THE EULER EQUATIONS AS A DIFFERENTIAL INCLUSION

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ABSTRACT. In this paper we propose a new point of view on weak solutions of the Euler equations, describing the motion of an ideal incompressible fluid in $\mathbb{R}^n$ with $n \geq 2$. We give a reformulation of the Euler equations as a differential inclusion, and in this way we obtain transparent proofs of several celebrated results of V. Scheffer and A. Shnirelman concerning the non-uniqueness of weak solutions and the existence of energy–decreasing solutions. Our results are stronger because they work in any dimension and yield bounded velocity and pressure.

1. Introduction

Consider the Euler equations in n space dimensions, describing the motion of an ideal incompressible fluid,

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - f &= 0 \\ \operatorname{div} v &= 0\,. \end{aligned} \tag{1}$$

Classical (i.e. sufficiently smooth) solutions of the Cauchy problem exist locally in time for sufficiently regular initial data and driving forces (see Chapter 3.2 in [16]). In two dimensions such existence results are available also for global solutions (e.g. Chapters 3.3 and 8.2 in [16] and the references therein). Classical solutions of Euler's equations with $f = 0$ conserve the energy, that is $t \mapsto \int |v(x,t)|^2\,dx$ is a constant function. Hence the energy space for (1) is $L^\infty_t(L^2_x)$.

A recurrent issue in the modern theory of PDEs is that one needs to go beyond classical solutions, in particular down to the energy space (see for instance [6, 8, 16, 25]). A divergence–free vector field $v \in L^2_{loc}$ is a *weak solution* of (1) if

$$\int \big(v\partial_t\varphi + \langle v \otimes v, \nabla\varphi\rangle + \varphi \cdot f\big)\,dx\,dt \;=\; 0 \tag{2}$$

for every test function $\varphi \in C^\infty_c(\mathbb{R}^n_x \times \mathbb{R}_t, \mathbb{R}^n)$ with $\operatorname{div}\varphi = 0$. It is well–known that then the pressure is determined up to a function depending only on time (see [28]). In the case of Euler strong motivation for considering weak solutions comes also from mathematical physics, especially the theory of turbulence laid down by Kolmogorov in 1941 [3, 11]. A celebrated criterion of Onsager related to Kolmogorov's theory says, roughly speaking, that dissipative weak solutions cannot have a Hölder exponent greater than 1/3 (see [4, 9, 10, 19]). It is therefore of interest to construct weak solutions with limited regularity.

Weak solutions are not unique. In a well–known paper [21] Scheffer constructed a surprising example of a weak solution to (1) with compact support in space and time when $f = 0$ and $n = 2$. Scheffer's proof is very long and complicated and a simpler construction was later given by Shnirelman in [22]. However, Shnirelman's proof is still quite difficult. In this paper we obtain a short and elementary proof of the following theorem.

Theorem 1.1. *Let $f = 0$. There exists $v \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n)$ and $p \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solving (1) in the sense of distributions, such that v is not identically zero, and* $\operatorname{supp} v$ *and* $\operatorname{supp} p$ *are compact in space-time $\mathbb{R}^n_x \times \mathbb{R}_t$.*

In mathematical physics weak solutions to the Euler equations that dissipate energy underlie the Kolmogorov theory of turbulence. In another groundbreaking paper [23] Shnirelman proved the existence of L^2 distributional solutions with $f = 0$ and energy which decreases in time. His methods are completely unrelated to those in [21] and [22]. In contrast, the following extension of his existence theorem is a simple corollary of our construction.

Theorem 1.2. *There exists (v, p) as in Theorem 1.1 such that, in addition:*
- *$\int |v(x,t)|^2\,dx = 1$ for almost every $t \in]-1, 1[$,*
- *$v(x,t) = 0$ for $|t| > 1$.*

Our method has several interesting features. First of all, our approach fits nicely in the well–known framework of L. Tartar for the analysis of oscillations in linear partial differential systems coupled with nonlinear pointwise constraints [7, 15, 26, 27]. Roughly speaking, Tartar's framework amounts to a plane–wave analysis localized in physical space, in contrast with Shnirelman's method in [22], which is based rather on a wave analysis in Fourier space. In combination with Gromov's convex integration or with Baire category arguments, Tartar's approach leads to a well understood mechanism for generating irregular oscillatory solutions to differential inclusions (see [14, 15, 17]).

Secondly, the velocity field we construct belongs to the energy space $L^\infty_t(L^2_x)$. This was not the case for the solutions in [21, 22], and it was a natural question whether weak solutions in the energy space were unique. Our first theorem shows that even higher summability assumptions of v do not rule out such pathologies. The pressure in [21, 22] is only a distribution solving (1). In our construction p is actually the potential–theoretic solution of

$$-\Delta p \;=\; \partial^2_{x_i x_j}(v^i v^j) - \partial_{x_i} f_i \,. \tag{3}$$

However, being bounded, it has slightly better regularity than the BMO given by the classical estimates for (3).

Next, our point of view reveals connections between the apparently unrelated constructions of Scheffer and Shnirelman. Shnirelman considers sequences of driving forces f_k converging to 0 in some negative Sobolev space.

In particular he shows that for a suitable choice of f_k the corresponding solutions of (1) converge in L^2 to a nonzero solution of (1) with $f = 0$. Scheffer builds his solution by iterating a certain piecewise constant construction at small scales. On the one hand both our proof and Scheffer's proof are based on oscillations localized in physical space. On the other hand, our proof gives as an easy byproduct the following approximation result in Shnirelman's spirit.

Theorem 1.3. *All the solutions (v, p) constructed in the proofs of Theorem 1.1 and in Theorem 1.2 have the following property. There exist three sequences $\{v_k\}, \{f_k\}, \{p_k\} \subset C_c^\infty$ solving (1) such that*

- *f_k converges to 0 in H^{-1},*
- *$\|v_k\|_\infty + \|p_k\|_\infty$ is uniformly bounded,*
- *$(v_k, p_k) \to (v, p)$ in L^q for every $q < \infty$.*

Our results give interesting information on which kind of additional (entropy) condition could restore uniqueness of solutions. As already remarked, belonging to the energy space is not sufficient. In fact, in view of our method of construction, there is strong evidence that neither energy–decreasing nor energy–preserving solutions are unique. In a forthcoming paper we plan to investigate this issue, and also the class of initial data for which our method yields energy–decreasing solutions.

The rest of the paper is organized as follows. In Section 2 we carry out the plane wave analysis of the Euler equations in the spirit of Tartar, and we formulate the core of our construction (Proposition 2.2). In Section 3 we prove Proposition 2.2. In Section 4 we show how our main results follow from the Proposition. We emphasize that the concluding argument in Section 4 appeals to the – by now standard – methods for solving differential inclusions, either by appealing to the Baire category theorem [1, 2, 5, 13], or by the more explicit convex integration method [12, 17, 18]. In our opinion, the Baire category argument developed in [14] and used in Section 4 is, for the purposes of this paper, the most efficient and elegant tool. However, we include in Section 5 an alternative proof which follows the convex integration approach, as it makes easier to "visualize" the solutions constructed in this paper.

In fact we believe that for $n \geq 3$ a suitable modification of the original approach of Gromov (see [12]) would also work, yielding solutions which are even continuous (work in progress).

2. Plane wave analysis of Euler's equations

We start by briefly explaining Tartar's framework [26]. One considers nonlinear PDEs that can be expressed as a system of linear PDEs (conservation laws)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{4}$$

coupled with a pointwise nonlinear constraint (constitutive relations)

$$z(x) \in K \subset \mathbb{R}^d \text{ a.e.}, \tag{5}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable. The idea is then to consider *plane wave* solutions to (4), that is, solutions of the form

$$z(x) = ah(x \cdot \xi), \tag{6}$$

where $h : \mathbb{R} \to \mathbb{R}$. The *wave cone* Λ is given by the states $a \in \mathbb{R}^d$ such that for any choice of the profile h the function (6) solves (4), that is,

$$\Lambda := \left\{ a \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \setminus \{0\} \quad \text{with} \quad \sum_{i=1}^m \xi_i A_i a = 0 \right\}. \tag{7}$$

The oscillatory behavior of solutions to the nonlinear problem is then determined by the compatibility of the set K with the cone Λ.

The Euler equations can be naturally rewritten in this framework. The domain is $\mathbb{R}^m = \mathbb{R}^{n+1}$, and the state variable z is defined as $z = (v, u, q)$, where

$$q = p + \frac{1}{n}|v|^2, \text{ and } u = v \otimes v - \frac{1}{n}|v|^2 I_n,$$

so that u is a symmetric $n \times n$ matrix with vanishing trace and I_n denotes the $n \times n$ identity matrix. From now on the linear space of symmetric $n \times n$ matrices will be denoted by $\mathcal{S}^n$ and the subspace of trace–free symmetric matrices by $\mathcal{S}^n_0$. The following lemma is straightforward.

Lemma 2.1. *Suppose* $v \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n)$, $u \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathcal{S}^n_0)$, *and* $q \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ *solve*

$$\begin{aligned} \partial_t v + div\ u + \nabla q &= 0, \\ div\ v &= 0, \end{aligned} \tag{8}$$

in the sense of distributions. If in addition

$$u = v \otimes v - \frac{1}{n}|v|^2 I_n \qquad a.e. \text{ in } \mathbb{R}^n_x \times \mathbb{R}_t, \tag{9}$$

then v *and* $p := q - \frac{1}{n}|v|^2$ *are a solution to* (1) *with* $f \equiv 0$. *Conversely, if* v *and* p *solve* (1) *distributionally, then* v, $u := v \otimes v - \frac{1}{n}|v|^2 I_n$ *and* $q := p + \frac{1}{n}|v|^2$ *solve* (8) *and* (9).

Consider the $(n+1) \times (n+1)$ symmetric matrix in block form

$$U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix}, \tag{10}$$

where I_n is the $n \times n$ identity matrix. Notice that by introducing new coordinates $y = (x, t) \in \mathbb{R}^{n+1}$ the equation (8) becomes simply

$$\text{div}_y U = 0.$$

Here, as usual, a divergence–free matrix field is a matrix of functions with rows that are divergence–free vectors. Therefore the wave cone corresponding to (8) is given by

$$\Lambda = \left\{ (v, u, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} : \det \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} = 0 \right\}.$$

Remark 1. *A simple linear algebra computation shows that for every $v \in \mathbb{R}^n$ and $u \in \mathcal{S}_0^n$ there exists $q \in \mathbb{R}$ such that $(v, u, q) \in \Lambda$, revealing that the wave cone is very large. Indeed, let $V^\perp \subset \mathbb{R}^n$ be the linear space orthogonal to v and consider on $V^\perp$ the quadratic form $\xi \mapsto \xi \cdot u\xi$. Then, $\det U = 0$ if and only if $-q$ is an eigenvalue of this quadratic form.*

In order to exploit this fact for constructing irregular solutions to the nonlinear system, one needs plane wave–like solutions to (8) which are localized in space. Clearly an exact plane–wave as in (6) has compact support only if it is identically zero. Therefore this can only be done by introducing an error in the range of the wave, deviating from the line spanned by the wave state $a \in \mathbb{R}^d$. However, this error can be made arbitrarily small. This is the content of the following proposition, which is the building block of our construction.

Proposition 2.2 (Localized plane waves)**.** *Let $a = (v_0, u_0, q_0) \in \Lambda$ with $v_0 \neq 0$, and denote by σ the line segment in $\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}$ joining the points $-a$ and a. For every $\varepsilon > 0$ there exists a smooth solution (v, u, q) of* (8) *with the properties:*

- *the support of (v, u, q) is contained in $B_1(0) \subset \mathbb{R}^n_x \times \mathbb{R}_t$,*
- *the image of (v, u, q) is contained in the ε–neighborhood of σ,*
- $\int |v(x,t)|\, dx\, dt \geq \alpha |v_0|$,

where $\alpha > 0$ is a dimensional constant.

3. Localized plane waves

For the proof of Proposition 2.2 there are two main points. Firstly, we appeal to a particular large group of symmetries of the equations in order to reduce the problem to some special Λ-directions. Secondly, to achieve a cut-off which preserves the linear equations (8), we introduce a suitable potential.

Definition 3.1. *We denote by $\mathcal{M}$ the set of symmetric $(n+1) \times (n+1)$ matrices A such that $A_{(n+1)(n+1)} = 0$. Clearly, the map*

$$\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \ni (v, u, q) \quad \mapsto \quad U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} \in \mathcal{M} \tag{11}$$

is a linear isomorphism.

As already observed, in the variables $y = (x, t) \in \mathbb{R}^{n+1}$, the equation (8) is equivalent to $\operatorname{div} U = 0$. Therefore Proposition 2.2 follows immediately from

Proposition 3.2. *Let $\overline{U} \in \mathcal{M}$ be such that $\det \overline{U} = 0$ and $\overline{U} e_{n+1} \neq 0$, and consider the line segment σ with endpoints $-\overline{U}$ and $\overline{U}$. Then there exists a constant $\alpha > 0$ such that for any $\varepsilon > 0$ there exists a smooth divergence–free matrix field $U : \mathbb{R}^{n+1} \to \mathcal{M}$ with the properties*

(p1) $\operatorname{supp} U \subset B_1(0)$,
(p2) $\operatorname{dist}\,(U(y), \sigma) < \varepsilon$ *for all* $y \in B_1(0)$,
(p3) $\int |U(y) e_{n+1}| dy \geq \alpha |\overline{U} e_{n+1}|$,

where $\alpha > 0$ is a dimensional constant.

The proof of Proposition 3.2 relies on two lemmas. The first deals with the symmetries of the equations.

Lemma 3.3 (The Galilean group). *Let $\mathcal{G}$ be the subgroup of $GL_{n+1}(\mathbb{R})$ defined by*

$$\left\{ A \in \mathbb{R}^{(n+1)\times(n+1)} : \det A \neq 0, A e_{n+1} = e_{n+1} \right\}. \tag{12}$$

For every divergence–free map $U : \mathbb{R}^{n+1} \to \mathcal{M}$ and every $A \in \mathcal{G}$ the map

$$V(y) := A^t \cdot U(A^{-t} y) \cdot A$$

is also a divergence–free map $V : \mathbb{R}^{n+1} \to \mathcal{M}$.

The second deals with the potential.

Lemma 3.4 (Potential in the general case). *Let $E_{ij}^{kl} \in C^\infty(\mathbb{R}^{n+1})$ be functions for $i, j, k, l = 1, \ldots, n+1$ so that the tensor E is skew–symmetric in ij and kl, that is*

$$E_{ij}^{kl} = -E_{ij}^{lk} = -E_{ji}^{kl} = E_{ji}^{lk}. \tag{13}$$

Then

$$U_{ij} = \mathcal{L}(E) = \frac{1}{2} \sum_{k,l} \partial_{kl}^2 (E_{kj}^{il} + E_{ki}^{jl}) \tag{14}$$

is symmetric and divergence–free. If in addition

$$E_{(n+1)i}^{(n+1)j} = 0 \qquad \text{for every } i \text{ and } j, \tag{15}$$

then U takes values in $\mathcal{M}$.

Remark 2. *A suitable potential in the case $n = 2$ can be obtained in a more direct way. Indeed, let $w \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be a divergence–free vector field and consider the map $U : \mathbb{R}^3 \to \mathcal{M}$ given by*

$$U = \begin{pmatrix} \partial_2 w_1 & \frac{1}{2}\partial_2 w_2 - \frac{1}{2}\partial_1 w_1 & \frac{1}{2}\partial_2 w_3 \\ \frac{1}{2}\partial_2 w_2 - \frac{1}{2}\partial_1 w_1 & -\partial_1 w_2 & -\frac{1}{2}\partial_1 w_3 \\ \frac{1}{2}\partial_2 w_3 & -\frac{1}{2}\partial_1 w_3 & 0 \end{pmatrix}. \tag{16}$$

Then it can be readily checked that U is divergence–free. Moreover, w is the curl of a vector field ω. However, this is just a particular case of Lemma 3.4. Indeed, given E as in the Lemma define the tensor $D_{ij}^k = \sum_l \partial_l E_{ij}^{kl}$. Note *that D is skew–symmetric in ij and for each ij, the vector $(D^k_{ij})_{k=1,\dots,n+1}$ is divergence–free. Moreover,*

$$U_{ij} = \frac{1}{2}\sum_k \partial_k (D^i_{kj} + D^j_{ki})\,.$$

Then the vector field w above is simply the special choice where $D^k_{12} = -D^k_{21} = w_k$ and all other D's are zero, and a corresponding relation can be found for E and ω.

The proofs of the two Lemmas will be postponed until the end of the section and we now come to the proof of the Proposition.

Proof of Proposition 3.2. **Step 1.** First we treat the case when $\overline{U} \in \mathcal{M}$ is such that

$$\overline{U}e_1 = 0, \quad \overline{U}e_{n+1} \neq 0. \tag{17}$$

Let

$$E^{j1}_{i1} = -E^{j1}_{1i} = -E^{1j}_{i1} = E^{1j}_{1i} = \overline{U}_{ij}\frac{\sin(Ny_1)}{N^2} \tag{18}$$

and all the other entries equal to 0. Note that by our assumption $\overline{U}_{ij} = 0$ whenever one index is 1 or both of them are $n+1$. This ensures that the tensor E is well defined and satisfies the properties of Lemma 3.4.

We remark that in the case $n = 2$ the matrix $\overline{U}$ takes necessarily the form

$$\overline{U} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & 0 \end{pmatrix} \tag{19}$$

with $b \neq 0$, and we can use the potential of Remark 2 by simply setting

$$\begin{aligned} w &= \frac{1}{N}(0, a\cos(Ny_1), 2b\cos(Ny_1))\,, \\ \omega &= \frac{1}{N^2}(0, 2b\sin(Ny_1), -a\sin(Ny_1))\,. \end{aligned}$$

We come back to the general case. Let E be defined as in (18), fix a smooth cutoff function φ such that

- $|\varphi| \leq 1$,
- $\varphi = 1$ on $B_{1/2}(0)$,
- $\mathrm{supp}\,(\varphi) \subset B_1(0)$,

and consider the map

$$U = \mathcal{L}(\varphi E).$$

Clearly, U is smooth and supported in $B_1(0)$. By Lemma 3.4, U is $\mathcal{M}$–valued and divergence–free. Moreover

$$U(y) = \overline{U}\sin(Ny_1) \text{ for } y \in B_{1/2}(0),$$

and in particular

$$\int |U(y)e_{n+1}|dy \geq |\overline{U}e_{n+1}| \int_{B_{1/2}(0)} |\sin(Ny_1)|\,dy \geq 2\alpha|\overline{U}e_{n+1}|,$$

for some positive dimensional constant $\alpha = \alpha(n)$ for sufficiently large N.

Finally, observe that

$$U - \varphi\tilde{U} \;=\; \mathcal{L}(\varphi E) - \varphi\mathcal{L}(E)$$

is a sum of products of first–order derivatives of φ with first–order derivatives of components of E and of second–order derivatives of φ with components of E. Thus,

$$\|U - \varphi\tilde{U}\|_\infty \;\leq\; C\|\varphi\|_{C^2}\|E\|_{C^1} \;\leq\; \frac{C'}{N}\|\varphi\|_{C^2}\,,$$

and by choosing N sufficiently large we obtain $\|U - \varphi\tilde{U}\|_\infty < \varepsilon$. On the other hand, since $|\varphi| \leq 1$ and $\tilde{U}$ takes values in σ, the image of $\varphi\tilde{U}$ is also contained in σ. This shows that the image of U is contained in the ε–neighborhood of σ.

Step 2. We treat the general case by reducing to the situation above. Let $\overline{U} \in \mathcal{M}$ be as in the Proposition, so that

$$\overline{U}f = 0, \quad \overline{U}e_{n+1} \neq 0,$$

where $f \in \mathbb{R}^{n+1} \setminus \{0\}$ is such that $\{f, e_{n+1}\}$ are linearly independent. Let $f_1, \ldots, f_{n+1}$ be a basis for $\mathbb{R}^{n+1}$ such that $f_1 = f$ and $f_{n+1} = e_{n+1}$ and consider the matrix A such that

$$Ae_i = f_i \text{ for } i = 1, \ldots, n+1.$$

Then $A \in \mathcal{G}$ (cf. with the definition of $\mathcal{G}$ given in Lemma 3.3), and the map

$$T : X \mapsto (A^{-1})^t X A^{-1} \tag{20}$$

is a linear isomorphism of $\mathbb{R}^{n+1}$. Set

$$\overline{V} = A^t\overline{U}A, \tag{21}$$

so that $\overline{V} \in \mathcal{M}$ satisfies

$$\overline{V}e_1 = 0, \quad \overline{V}e_{n+1} \neq 0.$$

Given $\varepsilon > 0$, using Step 1 we construct a smooth map $V : \mathbb{R}^{n+1} \to \mathcal{M}$ supported in $B_1(0)$ with the image lying in the $\|T\|^{-1}\varepsilon$–neighborhood of the line segment τ with endpoints $-\overline{V}$ and $\overline{V}$, and such that

$$V(y) = \overline{V}\sin(Ny_1).$$

Let U be the $\mathcal{M}$–valued map

$$U(y) = (A^{-1})^t V(A^t y) A^{-1}.$$

By our discussion above the isomorphism $T : X \mapsto (A^{-1})^t X A^{-1}$ maps the line segment τ onto σ. Therefore:

- U is supported in $A^{-t}(B_1(0))$ and it is smooth,
- U is divergence–free thanks to Lemma 3.3,
- U takes values in an ε–neighborhood of the segment σ,

and furthermore

$$\begin{aligned}
\int_{A^{-t}(B_1(0))} |U(y)e_{n+1}|dy &= \int_{A^{-t}(B_1(0))} |A^{-t}V(A^t y)e_{n+1}|dy \\
&= \int_{B_1(0)} |A^{-t}V(z)e_{n+1}| \frac{dz}{|\det A^t|} \\
&\geq \frac{2\alpha|A^{-t}\overline{V}e_{n+1}|}{|\det A|} = \frac{2\alpha}{|\det A|}|\overline{U}e_{n+1}|. \quad (22)
\end{aligned}$$

To complete the proof we appeal to a standard covering/rescaling argument. That is, we can find a finite number of points $y_k \in B_1(0)$ and radii $r_k > 0$ so that the rescaled and translated sets $A^{-t}(B_{r_k}(y_k))$ are pairwise disjoint, all contained in $B_1(0)$, and

$$\bigcup_k \left|A^{-t}(B_{r_k}(y_k))\right| \geq \frac{1}{2}|B_1(0)|. \quad (23)$$

Let $U_k(y) = U(\frac{y-y_k}{r_k})$ and $\tilde{U} = \sum_k U_k$. Then $\tilde{U} : \mathbb{R}^{n+1} \to \mathcal{M}$ is smooth, clearly satisfies (p1) and (p2), and

$$\begin{aligned}
\int |\tilde{U}(y)e_{n+1}|dy &= \sum_k \int_{A^{-t}B_{r_k}(y_k)} |U_k(y)e_{n+1}|dy \\
&\overset{(22)}{\geq} \sum_k 2\alpha|\overline{U}e_{n+1}||\det A|^{-1}\frac{|B_{r_k}(y_k)|}{|B_1(0)|} \\
&= 2\alpha|\overline{U}e_{n+1}|\frac{\sum_k \left|A^{-t}(B_{r_k}(y_k))\right|}{|B_1(0)|} \overset{(23)}{\geq} \alpha|\overline{U}e_{n+1}|.
\end{aligned}$$

This completes the proof. □

Proof of Lemma 3.3. First of all we check that whenever $B \in \mathcal{M}$, then $A^tBA \in \mathcal{M}$ for all $A \in \mathcal{G}$. Indeed, A^tBA is symmetric, and since A satisfies $Ae_{n+1} = e_{n+1}$, we have

$$\begin{aligned}
(A^tBA)_{(n+1)(n+1)} &= e_{n+1} \cdot A^tBAe_{n+1} = Ae_{n+1} \cdot BAe_{n+1} \\
&= e_{n+1} \cdot Be_{n+1} = B_{(n+1)(n+1)} = 0. \quad (24)
\end{aligned}$$

Now, let A, U and V be as in the statement. The argument above shows that V is $\mathcal{M}$–valued. It remains to check that if U is divergence–free, then V is also divergence–free. To this end let $\phi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ be a compactly supported test function and consider $\tilde{\phi} \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ defined by

$$\tilde{\phi}(x) = A\phi(A^tx).$$

Then $\nabla\tilde{\phi}(x) = A\nabla\phi(A^t x)A^t$, and by a change of variables we obtain

$$\begin{aligned}\int \mathrm{tr}\big(\mathrm{V}(\mathrm{y})\nabla\phi(\mathrm{y})\big)\mathrm{dy} &= \int \mathrm{tr}\big(\mathrm{A^t U(A^{-t}y)A}\nabla\phi(\mathrm{y})\big)\mathrm{dy}\\ &= \int \mathrm{tr}\big(\mathrm{U(A^{-t}y)A}\nabla\phi(\mathrm{y})\mathrm{A^t}\big)\mathrm{dy}\\ &= \int \mathrm{tr}\big(\mathrm{U(x)A}\nabla\phi(\mathrm{A^t x})\mathrm{A^t}\big)(\det \mathrm{A})^{-1}\mathrm{dx}\\ &=(\det A)^{-1}\int \mathrm{tr}\big(\mathrm{U(x)}\nabla\tilde{\phi}(\mathrm{x})\big)\mathrm{dx} = 0,\end{aligned}$$

since U is divergence–free. But this implies that V is also divergence-free. □

Proof of Lemma 3.4. First of all, U is clearly symmetric and $U_{(n+1)(n+1)} = 0$. Hence U takes values in $\mathcal{M}$. To see that U is divergence–free, we calculate

$$\begin{aligned}\sum_j \partial_j U_{ij} &= \frac{1}{2}\sum_{k,l}\partial^3_{jkl}(E^{il}_{kj} + E^{jl}_{ki})\\ &= \frac{1}{2}\sum_l \partial_l\Big(\sum_{jk}\partial^2_{jk}E^{il}_{kj}\Big) + \frac{1}{2}\sum_k \partial_k\Big(\sum_{jl}\partial^2_{jl}E^{jl}_{ki}\Big) \overset{(13)}{=} 0\,.\end{aligned}$$

This completes the proof of the lemma. □

4. Proof of the main results

For clarity we now state the precise form of our main result. Theorems 1.1, 1.2 and 1.3 are direct corollaries.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n_x \times \mathbb{R}_t$ be a bounded open domain. There exists $(v,p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solving the Euler equations*

$$\begin{aligned}\partial_t v + \mathrm{div}\,(v\otimes v) + \nabla p &= 0\\ \mathrm{div}\, v &= 0\,,\end{aligned}$$

such that

- $|v(x,t)| = 1$ *for a.e.* $(x,t) \in \Omega$,
- $v(x,t) = 0$ *and* $p(x,t) = 0$ *for a.e.* $(x,t) \in (\mathbb{R}^n_x \times \mathbb{R}_t) \setminus \Omega$.

Moreover, there exists a sequence of functions $(v_k, p_k, f_k) \in C^\infty_c(\Omega)$ such that

$$\begin{aligned}\partial_t v_k + \mathrm{div}\,(v_k\otimes v_k) + \nabla p_k &= f_k\\ \mathrm{div}\, v_k &= 0\,,\end{aligned}$$

and

- f_k *converges to* 0 *in* H^{-1},
- $\|v_k\|_\infty + \|p_k\|_\infty$ *is uniformly bounded,*
- $(v_k, p_k) \to (v,p)$ *in* L^q *for every* $q < \infty$.

We remark that the statements of Theorem 1.1 and Theorem 1.3 are just subsets of the statement of Theorem 4.1. As for Theorem 1.2, note that it suffices to choose, for instance, $\Omega = B_r(0)\times]-1,1[$, where $B_r(0)$ is the ball of $\mathbb{R}^n$ with volume 1.

We recall from Lemma 2.1 that for the first half of the theorem it suffices to prove that there exist

$$(v,u,q) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R})$$

with support in Ω, such that $|v| = 1$ a.e. in Ω and (8) and (9) are satisfied. In Proposition 2.2 we constructed compactly supported solutions (v,u,q) to (8). The point is thus to find solutions which satisfy in addition the pointwise constraint (9). The main idea is to consider the sets

$$K = \left\{(v,u) \in \mathbb{R}^n \times \mathcal{S}^n_0 : \, u = v\otimes v - \frac{1}{n}|v|^2 I_n \, , \; |v| = 1\right\}, \tag{25}$$

and

$$\mathcal{U} = \text{int } (K^{co} \times [-1,1]), \tag{26}$$

where int denotes the topological interior of the set in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}$, and K^{co} denotes the convex hull of K. Thus, a triple (v,u,q) solving (8) and taking values in the convex extremal points of $\overline{\mathcal{U}}$ is indeed a solution to (9). We will prove that $0 \in \mathcal{U}$, and therefore there exist plane waves taking values in $\mathcal{U}$. The goal is to add them so to get an infinite sum

$$(v,u,q) = \sum_{i=1}^{\infty} (v_i, u_i, q_i)$$

with the properties that

- the partial sums $\sum_{i=0}^k (v_i,u_i,q_i)$ take values in $\mathcal{U}$,
- (v,u,q) is supported in Ω,
- (v,u,q) takes values in the convex extremal points of $\overline{U}$ a.e. in Ω,
- (v,u,q) solves the linear partial differential equations (8).

There are two important reasons why this construction is possible. First of all, since the wave cone Λ is very large, we can always get closer and closer to the extremal point of $\mathcal{U}$ with the sequence (v_k, u_k, p_k). Secondly, because the waves are localized in space–time, by choosing the supports smaller and smaller we can achieve strong convergence of the sequence. In view of Lemma 2.1 this gives the solution of Euler that we we are looking for. The partial sums give the approximating sequence of the theorem.

This sketch of the proof is philosophically closer to the method of convex integration, where the difficulty is to ensure strong convergence of the partial sums. The Baire category argument avoids this difficulty by introducing a metric for the space of solutions to (8) with values in $\mathcal{U}$, and proving that in its closure a generic element takes values in the convex extreme points. An interesting corollary of the Baire category argument is that, within the class of solutions to the Euler equations with driving force in some particular bounded subset of H^{-1}, the typical (in the sense of category) element has the properties of Theorem 4.1 .

We split the proof of Theorem 4.1 into several lemmas and a short concluding argument, which is given at the beginning of Section 4.3. For the purpose of this section, we could have presented a shorter proof, avoiding Lemma 4.3 and without giving the explicit bound (30) of Lemma 4.6. However, these statements will be needed in the convex integration proof of Section 5.

4.1. **The geometric setup.**

Lemma 4.2. *Let K and $\mathcal{U}$ be defined as in* (25) *and* (26)*, i.e.*

$$K = \left\{(v,u) \in \mathbb{S}^{n-1} \times \mathcal{S}_0^n \,:\, u = v\otimes v - \frac{I_n}{n}\right\}.$$

Then $0 \in int\, K^{co}$ and hence $0 \in \mathcal{U}$.

Proof. Let μ be the Haar measure on $\mathbb{S}^{n-1}$ and consider the linear map

$$T : C(\mathbb{S}^{n-1}) \to \mathbb{R}^n \times \mathcal{S}_0^n, \quad \phi \mapsto \int_{\mathbb{S}^{n-1}} \left(v, v\otimes v - \frac{I_n}{n}\right)\phi(v)\, d\mu\,.$$

Clearly, if

$$\phi \geq 0 \qquad \text{and} \qquad \int_{\mathbb{S}^{n-1}} \phi\, d\mu = 1\,, \tag{27}$$

then $T(\phi) \in K^{co}$. Notice that

$$T(1) \;=\; \int_{\mathbb{S}^{n-1}} \left(v, v\otimes v - \frac{I_n}{n}\right) d\mu \;=\; 0,$$

and hence $0 \in K^{co}$. Moreover, whenever $\psi \in C(\mathbb{S}^{n-1})$ is such that

$$\alpha \;=\; 1 - \int_{\mathbb{S}^{n-1}} \psi\, d\mu \;\geq\; \|\psi\|_{C(\mathbb{S}^{n-1})}, \tag{28}$$

$\phi = \alpha + \psi$ satisfies (27) and hence $T(\psi) = T(\phi) \in K^{co}$. Since (28) holds whenever $\|\psi\|_{C(\mathbb{S}^{n-1})} < 1/2$, it suffices to show that T is surjective to prove that K^{co} contains a neighborhood of 0.

The surjectivity of T follows from orthogonality in $L^2(\mathbb{S}^{n-1})$. Indeed, letting $\phi = v_i$ for each i, we obtain

$$T(\phi) = \beta_1(e_i, 0), \text{ where } \beta_1 = \int_{\mathbb{S}^{n-1}} v_1^2 d\mu.$$

Furthermore, setting $\phi = v_i v_j$ with $i \neq j$, we obtain

$$T(\phi) = \beta_2\big(0, e_i\otimes e_j + e_j \otimes e_i\big), \text{ where } \beta_2 = \int_{\mathbb{S}^{n-1}} v_1^2 v_2^2 d\mu.$$

Finally, setting $\phi = v_i^2 - \frac{1}{n}$ we obtain

$$T(\phi) = \beta_3\Big(0, e_i\otimes e_i - \frac{1}{(n-1)}\sum_{j\neq i} e_j \otimes e_j\Big),$$

where

$$\beta_3 = \int_{\mathbb{S}^{n-1}} \left(v_1^2 - \frac{1}{n} \right)^2 d\mu.$$

This shows that the image of T contains $n+\frac{1}{2}n(n+1)-1$ linearly independent elements, hence a basis for $\mathbb{R}^n \times \mathcal{S}_0^n$.

□

Lemma 4.3. *There exists a dimensional constant $C > 0$ such that for any $(v, u, q) \in \mathcal{U}$ there exists $(\bar{v}, \bar{u}) \in \mathbb{R}^n \times \mathcal{S}_0^n$ such that $(\bar{v}, \bar{u}, 0) \in \Lambda$, the line segment with endpoints $(v, u, q) \pm (\bar{v}, \bar{u}, 0)$ is contained in $\mathcal{U}$, and*

$$|\bar{v}| \geq C(1 - |v|^2).$$

Proof. Let $z = (v, u) \in \text{int } K^{co}$. By Carathéodory's theorem (v, u) lies in the interior of a simplex in $\mathbb{R}^n \times \mathcal{S}_0^n$ spanned by elements of K. In other words

$$z = \sum_{i=1}^{N+1} \lambda_i z_i,$$

where $\lambda_i \in]0, 1[$, $z_i = (v_i, u_i) \in K$, $\sum_{i=1}^{N+1} \lambda_i = 1$, and $N = n(n+3)/2 - 1$ is the dimension of $\mathbb{R}^n \times \mathcal{S}_0^n$. Assume that the coefficients are ordered so that $\lambda_1 = \max_i \lambda_i$. Then for any $j > 1$

$$z \pm \frac{1}{2}\lambda_j(z_j - z_1) \in \text{int } K^{co}.$$

Indeed,

$$z \pm \frac{1}{2}\lambda_j(z_j - z_1) = \sum_i \mu_i z_i,$$

where $\mu_1 = \lambda_1 \mp \frac{1}{2}\lambda_j$, $\mu_j = \lambda_j \pm \frac{1}{2}\lambda_j$ and $\mu_i = \lambda_i$ for $i \notin \{1, j\}$. It is easy to see that $\mu_i \in]0, 1[$ for all $i = 1 \dots N+1$.

On the other hand $z - z_1 = \sum_{i=2}^{N+1} \lambda_i(z_i - z_1)$, so that in particular

$$|v - v_1| \leq N \max_{i=2\dots N+1} \lambda_i |v_i - v_1|$$

Let $j > 1$ be such that $\lambda_j |v_j - v_1| = \max_{i=2\dots N+1} \lambda_i |v_i - v_1|$, and let

$$(\bar{v}, \bar{u}) = \frac{1}{2}\lambda_j(z_j - z_1).$$

The line segment with endpoints $(v, u) \pm (\bar{v}, \bar{u})$ is contained in the interior of K^{co} and hence also the line segment $(v, u, q) \pm (\bar{v}, \bar{u}, 0)$ is contained in $\mathcal{U}$. Furthermore

$$\frac{1}{4N}(1 - |v|^2) \leq \frac{1}{2N}(1 - |v|) \leq \frac{1}{2N}(|v - v_1|) \leq |\bar{v}|.$$

Finally, we show that $(\bar v, \bar u, 0) \in \Lambda$. This amounts to showing that whenever $a, b \in \mathbb{S}^{n-1}$, the matrix

$$\begin{pmatrix} a\otimes a - \frac{I_n}{n} & a \\ a & 0 \end{pmatrix} - \begin{pmatrix} b\otimes b - \frac{I_n}{n} & b \\ b & 0 \end{pmatrix}$$

has zero determinant and hence lies in the wave cone Λ defined in (7). Let $P \in GL_n(\mathbb{R})$ with $Pa = e_1$ and $Pb = e_2$. Note that

$$\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a\otimes a & a \\ a & 0 \end{pmatrix}\begin{pmatrix} P^t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Pa\otimes Pa & Pa \\ Pa & 1 \end{pmatrix},$$

so that it suffices to check the determinant of

$$\begin{pmatrix} e_1\otimes e_1 & e_1 \\ e_1 & 0 \end{pmatrix} - \begin{pmatrix} e_2\otimes e_2 & e_2 \\ e_2 & 0 \end{pmatrix}.$$

Since $e_1 + e_2 - e_{n+1}$ is in the kernel of this matrix, it has indeed determinant zero. This completes the proof.

□

4.2. **The functional setup.** We define the complete metric space X as follows. Let

$$X_0 := \big\{(v, u, q) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t) : \text{ (i), (ii) and (iii) below hold}\big\}$$

(i) $\operatorname{supp}(v, u, q) \subset \Omega$,
(ii) (v, u, q) solves (8) in $\mathbb{R}^n_x \times \mathbb{R}_t$,
(iii) $(v(x,t), u(x,t), q(x,t)) \in \mathcal{U}$ for all $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t$.

We equip X_0 with the topology of L^∞-weak* convergence of (v, u, q) and we let X be the closure of X_0 in this topology.

Lemma 4.4. *The set X with the topology of L^∞ weak* convergence is a nonempty compact metrizable space. Moreover, if $(v, u, q) \in X$ is such that*

$$|v(x,t)| = 1 \quad \text{for almost every } (x,t) \in \Omega,$$

then v and $p := q - \frac{1}{n}|v|^2$ is a weak solution of (1) in $\mathbb{R}^n_x \times \mathbb{R}_t$ such that

$$v(x,t) = 0 \text{ and } p(x,t) = 0 \text{ for all } (x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t \setminus \Omega.$$

Proof. In Lemma 4.2 we showed that $0 \in \mathcal{U}$, hence X is nonempty. Moreover, X is a bounded and closed subset of $L^\infty(\Omega)$, hence with the weak* topology it becomes a compact metrizable space. Since $\overline{\mathcal{U}}$ is a compact convex set, any $(v, u, q) \in X$ satisfies

$$\operatorname{supp}(v, u, q) \subset \overline{\Omega}, \quad (v, u, q) \text{ solves (8) and takes values in } \overline{\mathcal{U}}.$$

In particular $(v, u)(x,t) \in K^{co}$ almost everywhere. Finally, observe also that if $(v, u)(x,t) \in K^{co}$, then

$$(v, u)(x,t) \in K \text{ if and only if } |v(x,t)| = 1.$$

In light of Lemma 2.1 this concludes the proof. □

Fix a metric d^*_∞ inducing the weak* topology of L^∞ in X, so that (X, d^*_∞) is a complete metric space.

Lemma 4.5. *The identity map*

$$I : (X, d^*_\infty) \to L^2(\mathbb{R}^n_x \times \mathbb{R}_t) \text{ defined by } (v, u, q) \mapsto (v, u, q)$$

is a Baire-1 map and therefore the set of points of continuity is residual in (X, d^*_∞).

Proof. Let $\phi_r(x,t) = r^{-(n+1)}\phi(rx, rt)$ be any regular space-time convolution kernel. For each fixed $(v, u, q) \in X$ we have

$$(\phi_r * v, \phi_r * u, \phi_r * q) \to (v, u, q) \text{ strongly in } L^2 \text{ as } r \to 0.$$

On the other hand, for each $r > 0$ and $(v^k, u^k, q^k) \in X$

$$(v^k, u^k, q^k) \overset{*}{\rightharpoonup} (v, u, q) \text{ in } L^\infty \implies \phi_r * (v^k, u^k, q^k) \to \phi_r * (v, u, q) \text{ in } L^2.$$

Therefore each map $I_r : (X, d^*_\infty) \to L^2$ defined by

$$I_r : (u, v, q) \mapsto (\phi_r * v, \phi_r * u, \phi_r * q)$$

is continuous, and

$$I(v, u, q) = \lim_{r \to 0} I_r(v, u, q) \qquad \text{for all } (v, u, q) \in X\,.$$

This shows that $I : (X, d^*_\infty) \to L^2$ is a pointwise limit of continuous maps, hence it is a Baire-1 map. Therefore the set of points of continuity of I is residual in (X, d^*_∞), see [20]. □

4.3. **Points of continuity of the identity map.** The proof of Theorem 4.1 will follow from Lemmas 4.4 and 4.5 once we prove the following

Claim: If $(v, u, q) \in X$ is a point of continuity of I, then

$$|v(x,t)| = 1 \text{ for almost every } (x,t) \in \Omega\,. \tag{29}$$

Indeed, if the claim is true, then the set of $(v, u, q) \in X$ such that $|v| = 1$ a.e. is nonempty, yielding solutions of (1). Furthermore, any such (v, u, q) must be the strong L^2 limit of some sequence $\{(v_k, u_k, q_k)\} \subset X_0$. Therefore, with $p_k = q_k - \frac{1}{n}|v_k|^2$, and

$$f_k = \operatorname{div}\left(v_k \otimes v_k - \frac{1}{n}|v_k|^2 Id - u_k\right),$$

we obtain $\operatorname{div} v_k = 0$ and

$$\partial_t v_k + \operatorname{div} v_k \otimes v_k + \nabla p_k = f_k.$$

Moreover, $f_k \to 0$ in H^{-1}.

Therefore it remains to prove our claim. Observe that since $|v(x,t)| \le 1$ a.e. $(x,t) \in \Omega$, (29) is equivalent to

$$\|v\|_{L^2(\Omega)} = |\Omega|,$$

where $|\Omega|$ denotes the $(n+1)$-dimensional Lebesgue measure of Ω. To prove the claim we prove the following lemma, from which the claim immediately follows:

Lemma 4.6. *There exists a dimensional constant $\beta > 0$ with the following property. Given $(v_0, u_0, q_0) \in X_0$ there exists a sequence $(v_k, u_k, q_k) \in X_0$ such that*

$$\|v_k\|^2_{L^2(\Omega)} \geq \|v_0\|^2_{L^2(\Omega)} + \beta \left(|\Omega| - \|v_0\|^2_{L^2(\Omega)}\right)^2, \tag{30}$$

and

$$(v_k, u_k, q_k) \overset{*}{\rightharpoonup} (v_0, u_0, q_0) \text{ in } L^\infty(\Omega).$$

Indeed, assume for a moment that (v, u, q) is a point of continuity of I. Fix a sequence $\{(v_k, u_k, q_k\} \subset X_0$ converges weakly* to (v, u, q). Using Lemma 4.6 and a standard diagonal argument, we can produce a second sequences $(\tilde{v}_k, \tilde{u}_k, \tilde{q}_k)$ which converges weakly* to (v, u, q) and such that

$$\liminf_{k\to\infty} \|\tilde{v}_k\|_2^2 \;\geq\; \liminf_{k\to\infty} \left(\|v_k\|_2^2 + \beta\left(|\Omega| - \|v_k\|_2^2\right)^2\right). \tag{31}$$

Since I is continuous at (v, u, q), both v_k and $\tilde{v}_k$ converge strongly to v. Therefore

$$\|v\|_2^2 \;\geq\; \|v\|_2^2 + \beta\left(|\Omega| - \|v\|_2^2\right)^2. \tag{32}$$

Therefore, $\|v\|_2^2 = |\Omega|$. On the other hand, since $v = 0$ a.e. outside Ω and $|v| \leq 1$ a.e. on Ω, this implies (29).

Proof of Lemma 4.6. **Step 1.** Let $(v_0, u_0, q_0) \in X_0$. By Lemma 4.3 for any $(x, t) \in \Omega$ there exists a direction

$$\big(\bar{v}(x,t), \bar{u}(x,t)\big) \in \mathbb{R}^n \times \mathcal{S}_0^n$$

such that the line segment with endpoints

$$\big(v_0(x,t), u_0(x,t), q_0(x,t)\big) \pm \big(\bar{v}(x,t), \bar{u}(x,t), 0\big)$$

is contained in $\mathcal{U}$, and

$$|\bar{v}(x,t)| \geq C(1 - |v_0(x,t)|^2).$$

Moreover, since (v_0, u_0, q_0) is uniformly continuous, there exists $\varepsilon > 0$ such that for any $(x,t), (x_0, t_0) \in \Omega$ with $|x-x_0|+|t-t_0| < \varepsilon$, the ε-neighbourhood of the line segment with endpoints

$$\big(v_0(x,t), u_0(x,t), q_0(x,t)\big) \pm \big(\bar{v}(x_0,t_0), \bar{u}(x_0,t_0), 0\big)$$

is also contained in $\mathcal{U}$.

Step 2. Fix $(x_0, t_0) \in \Omega$ for the moment. Use Proposition 2.2 with

$$a = (\bar{v}(x_0,t_0), \bar{u}(x_0,t_0), 0) \in \Lambda$$

and $\varepsilon > 0$ to obtain a smooth solution (v, u, q) of (8) with the properties stated in the Proposition, and for any $r < \varepsilon$ let

$$(v_r, u_r, q_r)(x,t) = (v, u, q)\left(\frac{x - x_0}{r}, \frac{t - t_0}{r}\right).$$

Then (v_r, u_r, q_r) is also a smooth solution of (8), with the properties

- the support of (v_r, u_r, q_r) is contained in $B_r(x_0, t_0) \subset \mathbb{R}^n_x \times \mathbb{R}_t$,
- the image of (v_r, u_r, q_r) is contained in the ε–neighborhood of the line-segment with endpoints $\pm(\bar{v}(x,t), \bar{u}(x,t), 0)$,
- and
$$\int |v_r(x,t)|\, dx\, dt \geq \alpha |\bar{v}(x_0, t_0)||B_r(x_0, r_0)|.$$

In particular, for any $r < \varepsilon$ we have $(v_0, u_0, q_0) + (v_r, u_r, q_r) \in X_0$.

Step 3. Next, observe that since v_0 is uniformly continuous, there exists $r_0 > 0$ such that for any $r < r_0$ there exists a finite family of pairwise disjoint balls $B_{r_j}(x_j, t_j) \subset \Omega$ with $r_j < r$ such that

$$\int_\Omega (1 - |v_0(x,t)|^2) dxdt \leq 2 \sum_j (1 - |v_0(x_j, t_j)|^2)|B_r(x_j, t_j)| \tag{33}$$

Fix $k \in \mathbb{N}$ with $\frac{1}{k} < \min\{r_0, \varepsilon\}$ and choose a finite family of pairwise disjoint balls $B_{r_{k,j}}(x_{k,j}, t_{k,j}) \subset \Omega$ with radii $r_{k,j} < \frac{1}{k}$ such that (33) holds. In each ball $B_{r_{k,j}}(x_{k,j}, t_{k,j})$ we apply the construction above to obtain $(v_{k,j}, u_{k,j}, q_{k,j})$, and in particular we then have

$$(v_k, u_k, q_k) := (v_0, u_0, q_0) + \sum_j (v_{k,j}, u_{k,j}, q_{k,j}) \in X_0,$$

and

$$\begin{aligned}
\int |v_k(x,t) - v_0(x,t)| dxdt &= \sum_j \int |v_{k,j}(x,t)| dxdt \\
&\geq \alpha \sum_j |\bar{v}(x_{k,j}, t_{k,j})||B_{r_{k,j}}(x_{k,j}, t_{k,j})| \\
&\geq C\alpha \sum_j (1 - |v_0(x_{k,j}, t_{k,j})|^2)|B_{r_{k,j}}(x_{k,j}, t_{k,j})| \\
&\geq \frac{1}{2} C\alpha \int_\Omega (1 - |v_0(x,t)|^2) dxdt. \qquad (34)
\end{aligned}$$

Finally observe that by letting $k \to \infty$, the above construction yields a sequence $(v_k, u_k, q_k) \in X_0$ such that

$$(v_k, u_k, q_k) \overset{*}{\rightharpoonup} (v_0, u_0, q_0). \tag{35}$$

Hence,

$$\begin{aligned}
\liminf_{k\to\infty} \|v_k\|_{L^2(\Omega)} &= \|v_0\|_2^2 + \liminf_{k\to\infty} \left(\langle v_0, (v_k - v_0)\rangle_2 + \|v_k - v_0\|_2^2 \right) \\
&\overset{(35)}{=} |v_0\|_2^2 + \liminf_{k\to\infty} \|v_k - v_0\|_2^2 \\
&\geq \|v_0\|_2^2 + |\Omega| \liminf_{k\to\infty} \left(\|v_k - v_0\|_{L^1(\Omega)} \right)^2 \qquad (36)
\end{aligned}$$

Combining (34) and (36) we get

$$\liminf_{k\to\infty} \|v_k\|_{L^2(\Omega)} \quad \geq \quad \|v_0\|^2_{L^2(\Omega)} + \frac{|\Omega|C^2\alpha^2}{4}\left(|\Omega| - \|v_0\|^2_{L^2(\Omega)}\right)^2,$$

which gives (30) with $\beta = \frac{1}{4}|\Omega|C^2\alpha^2$. □

5. A proof of Theorem 4.1 using convex integration

In this section we provide an alternative, more direct proof for Theorem 4.1, following the method of convex integration as presented for example in [17].

In fact the two approaches (i.e. Baire category methods and convex integration) can be unified to a large extent. For a discussion comparing the two approaches we refer to the end of Section 3.3 in [14], see also the paper [24] for a different point of view. Nevertheless, in order to get a feeling for the type of solution that Theorem 4.1 produces, it helps to see the direct construction of the convex integration method.

We will freely refer to the notation of the previous sections. In particular the proof relies on Lemmas 4.2, 4.3, 4.4 and 4.6. These results enable us to construct an approximating sequence, as explained briefly at the beginning of Section 4, by adding (almost-)plane-waves on top of each other. It is only the limiting step that is more explicit in this approach. The following argument is essentially from Section 3.3 in [17].

Alternative proof of Theorem 4.1. Using Lemma 4.6, we construct inductively a sequence $(v_k, u_k, q_k) \in X_0$ and a sequence of numbers $\eta_k > 0$ as follows. Let ρ_ε be a standard mollifying kernel in $\mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$ and set $(v_1, u_1, q_1) \equiv 0$ in $\mathbb{R}^n_x \times \mathbb{R}_t$.

Having obtained $z_j := (v_j, u_j, q_j)$ for $j \leq k$ and $\eta_1, \dots, \eta_{k-1}$ we choose

$$\eta_k < 2^{-k} \tag{37}$$

in such a way that

$$\|z_k - z_k * \rho_{\eta_k}\|_{L^2(\Omega)} < 2^{-k}. \tag{38}$$

Then we apply Lemma 4.6 to obtain $z_{k+1} = (v_{k+1}, u_{k+1}, q_{k+1}) \in X_0$ such that

$$\|v_{k+1}\|^2_{L^2(\Omega)} \geq \|v_k\|^2_{L^2(\Omega)} + \beta\left(|\Omega| - \|v_k\|^2_{L^2(\Omega)}\right)^2, \tag{39}$$

and

$$\|(z_{k+1} - z_k) * \rho_{\eta_j}\|_{L^2(\Omega)} < 2^{-k} \quad \text{for all } j \leq k. \tag{40}$$

The sequence $\{z_k\}$ is bounded in $L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$, therefore by passing to a suitable subsequence we may assume without loss of generality that

$$z_k \overset{*}{\rightharpoonup} z \quad \text{in } L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$$

for some $z = (v, u, q) \in X$, and that the sequence $\{z_k\}$ and the corresponding sequence $\{\eta_k\}$ satisfies the properties (37),(38),(39) and (40). Then for every $k \in \mathbb{N}$

$$\begin{aligned} \|z_k * \rho_{\eta_k} - z * \rho_{\eta_k}\|_{L^2(\Omega)} &\leq \sum_{j=0}^{\infty} \|z_{k+j} * \rho_{\eta_k} - z_{k+j+1} * \rho_{\eta_k}\|_{L^2(\Omega)} \\ &\leq \sum_{j=0}^{\infty} 2^{-(k+j)} \leq 2^{-k+1}, \end{aligned}$$

and since

$$\begin{aligned} \|z_k - z\|_{L^2(\Omega)} \leq &\|z_k - z_k * \rho_{\eta_k}\|_{L^2(\Omega)} + \|z_k * \rho_{\eta_k} - z * \rho_{\eta_k}\|_{L^2(\Omega)} \\ &+ \|z * \rho_{\eta_k} - z\|_{L^2(\Omega)}, \end{aligned}$$

we deduce that $v_k \to v$ strongly in $L^2(\Omega)$.

Therefore, passing into the limit in (39) we conclude

$$\|v\|^2_{L^2(\Omega)} \geq \|v\|^2_{L^2(\Omega)} + \beta \left(|\Omega| - \|v\|^2_{L^2(\Omega)}\right)^2 \tag{41}$$

and hence $\|v\|_2^2 = |\Omega|$. Since v vanishes outside Ω and $|v| \leq 1$ in Ω, we conclude that $|v| = \mathbf{1}_\Omega$. Since $(v, u, q) \in X$, we also have that $(v, u)(x, t) \in K^{co}$ for a.e. $(x, t) \in \Omega$. From this we deduce that $(v, u)(x, t) \in K$ for a.e. $(x, t) \in \Omega$, thus concluding the proof. □

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