tilde{s}_H(\mathcal{A}) = 1 + \frac{2 \log n}{\log(np)}$$

for any Heisenberg lift $\mathcal{F}_H(\tau)$ of $\mathcal{F}$. Note that $\tilde{s}_H(\mathcal{A}) = 2$ only in the self-similar case $n = p$.

From (5.5) it easily follows that $\tilde{s}_E(\mathcal{A}) = s_E(\mathcal{A}) = 2$. Thus

$$\dim_E K_H(\tau) = 2$$

for all τ.

Remark 5.14. In a subsequent paper [11], Falconer derived lower bounds for $\dim_E K(b)$ which hold for every b. Let $s_- = s_-(A_1, \dots, A_M)$ be the unique nonnegative solution to the equation

$$\lim_{m\to\infty} \left(\sum_{w \in W_m} \varphi^s(A_w^{-1})^{-1} \right)^{1/m} = 1. \tag{5.15}$$

Then [11, Proposition 2] reads as follows:

Proposition 5.16. *If $\mathcal{F}$ satisfies the disjointness condition $f_i(K(b)) \cap f_j(K(b)) = \emptyset$ for every $i \neq j$, then*

$$\dim_E K(b) \geq s_-. \tag{5.17}$$

Note that the open set condition does not suffice to imply (5.17); see [11, Example 2] for an example of an AIFS $\mathcal{F}$ in $\mathbb{R}^2$ such that $s_- > 0$ but $K(b)$ is a single point.

The claim regarding the Euclidean dimension of the horizontal lift in Example 3.2 may be proved using Proposition 5.16.

6. The self-affine case II

In this section, we give the proofs of Theorems 1.15 and 1.16. To simplify the exposition, we will present the proofs of the first parts of both theorems together, followed by the proofs of the second parts. In each case, we present in detail the proof for the Heisenberg dimension (Theorem 1.16) and only sketch how this proof should be modified for the Euclidean dimension (Theorem 1.15).

Proofs of the first parts in Theorems 1.15 and 1.16. Fix $b \in \mathbb{R}^{2M}$ and $\tau \in \mathbb{R}^M$ and let $s > \tilde{s}_H(\mathcal{A})$. We will show that

$$\mathcal{H}^s_H(K_H(b,\tau)) \le C\widetilde{\mathcal{M}}^s_H(\Sigma) \tag{6.1}$$

for some absolute constant C. Since $\widetilde{\mathcal{M}}^s_H(\Sigma) = 0$ by Proposition 5.10, this suffices to complete the proof of Theorem 1.16(i).

Let $0 < \alpha < 1$ be so that

$$d_H(F_i(p), F_i(q)) < \alpha d_H(p,q)$$

for $p, q \in \mathbb{H}$ and $i = 1, \ldots, M$. Let $B = B_H(0,R) \subset \mathbb{H}$ be a Heisenberg ball centered at the origin of radius R, chosen so large that $F_i(B) \subset B$ for all i. Given $\delta > 0$, choose m so large that $\alpha^m < \delta$.

Let Λ be an arbitrary partition of Σ by words of length at least m. By the choice of m, $\operatorname{diam}_H F_w(B) < \delta$ for all $w \in \Lambda$. For each $w \in \Lambda$, we may write $F_w(x,t) = \tilde{A}_{w,b_w}(x,t) + \tilde{b}_w$, where $\tilde{A}_{w,b_w}$ and $\tilde{b}_w$ are given by the formulas in (2.1). Denoting by $\alpha_{i,1} \ge \alpha_{i,2}$ the singular values of A_i, $i = 1, \ldots, M$, the singular values of A_w are $\alpha_{w,1} \ge \alpha_{w,2}$, where

$$\alpha_{w,j} \le \prod_{i=1}^m \alpha_{w_i,j} \le \alpha^m$$

for any word w of length m.

In what follows we fix a word w and write $\alpha_j = \alpha_{w,j}$, $j = 1, 2$. Let Q_w be a rectangle containing $\pi(F_w(B)) = f_w(\pi(B))$ with sides of length $3R\alpha_1$ and $3R\alpha_2$. Observe that $b_w = f_w(0) \in Q_w$. Then $F_w(B) \subset \tilde{Q}_w$, where

$$\tilde{Q}_w = \left\{ (x,t) : \begin{array}{c} x \in Q_w, \\ |t + 2\langle x, Jb_w\rangle - \tau_w| < R^2\alpha_1\alpha_2 \end{array} \right\}$$

is a parallelpiped with base $Q_w \in \mathbb{R}^2$ and (Euclidean) height $2R^2\alpha_1\alpha_2$.

If (x,t) and (x',t') are elements of $\tilde{Q}_w$, then

$$|x' - x| \le \operatorname{diam} Q_w \le 3\sqrt{2}R\alpha_1 \tag{6.2}$$

and

$$|t' - t - 2\langle x, Jx'\rangle| \le 2R^2\alpha_1\alpha_2 + 2|\langle x' - x, J(b_w - x)\rangle|. \tag{6.3}$$

We distinguish three cases according to the value of s.

Case 1 ($0 \le s \le 1$). From (6.2) we have that $|x - x'| \le CR\alpha_1$. Since $b_w \in Q_w$ we obtain likewise that $|b_w - x| \le CR\alpha_1$. Using (6.3) we deduce

$$|t' - t - 2\langle x, Jx'\rangle| \le CR^2\alpha_1^2.$$

Using this and (6.2) in (1.2) we obtain

$$\mathrm{diam}_H \tilde{Q}_w \le CR\alpha_1.$$

The sets $\tilde{Q}_w$, $w \in \Lambda$, cover $K_H(b,\tau)$ and we obtain

$$\mathcal{H}^s_{H,CR\delta}(K_H(b,\tau)) \le C(R,s)\sum_{w\in\Lambda}\alpha_1^s = C(R,s)\sum_{w\in\Lambda}\psi^s(A_w)$$

by the choice of m.

Observe that in this estimate the dependence of b does not appear at all; this will also happen in the other cases.

Case 2 ($1 \le s \le 3$). In this case we divide $\tilde{Q}_w$ into at most $K := 2\alpha_1/\sqrt{\alpha_1\alpha_2}$ smaller parallelpipeds $\tilde{P}_j$ whose base is a rectangle P_j in $\mathbb{R}^2$ with sides of length $3R\sqrt{\alpha_1\alpha_2}$ and $3R\alpha_2$ and whose (Euclidean) height is still $2R^2\alpha_1\alpha_2$.

Our task is to estimate the Heisenberg diameter of such a parallelpiped $\tilde{P}_j$. Let (x,t) and (x',t') be elements of $\tilde{P}_j$. Then

$$|x' - x| \le 3\sqrt{2}\,R\sqrt{\alpha_1\alpha_2}$$

and (by (6.3))

$$|t' - t - 2\langle x, Jx'\rangle| \le 2R^2\alpha_1\alpha_2 + 2|\langle x' - x, J(b_w - x)\rangle|.$$

Observe that the expression $|\langle x' - x, J(b_w - x)\rangle|$ equals twice the area of the planar triangle with vertices x', x, b_w which lies in Q_w. This yields

$$|\langle x' - x, J(b_w - x)\rangle| \le 2\alpha_1\alpha_2,$$

and so

$$\mathrm{diam}_H \tilde{P}_j \le CR\sqrt{\alpha_1\alpha_2}$$

using (1.2).

The sets $\tilde{P}_j$, $j = 1, \dots, K$, associated with each $\tilde{Q}_w$, $w \in \Lambda$, cover $K_H(b,\tau)$ and we obtain

$$\mathcal{H}^s_{H,CR\delta}(K_H(b,\tau)) \le C(R,s)\sum_{w\in\Lambda}\sum_{j=1}^{K}(\sqrt{\alpha_1\alpha_2})^s \le C(R,s)\sum_{w\in\Lambda}\psi^s(A_w)$$

by the choices of m and K.

Case 3 ($3 \le s \le 4$). As in the previous case we begin by dividing $\tilde{Q}_w$ into at most $N := 2\alpha_1/\alpha_2$ smaller parallelpipeds $\tilde{P}_j$ whose base is a square P_j in $\mathbb{R}^2$ with side length $3R\alpha_2$ and whose (Euclidean) height is at most $40R^2\alpha_1\alpha_2$. Explicitly, let c_{wj} be the center of the square P_j and let $\tilde{P}_j$ be the set of points $(x,t) \in \tilde{Q}_w$ for which $x \in P_j$ and

$$|t + 2\langle x - b_w, Jc_{wj}\rangle - \tau_w| < 40R^2\alpha_1\alpha_2. \tag{6.4}$$

Observe that

$$\tilde{Q}_w \subset \bigcup_{j=1}^{N} \tilde{P}_j.$$

Indeed, $(x,t) \in \tilde{Q}_w$ and $x \in P_j$ imply that

$$\begin{aligned} |t + 2\langle x - b_w, Jc_{wj}\rangle - \tau_w| &\le |t + 2\langle x, Jb_w\rangle - \tau_w| + 2|\langle x - b_w, J(c_{wj} - x)\rangle| \\ &\le R^2\alpha_1\alpha_2 + 2(3\sqrt{2}R\alpha_1)(3\sqrt{2}R\alpha_2) \\ &< 40R^2\alpha_1\alpha_2. \end{aligned}$$

Next we show that we can cover $\tilde{P}_j$ by at most $2N$ Heisenberg balls of the form $B_H(p_{jk}, CR\alpha_2)$ with centers

$$p_{jk} = (c_{wj}, t_{jk}),$$

where

$$t_{jk} = \tau_w + 2\langle b_w, Jc_{wj}\rangle + 20kR^2\alpha_2^2, \tag{6.5}$$

for $k = -N, \ldots, N$.

Indeed, if $(x,t) \in \tilde{P}_j$ then $x \in P_j$ and so $|x - c_{wj}| \le CR\alpha_2$.

By (6.4) there exists an integer $k \in [-N, N]$ such that

$$20 \cdot (k-1)R^2\alpha_2^2 \le t + 2\langle x - b_w, Jw_j\rangle - \tau_w \le 20 \cdot kR^2\alpha_2^2.$$

Using (6.5) this implies that for large enough $C > 0$ we have

$$|t - t_{jk} + 2\langle x - c_{wj}, Jc_{wj}\rangle| \le CR^2\alpha_2^2$$

and so

$$|t - t_{jk} + 2\langle x, Jc_{wj}\rangle| \le CR^2\alpha_2^2.$$

From (1.2) we deduce

$$d_H((x,t), (c_{wj}, t_{jk})) \le CR\alpha_2,$$

as required.

The balls $B_H(p_{jk}, CR\alpha_2)$, $j = 1, \dots, N$, $k = -N, \dots, N$, associated with each $\tilde{Q}_w$, $w \in \Lambda$, cover $K_H(b, \tau)$ and we obtain

$$\begin{aligned}\mathcal{H}^s_{H,CR\delta}(K_H(b,\tau)) &\le C(R,s) \sum_{w\in\Lambda} \sum_{j=1}^{N} \sum_{k=-N}^{N} \alpha_2^s \\ &\le C(R,s) \sum_{w\in\Lambda} \psi^s(A_w)\end{aligned}$$

by the choices of m and N.

In all cases, we have shown that

$$\mathcal{H}^s_{H,CR\delta}(K_H(b,\tau)) \le C(R,s) \sum_{w\in\Lambda} \psi^s(A_w).$$

Taking the infimum over partitions Λ followed by the limit as $m \to \infty$ yields (6.1). This completes the proof of Theorem 1.16(i).

For Theorem 1.15(i) only a few modifications need to be made in the above reasoning. In case 1, the Euclidean diameter of $\tilde{Q}_w$ is at most $CR\alpha_1$ and so

$$\mathcal{H}^s_{E,CR\delta}(K_H(b,\tau)) \le C(R,s) \sum_{w\in\Lambda} \alpha_1^s \le C(R,s) \sum_{w\in\Lambda} \varphi^s(\tilde{A}_{w,b_w}).$$

Case 2 must be split into two subcases: either $1 \le s \le 2$ or $2 \le s \le 3$. In the former case we divide $\tilde{Q}_w$ into at most $N = 2\alpha_1/\alpha_2$ parallelpipeds whose base is a square of side length $3R\alpha_2$ and estimate

$$\mathcal{H}^s_{E,CR\delta}(K_H(b,\tau)) \le C(R,s) \sum_{w\in\Lambda} \sum_{j=1}^{N} \alpha_2^s \le C(R,s) \sum_{w\in\Lambda} \varphi^s(\tilde{A}_{w,b_w}).$$

In the latter case we divide $\tilde{Q}_w$ into at most $P = 2/\alpha_1\alpha_2$ parallelpipeds whose base is a square of side length $3R\alpha_1\alpha_2$ and estimate

$$\mathcal{H}^s_{E,CR\delta}(K_H(b,\tau)) \le C(R,s) \sum_{w\in\Lambda} \sum_{j=1}^{P} (\alpha_1\alpha_2)^s \le C(R,s) \sum_{w\in\Lambda} \varphi^s(\tilde{A}_{w,b_w}).$$

As before, these estimates suffice to complete the proof of Theorem 1.15(i).

Proofs of the second parts in Theorems 1.15 and 1.16. As is typical with problems involving the computation of Hausdorff dimension, obtaining lower bounds is the more difficult case. Following the technique employed by Falconer in the Euclidean case (Theorem 1.13), we use potential-theoretic arguments to obtain almost sure lower bounds.

We begin with a simple lemma which illustrates a geometric interpretation of the Heisenberg singular value function. Compare Lemma 2.2 of [10].

Lemma 6.6. *Let $0 < s < 4$, $s \neq 1,3$. For each $R > 0$ there exists a constant C depending only on R and s so that*

$$\int_{B_H(0,R)} \frac{dp}{|\tilde{A}_b(p)|_H^s} \leq \frac{C}{\psi^s(A)}, \tag{6.7}$$

where, for $p \in \mathbb{H}$, $B_H(p,r)$ denotes the ball in the Heisenberg metric d_H of radius r and $|p|_H = d_H(p,0)$.

In the proof of Lemma 6.6 the following fact (whose proof is an easy exercise) will be used several times.

Lemma 6.8. *Let $F : \mathbb{R} \to \mathbb{R}$ be an even function which is decreasing for $\tau > 0$. Let $c \in \mathbb{R}$ and $h > 0$. Then $\int_{c-h}^{c+h} F(\tau)\,d\tau \leq \int_{-h}^{h} F(\tau)\,d\tau$.*

Proof of Lemma 6.6. From the expression for $\tilde{A}_b$ in (2.1) we observe that the integrand in (6.7) equals

$$\left(|Ax|^4 + (t \det A - 2\langle Ax, Jb\rangle)^2\right)^{-s/4}, \tag{6.9}$$

where $p = (x,t)$. Choose coordinates in the base space $\mathbb{R}^2$ so that $|Ax|^2 = \alpha_1^2 x_1^2 + \alpha_2^2 x_2^2$. Writing $b = (p,q)$ in these coordinates, we express (6.9) in the form

$$\left((\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2)^2 + (\alpha_1\alpha_2 t + 2\alpha_1 x_1 q - 2\alpha_2 x_2 p)^2\right)^{-s/4}.$$

Make the change of variables $y_i = \alpha_i x_i / R$, $i = 1,2$ and

$$\tau = \frac{\alpha_1\alpha_2 t + 2\alpha_1 x_1 q - 2\alpha_2 x_2 p}{R^2}$$

in the integral in (6.7) to obtain

$$\frac{R^{4-s}}{\alpha_1^2\alpha_2^2} \int_E \int_{I_y} \frac{d\tau\,dy}{(|y|^4 + \tau^2)^{s/4}}, \tag{6.10}$$

where $E = \{y : (y_1/\alpha_1)^2 + (y_2/\alpha_2)^2 < 1\}$,

$$I_y = \{\tau : |\tau + uy_2 - vy_1| < \alpha_1\alpha_2\}$$

and $(u,v) = (2/R)(p,q)$.

As before we distinguish three cases according to the value of s.

Case 1 $(0 < s < 1)$**.** The integral in (6.10) may be estimated from above by

$$\frac{C(R,s)}{\alpha_1^2\alpha_2^2}(\alpha_2)(\alpha_1\alpha_2)\int_0^{\alpha_1}\frac{dy_1}{y_1^s} = \frac{C(R,s)}{\psi^s(A)}.$$

Case 2 $(1 < s < 3)$**.** The region of integration in (6.10) may be written as $\cup_{y\in E} I_y = P_1 \cup P_2$, where

$$P_1 = \{(y,\tau) \in \cup_{y\in E} I_y : y_1^4 + (\tau + uy_2 - vy_1)^2 \le 2\alpha_1^2\alpha_2^2\}$$

and

$$P_2 = \{(y,\tau) \in \cup_{y\in E} I_y : y_1 > \sqrt{\alpha_1\alpha_2}\}.$$

Write the integral in (6.10) in the form $(R^{4-s}\alpha_1^{-2}\alpha_2^{-2})(I_1 + I_2)$, where

$$I_j = \int_{P_j} (|y|^4 + \tau^2)^{-s/4} dy d\tau \ , \ j = 1, 2 \ .$$

For the first term, we use Lemma 6.8 to estimate

$$\begin{aligned} I_1 &\le \int_{-2\sqrt{\alpha_1\alpha_2}}^{2\sqrt{\alpha_1\alpha_2}} dy_1 \int_{-\alpha_2}^{\alpha_2} dy_2 \int_{-\alpha_1\alpha_2}^{\alpha_1\alpha_2} d\tau\, (y_1^4 + \tau^2)^{-s/4} \\ &\le C\,\alpha_2 \int_0^{2\sqrt{\alpha_1\alpha_2}} dy_1 \int_0^{\alpha_1\alpha_2} d\tau (y_1^4 + \tau^2)^{-s/4}. \end{aligned}$$

(Note that $|y_1| \le 2\sqrt{\alpha_1\alpha_2}$ for any $(y,\tau) \in P_1$.)

Making the following Heisenberg change of variables $y_1 = r\sqrt{\cos\phi}$, $\tau = r^2 \sin\phi$ (cf. the polar coordinates in [6]) we find

$$I_1 \le C\alpha_2 \int_0^{2\sqrt{\alpha_1\alpha_2}} r^{2-s}\, dr = C\alpha_1^{(3-s)/2}\alpha_2^{(5-s)/2}.$$

Similarly, we obtain the estimate

$$I_2 \le C\,\alpha_2(\alpha_1\alpha_2)\int_{\sqrt{\alpha_1\alpha_2}}^{\infty} y_1^{-s}\, dy_1 = C\alpha_1^{(3-s)/2}\alpha_2^{(5-s)/2}.$$

Returning to (6.10) we see that

$$\frac{R^{4-s}}{\alpha_1^2\alpha_2^2}\int_E\int_{I_y}\frac{d\tau\, dy}{(|y|^4 + \tau^2)^{s/4}} \le \frac{C(R,s)}{\psi^s(A)}$$

as desired.

Case 3 $(3 < s < 4)$**.** This is similar to the previous case. We write $\cup_{y\in E} I_y = P_1 \cup P_2$, where

$$P_1 = \{(y,\tau) \in \cup_{y\in E} I_y : |y|^4 + (\tau + uy_2 - vy_1)^2 \le 4\alpha_2^4\}$$

and

$$P_2 = \{(y,\tau) \in \cup_{y\in E} I_y : y_1^4 + (\tau + uy_2 - vy_1)^2 > \alpha_2^4\},$$

and decompose the integral in (6.10) as before into I_1 and I_2 terms. In this case, another application of Lemma 6.8 gives

$$I_1 \le \int_{-2\alpha_2}^{2\alpha_2} dy_1 \int_{-2\alpha_2}^{2\alpha_2} dy_2 \int_{-2\alpha_2}^{2\alpha_2} d\tau\, (|y|^4 + \tau^2)^{-s/4},$$

and making the change of variables for integration in Heisenberg polar coordinates $y_1 = r\sqrt{\cos\phi}\cos\theta$, $y_2 = r\sqrt{\cos\phi}\sin\theta$, $\tau = r^2\sin\phi$ yields

$$I_1 \le C\int_0^{2\alpha_2} r^{3-s}\, dr = C\alpha_2^{4-s}.$$

In a similar manner we obtain the estimate

$$I_2 \le C\,\alpha_2 \int_{\alpha_2}^{\infty} r^{2-s}\, dr = C\alpha_2^{4-s}$$

and hence

$$\frac{R^{4-s}}{\alpha_1^2\alpha_2^2}\int_E\int_{I_y}\frac{dy\, d\tau}{(|y|^4+\tau^2)^{s/4}} \le \frac{C(R,s)}{\psi^s(A)}$$

as desired. □

Next, we consider products of matrices indexed by words in Σ. We use the following notation: for an IFS $\mathcal{F}(b)$ in $\mathbb{R}^2$ with horizontal lift $\mathcal{F}_H(\tilde{b})$ we write

$$p_H(\tilde{b}) : \Sigma \to K_H(\tilde{b})$$

and

$$p_E(b) : \Sigma \to K(b)$$

for the canonical surjections from Σ to the invariant sets. Thus

$$p_H(\tilde{b}, w) = \bigcap_{m=1}^{\infty} F_{w_1} \circ \cdots \circ F_{w_m}(K_H(\tilde{b}))$$

and

$$p_E(b, w) = \bigcap_{m=1}^{\infty} f_{w_1} \circ \cdots \circ f_{w_m}(K(b)),$$

where $w = w_1 w_2 \cdots \in \Sigma$ and $f_i(x) = A_i x + b_i$, $F_i(x,t) = \tilde{A}_{i,b_i}(x,t) + \tilde{b}_i$.

Observe that

$$p_H(\tilde{b}, w) = \tilde{b}_{w_1} + \tilde{A}_{w_1,b_{w_1}} \cdot \tilde{b}_{w_2} + \tilde{A}_{w_1,b_{w_1}} \cdot \tilde{A}_{w_2,b_{w_2}} \cdot \tilde{b}_{w_3} + \cdots . \tag{6.11}$$

For w and w' in Σ denote by $w \wedge w'$ the maximal finite word which is a subword of both w and w'.

Lemma 6.12. *Assume that* $||A_i|| < 1/2$ *for each* i*. For* $0 < s < 4$*,* $s \neq 1, 3$*, and* $R > 0$ *there exists a constant* $C = C(R, s)$ *so that*

$$\int_{B_H(0,R)^M} \frac{d\tilde{b}}{d_H(p_H(\tilde{b}, w), p_H(\tilde{b}, w'))^s} \leq \frac{C}{\psi^s(A_{w \wedge w'})}$$

for all $w, w' \in \Sigma$*. Here* $B_H(0, R)^M = B_H(0, R) \times \ldots \times B_H(0, R) \subseteq \mathbb{R}^{3M}$ *and* $\tilde{b} = \tilde{b}_1, \ldots, \tilde{b}_2 \in \mathbb{R}^{3M}$*.*

Proof. Write $w \wedge w' = \alpha \in W$ and set $w = \alpha v$ and $w' = \alpha v'$, $v, v' \in \Sigma$. Then

$$\begin{aligned} &\int_{B_H(0,R)^M} \frac{d\tilde{b}}{d_H(p_H(\tilde{b}, w), p_H(\tilde{b}, w'))^s} \\ &\qquad = \int_{B_H(0,R)^M} \frac{d\tilde{b}}{|\tilde{A}_{\alpha, b_\alpha}(p_H(\tilde{b}, v)^{-1} * p_H(\tilde{b}, v'))|_H^s}. \end{aligned} \tag{6.13}$$

By the choice of α, $v_1 \neq v_1'$. Without loss of generality we may assume $v_1 = 2$ and $v_1' = 1$.

With (6.11) in mind we make the change of variable

$$\begin{aligned} q = p_H(\tilde{b}, v)^{-1} * p_H(\tilde{b}, v') &= \begin{pmatrix} b_1 - b_2 + E(b) \\ \tau_1 - \tau_2 + F(\tau) + G(b) \end{pmatrix}, \\ \tilde{b}_2 &= \tilde{b}_2, \\ &\vdots \\ \tilde{b}_M &= \tilde{b}_M \end{aligned} \tag{6.14}$$

where $E : \mathbb{R}^{2M} \to \mathbb{R}^2$, $E(b) = E_1(b_1) + \cdots + E_M(b_M)$ and $F : \mathbb{R}^M \to \mathbb{R}$ are linear maps and $G : \mathbb{R}^{2M} \to \mathbb{R}$ is a quadratic map.

We claim that

$$||E_\nu|| < 1 \tag{6.15}$$

for some $\nu = 1, 2$ and

$$||F|| < 1. \tag{6.16}$$

Taking (6.15) and (6.16) for granted observe that the preceding change of variables is invertible. Consequently we obtain

$$\begin{aligned}\int_{B_H(0,R)^M} &\frac{d\tilde{b}}{d_H(p_H(\tilde{b},w),p_H(\tilde{b},w'))^s} \\ &\le C\int_{B_H(0,(2+M)R)} dq \int_{B_H(0,R)^{M-1}} d\tilde{b}_2\cdots d\tilde{b}_M \frac{1}{|\tilde{A}_{\alpha,b_\alpha}(q)|_H^s} \\ &\le \frac{C(R,s)}{\psi^s(A_\alpha)}\end{aligned}$$

by Lemma 6.6.

It remains to describe the maps E, F, G explicitly and show (6.15) and (6.16).

Using (6.11) a direct computation yields

$$p_H(\tilde{b},v)^{-1} * p_H(\tilde{b},v') = \begin{pmatrix} X \\ T \end{pmatrix},$$

where

(6.17)

$$\begin{aligned}X &= p_E(b,v') - p_E(b,v) \\ &= b_{v_1'} - b_{v_1} + (A_{v_1'}b_{v_2'} - A_{v_1}b_{v_2} + A_{v_1'}A_{v_2'}b_{v_3'} - A_{v_1}A_{v_2}b_{v_3} + \cdots)\end{aligned}$$

(see equation (3.7) in [10]) and

(6.18)

$$\begin{aligned}T &= (\tau_{v_1'} - \tau_{v_1}) + (\lambda_{v_1'}\tau_{v_2'} - \lambda_{v_1}\tau_{v_2}) + (\lambda_{v_1'}\lambda_{v_2'}\tau_{v_3'} - \lambda_{v_1}\lambda_{v_2}\tau_{v_3}) + \cdots \\ &\quad - 2\left(\langle A_{v_1'}b_{v_2'}, Jb_{v_1'}\rangle - \langle A_{v_1}b_{v_2}, Jb_{v_1}\rangle + \cdots\right) \\ &\quad - 2\left(\langle b_{v_1'}, Jb_{v_1}\rangle + \langle A_{v_1'}b_{v_2'}, Jb_{v_1}\rangle + \langle b_{v_1'}, JA_{v_1}b_{v_2}\rangle + \cdots\right),\end{aligned}$$

where $\lambda_{v_i} = \det A_i$. Observe that the last term in equation (6.18) is the contribution to T from the term $-2\langle p_E(\tilde{b},v), Jp_E(\tilde{b},v')\rangle$.

We may choose ν equal to either 1 or 2 and an index $2 \le m \le \infty$ so that the following conditions hold: (i) for each $k < m$, both v_k and v_k' are not equal to ν, and (ii) if $m < \infty$, then $v_m \ne \nu$ and $v_m' \ne \nu$.

From (6.17) and (6.18) we have

$$X = b_1 - b_2 + E(b) = b_1 - b_2 + E_1(b_1) + \cdots E_M(b_M)$$

and

$$T = \tau_1 - \tau_2 + F(\tau) + G(b),$$

where the E_i are linear maps on $\mathbb{R}^{2M}$ with values in $\mathbb{R}^2$, F is a real-valued linear map on $\mathbb{R}^M$, and G is a real-valued quadratic map on

$\mathbb{R}^{2M}$. With the choice $\eta = \max_{i=1,\dots,M} ||A_i||$ we find

$$||E_\nu|| \le \sum_{k=2}^{m-1} \eta^{k-1} + \sum_{k=m+1}^{\infty} 2\eta^{k-1} \le \frac{\eta}{1-\eta}$$

and

$$||F|| \le \sum_{k=1}^{\infty} 2\eta^{2k} = \frac{2\eta^2}{1-\eta^2}.$$

(Observe that it is only necessary to ensure the invertibility of E and F in order to perform the change of variables (6.14) in (6.13). No restriction on G is needed.) The restriction $\eta < 1/2$ guarantees (6.15) and (6.16) and completes the proof of the lemma. □

To obtain the lower bounds for the dimension in Theorems 1.15(ii) and 1.16(ii) we use the following well-known connection between Hausdorff dimension and measures with finite energy:

> If A is a subset of a complete metric space $X = (X, d)$ which supports a Borel measure ν with $0 < \nu(A) < \infty$ whose s-energy $\int\int d(x,y)^{-s}\, d\nu(x) d\nu(y)$ is finite, then $\dim A \ge s$.

For the case $X = \mathbb{R}^n$ see e.g. [19, Theorem 8.7] or [9, Corollary 6.6]. The general case is similar.

The following lemma is the Heisenberg version of Lemma 5.2 in [10]. Its proof is entirely analogous to the proof of the result from [10] and will be omitted.

Lemma 6.19. *Let μ be a Borel measure on Σ with $0 < \mu(\Sigma) < \infty$ for which*

$$\int_\Sigma \int_\Sigma \int_{B_H(0,R)^M} \frac{d\tilde{b}\, d\mu(w)\, d\mu(w')}{d_H(p_H(\tilde{b}, w), p_H(\tilde{b}, w'))^t} < \infty$$

for some $t < 4$ and some $R < \infty$. Then

$$\dim_H K_H(b, \tau) \ge t$$

for almost every $\tilde{b} = (b, \tau) \in B_H(0, R)^M \subset \mathbb{H}^M$.

Proof of Theorem 1.15(ii). Fix $R > 0$. Let t be a real number so that $0 < t < \min\{4, \tilde{s}_H(\mathcal{A})\}$, $t \ne 1, 3$. We will verify the assumptions of Lemma 6.19 for such choice of t.

Fix s so that $t < s < \min\{4, \tilde{s}_H(\mathcal{A})\}$. Then $\widetilde{\mathcal{M}}^s_H(\Sigma) = \infty$. By Lemma 5.11, there is a compact set $C_0 \subset \Sigma$ and a constant $C < \infty$ so that $0 < \widetilde{\mathcal{M}}^s_H(C_0) < \infty$ and

$$\mu(\Sigma_w) \le C\psi^s(A_w), \quad w \in W, \tag{6.20}$$

where μ is the Borel measure on Σ given by $\mu(A) := \widetilde{\mathcal{M}}^s_H(C_0 \cap A)$. By Lemma 6.12,

$$\begin{aligned} I &:= \int_\Sigma \int_\Sigma \int_{B_H(0,R)^M} \frac{d\tilde{b}\, d\mu(w)\, d\mu(w')}{d_H(p_H(\tilde{b},w), p_H(\tilde{b},w'))^t} \\ &\le C \int_\Sigma \int_\Sigma \frac{d\mu(w)\, d\mu(w')}{\psi^t(A_{w\wedge w'})}, \end{aligned}$$

and by the definition of the cylinder sets Σ_w and (6.20) we obtain

$$\begin{aligned} I &\le C \sum_{\alpha\in W} \sum_{i\ne j} \frac{\mu(\Sigma_{\alpha i})\mu(\Sigma_{\alpha j})}{\psi^t(A_\alpha)} \le C \sum_{w\in W} \frac{\mu(\Sigma_w)^2}{\psi^t(A_w)} \\ &\le C \sum_{m=1}^\infty \sum_{w\in W_m} \frac{\psi^s(A_w)\mu(\Sigma_w)}{\psi^t(A_w)}. \end{aligned}$$

From the definition (5.7) of the Heisenberg singular value function, we see that

$$\psi^s(A_w) \le \psi^t(A_w)\alpha_1(A_w)^{s-t}$$

for all contractive linear A_w. The submultiplicativity of the singular value function gives

$$\alpha_1(A_w) \le \alpha_1(A_{w_1}) \cdots \alpha_1(A_{w_m}), \quad w = w_1 \cdots w_m.$$

Fix $a < 1$ so that $\alpha_1(A_i) \le a$ for all i. Then

$$\psi^s(A_w) \le a^{m(s-t)}\psi^t(A_w)$$

and

$$I \le C \sum_{m=1}^\infty a^{m(s-t)} \sum_{w\in W_m} \mu(\Sigma_w) \le C\mu(\Sigma) < \infty$$

since $a < 1$, $t < s$ and $\mu(\Sigma) = \widetilde{\mathcal{M}}^s_H(C_0) < \infty$. By Lemma 6.19 we conclude that $\dim_H K_H(b,\tau) \ge t$ for almost every $\tilde{b} = (b,\tau) \in B_H(0,R)^M$. Letting $R \to \infty$ gives the result for almost every $\tilde{b} \in \mathbb{H}^M$, and letting $t \nearrow \min\{4, \tilde{s}_H(\mathcal{A})\}$ through values in $(0,4) \setminus \{1,3\}$ finishes the proof. □

For the Euclidean case (Theorem 1.15(ii)) the derivation is very similar. It uses the following modified versions of Lemmas 6.6 and 6.12.

Lemma 6.21. *Let $0 < s < 3$ be nonintegral. For each $R > 0$ there exists a constant C depending only on R and s so that*

$$\int_{B_E(0,R)} \frac{dp}{|\tilde{A}_b(p)|_E^s} \le \frac{C}{\varphi^s(\tilde{A}_b)}.$$

Lemma 6.22. *Assume that* $||A_i|| < 1/2$ *for each* i. *For* $0 < s < 3$ *nonintegral and* $R > 0$, *there exists a constant* $C = C(R, s)$ *so that*

$$\int_{B_E(0,R)^M} \frac{d\tilde{b}}{|p_H(\tilde{b}, w) - p_H(\tilde{b}, w')|_E^s} \le \frac{C}{\varphi^s(\tilde{A}_{w\wedge w', b_{w\wedge w'}})}$$

for all $w, w' \in \Sigma$.

The proofs of these lemmas are easy variations on the earlier proofs for the Heisenberg versions.

With these lemmas in hand the remainder of the proof proceeds in a manner similar to the previous case. We omit the details.

7. Appendix: eigenvalues of rank one perturbations of block diagonal matrices

This appendix is devoted to discussion and a sketch of an elementary proof of (5.6) using only techniques from linear algebra. The key result (Theorem 7.4) concerns deformation of the singular values of block diagonal matrices under rank one perturbations. We begin with a classical theorem of Golub.

Theorem 7.1 (Golub [14]). *Let* $D \in \mathbb{R}^{n\times n}$ *be a diagonal matrix, with diagonal entries* $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. *Let* $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ *be a complex* n*-tuple with nonzero entries, and let* $w \otimes w$ *be the associated rank one matrix whose* (i, j)*th entry is* $\overline{w_i} w_j$. *Finallly, let* $\epsilon \neq 0$ *be real.*

Then the eigenvalues of the Hermitian matrix $P(\epsilon) = D + \epsilon w \otimes w$ *are the solutions to the equation*

$$1 + \epsilon \sum_{i=1}^{n} \frac{|w_i|^2}{\lambda_i - \lambda} = 0.$$

Corollary 7.2. *If* $\epsilon > 0$ *then the eigenvalues* $\lambda_1' > \cdots > \lambda_n'$ *of* $P(\epsilon)$ *interlace with the eigenvalues of* D *in the following sense:*

$$\lambda_n < \lambda_n' < \ldots < \lambda_1 < \lambda_1'. \tag{7.3}$$

For extensions and further discussion of Golub's theorem, see Anderson [2].

Assume that the conditions in Corollary 7.2 hold and assume also that $\lambda_n = 0$. Then $P = \tilde{A}^T \tilde{A}$, where

$$\tilde{A} = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ \sqrt{\epsilon} w_1 & \sqrt{\epsilon} w_2 & \cdots & \sqrt{\epsilon} w_n \end{pmatrix}$$

and α_i denotes the positive square root of λ_i. The values $\alpha_1, \ldots, \alpha_{n-1}, 0$ are the singular values for $A = \sqrt{D}$. Denote by $\alpha_i' = \sqrt{\lambda_i'}$, $i = 1, \ldots, n$, the singular values for P. Then (7.3) implies the following inequality between the singular value functions for A and $\tilde{A}$:

$$\varphi^s(A) \le \varphi^s(\tilde{A}), \qquad 0 \le s \le n.$$

This gives some indication of how Theorem 7.1 may be applied in the context of Falconer's theory. For the specific application to (5.6), however, we require a version of Golub's result for block diagonal matrices. As we are interested in applications to the first Heisenberg group, we give the following result only in the case $n = 3$.

Theorem 7.4. *Let $A \in \mathbb{R}^{2\times 2}$ with distinct singular values $0 < \alpha_2 < \alpha_1$, let $b \in \mathbb{R}^2$ and let $\epsilon \ne 0$. Set*

$$D = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \qquad w = w(A,b) = \begin{pmatrix} -2(Jb)^T A & \det A \end{pmatrix},$$

and

$$\tilde{A}_b(\epsilon) = D + \sqrt{\epsilon}\, e_3 \otimes w = \begin{pmatrix} & A & \\ -2\sqrt{\epsilon}(Jb)^T A & & \sqrt{\epsilon} \det A \end{pmatrix},$$

where $e_3 = (0,0,1)$.

Let λ be a complex number which is not equal to one of the eigenvalues $\lambda_1 = \alpha_1^2$, $\lambda_2 = \alpha_2^2$ for $A^T A$. Then λ is an eigenvalue for $P(\epsilon) = \tilde{A}_b(\epsilon)^T \tilde{A}_b(\epsilon)$ if and only if λ solves the equation

$$F(\lambda) := 1 + \epsilon \left\{ \frac{(\det A)^2}{-\lambda} + \frac{C_1}{\lambda_1 - \lambda} + \frac{C_2}{\lambda_2 - \lambda} \right\} = 0, \tag{7.5}$$

where

$$C_1 = \frac{4\lambda_1(|A^T Jb|^2 - \lambda_2 |b|^2)}{\lambda_1 - \lambda_2}$$

and

$$C_2 = \frac{4\lambda_2(\lambda_1 |b|^2 - |A^T Jb|^2)}{\lambda_1 - \lambda_2}.$$

Observe that $C_1, C_2 \ge 0$.

The proof of this theorem is by direct computation of the characteristic polynomial of $P(\epsilon)$. We omit the details.

A modified version of Theorem 7.4 holds in the orthogonal case $\lambda_1 = \lambda_2$; we omit the details. More general versions are presumably true for block diagonal matrices in higher dimensions, but we do not pursue this here.

We now sketch the application of Theorem 7.4 to the proof of (5.6). Choose $\epsilon = 1$. Then the matrix $\tilde{A}_b(\epsilon)$ is precisely the matrix $\tilde{A}_b$ from the introduction. We assume that $b \neq 0$. As before, denote by $\lambda_1' > \lambda_2' > \lambda_3'$ the eigenvalues of $P(1) = \tilde{A}_b^T \tilde{A}_b$.

If Jb is not an eigenvector for $A^T A$, then C_1 and C_2 are both positive. Consequently, we may conclude as in Corollary 7.2 that the eigenvalues of $P(1)$ and D interlace:

$$0 < \lambda_3' < \lambda_2 < \lambda_2' < \lambda_1 < \lambda_1'.$$

If $|A^T Jb| = \alpha_2 |b|$ then $C_1 = 0$, $C_2 = 4\lambda_2 |b|^2 > 0$ and

$$0 < \lambda_3' \leq \lambda_2 \leq \lambda_2' < \lambda_1 < \lambda_1'.$$

Similarly, if $|A^T Jb| = \alpha_1 |b|$ then $C_2 = 0$, $C_1 = 4\lambda_1 |b|^2 > 0$ and

$$0 < \lambda_3' < \lambda_2 < \lambda_2' \leq \lambda_1 \leq \lambda_1'.$$

In all cases, we find that

$$\lambda_1 \leq \lambda_1' \qquad \text{and} \qquad \lambda_2 \leq \lambda_2'. \tag{7.6}$$

The function $F(\lambda)$ in (7.5) is equal to

$$\frac{\det(P(\epsilon) - \lambda I)}{\det(D - \lambda I)} = \frac{\det(P - \lambda I)}{(-\lambda)(\lambda_1 - \lambda)(\lambda_2 - \lambda)}$$

From the expression for $F(\lambda)$ in (7.5), the product of the eigenvalues of $P(1)$ is

$$\lambda_1' \lambda_2' \lambda_3' = (\det A)^2 = \lambda_1^2 \lambda_2^2. \tag{7.7}$$

Since the singular values of $\tilde{A}_0$ are α_1, α_2 and $\alpha_1 \alpha_2$, it follows from (7.6), (7.7) and the definition of the singular value function in (5.5) that

$$\varphi^s(\tilde{A}_0) \leq \varphi^s(\tilde{A}_b)$$

for all $s \geq 0$. Thus the critical exponents satisfy the inequality

$$\tilde{s}_E(\mathcal{A}) \leq \tilde{s}_E(b; \mathcal{A})$$

as desired. This concludes the proof of (5.6).

References

[1] Ambrosio, L., and Kirchheim, B. Rectifiable sets in metric and Banach spaces. *Math. Ann. 318*, 3 (2000), 527–555.

[2] Anderson, J. A secular equation for the eigenvalues of a diagonal matrix perturbation. *Linear Algebra Appl. 246* (1996), 49–70.

[3] Balogh, Z. M. Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group. *J. Anal. Math. 83* (2001), 289–312.

[4] Balogh, Z. M., Hofer-Isenegger, R., and Tyson, J. T. Lifts of Lipschitz maps and horizontal fractals in the Heisenberg group. Preprint 2003.

[5] Balogh, Z. M., Rickly, M., and Serra-Cassano, F. Comparison of Hausdorff measures with respect to the Euclidean and Heisenberg metric. *Publ. Mat. 47* (2003), 237–259.

[6] Balogh, Z. M., and Tyson, J. T. Polar coordinates on Carnot groups. *Math. Z. 241*, 4 (2002), 697–730.

[7] Bandt, C., and Graf, S. Self-similar sets. VII. A characterization of self-similar fractals with positive Hausdorff measure. *Proc. Amer. Math. Soc. 114*, 4 (1992), 995–1001.

[8] Edgar, G. A. Fractal dimension of self-affine sets: some examples. *Rend. Circ. Mat. Palermo (2) Suppl.*, 28 (1992), 341–358. Measure theory (Oberwolfach, 1990).

[9] Falconer, K. J. *The geometry of fractal sets*, vol. 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1986.

[10] Falconer, K. J. The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc. 103*, 2 (1988), 339–350.

[11] Falconer, K. J. The dimension of self-affine fractals. II. *Math. Proc. Cambridge Philos. Soc. 111*, 1 (1992), 169–179.

[12] Franchi, B., Serapioni, R., and Serra-Cassano, F. Rectifiability and perimeter in the Heisenberg group. *Math. Ann. 321*, 3 (2001), 479–531.

[13] Garofalo, N., and Nhieu, D.-M. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math. 49* (1996), 1081–1144.

[14] Golub, G. H. Some modified matrix eigenvalue problems. *SIAM Rev. 15* (1973), 318–334.

[15] Gromov, M. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian Geometry*, vol. 144 of *Progress in Mathematics*. Birkhäuser, Basel, 1996, pp. 79–323.

[16] Hutchinson, J. E. Fractals and self-similarity. *Indiana Univ. Math. J. 30* (1981), 713–747.

[17] Kigami, J. *Analysis on fractals*, vol. 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[18] Kircheim, B., and Serra Cassano, F. Rectifiability and parametrization of intrinsic regular surfaces in the Heisenberg group. Preprint.

[19] Mattila, P. *Geometry of sets and measures in Euclidean spaces*, vol. 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[20] Moran, P. A. P. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc. 42* (1946), 15–23.

[21] Rogers, C. A. *Hausdorff Measures*. Cambridge University Press, London, 1970.

[22] Schief, A. Separation properties for self-similar sets. *Proc. Amer. Math. Soc. 122*, 1 (1994), 111–115.

[23] Solomyak, B. Measure and dimension for some fractal families. *Math. Proc. Camb. Phil. Soc. 124*, 3 (1998), 531–546.

[24] Strichartz, R. S. Self-similarity on nilpotent Lie groups. In *Geometric analysis (Philadelphia, PA, 1991)*, vol. 140 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1992, pp. 123–157.

[25] Strichartz, R. S. Self-similarity in harmonic analysis. *J. Fourier Anal. Appl. 1*, 1 (1994), 1–37.

Department of Mathematics, University of Bern, Sidlerstrasse 5, 3012 Bern Switzerland
E-mail address: zoltan.balogh@math-stat.unibe.ch

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801
E-mail address: tyson@math.uiuc.edu