# ASYMPTOTIC ANALYSIS OF NON-SYMMETRIC LINEAR OPERATORS VIA Γ-CONVERGENCE

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**Abstract.** We study the asymptotic behavior of a sequence of Dirichlet problems for second order linear operators in divergence form

$$\begin{cases} -\mathrm{div}(\sigma_\varepsilon \nabla u) = f & \text{in } \Omega, \\ u \in H^1_0(\Omega), \end{cases}$$

where $(\sigma_\varepsilon) \subset L^\infty(\Omega; \mathbb{R}^{n\times n})$ is uniformly elliptic and possibly non-symmetric. On account of the variational principle of Cherkaev and Gibiansky [1], we are able to prove a variational characterization of the $H$-convergence of $(\sigma_\varepsilon)$ in terms of the $\Gamma$-convergence of suitably associated quadratic forms.

**Keywords**: linear elliptic operators, $H$-convergence, $\Gamma$-convergence.

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**1. Introduction.** An ever larger number of applications witnesses the importance of $\Gamma$, $G$ and $H$-convergence as tools to study a wide variety of physical and mechanical models. The well-known examples include composites (fibered, layered, porous materials, *etc.*), elastic thin bodies (films, rods, *etc.*), lattice systems with characteristic atomic scales and, in general, a range of models with a microstructure or exhibiting some kind of microscopic phenomenon (phase transitions, internal boundary layers, *etc.*). Therefore, the investigation of the possible connections between these different types of convergences is an interesting task which may boost the development of new and more effective techniques, thus further increasing the number of applications.

The aim of the present paper is to provide a variational characterization of $H$-convergence of linear elliptic operators in terms of the $\Gamma$-convergence of suitably associated functionals.

A prototypical model for heterogeneous materials in electrostatics is given by a sequence of linear Dirichlet problems of the following type

$$\begin{cases} -\mathrm{div}(\sigma_\varepsilon(x) \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon \in H^1_0(\Omega), \end{cases} \tag{1.1}$$

where the conductivity matrices $\sigma_\varepsilon$ are uniformly elliptic with respect to $\varepsilon > 0$.

Spagnolo's $G$-convergence and Murat and Tartar's $H$-convergence provide a suitable notion of convergence of $(\sigma_\varepsilon)$ which permits to reduce the study of the sequence of problems (1.1) to that of a single "effective" problem, of the same type as (1.1), independent of $\varepsilon$. In fact, it is well know [12, 13, 14, 15, 16] that there exists a subsequence, still denoted by $(\sigma_\varepsilon)$, and an elliptic matrix $\sigma_0$, called the $H$-limit of $(\sigma_\varepsilon)$ (or the $G$-limit in the symmetric case), such that for every $f \in H^{-1}(\Omega)$ the solutions $u_\varepsilon$ to (1.1) converge weakly in $H^1_0(\Omega)$ to the solution $u_0$ of

$$\begin{cases} -\mathrm{div}(\sigma_0(x) \nabla u_0) = f & \text{in } \Omega, \\ u_0 \in H^1_0(\Omega), \end{cases}$$

and satisfy also

$$\sigma_\varepsilon(x)\nabla u_\varepsilon \rightharpoonup \sigma_0(x)\nabla u_0 \quad \text{in } L^2(\Omega; \mathbb{R}^n).$$

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In the special case $\sigma_\varepsilon = \sigma_\varepsilon^T$ equation (1.1) has a variational structure as it can be seen as the Euler-Lagrange equation associated with

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega \sigma_\varepsilon(x)\nabla u \cdot \nabla u \, dx - \int_\Omega f \, u \, dx \, ,$$

or, equivalently, as the solution to the minimization problem

$$\min\{F_\varepsilon(u) : \; u \in H_0^1(\Omega)\}. \tag{1.2}$$

Therefore, (1.2) provides a variational principle for the Dirichlet problem (1.1) and the convergence of the solutions of (1.1) can be equivalently studied by means of De Giorgi's $\Gamma$-convergence [3]. In fact, in [6] De Giorgi and Spagnolo prove that the $G$-convergence of uniformly elliptic, symmetric matrices is equivalent to the $\Gamma$-convergence of the associated functionals $F_\varepsilon$. Specifically, $(\sigma_\varepsilon)$ $G$-converges to $\sigma_0$ (which now is symmetric) if and only if the quadratic forms

$$Q_\varepsilon(u) = \int_\Omega \sigma_\varepsilon(x)\nabla u \cdot \nabla u \, dx$$

$\Gamma$-converge, with respect to the weak topology of $H_0^1(\Omega)$, to the quadratic form defied through $\sigma_0$ as

$$Q_0(u) = \int_\Omega \sigma_0(x)\nabla u \cdot \nabla u \, dx.$$

In this paper we generalize the above equivalence to the *non-symmetric* setting providing a variational definition of $H$-convergence of matrices $\sigma_\varepsilon$ in terms of the $\Gamma$-convergence of suitably associated quadratic forms.

We notice that if $\sigma_\varepsilon$ is non-symmetric, (1.2) does not provide a variational principle for (1.1) and $u_\varepsilon$ is not the solution of the Euler-Lagrange equation associated with $F_\varepsilon$. Then, trying to establish a connection between $H$-convergence and $\Gamma$-convergence as in the symmetric case, the idea is to complement a problem of type (1.1) combining it with an analogous Dirichlet problem involving the transpose matrix $\sigma^T$ (which is elliptic if $\sigma$ is so). Then, for $f_1, f_2 \in H^{-1}(\Omega)$ we consider the following Dirichlet *system* of equations

$$\begin{cases} -\mathrm{div}(\sigma(x)\nabla u) = f_1 & \text{in } \Omega, \\ u \in H_0^1(\Omega), & \\ & \\ -\mathrm{div}(\sigma^T(x)\nabla v) = f_2 & \text{in } \Omega, \\ v \in H_0^1(\Omega). & \end{cases} \tag{1.3}$$

Therefore, suitably coupling the solutions $u, v$ to (1.3) one expects to recover enough structure to see (1.3) as the Euler-Lagrange system associated with a quadratic functional defined on pairs of functions. In [1] Cherkaev and Gibiansky (see also Fannjiang and Papanicolaou [7] and Milton [8]) observe that the variational principle that follows solving equations (1.3) with respect to the "natural" coupling $u+v$ and $u-v$ is a mini-max variational principle (see [1, Section II B] or Section 3, for more details), and the solution to (1.3) is a saddle-point of a quadratic functional defined through the following symmetric (but non-positive definite) $(2n \times 2n)$-matrix-valued function

$$\begin{pmatrix} \sigma^s & \sigma^a \\ -\sigma^a & -\sigma^s \end{pmatrix},$$

where $\sigma^s$ and $\sigma^a$ are the symmetric and skew-symmetric part of $\sigma$, respectively. Then, the key observation in [1] is that by means of a partial Legendre transform (which amounts to solving (1.3) with respect to the sum of the momenta $\sigma\nabla u+\sigma^T\nabla v$ and to $u-v$) it is possible to recast the above mini-max problem into a minimization problem for a quadratic functional associated with the following matrix-valued function

$$\mathbf{\Sigma}=\begin{pmatrix} (\sigma^s)^{-1} & -(\sigma^s)^{-1}\sigma^a \\ \sigma^a(\sigma^s)^{-1} & \sigma^s-\sigma^a(\sigma^s)^{-1}\sigma^a \end{pmatrix}, \tag{1.4}$$

which is symmetric and positive definite a.e. in $\Omega$.

In fact, in [1] Cherkaev and Gibiansky derived the aforementioned variational principles to provide bounds on the effective conductivity matrix for media that can be described by linear elliptic equations with complex coefficients. Therefore, in [1] the authors deal with a complex conductivity matrix $\sigma$ and, instead of $\sigma^s$ and $\sigma^a$, they consider $\sigma_R$ and $\sigma_I$ (real and imaginary part of $\sigma$, respectively), with $\sigma_R$ and $\sigma_I$ symmetric and $\sigma_R$ positive definite. In [8], among other, Milton extends Cherkaev and Gibiansky's approach to study a non-symmetric conductivity problem when a magnetic field is present (as for conduction in a fixed magnetic field the conductivity matrix is real but not symmetric) and generalizes their variational principles thus considering $\sigma^s$ and $\sigma^a$. To underline the relevance in the applications of the Cherkaev and Gibiansky variational principle, we quote [4, 9, 5] where the authors employ this variational principle to investigate the problem of deriving rigorous bounds on the moduli of viscoelastic two-phase composites, and the more recent [10] where Cherkaev and Gibiansky's ideas are extended to the setting of acoustic, elastodynamics, and electromagnetism in lossy heterogeneous bodies.

In the present paper we start by providing a rigorous rephrasing of Cherkaev and Gibiansky's ideas, suitable for our variational setting. Specifically, for $h\in L^2(\Omega;\mathbf{R}^n)$ and $\sigma\in L^\infty(\Omega;\mathbf{R}^{n\times n})$ elliptic, in Section 3 we consider the functional $F(\sigma)\colon L^2(\Omega;\mathbf{R}^n)\times H^1_0(\Omega)\to\mathbf{R}$ defined by

$$F(\sigma)(j,\psi)=\frac{1}{2}\int_\Omega\langle\mathbf{\Sigma}\begin{pmatrix} j \\ \nabla\psi\end{pmatrix},\begin{pmatrix} j \\ \nabla\psi\end{pmatrix}\rangle\,dx-\int_\Omega\nabla\psi\cdot h\,dx, \tag{1.5}$$

with $\mathbf{\Sigma}$ as in (1.4), and we show that the minimization problem

$$\min\left\{F(\sigma)(j,\psi):\ (j,\psi)\in L^2(\Omega;\mathbf{R}^n)\times H^1_0(\Omega),\ \mathrm{div}j=f\right\} \tag{1.6}$$

provides a variational principle for the system (1.3), up to choosing $f_1=-(f+\mathrm{div}h)/2$ and $f_2=-(f-\mathrm{div}h)/2$, with $f\in H^{-1}(\Omega)$. We also prove an interesting characterization for matrices "of type $\mathbf{\Sigma}$", showing that any symmetric matrix $\mathbf{M}\in\mathbf{R}^{2n\times 2n}$ has the same form as $\mathbf{\Sigma}$ (for some $\sigma\in\mathbf{R}^{n\times n}$) if and only if $\mathbf{M}$ belongs to the *indefinite special orthogonal group* $SO(n,n)$ (see Proposition 3.1). This algebraic characterization allows us to show that if we consider a functional as in (1.5), defined through an arbitrary symmetric and positive definite matrix $\mathbf{M}$, then the associated minimization problem provides a variational principle for (1.3) if and only if $\mathbf{M}\in SO(n,n)$ a.e. in $\Omega$ (see Remark 1).

On account of the variational principle discussed in Section 3 and by virtue of the properties of $H$-convergence, in Section 4 we finally prove the main results of this paper, Theorem 4.1 and Theorem 4.4. Namely, we establish an equivalence between $H$-convergence and $\Gamma$-convergence proving that a sequence of uniformly elliptic matrices $(\sigma_\varepsilon)$ $H$-converges to some elliptic matrix $\sigma_0$ if and only if the quadratic forms

$$Q^f_\varepsilon(j,\psi)=\begin{cases}\displaystyle\int_\Omega\langle\mathbf{\Sigma}_\varepsilon\begin{pmatrix} j \\ \nabla\psi\end{pmatrix},\begin{pmatrix} j \\ \nabla\psi\end{pmatrix}\rangle\,dx & \text{if}\quad \mathrm{div}j=f,\\ +\infty & \text{otherwise in } L^2(\Omega;\mathbf{R}^n)\times H^1_0(\Omega),\end{cases}$$

$\Gamma$-converge with respect to the weak topology of $L^2(\Omega;\mathbf{R}^n)\times H^1_0(\Omega)$ to

$$Q^f_0(j,\psi)=\begin{cases}\displaystyle\int_\Omega\langle\mathbf{\Sigma}_0\begin{pmatrix} j \\ \nabla\psi\end{pmatrix},\begin{pmatrix} j \\ \nabla\psi\end{pmatrix}\rangle\,dx & \text{if } \mathrm{div}j=f,\\ +\infty & \text{otherwise in } L^2(\Omega;\mathbf{R}^n)\times H^1_0(\Omega),\end{cases}$$

for every $f \in H^{-1}(\Omega)$, where $\mathbf{\Sigma}_\varepsilon$, $\mathbf{\Sigma}_0 \in L^\infty(\Omega; \mathbf{R}^{2n\times 2n})$ are as in (1.4) and correspond to the choice $\sigma = \sigma_\varepsilon$ and $\sigma = \sigma_0$, respectively.

We finally remark that the above equivalence result provides two immediate advantages. On the one hand, the implication "$H$-convergence $\Rightarrow$ $\Gamma$-convergence" yields interesting information on the structure of the $\Gamma$-limit that cannot be directly derived from the $\Gamma$-convergence itself. In particular, as a consequence of Theorem 4.1 (see also Proposition 3.1 and Remark 2) we find that the class of symmetric matrices in $SO(n, n)$ is closed with respect to $\Gamma$-convergence. On the other hand, an advantage of working with $\Gamma$-convergence instead of $H$-convergence (hence the advantage of knowing that "$\Gamma$-convergence $\Rightarrow$ $H$-convergence") is that $\Gamma$-convergence is stable with respect to continuos perturbations. If these perturbations are suitably chosen, the minimizers of the perturbed functionals solve some Euler-Lagrange equation. Therefore, thanks to the fundamental property of $\Gamma$-convergence, working with the quadratic forms $Q^f_\varepsilon$ may easily yield the convergence of the solutions of a wider class of elliptic equations than (1.1).

**2. Notation and preliminaries.** In this section we introduce a few notation and we recall some preliminaries we employ in the following. For the general theory of $H$-convergence we refer the reader to [16, 17], while we refer to [2] for a comprehensive introduction to $\Gamma$-convergence.

For any $\xi, \eta \in \mathbf{R}^n$, $\xi \cdot \eta$ denotes the scalar product on $\mathbf{R}^n$, while the scalar product of any given pair of vectors $\mathrm{v}, \mathrm{w} \in \mathbf{R}^{2n}$ is denoted by $\langle \mathrm{v}, \mathrm{w}\rangle$. For any $A \in \mathbf{R}^{n\times n}$ we denote by $A^s$ and $A^a$ the symmetric and the skew-symmetric part of $A$, respectively; *i.e.,*

$$A^s := \frac{A + A^T}{2}, \quad A^a := \frac{A - A^T}{2},$$

where $A^T$ is the transpose matrix of $A$.

Let $\Omega$ be an open bounded subset of $\mathbf{R}^n$. For $0 < \alpha \leq \beta < +\infty$, $\mathcal{M}(\alpha, \beta, \Omega)$ denotes the set of matrix-valued functions $\sigma \in L^\infty(\Omega; \mathbf{R}^{n\times n})$ satisfying

$$\sigma(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad \sigma^{-1}(x)\xi \cdot \xi \geq \beta^{-1}|\xi|^2, \qquad \text{for every } \xi \in \mathbf{R}^n, \ \text{ for a.e. } x \in \Omega. \tag{2.1}$$

Note that (2.1) implies that

$$|\sigma(x)| \leq \beta \qquad \text{for a.e. } x \in \Omega,$$

and that necessarily $\alpha \leq \beta$. Not to overburden notation, in all that follows we always write $\sigma$ in place of $\sigma(x)$.

Throughout the paper the parameter $\varepsilon$ varies in a strictly decreasing sequence of positive real numbers converging to zero.

**2.1. $\Gamma$-convergence of quadratic forms.** For the reader's sake, in this subsection we briefly discuss the connection between $\Gamma$-convergence of quadratic forms and convergence of minimization problems (see also [2, Theorem 13.5]).

Let $H$ be a real Hilbert space with norm $\|\cdot\|$. Let $H^*$ be the dual space of $H$ and denote by $\langle\cdot,\cdot\rangle_{H^*,H}$ the duality pairing between $H^*$ and $H$.

Given $\lambda > 0$, let $Q\colon H \to [0, +\infty)$ and $Q_\varepsilon\colon H \to [0, +\infty)$, with $\varepsilon > 0$, be quadratic forms satisfying $Q(x) \geq \lambda\|x\|^2$, $Q_\varepsilon(x) \geq \lambda\|x\|^2$ for every $x \in H$ and for every $\varepsilon > 0$.

Let $K$ be a convex closed subset of $H$; for any fixed $F^* \in H^*$ we consider the two following minimization problems

$$M^{F^*}_\varepsilon := \min_{x\in K} \Big(\frac{1}{2}Q_\varepsilon(x) - \langle F^*, x\rangle_{H^*,H}\Big), \tag{2.2}$$

and

$$M^{F^*} := \min_{x\in K} \Big(\frac{1}{2}Q(x) - \langle F^*, x\rangle_{H^*,H}\Big). \tag{2.3}$$

The following theorem holds true.

THEOREM 2.1. *For every fixed $F^* \in H^*$ and for every fixed $\varepsilon > 0$, let $\tilde{x}_\varepsilon \in K$ be the unique solution to (2.2) and let $\tilde{x} \in K$ be the unique solution to (2.3).*

*Suppose that*

$$M_\varepsilon^{F^*} \to M^{F^*} \quad \text{and} \quad \tilde{x}_\varepsilon \rightharpoonup \tilde{x} \quad \text{for every} \quad F^* \in H^*. \tag{2.4}$$

*Then, the sequence of functionals $G_\varepsilon : H \to [0, +\infty]$ defined as*

$$G_\varepsilon(x) := \begin{cases} Q_\varepsilon(x) & \text{if } x \in K, \\ +\infty & \text{if } x \in H \setminus K, \end{cases}$$

*$\Gamma$-converges, with respect to the weak topology of $H$, to the functional $G : H \to [0, +\infty]$ defined by*

$$G(x) := \begin{cases} Q(x) & \text{if } x \in K, \\ +\infty & \text{if } x \in H \setminus K. \end{cases}$$

*Proof.* By hypothesis, the sequence $(G_\varepsilon)$ is equi-coercive with respect to the weak topology of $H$. Then, if $(x_\varepsilon) \subset K$ is such that $\sup_\varepsilon G_\varepsilon(x_\varepsilon) < +\infty$ we immediately deduce $x_\varepsilon \rightharpoonup x$, for some $x \in K$.

Let $\tilde{x} \in K$; we can find $\tilde{F} \in H^*$ such that $\tilde{x}$ realizes $M^{\tilde{F}}$. Indeed, denoting by $Q'(\tilde{x})$ the Gâteaux differential of $Q$ at $\tilde{x}$, by the convexity of $Q$ we have

$$Q(x) \geq Q(\tilde{x}) + \langle Q'(\tilde{x}), x - \tilde{x}\rangle_{H^*,H},$$

or equivalently

$$Q(x) - \langle Q'(\tilde{x}), x\rangle_{H^*,H} \geq Q(\tilde{x}) - \langle Q'(\tilde{x}), \tilde{x}\rangle_{H^*,H}$$

for every $x \in K$, which entails the minimality of $\tilde{x}$ up to choosing $\tilde{F} = \frac{1}{2}Q'(\tilde{x})$.

For any fixed $\varepsilon > 0$ let $\tilde{x}_\varepsilon \in K$ be the unique solution to (2.2) with $F^* = \tilde{F} = \frac{1}{2}Q'(\tilde{x})$. By hypothesis $\tilde{x}_\varepsilon \rightharpoonup \tilde{x}$. Moreover, by the minimality of $\tilde{x}_\varepsilon$, for every $(x_\varepsilon) \subset K$ such that $x_\varepsilon \rightharpoonup \tilde{x}$, we have

$$\frac{1}{2}Q_\varepsilon(x_\varepsilon) \geq \frac{1}{2}Q_\varepsilon(\tilde{x}_\varepsilon) + \langle \tilde{F}, x_\varepsilon - \tilde{x}_\varepsilon\rangle_{H^*,H},$$

hence

$$\liminf_{\varepsilon\to 0} Q_\varepsilon(x_\varepsilon) \geq \liminf_{\varepsilon\to 0} Q_\varepsilon(\tilde{x}_\varepsilon). \tag{2.5}$$

Moreover, since

$$Q_\varepsilon(\tilde{x}_\varepsilon) = 2(M_\varepsilon^{\tilde{F}} + \langle \tilde{F}, \tilde{x}_\varepsilon\rangle_{H^*,H}),$$

by the convergence of minimum values, $M_\varepsilon^{\tilde{F}} \to M^{\tilde{F}}$, we deduce

$$\lim_{\varepsilon\to 0} Q_\varepsilon(\tilde{x}_\varepsilon) = 2(M^{\tilde{F}} + \langle \tilde{F}, \tilde{x}\rangle_{H^*,H}) = Q(\tilde{x});$$

this yields the limsup-inequality and, by (2.5), the liminf-inequality as well. □

**3. A variational principle for non symmetric linear operators.** In this section we review the variational principle due to Cherkaev and Gibiansky [1] (see also Milton's variant [8]) to construct a functional whose associated Euler-Lagrange equations provide a solution to (1.3). We proceed in two steps. Starting from equations (1.3), and thus form a given elliptic $\sigma$, we first construct a symmetric positive-definite matrix $\mathbf{\Sigma} \in L^\infty(\Omega; \mathbf{R}^{2n\times 2n})$. Then, we exhibit a strictly convex functional, associated with $\mathbf{\Sigma}$, whose unique minimizer provide a solution to (1.3).

**3.1. From equations (1.3) to the matrix** $\Sigma$**.** Let $\sigma \in \mathcal{M}(\alpha, \beta, \Omega)$ and $f_1, f_2 \in H^{-1}(\Omega)$ we consider the system

$$\begin{cases} -\mathrm{div}(\sigma\nabla u) = f_1 \\ -\mathrm{div}(\sigma^T\nabla v) = f_2, \end{cases} \tag{3.1}$$

with $u, v \in H_0^1(\Omega)$. This system can be expressed as

$$\begin{cases} -\mathrm{div} j_u = f_1 \\ -\mathrm{div} j_v = f_2, \end{cases}$$

where

$$\begin{cases} j_u := \sigma\nabla u \\ j_v := \sigma^T\nabla v\,. \end{cases} \tag{3.2}$$

Let $\sigma = \sigma^s + \sigma^a$, being $\sigma^s$ and $\sigma^a$ the symmetric and the skew-symmetric part of $\sigma$, respectively; equations (3.2) can be rewritten as

$$\begin{cases} (\sigma^s + \sigma^a)\nabla u = j_u \\ (\sigma^s - \sigma^a)\nabla v = j_v. \end{cases} \tag{3.3}$$

If we set $\varphi := u + v$ and $\psi := u - v$, by summing and subtracting the two equations in (3.3) we get

$$\begin{cases} \sigma^s\nabla\varphi + \sigma^a\nabla\psi = j_u + j_v \\ -\sigma^a\nabla\varphi - \sigma^s\nabla\psi = j_v - j_u, \end{cases} \tag{3.4}$$

whose compact form is

$$\mathbf{\Xi}\begin{pmatrix} \nabla\varphi \\ \nabla\psi \end{pmatrix} = \begin{pmatrix} j_u + j_v \\ j_v - j_u \end{pmatrix},$$

where $\mathbf{\Xi} \in L^\infty(\Omega; \mathbf{R}^{2n\times 2n})$ has the following block structure

$$\mathbf{\Xi} := \begin{pmatrix} \sigma^s & \sigma^a \\ -\sigma^a & -\sigma^s \end{pmatrix}. \tag{3.5}$$

Notice that $\mathbf{\Xi}$ is symmetric but it is not positive definite. As we are concerned with minimum problems, we rearrange (3.4) in order to get a linear system whose associate matrix is both symmetric and positive definite. To this end, we solve (3.4) with respect to $\nabla\varphi$ and $j_u - j_v$ obtaining

$$\begin{cases} \nabla\varphi = (\sigma^s)^{-1}(j_u + j_v) - (\sigma^s)^{-1}\sigma^a\nabla\psi \\ j_u - j_v = \sigma^a(\sigma^s)^{-1}(j_u + j_v) + (\sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a)\nabla\psi, \end{cases} \tag{3.6}$$

or in compact notation

$$\mathbf{\Sigma}\begin{pmatrix} j_u + j_v \\ \nabla\psi \end{pmatrix} = \begin{pmatrix} \nabla\varphi \\ j_u - j_v \end{pmatrix}, \tag{3.7}$$

with

$$\boldsymbol{\Sigma} := \begin{pmatrix} (\sigma^s)^{-1} & -(\sigma^s)^{-1}\sigma^a \\ \sigma^a(\sigma^s)^{-1} & \sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a \end{pmatrix}. \tag{3.8}$$

Notice that $\boldsymbol{\Sigma}$ is symmetric and positive definite.

**3.1.1. Properties of $\Sigma$.** The matrix-valued function $\boldsymbol{\Sigma} \in L^\infty(\Omega; \mathbf{R}^{2n\times 2n})$ satisfies the following properties:

*i)* $\boldsymbol{\Sigma}$ is coercive; *i.e.,* there exists a constant $C = C(\alpha, \beta) > 0$ such that

$$\langle \boldsymbol{\Sigma}\mathrm{w}, \mathrm{w}\rangle \geq C|\mathrm{w}|^2, \quad \forall \mathrm{w} \in \mathbf{R}^{2n}, \text{ a.e. in } \Omega. \tag{3.9}$$

*Proof.* Let $\mathrm{w} := (w_1, w_2) \in \mathbf{R}^{2n}$. Since by hypothesis $\sigma \in \mathcal{M}(\alpha, \beta, \Omega)$, we deduce $\sigma^s \in \mathcal{M}(\alpha, \beta, \Omega)$, and therefore

$$\begin{aligned} \langle \boldsymbol{\Sigma}\mathrm{w}, \mathrm{w}\rangle &= (\sigma^s)^{-1}(w_1 - \sigma^a w_2)\cdot(w_1 - \sigma^a w_2) + \sigma^s w_2 \cdot w_2 \\ &\geq \beta^{-1}|w_1 - \sigma^a w_2|^2 + \alpha |w_2|^2 \\ &\geq \min\{\beta^{-1}, \alpha\}\,(|w_1 - \sigma^a w_2|^2 + |w_2|^2). \end{aligned} \tag{3.10}$$

As we have

$$\begin{aligned} |w_1|^2 &\leq 2|w_1 - \sigma^a w_2|^2 + 2|\sigma^a w_2|^2 \\ &\leq 2|w_1 - \sigma^a w_2|^2 + 8\beta^2 |w_2|^2 \\ &\leq \max\{2, 8\beta^2\}\,(|w_1 - \sigma^a w_2|^2 + |w_2|^2), \end{aligned}$$

by (3.10) we get

$$\langle \boldsymbol{\Sigma}\mathrm{w}, \mathrm{w}\rangle \geq \frac{\min\{\beta^{-1}, \alpha\}}{\max\{2, 8\beta^2\}}\,|w_1|^2 \quad \text{a.e. in } \Omega. \tag{3.11}$$

Then, again invoking (3.10) we also find

$$\langle \boldsymbol{\Sigma}\mathrm{w}, \mathrm{w}\rangle \geq \alpha |w_2|^2 \quad \text{a.e. in } \Omega. \tag{3.12}$$

Eventually, gathering (3.11) and (3.12) gives (3.9). □

*ii)* $\boldsymbol{\Sigma} \in SO(n, n)$ a.e. in $\Omega$, where $SO(n, n)$ is the *indefinite special orthogonal group*; *i.e.,* $\boldsymbol{\Sigma}$ is such that

$$(\boldsymbol{\Sigma})^{-1} = \mathbf{J}\boldsymbol{\Sigma}\mathbf{J} \text{ a.e. in } \Omega, \quad \text{with } \mathbf{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{3.13}$$

where $I \in \mathbf{R}^{n\times n}$ is the identity matrix.

*Proof.* Solving (3.4) with respect to $\nabla\psi$ and $j_u + j_v$ (instead of $\nabla\varphi$ and $j_u - j_v$) gives

$$\begin{cases} \nabla\psi = (\sigma^s)^{-1}(j_u - j_v) - (\sigma^s)^{-1}\sigma^a \nabla\varphi \\ j_u + j_v = \sigma^a(\sigma^s)^{-1}(j_u - j_v) + (\sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a)\nabla\varphi, \end{cases}$$

or, equivalently,

$$\boldsymbol{\Sigma}\begin{pmatrix} j_u - j_v \\ \nabla\varphi \end{pmatrix} = \begin{pmatrix} \nabla\psi \\ j_u + j_v \end{pmatrix}. \tag{3.14}$$

Hence, gathering (3.7) and (3.14) easily yields

$$\big(\mathbf{J\Sigma J} - \mathbf{\Sigma}^{-1}\big)\begin{pmatrix} \nabla\varphi \\ j_u - j_v \end{pmatrix} = 0.$$

Then, since in (3.1) $f_1$ and $f_2$ can be arbitrarily chosen in $H^{-1}(\Omega)$, taking $f_1 = -\mathrm{div}(\sigma\nabla u^*)$, with $u^* \in H^1_0(\Omega)$, and $f_2 = 0$ we deduce that

$$\big(\mathbf{J\Sigma J} - \mathbf{\Sigma}^{-1}\big)\begin{pmatrix} \nabla u^* \\ \sigma\nabla u^* \end{pmatrix} = 0 \quad \text{for every } u^* \in H^1_0(\Omega).$$

Finally, (3.13) is achieved taking $u^*$ to be any affine function in an open set $\omega$ with $\overline{\omega} \subset \Omega$ and recalling that $\sigma$ is invertible. □

*iii)* $\mathbf{\Sigma}$ is *unimodal*; *i.e.,* $\det \mathbf{\Sigma} = 1$.

*Proof.* This property is a straightforward consequence of (3.13). Indeed, (3.13) entails

$$(\det \mathbf{\Sigma})^2 = (\det \mathbf{J})^2 = 1,$$

therefore we conclude by the positivity of $\mathbf{\Sigma}$. □

Finally, the following algebraic proposition shows that (3.13) actually gives a characterization of matrices "of type $\mathbf{\Sigma}$".

PROPOSITION 3.1. *Let $\mathbf{M} \in \mathbf{R}^{2n\times 2n}$ be a symmetric matrix; i.e.,*

$$\mathbf{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

*for some $A, B, C \in \mathbf{R}^{n\times n}$ with $A = A^T$ and $C = C^T$. Suppose moreover that $\det A \neq 0$. Then, $\mathbf{M} = \mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is as in (3.8) for some $\sigma \in \mathbf{R}^{n\times n}$, with $\det \sigma^s \neq 0$, if and only if $\mathbf{M} \in SO(n,n)$.*

*Proof.* We have already proved that $\mathbf{\Sigma} \in SO(n,n)$; hence, it is enough to show the other implication. To this end, let $\mathbf{M} \in \mathbf{R}^{2n\times 2n}$ be such that

$$(\mathbf{M})^{-1} = \mathbf{JMJ}\,, \quad \text{with} \quad \mathbf{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.15}$$

Since $\mathbf{J}^2 = \mathbf{I}$, it is immediate to show that (3.15) is equivalent to

$$\mathbf{MJM} = \mathbf{J},$$

which, in its turn, is equivalent to the following system

$$\begin{cases} BA + AB^T = 0 \\ AC + B^2 = I. \end{cases} \tag{3.16}$$

Finally, by virtue of (3.16), we get the thesis by choosing $\sigma^s = A^{-1}$ and $\sigma^a = -A^{-1}B$. □

**3.2. From the matrix $\Sigma$ to the variational principle..** Let $\mathbf{\Sigma} \in L^\infty(\Omega; \mathbf{R}^{2n\times 2n})$ be as in (3.8) for some $\sigma \in \mathcal{M}(\alpha, \beta, \Omega)$. For every fixed $h, k \in L^2(\Omega; \mathbf{R}^n)$ we define the following functional

$$F^{h,k}(\sigma)(j,\psi) := \frac{1}{2}\int_\Omega \langle \mathbf{\Sigma}\begin{pmatrix} j \\ \nabla\psi \end{pmatrix}, \begin{pmatrix} j \\ \nabla\psi \end{pmatrix}\rangle\, dx - \int_\Omega (j \cdot 2k + \nabla\psi \cdot h)\, dx, \tag{3.17}$$

for every pair $(j,\psi) \in L^2(\Omega; \mathbf{R}^n) \times H^1_0(\Omega)$. We are now able to prove that, for every fixed $h \in L^2(\Omega; \mathbf{R}^n)$ and $f \in H^{-1}(\Omega)$, the following minimization problem

$$\min\big\{F^{h,k}(\sigma)(j,\psi) :\ (j,\psi) \in L^2(\Omega; \mathbf{R}^n) \times H^1_0(\Omega),\ \mathrm{div} j = f\big\} \tag{3.18}$$

provides a variational principle for equations (3.1) when $k = 0$. Using standard arguments (that we briefly recall for the reader's convenience) we start by computing the Euler-Lagrange system of equations associated with $F^{h,k}(\sigma)$. To this end, we consider the following subspaces of $L^2(\Omega; \mathbf{R}^n)$

$$Y := \{\eta \in L^2(\Omega; \mathbf{R}^n) \colon \operatorname{div} \eta = 0\}$$

and

$$Y^{\perp} = \{\nabla\varphi \colon \varphi \in H_0^1(\Omega)\}.$$

Let $(j, \psi)$ be a solution to (3.18); requiring that the first variation of $F^{h,k}(\sigma)$ at $(j, \psi)$ is zero, we find

$$\int_\Omega \langle \mathbf{\Sigma} \begin{pmatrix} j \\ \nabla\psi \end{pmatrix} - \begin{pmatrix} 2k \\ h \end{pmatrix}, \begin{pmatrix} \eta \\ \nabla\varphi \end{pmatrix} \rangle \, dx = 0,$$

for every $\eta \in Y$ and $\varphi \in H_0^1(\Omega)$. Therefore, in view of (3.8) we get

$$\begin{cases} (\sigma^s)^{-1} j - (\sigma^s)^{-1}\sigma^a \nabla\psi - 2k \in Y^{\perp} \\ \sigma^a(\sigma^s)^{-1} j + (\sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a)\nabla\psi - h \in Y \end{cases}$$

or, equivalently,

$$\begin{cases} (\sigma^s)^{-1} j - (\sigma^s)^{-1}\sigma^a \nabla\psi - 2k = \nabla\varphi \\ \operatorname{div}\big(\sigma^a(\sigma^s)^{-1} j + (\sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a)\nabla\psi\big) = \operatorname{div} h, \end{cases} \tag{3.19}$$

with $\varphi \in H_0^1(\Omega)$.

By means of (3.19), we may associate to the minimizer $(j, \psi)$ of (3.18) a pair of functions $(u, v)$ by setting

$$u := \frac{\varphi + \psi}{2}, \qquad v := \frac{\varphi - \psi}{2}. \tag{3.20}$$

Clearly $u, v \in H_0^1(\Omega)$ and, by definition, we have

$$\nabla\psi = \nabla(u - v) \qquad \nabla\varphi = \nabla(u + v)\,. \tag{3.21}$$

Solving the first equation in (3.19) with respect to $j$ and taking into account (3.21) we get

$$\begin{aligned} j &= \sigma^a(\nabla u - \nabla v) + \sigma^s(\nabla u + \nabla v) + 2\sigma^s k \\ &= \sigma\nabla u + \sigma^T\nabla v + 2\sigma^s k, \end{aligned} \tag{3.22}$$

moreover, the constraint $\operatorname{div} j = f$ yields

$$\operatorname{div}(\sigma\nabla u + \sigma^T\nabla v + 2\sigma^s k) = f\,. \tag{3.23}$$

We now define

$$j' := \sigma^a(\sigma^s)^{-1} j + (\sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a)\nabla\psi - h\,; \tag{3.24}$$

by (3.19) we deduce that $\operatorname{div} j' = 0$. Furthermore, pre-multiplying the first equation in (3.19) by $\sigma^a$ and subtracting it to (3.24) we find

$$j' + h = \sigma^a(\nabla u + \nabla v) + \sigma^s(\nabla u - \nabla v) + 2\sigma^a k,$$

which entails

$$\operatorname{div}(\sigma\nabla u - \sigma^T\nabla v + 2\sigma^a k) = \operatorname{div}h\,. \tag{3.25}$$

Finally, gathering (3.23) and (3.25) we have

$$\begin{cases} \operatorname{div}(\sigma\nabla u + \sigma^T\nabla v + 2\sigma^s k) = f \\ \operatorname{div}(\sigma\nabla u - \sigma^T\nabla v + 2\sigma^a k) = \operatorname{div}h\,, \end{cases}$$

or equivalently

$$\begin{cases} \operatorname{div}(\sigma(k + \nabla u)) = (f + \operatorname{div}h)/2 \\ \operatorname{div}(\sigma^T(k + \nabla v)) = (f - \operatorname{div}h)/2\,. \end{cases} \tag{3.26}$$

Therefore, for every $h \in L^2(\Omega;\mathbf{R}^n)$, $f \in H^{-1}(\Omega)$ and for $k = 0$ the minimization problem (3.18) associated with the functional $F^{h,0}(\sigma)$ provides a variational principle for the system of equations (3.1), with $f_1 = -(f + \operatorname{div}h)/2$ and $f_2 = -(f - \operatorname{div}h)/2$.

Remark 1. Let $\mathbf{M} \in L^\infty(\Omega;\mathbf{R}^{2n\times 2n})$ be a symmetric, positive definite matrix of the form

$$\mathbf{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

for some $A, B, C \in L^\infty(\Omega;\mathbf{R}^{n\times n})$ with $A = A^T$, $C = C^T$. Notice that, in particular, the matrix $A$ is also positive definite hence there exists the inverse matrix $A^{-1}$.

Assume that $(A^{-1} - A^{-1}B)$, $(C - B^TA^{-1}B + B^TA^{-1}) \in \mathcal{M}(\alpha,\beta,\Omega)$.

Let $h \in L^2(\Omega;\mathbf{R}^n)$; for every $(j,\psi) \in L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega)$ define

$$G(j,\psi) := \frac{1}{2}\int_\Omega \langle \mathbf{M}\begin{pmatrix} j \\ \nabla\psi \end{pmatrix}, \begin{pmatrix} j \\ \nabla\psi \end{pmatrix}\rangle\,dx - \int_\Omega \nabla\psi\cdot h\,dx.$$

Let $f \in H^{-1}(\Omega)$ and consider the minimization problem

$$\min\big\{G(j,\psi):\ (j,\psi) \in L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega),\ \operatorname{div}j = f\big\}. \tag{3.27}$$

Then, (3.27) provides a variational principle for the system (3.1) if and only if $\mathbf{M} \in SO(n,n)$ a.e in $\Omega$. Indeed, if $\mathbf{M} \in SO(n,n)$ a.e. in $\Omega$ by virtue of Proposition 3.1 $\mathbf{M} = \mathbf{\Sigma}$ a.e. in $\Omega$ and $G = F^{h,0}(\sigma)$, for some $\sigma \in \mathcal{M}(\alpha,\beta,\Omega)$.

On the other hand, let $(j,\psi) \in L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega)$, with $\operatorname{div} j = f$, be the unique solution to (3.27); *i.e.,*

$$\begin{cases} Aj + B\nabla\psi = \nabla\varphi \\ \operatorname{div}(B^Tj + C\nabla\psi) = \operatorname{div} h, \end{cases} \tag{3.28}$$

for some $\varphi \in H^1_0(\Omega)$ (cf. (3.19)).

Let $u, v \in H^1_0(\Omega)$ be such that $\psi = u - v$, $\varphi = u + v$. Then, rewriting (3.28) in terms of $u, v$ entails

$$\begin{cases} \operatorname{div}\Big((-A^{-1}B + A^{-1})\nabla u + (A^{-1}B + A^{-1})\nabla v\Big) = f \\ \operatorname{div}\Big((-B^TA^{-1}B + C + B^TA^{-1})\nabla u + (B^TA^{-1}B - C + B^TA^{-1})\nabla v\Big) = \operatorname{div} h\,. \end{cases} \tag{3.29}$$

Since by assumption equations (3.29) have to be of the form

$$\begin{cases} \operatorname{div}(\sigma\nabla u + \sigma^T\nabla v) = f \\ \operatorname{div}(\sigma\nabla u - \sigma^T\nabla v) = \operatorname{div}h\,, \end{cases} \tag{3.30}$$

for some $\sigma \in \mathcal{M}(\alpha,\beta,\Omega)$, by comparing (3.29) and (3.30) we deduce that the two following conditions must be fulfilled

$$\begin{cases} (A^{-1} - A^{-1}B)^T = A^{-1}B + A^{-1} \\ (-B^T A^{-1} B + C + B^T A^{-1})^T = B^T A^{-1} B - C + B^T A^{-1} \end{cases} \quad \text{a.e. in } \Omega,$$

or equivalently

$$\begin{cases} A^{-1}B + B^T A^{-1} = 0 \\ AC + B^2 = I \end{cases} \quad \text{a.e. in } \Omega.$$

Finally, noticing that the above system is equivalent to (3.16) yields $\mathbf{M} \in SO(n,n)$ a.e. in $\Omega$ (cf. proof of Proposition 3.1).

**4. Equivalence between $H$ and $\Gamma$ convergence..** On account of the variational principle stated in the previous section, here we prove the main result of this paper. Namely, we establish a connection between $H$-convergence and $\Gamma$-convergence showing that the $H$-convergence of uniformly elliptic linear operators is equivalent to the $\Gamma$-convergence of suitably associated quadratic forms.

We introduce some preliminary notation. Let $f \in H^{-1}(\Omega)$ and $\sigma_\varepsilon, \sigma_0 \in \mathcal{M}(\alpha,\beta,\Omega)$; we define the following convex closed subset of $L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega)$

$$K(f) := \{(j,\psi) \in L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega) \colon \ \mathrm{div} j = f\},$$

and the two quadratic forms $Q^f_\varepsilon, Q^f \colon L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega) \to [0,+\infty]$ defined by

$$Q^f_\varepsilon(j,\psi) := \begin{cases} \displaystyle\int_\Omega \langle \mathbf{\Sigma}_\varepsilon \begin{pmatrix} j \\ \nabla\psi \end{pmatrix}, \begin{pmatrix} j \\ \nabla\psi \end{pmatrix} \rangle \, dx & \text{if } (j,\psi) \in K(f), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1}$$

and

$$Q^f_0(j,\psi) := \begin{cases} \displaystyle\int_\Omega \langle \mathbf{\Sigma}_0 \begin{pmatrix} j \\ \nabla\psi \end{pmatrix}, \begin{pmatrix} j \\ \nabla\psi \end{pmatrix} \rangle \, dx & \text{if } (j,\psi) \in K(f), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.2}$$

where $\mathbf{\Sigma}_\varepsilon$, $\mathbf{\Sigma}_0 \in L^\infty(\Omega;\mathbf{R}^{2n\times 2n})$ are as in (3.8) and correspond to the choice $\sigma = \sigma_\varepsilon$ and $\sigma = \sigma_0$, respectively.

We start by proving that $H$-convergence implies $\Gamma$-convergence.

THEOREM 4.1. *Let $\sigma_\varepsilon, \sigma_0 \in \mathcal{M}(\alpha,\beta,\Omega)$, $f \in H^{-1}(\Omega)$ and let $Q^f_\varepsilon, Q^f_0 \colon L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega) \to [0,+\infty]$ be as in (4.1) and (4.2), respectively. If $(\sigma_\varepsilon)$ $H$-converges to $\sigma_0$ then $(Q^f_\varepsilon)$ $\Gamma$-converges to $Q^f_0$, for every $f \in H^{-1}(\Omega)$, with respect to the weak topology of $L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega)$.*

The proof of Theorem 4.1 will follow by application of Theorem 2.1 with $H = L^2(\Omega;\mathbf{R}^n) \times H^1_0(\Omega)$, $K = K(f)$, $G_\varepsilon = Q^f_\varepsilon$ and $G = Q^f_0$. Since by (3.9) $Q^f_\varepsilon$ and $Q^f_0$ satisfy the required coercivity property, it remains only to prove the convergence of minima and minimizers corresponding to assumption (2.4) in Theorem 2.1. To this end, we prove Proposition 4.2 and Theorem 4.3 below.

PROPOSITION 4.2. *Let $(\sigma_\varepsilon) \subset \mathcal{M}(\alpha,\beta,\Omega)$ be a sequence which $H$-converges to $\sigma_0 \in \mathcal{M}(\alpha,\beta,\Omega)$; let $f \in H^{-1}(\Omega)$, $k \in L^2(\Omega;\mathbf{R}^n)$, and for every fixed $\varepsilon > 0$ let $u_\varepsilon$ be the solution to*

$$\begin{cases} -\mathrm{div}(\sigma_\varepsilon(k + \nabla u_\varepsilon)) = f & \text{in } \Omega, \\ u_\varepsilon \in H^1_0(\Omega)\,. \end{cases} \tag{4.3}$$

*Then,*

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } H_0^1(\Omega), \\ \sigma_\varepsilon(k+\nabla u_\varepsilon) \rightharpoonup \sigma_0(k+\nabla u) & \text{in } L^2(\Omega;\mathbf{R}^n), \end{cases} \tag{4.4}$$

*where* $u \in H_0^1(\Omega)$ *is the solution to* $-\mathrm{div}(\sigma_0(k+\nabla u)) = f$.

*Proof.* For every fixed $\varepsilon > 0$ let $u_\varepsilon \in H_0^1(\Omega)$ be the solution to (4.3); then

$$-\mathrm{div}(\sigma_\varepsilon \nabla u_\varepsilon) = f + \mathrm{div}(\sigma_\varepsilon k).$$

By the uniform ellipticity of $(\sigma_\varepsilon) \subset \mathcal{M}(\alpha,\beta,\Omega)$ we deduce

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\big(\|f\|_{H^{-1}(\Omega)} + \beta\|k\|_{L^2(\Omega;\mathbf{R}^n)}\big),$$

therefore, up to subsequences (not relabeled),

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } H_0^1(\Omega), \\ \sigma_\varepsilon(k+\nabla u_\varepsilon) \rightharpoonup m & \text{in } L^2(\Omega;\mathbf{R}^n). \end{cases}$$

We have to show that $m = \sigma_0(k+\nabla u)$. We start by proving this equality for $k \in L^2(\Omega;\mathbf{R}^n)$ piecewise constant in $\Omega$. This means that there exists a finite partition of $\Omega$ into open sets $\Omega_i \subset \Omega$, $i = 1,\dots,N$ such that $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$, $i,j = 1,\dots,N$, and $k_{|\Omega_i} = k_i$, with $k_i \in \mathbf{R}^n$, for every $i = 1,\dots,N$.

For every $i = 1,\dots,N$, let $z_\varepsilon^i := k_i \cdot x + u_\varepsilon$; clearly $z_\varepsilon^i \in H^1(\Omega_i)$. We have

$$\begin{cases} -\mathrm{div}(\sigma_\varepsilon \nabla z_\varepsilon^i) = -\mathrm{div}(\sigma_\varepsilon(k_i+\nabla u_\varepsilon)) = f & \text{in } \Omega_i, \\ z_\varepsilon^i \rightharpoonup k_i\cdot x + u & \text{in } H^1(\Omega_i), \end{cases} \quad \text{for } i = 1,\dots,N.$$

Therefore, by the locality of $H$-convergence [17, Lemma 10.5] and by [17, Lemma 10.3] we deduce that $m = \sigma_0(k_i + \nabla u)$ in $\Omega_i$, for every $i = 1,\dots,N$. Hence, $m = \sigma_0(k+\nabla u)$ in $\Omega$ and the result for piecewise constant functions $k$ is accomplished.

Now, let $k \in L^2(\Omega;\mathbf{R}^n)$. For every $\delta > 0$ there exists $k^\delta$ piecewise constant such that $\|k - k^\delta\|_{L^2(\Omega;\mathbf{R}^n)} \leq \delta$.

For fixed $\varepsilon, \delta > 0$ let $u_\varepsilon^\delta \in H_0^1(\Omega)$ be the solution to

$$-\mathrm{div}(\sigma_\varepsilon(k^\delta + \nabla u_\varepsilon^\delta)) = f \quad \text{in } \Omega.$$

Then $u_\varepsilon - u_\varepsilon^\delta \in H_0^1(\Omega)$ satisfies

$$-\mathrm{div}(\sigma_\varepsilon(k - k^\delta + \nabla(u_\varepsilon - u_\varepsilon^\delta)) = 0 \quad \text{in } \Omega;$$

which yields

$$\|u_\varepsilon - u_\varepsilon^\delta\|_{H_0^1(\Omega)} \leq \frac{\beta}{\alpha}\|k - k^\delta\|_{L^2(\Omega;\mathbf{R}^n)}. \tag{4.5}$$

Since for $k_\delta$ the thesis holds true, we find

$$\begin{cases} u_\varepsilon^\delta \rightharpoonup u^\delta & \text{in } H_0^1(\Omega), \\ \sigma_\varepsilon(k^\delta+\nabla u_\varepsilon^\delta) \rightharpoonup \sigma_0(k^\delta+\nabla u^\delta) & \text{in } L^2(\Omega;\mathbf{R}^n), \end{cases}$$

where $u^\delta \in H_0^1(\Omega)$ is the solution to $-\mathrm{div}(\sigma_0(k^\delta + \nabla u^\delta)) = f$. Notice that by (4.5) we also deduce $u^\delta \to u$ in $H_0^1(\Omega)$ as $\delta \to 0$, hence, $u$ satisfies $-\mathrm{div}(\sigma_0(k+\nabla u)) = f$. Then, it remains to prove that

$$\sigma_\varepsilon(k+\nabla u_\varepsilon) \rightharpoonup \sigma_0(k+\nabla u) \quad \text{in} \quad L^2(\Omega;\mathbf{R}^n). \tag{4.6}$$

For every $g \in L^2(\Omega;\mathbf{R}^n)$ we have

$$
\begin{aligned}
&\Big|\int_\Omega (\sigma_\varepsilon(k+\nabla u_\varepsilon) - \sigma_0(k+\nabla u))g\,dx\Big| \\
&\le \Big|\int_\Omega (\sigma_\varepsilon(k+\nabla u_\varepsilon) - \sigma_\varepsilon(k^\delta+\nabla u_\varepsilon^\delta))g\,dx\Big| + \Big|\int_\Omega (\sigma_\varepsilon(k^\delta+\nabla u_\varepsilon^\delta) - \sigma_0(k+\nabla u))g\,dx\Big| \\
&\le \beta\Big(\|k-k^\delta\|_{L^2(\Omega;\mathbf{R}^n)} + \frac{\beta}{\alpha}\|k-k^\delta\|_{L^2(\Omega;\mathbf{R}^n)}\Big)\|g\|_{L^2(\Omega;\mathbf{R}^n)} \\
&\quad + \Big|\int_\Omega (\sigma_0(k^\delta+\nabla u^\delta) - \sigma_\varepsilon(k^\delta+\nabla u_\varepsilon^\delta))g\,dx\Big| + \Big|\int_\Omega (\sigma_0(k^\delta+\nabla u^\delta) - \sigma_0(k+\nabla u))g\,dx\Big|.
\end{aligned}
$$

Thus, we obtain (4.6) first letting $\varepsilon$ and then $\delta$ go to zero. Finally, the uniqueness of the limit and of the solution to $-\mathrm{div}(\sigma_0(k+\nabla u)) = f$ imply that the whole sequences $(u_\varepsilon)$ and $(\sigma_\varepsilon(k+\nabla u_\varepsilon))$ converge and this yields the thesis. ☐

Proposition 4.2 allows us to prove the following result on convergence of minimization problems associated with $F^{h,k}(\sigma_\varepsilon)$.

THEOREM 4.3. *Let $\sigma_\varepsilon, \sigma_0 \in \mathcal{M}(\alpha,\beta;\Omega)$, let $h,k \in L^2(\Omega;\mathbf{R}^n)$, and let $F^{h,k}(\sigma_\varepsilon), F^{h,k}(\sigma_0)$ be the corresponding functionals as in (3.17). Let $f \in H^{-1}(\Omega)$; for every fixed $\varepsilon > 0$ let $(\tilde{j}_\varepsilon, \tilde{\psi}_\varepsilon) \in L^2(\Omega;\mathbf{R}^n) \times H_0^1(\Omega)$ be the unique minimizer of $F_\varepsilon^{f,h,k}\colon L^2(\Omega;\mathbf{R}^n) \times H_0^1(\Omega) \to \mathbf{R} \cup \{+\infty\}$ where*

$$F_\varepsilon^{f,h,k}(j,\psi) := \begin{cases} F^{h,k}(\sigma_\varepsilon)(j,\psi) & \text{if } (j,\psi) \in K(f), \\ +\infty & \text{otherwise,} \end{cases}$$

*and let $(\tilde{j}, \tilde{\psi}) \in L^2(\Omega;\mathbf{R}^n) \times H_0^1(\Omega)$ be the unique minimizer of $F_0^{f,h,k}\colon L^2(\Omega;\mathbf{R}^n) \times H_0^1(\Omega) \to \mathbf{R} \cup \{+\infty\}$ where*

$$F_0^{f,h,k}(j,\psi) := \begin{cases} F^{h,k}(\sigma_0)(j,\psi) & \text{if } (j,\psi) \in K(f), \\ +\infty & \text{otherwise} \end{cases}.$$

*Then,*

$$\begin{cases} \tilde{j}_\varepsilon = \sigma_\varepsilon \nabla u_\varepsilon + \sigma_\varepsilon^T \nabla v_\varepsilon + 2\sigma_\varepsilon^s k, \\ \tilde{\psi}_\varepsilon = u_\varepsilon - v_\varepsilon, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{j} = \sigma_0 \nabla u + \sigma_0^T \nabla v + 2\sigma_0^s k, \\ \tilde{\psi} = u - v, \end{cases} \tag{4.7}$$

*where $u_\varepsilon, v_\varepsilon \in H_0^1(\Omega)$ solve*

$$\mathrm{div}(\sigma_\varepsilon(k+\nabla u_\varepsilon)) = (f + \mathrm{div}h)/2, \quad \mathrm{div}(\sigma_\varepsilon^T(k+\nabla v_\varepsilon)) = (f - \mathrm{div}h)/2, \tag{4.8}$$

*respectively, and $u, v \in H_0^1(\Omega)$ solve*

$$\mathrm{div}(\sigma_0(k+\nabla u)) = (f + \mathrm{div}h)/2, \quad \mathrm{div}(\sigma_0^T(k+\nabla v)) = (f - \mathrm{div}h)/2, \tag{4.9}$$

*respectively. Moreover, if $(\sigma_\varepsilon)$ $H$-converges to $\sigma_0$ then*

$$(\tilde{j}_\varepsilon, \tilde{\psi}_\varepsilon) \rightharpoonup (\tilde{j}, \tilde{\psi}) \quad \text{in} \quad L^2(\Omega;\mathbf{R}^n) \times H_0^1(\Omega) \tag{4.10}$$

*and*

$$\min_{(j,\psi)\in K(f)} F^{h,k}(\sigma_\varepsilon)(j,\psi) \to \min_{(j,\psi)\in K(f)} F^{h,k}(\sigma_0)(j,\psi), \tag{4.11}$$

*for every $f \in H^{-1}(\Omega)$, $h,k \in L^2(\Omega;\mathbf{R}^n)$.*

*Proof.* Reasoning as in Section 3.2, by (3.22), (3.20) and (3.26) we get (4.7), (4.8) and (4.9).

If $(\sigma_\varepsilon)$ $H$-converges to $\sigma_0$ then the convergence of minimizers (4.10) immediately follows by [17, Lemma 10.2] and Proposition 4.2 as

$$\tilde{j}_\varepsilon = \sigma_\varepsilon(k + \nabla u_\varepsilon) + \sigma_\varepsilon^T(k + \nabla v_\varepsilon) \rightharpoonup \sigma_0(k + \nabla u) + \sigma_0^T(k + \nabla v) = \tilde{j} \quad \text{in} \quad L^2(\Omega; \mathbf{R}^n).$$

We now prove the convergence of minimum values (4.11). By (4.7)-(4.9) we have

$$\begin{aligned}
\min_{(j,\psi)\in K(f)} F^{h,k}(\sigma_\varepsilon)(j,\psi) = F_\varepsilon^{f,h,k}(\tilde{j}_\varepsilon, \tilde{\psi}_\varepsilon)& \\
&= \int_\Omega \sigma_\varepsilon(k + \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx + \int_\Omega \sigma_\varepsilon^T(k + \nabla v_\varepsilon) \cdot \nabla v_\varepsilon \, dx \\
&\quad - \int_\Omega \left(\tilde{j}_\varepsilon \cdot k + \nabla \tilde{\psi}_\varepsilon \cdot h\right) dx \\
&= \int_\Omega \sigma_0(k + \nabla u) \cdot \nabla u_\varepsilon \, dx + \int_\Omega \sigma_0^T(k + \nabla v) \cdot \nabla v_\varepsilon \, dx \\
&\quad - \int_\Omega \left(\tilde{j}_\varepsilon \cdot k + \nabla \tilde{\psi}_\varepsilon \cdot h\right) dx.
\end{aligned}$$

Therefore, passing to the limit as $\varepsilon$ tends to zero we get

$$\begin{aligned}
\lim_{\varepsilon\to 0} \min_{(j,\psi)\in K(f)} F^{h,k}(\sigma_\varepsilon)(j,\psi) &= \int_\Omega \sigma_0(k + \nabla u) \cdot \nabla u \, dx + \int_\Omega \sigma_0^T(k + \nabla v) \cdot \nabla v \, dx \\
&\quad - \int_\Omega \left(\tilde{j} \cdot k + \nabla \tilde{\psi} \cdot h\right) dx \\
&= F_0^{f,h,k}(\tilde{j}, \tilde{\psi}) = \min_{(j,\psi)\in K(f)} F^{h,k}(\sigma_0)(j,\psi),
\end{aligned}$$

hence the complete proof is achieved. □

Remark 2. By virtue of Proposition 3.1 and Theorem 4.1 we may deduce that, under the assumption of $H$-convergence of $(\sigma_\varepsilon)$, the property of $\mathbf{\Sigma}_\varepsilon$ of being in $SO(n,n)$ for a.e. $x \in \Omega$ is stable under $\Gamma$-convergence; *i.e.,* $\Sigma_0 \in SO(n,n)$ for a.e. $x \in \Omega$.

We now come to prove the other implication; *i.e.,* we prove that the $\Gamma$-convergence of the sequence of quadratic forms $(Q_\varepsilon^f)$ to $Q_0^f$, for every $f \in H^{-1}(\Omega)$, implies the $H$-convergence of the associated sequence $(\sigma_\varepsilon)$ to $\sigma_0$. We give two different proofs of this result. The first one relies on the fundamental property of $\Gamma$-convergence. The second proof makes use of the compactness of $H$-convergence and of an interesting result (Theorem 4.5, below) concerning the identification of two matrices defining the same quadratic form on $L^2(\Omega; \mathbf{R}^n) \times H_0^1(\Omega)$.

Theorem 4.4. *Let $\sigma_\varepsilon, \sigma_0 \in \mathcal{M}(\alpha, \beta, \Omega)$, $f \in H^{-1}(\Omega)$ and let $Q_\varepsilon^f, Q_0^f \colon L^2(\Omega; \mathbf{R}^n) \times H_0^1(\Omega) \to [0, +\infty]$ be as in (4.1) and (4.2), respectively. If for every fixed $f \in H^{-1}(\Omega)$ $(Q_\varepsilon^f)$ $\Gamma$-converges to $Q_0^f$, with respect to the weak topology of $L^2(\Omega; \mathbf{R}^n) \times H_0^1(\Omega)$, then $(\sigma_\varepsilon)$ $H$-converges to $\sigma_0$.*

*Proof.* [First proof of Theorem 4.4] Let $h, k \in L^2(\Omega; \mathbf{R}^n)$; by the continuity of

$$(j,\psi) \mapsto \int_\Omega j \cdot 2k \, dx + \int_\Omega \nabla\psi \cdot h \, dx$$

with respect to the weak topology of $L^2(\Omega; \mathbf{R}^n) \times H_0^1(\Omega)$, we deduce that $(F_\varepsilon^{f,h,k})$ $\Gamma$-converges to $F_0^{f,h,k}$ for every $f \in H^{-1}(\Omega)$ and for every $h, k \in L^2(\Omega; \mathbf{R}^n)$. Moreover, notice that $(F_\varepsilon^{f,h,k})$ is equi-coercive with respect to the weak topology of $L^2(\Omega; \mathbf{R}^n) \times H_0^1(\Omega)$.

For every fixed $f \in H^{-1}(\Omega)$, we may choose $h \in L^2(\Omega; \mathbf{R}^n)$ such that $\mathrm{div} h = f$, and $k = 0$. Then, appealing to the equi-coercivity of $(F_\varepsilon^{f,h,k})$, to [2, Theorem 7.8], and recalling (4.7)-(4.8), we find that the unique solution to

$$\begin{cases} -\mathrm{div}(\sigma_\varepsilon \nabla u_\varepsilon) = f \\ u_\varepsilon \in H_0^1(\Omega) \end{cases}$$

is such that, up to subsequences,

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } H_0^1(\Omega) \\ \sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma_0 \nabla u & \text{in } L^2(\Omega; \mathbf{R}^n), \end{cases}$$

where $u \in H_0^1(\Omega)$ satisfies $-\mathrm{div}(\sigma_0 \nabla u) = f$. Finally, since the limit is independent of the subsequence, we conclude by the arbitrariness of $f \in H^{-1}(\Omega)$. $\square$

To prove the following result we employ a similar argument to that used by Spagnolo in [11] to characterize a certain class of continuous symmetric forms on $H^1(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$ (see also [2, Lemma 22.5]).

THEOREM 4.5. *Let $\mathbf{M}, \overline{\mathbf{M}} \in L^\infty(\Omega; \mathbf{R}^{2n \times 2n})$ be two symmetric matrices; i.e.,*

$$\mathbf{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \qquad \overline{\mathbf{M}} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B}^T & \bar{C} \end{pmatrix}$$

*for some $A, B, C, \bar{A}, \bar{B}, \bar{C} \in L^\infty(\Omega; \mathbf{R}^n)$ such that $A = A^T$, $C = C^T$, $\bar{A} = \bar{A}^T$, $\bar{C} = \bar{C}^T$. Suppose moreover that the matrices $\mathbf{M}$, $\overline{\mathbf{M}}$ are positive definite a.e. in $\Omega$.*

*Then, if*

$$\int_\Omega \langle \mathbf{M} \begin{pmatrix} \phi_1 \\ \nabla \phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \nabla \phi_2 \end{pmatrix} \rangle \, dx = \int_\Omega \langle \overline{\mathbf{M}} \begin{pmatrix} \phi_1 \\ \nabla \phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \nabla \phi_2 \end{pmatrix} \rangle \, dx,$$

*for every $\phi_1, \phi_2 \in L^2(\Omega; \mathbf{R}^n) \times H_0^1(\Omega)$, we have $\mathbf{M} = \overline{\mathbf{M}}$ a.e. in $\Omega$.*

*Proof.* Let $\omega \in C_c^\infty(\Omega)$ and choose

$$\phi_1(x) = \lambda \omega(x) \cos(\lambda \eta \cdot x) \, \eta \,, \qquad \phi_2(x) = \omega(x) \sin(\lambda \xi \cdot x),$$

with $\lambda \in \mathbf{R}$ and $\xi, \eta \in \mathbf{R}^n$. Clearly, $(\phi_1, \phi_2) \in L^2(\Omega; \mathbf{R}^n) \times H_0^1(\Omega)$.

We have

$$\begin{aligned} &\langle \mathbf{M} \begin{pmatrix} \phi_1 \\ \nabla \phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \nabla \phi_2 \end{pmatrix} \rangle \\ &= \lambda^2 \omega^2 \Big( \cos^2(\lambda \eta \cdot x) \, A\eta \cdot \eta + 2 \, \cos(\lambda \eta \cdot x) \cos(\lambda \xi \cdot x) \, B\xi \cdot \eta + \cos^2(\lambda \xi \cdot x) \, C\xi \cdot \xi \Big) \\ &\quad + 2\lambda \omega \Big( \cos(\lambda \eta \cdot x) \sin(\lambda \xi \cdot x) B^T \eta \cdot \nabla \omega + \cos(\lambda \xi \cdot x) \sin(\lambda \xi \cdot x) C\xi \cdot \nabla \omega \Big) \\ &\quad + \sin^2(\lambda \xi \cdot x) C \nabla \omega \cdot \nabla \omega. \end{aligned}$$

We make the same computation as above now choosing

$$\tilde{\phi}_1(x) = -\lambda \omega(x) \sin(\lambda \eta \cdot x) \eta \,, \qquad \tilde{\phi}_2(x) = \omega(x) \cos(\lambda \xi \cdot x);$$

we get

$$\begin{aligned} &\langle \mathbf{M} \begin{pmatrix} \tilde{\phi}_1 \\ \nabla \tilde{\phi}_2 \end{pmatrix}, \begin{pmatrix} \tilde{\phi}_1 \\ \nabla \tilde{\phi}_2 \end{pmatrix} \rangle \\ &= \lambda^2 \omega^2 \Big( \sin^2(\lambda \eta \cdot x) \, A\eta \cdot \eta + 2 \, \sin(\lambda \eta \cdot x) \sin(\lambda \xi \cdot x) \, B\xi \cdot \eta + \sin^2(\lambda \xi \cdot x) \, C\xi \cdot \xi \Big) \\ &\quad - 2\lambda \omega \Big( \sin(\lambda \eta \cdot x) \cos(\lambda \xi \cdot x) B^T \eta \cdot \nabla \omega + \sin(\lambda \xi \cdot x) \cos(\lambda \xi \cdot x) C\xi \cdot \nabla \omega \Big) \\ &\quad + \cos^2(\lambda \xi \cdot x) C \nabla \omega \cdot \nabla \omega. \end{aligned}$$

Therefore we find

$$\begin{aligned}
&\langle \mathbf{M}\begin{pmatrix} \phi_1 \\ \nabla\phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \nabla\phi_2 \end{pmatrix}\rangle + \langle \mathbf{M}\begin{pmatrix} \tilde\phi_1 \\ \nabla\tilde\phi_2 \end{pmatrix}, \begin{pmatrix} \tilde\phi_1 \\ \nabla\tilde\phi_2 \end{pmatrix}\rangle \\
&= \lambda^2\omega^2\Big(A\eta\cdot\eta + 2\cos(\lambda(\xi-\eta)\cdot x)B\xi\cdot\eta + C\xi\cdot\xi\Big) \\
&\quad + 2\lambda\omega\sin(\lambda(\xi-\eta)\cdot x)B\nabla\omega\cdot\eta + C\nabla\omega\cdot\nabla\omega,
\end{aligned}$$

hence

$$\begin{aligned}
&\int_\Omega \Big(\langle \mathbf{M}\begin{pmatrix} \phi_1 \\ \nabla\phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \nabla\phi_2 \end{pmatrix}\rangle + \langle \mathbf{M}\begin{pmatrix} \tilde\phi_1 \\ \nabla\tilde\phi_2 \end{pmatrix}, \begin{pmatrix} \tilde\phi_1 \\ \nabla\tilde\phi_2 \end{pmatrix}\rangle\Big)\,dx \\
&= \lambda^2\int_\Omega \omega^2\Big(A\eta\cdot\eta + 2\cos(\lambda(\xi-\eta)\cdot x)B\xi\cdot\eta + C\xi\cdot\xi\Big)\,dx \\
&\quad +2\lambda\int_\Omega \omega\sin(\lambda(\xi-\eta)\cdot)B\nabla\omega\cdot\eta\,dx \\
&\quad + \int_\Omega C\nabla\omega\cdot\nabla\omega\,dx\,.
\end{aligned} \tag{4.12}$$

Similarly, replacing in (4.12) $\mathbf{M}$ with $\overline{\mathbf{M}}$ we get

$$\begin{aligned}
&\int_\Omega \Big(\langle \overline{\mathbf{M}}\begin{pmatrix} \phi_1 \\ \nabla\phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \nabla\phi_2 \end{pmatrix}\rangle + \langle \overline{\mathbf{M}}\begin{pmatrix} \tilde\phi_1 \\ \nabla\tilde\phi_2 \end{pmatrix}, \begin{pmatrix} \tilde\phi_1 \\ \nabla\tilde\phi_2 \end{pmatrix}\rangle\Big)\,dx \\
&= \lambda^2\int_\Omega \omega^2\Big(\bar A\eta\cdot\eta + 2\cos(\lambda(\xi-\eta)\cdot x)\bar B\xi\cdot\eta + \bar C\xi\cdot\xi\Big)\,dx \\
&\quad +2\lambda\int_\Omega \omega\sin(\lambda(\xi-\eta)\cdot x)\bar B\nabla\omega\cdot\eta\,dx \\
&\quad + \int_\Omega \bar C\nabla\omega\cdot\nabla\omega\,dx\,.
\end{aligned} \tag{4.13}$$

By assumption, (4.12) is equal to (4.13) for every $\lambda \in \mathbf{R}$, $\omega \in C_c^\infty(\Omega)$, and $\xi,\eta \in \mathbf{R}^n$. This yields in particular

$$\begin{aligned}
&A\eta\cdot\eta + 2\cos(\lambda(\xi-\eta)\cdot x)B\xi\cdot\eta + C\xi\cdot\xi \\
&= \bar A\eta\cdot\eta + 2\cos(\lambda(\xi-\eta)\cdot x)\bar B\xi\cdot\eta + \bar C\xi\cdot\xi,
\end{aligned} \tag{4.14}$$

for every $\xi,\eta \in \mathbf{R}^n$, a.e. in $\Omega$. Letting $\xi = 0$ in (4.14) gives

$$A\eta\cdot\eta = \bar A\eta\cdot\eta, \tag{4.15}$$

for every $\eta \in \mathbf{R}^n$, a.e. in $\Omega$. Since the matrices $A$ and $\bar A$ are symmetric, from (4.15) and from the polarization identity we obtain

$$A\xi\cdot\eta = \bar A\xi\cdot\eta,$$

for a.e. $x \in \Omega$ and for every $\xi,\eta \in \mathbf{R}^n$. Hence choosing $\xi = e_i$ and $\eta = e_j$, for every $i,j = 1,\dots,n$ (where $e_1,\dots,e_n$ is the canonical basis of $\mathbf{R}^n$), entails $A = \bar A$ a.e. in $\Omega$. In the same way, choosing $\eta = 0$ and appealing to the symmetry of $C$ and $\bar C$, yield $C = \bar C$ a.e. in $\Omega$.

Finally, it remains to show that $B = \bar B$ a.e. in $\Omega$. To this end we choose

$$\phi_1(x) = (B-\bar B)\nabla\phi_2(x)\,, \qquad \phi_2(x) = \omega(x)\sin(\lambda\xi\cdot x),$$

and

$$\tilde\phi_1(x) = (B-\bar B)\nabla\tilde\phi_2(x)\,, \qquad \tilde\phi_2(x) = \omega(x)\cos(\lambda\xi\cdot x).$$

A straightforward calculation easily yields

$$(B - \bar{B})^T (B - \bar{B})\xi \cdot \xi = 0$$

for every $\xi \in \mathbf{R}^n$, a.e. in $\Omega$. Therefore, $(B - \bar{B})\xi = 0$, for every $\xi \in \mathbf{R}^n$, a.e. in $\Omega$, hence $B = \bar{B}$ a.e. in $\Omega$, and thus the thesis. □

*Proof.* [Second proof of Theorem 4.4] Let $(\sigma_\varepsilon) \subset \mathcal{M}(\alpha, \beta, \Omega)$ be the sequence of matrices associated with the sequence of quadratic forms $(Q^f_\varepsilon)$ through $\mathbf{\Sigma}_\varepsilon$.

By virtue of the compactness of $H$-convergence [17, Theorem 6.5], there exists a subsequence $(\sigma_{\varepsilon_j})$ of $(\sigma_\varepsilon)$ such that $(\sigma_{\varepsilon_j})$ $H$-converges to some $\bar{\sigma} \in \mathcal{M}(\alpha, \beta, \Omega)$, as $j \to +\infty$. Let $\bar{Q}^f$ be the quadratic form associated with $\bar{\sigma}$ through $\overline{\mathbf{\Sigma}}$; therefore by Theorem 4.1 we may deduce that $(Q^f_{\varepsilon_j})$ $\Gamma$-converges to $\bar{Q}^f$, as $j \to +\infty$, for every $f \in H^{-1}(\Omega)$. Then, by the uniqueness of the $\Gamma$-limit we have that $\bar{Q}^f = Q^f_0$ on $L^2(\Omega; \mathbf{R}^n) \times H^1_0(\Omega)$, for every $f \in H^{-1}(\Omega)$; *i.e.,*

$$\int_\Omega \langle \overline{\mathbf{\Sigma}} \begin{pmatrix} j \\ \nabla\psi \end{pmatrix}, \begin{pmatrix} j \\ \nabla\psi \end{pmatrix} \rangle \, dx = \int_\Omega \langle \mathbf{\Sigma}_0 \begin{pmatrix} j \\ \nabla\psi \end{pmatrix}, \begin{pmatrix} j \\ \nabla\psi \end{pmatrix} \rangle \, dx,$$

for every $(j, \psi) \in L^2(\Omega; \mathbf{R}^n) \times H^1_0(\Omega)$ such that $\mathrm{div} j = f$, and for every $f \in H^{-1}(\Omega)$. Hence, by the arbitrariness of $f \in H^{-1}(\Omega)$ we can argue as in the proof of Theorem 4.5 to deduce that $\overline{\mathbf{\Sigma}} = \mathbf{\Sigma}_0$ a.e. in $\Omega$. As a consequence, we immediately find that $\bar{\sigma} = \sigma_0$, a.e. in $\Omega$ and that the whole sequence $(\sigma_\varepsilon)$ $H$-converges to $\sigma_0$. □

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