Anisotropic inhomogeneous rectangular thin-walled beams

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Abstract

This paper is devoted to the asymptotic analysis of the problem of linear elasticity for an anisotropic and inhomogeneous body occupying, in its reference configuration, a cylindrical domain with a rectangular cross section with sides proportional to ε and ε^2 and clamped on one of its bases. The sequence of solutions u^ε of the equilibrium problem is shown to converge in an appropriate topology, as ε goes to zero, to the solution of a problem for a beam in which the extensional, flexural, and torsional effects are all coupled together.

Keywords: asymptotic analysis, calculus of variations, thin-walled beams, dimension reduction, variational convergence, linear elasticity

1 Introduction

Geometrically, a thin-walled beam is a slender structural element whose length is much larger than the diameter of the cross section which, on its hand, is larger than the thickness of the thin wall. These kinds of beams have been used for a long time in civil and mechanical engineering and, most of all, in flight vehicle structures because of their high ratio between maximum strength and weight. More recently, their importance has increased because of the introduction of fiber-reinforced composite materials in structural components. These materials are finding more and more applications for their high resistance to corrosion and high strength. Composite beams are usually made up by fiber-reinforced laminates and, hence, are anisotropic and inhomogeneous, even in cross-section planes. These peculiarities make classical thin-walled beam theories not applicable. The problem though has attracted the interest of several researchers

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and by now a huge number of articles can be found on the subject; see, for instance, [11] and the references therein. This is strongly remarked also in the first sentences of the abstract of [12]: "There is no lack of composite beam theories. Quite to the contrary, there might be too many of them. Different approaches, notations, etc., are used by the authors of those theories, so it is not always straightforward to compare the assumptions made and to assess the quantitative consequences of those assumptions."

The problem under study has a huge technological interest. One very suggestive, mentioned in [2], concerns the rotor blades of helicopters. The blades are composite beams and, hence, anisotropic and inhomogeneous. The anisotropy and the inhomogeneity introduce, as we shall also deduce, structural couplings between bending, extension, and twisting behaviors. It has been observed experimentally that these couplings have a powerful influence on blade dynamics including vibrations and the aeroelastic stability; see [2]. If a model of composite beams that accurately describes the structural couplings was at our disposal, then we could try to vary the anisotropy and inhomogeneity so as to minimize undesired effects like, for instance, vibrations. Through the control of lamination parameters (ply orientation and stacking sequence), it would then be possible for industry to minimize the undesired effects.

Our aim here is to deduce a "rigorous" model for a composite thin-walled beam, that is, a inhomogeneous and anisotropic beam. We shall achieve our goal by means of well-established asymptotic methods starting from the three-dimensional linear theory of elasticity.

This paper is devoted to the asymptotic analysis of the linearized system of equilibrium equations of a body which occupies, in its reference configuration, a cylindrical domain with a rectangular cross section with sides proportional to ε and ε^2 and clamped on one of its two bases. In particular, we study the compactness properties of the sequence of solutions u^ε of the equilibrium problems and, letting ε go to zero, we are concerned with the identification of the limit problem. The same problem has been studied from the point of view of Γ-convergence: in [4] in the simpler setting of homogeneous and isotropic material and in [3] in the case of an anisotropic material which is inhomogeneous only along the longitudinal axis and subject to residual stress. Trabucho and Viano [9] also studied the same problem by superimposing two asymptotic analyses where the lengths of the two sides of the cross section go to zero independently.

Besides the material properties of the body, our treatment differs from the preceding works also in the topology used in the passage from the three-dimensional problem to the one-dimensional: the one used in the present paper delivers much more information on the deformation of the beam. The approach is close to the one developed in a recent paper of Murat and Sili [8] for a thin cylinder of radius ε. The lack of isotropy or homogeneity assumptions leads to a limit problem where the extensional, flexural, and torsional effects are coupled together. In fact, we prove that the limit problem can be written as a system of five equations in a 5-tuple of unknowns (u, v, w, p, q) (see Theorem 6.1) and that $u^\varepsilon - (u + \varepsilon v + \varepsilon^2 w + \varepsilon^3 p + \varepsilon^4 q)$ converges strongly to zero in $H^1(\Omega)$, under some regularity assumptions on v, w, p, and q (see Corollary 6.1). We also derive the set of Euler equations of the variational limit problem, that is, the system of equilibrium differential equations, in the fully general case. Then we show that a strong simplification and a partial decoupling occurs when the material is homogeneous, and a complete decoupling is obtained for a homogeneous orthotropic material.

Notation. Throughout this paper Ω_1, Ω_2, and Ω_3 will denote the following three intervals:

$$\Omega_\alpha := (-a_\alpha/2, +a_\alpha/2) \quad \text{for } \alpha = 1, 2 \quad \text{and } \Omega_3 := (0, \ell),$$

where a_1, a_2, and ℓ are three positive real numbers. Also, for $i, j = 1, 2, 3$ we set

$$\Omega_{ij} := \Omega_i \times \Omega_j$$

and

$$\Omega := \Omega_1 \times \Omega_2 \times \Omega_3.$$

Unless otherwise specified, we use the Einstein summation convention. Moreover, we use the following convention for indexing vector and tensor components: Greek indices α and β take their values in the set $\{1, 2\}$ and Latin indices i, j, and k in the set $\{1, 2, 3\}$. With a little abuse of notation, and because this is a common practice and does not give rise to any mistakes, we call “sequences” even those families indicized by a continuous parameter $\varepsilon \in (0, 1)$. The component k of a vector v will be denoted either with $(v)_k$ or v_k, and an analogous notation will be used to denote tensor components. $\mathcal{E}_{\alpha\beta}$ denotes the Ricci's symbol, that is, $\mathcal{E}_{11} = \mathcal{E}_{22} = 0$, $\mathcal{E}_{12} = 1$, and $\mathcal{E}_{21} = -1$. Since usually $x = (x_1, x_2, x_3)$, we shall then denote by $x' := (x_1, x_2)$. A wide use will be made of vector valued distributions and Sobolev spaces; for a brief account of which and for the current notation we refer the reader to the book of Le Dret [5]. Throughout this paper C will denote a constant which may change line by line.

2 The three-dimensional problem

We consider a body which occupies, in its reference configuration, the region

$$\Omega_\varepsilon := \varepsilon^2\Omega_1 \times \varepsilon\Omega_2 \times \Omega_3 \subset \mathbb{R}^3.$$

We denote by $E(u)$ the strain of the displacement u, whose components are

$$E_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The elasticity tensor, with respect to the reference configuration Ω_ε, of the material will be denoted by $\mathbb{C}^\varepsilon$. We assume it to be essentially bounded,

$$\mathbb{C}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times3\times3\times3});$$

to have the minor symmetries,

$$\mathbb{C}^\varepsilon_{ijkl} = \mathbb{C}^\varepsilon_{jikl} = \mathbb{C}^\varepsilon_{ijlk};$$

and to be positive definite. That is, there exists a constant $c > 0$ such that

$$\mathbb{C}^\varepsilon A \cdot A \geq c|A|^2, \tag{1}$$

for all three by three symmetric matrices A and for all ε. We consider the body clamped on $\Gamma_b^\varepsilon := \partial\Omega_\varepsilon \cap \{x_3 = 0\}$ and we denote by

$$H^1_{dn}(\Omega_\varepsilon;\mathbb{R}^3) := \left\{\varphi \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : \varphi = 0 \text{ on } \Gamma_b^\varepsilon\right\}.$$

The weak form of the equilibrium problem can be written as

$$\left\{\begin{array}{l} \tilde{u}^\varepsilon \in H^1_{dn}(\Omega_\varepsilon;\mathbb{R}^3), \\ \displaystyle\int_{\Omega_\varepsilon} \mathbb{C}^\varepsilon E(\tilde{u}^\varepsilon) \cdot E(\varphi)\, dv = \int_{\Omega_\varepsilon} \tilde{F}^\varepsilon \cdot E(\varphi)\, dv \quad \forall \varphi \in H^1_{dn}(\Omega_\varepsilon;\mathbb{R}^3), \end{array}\right. \tag{2}$$

where the matrix field $\tilde{F}^\varepsilon$, which takes into account the presence of external forces, is assumed to be an element of $L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$.

If $\tilde{F}^\varepsilon$ is not just in $L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ but in

$$H(\mathrm{div},\Omega_\varepsilon) := \{T \in L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3}_{\mathrm{sym}}) : \mathrm{div}\, T \in L^2(\Omega_\varepsilon;\mathbb{R}^3)\},$$

then the previous problem can be seen as the weak form of the following problem:

$$\left\{\begin{array}{ll} \mathrm{div}\, \mathbb{C}^\varepsilon E(\tilde{u}^\varepsilon) + \tilde{b}^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \mathbb{C}^\varepsilon E(\tilde{u}^\varepsilon) n^\varepsilon = \tilde{c}^\varepsilon & \text{on } \Gamma_c^\varepsilon, \\ \tilde{u}^\varepsilon = 0 & \text{on } \Gamma_b^\varepsilon, \end{array}\right. \tag{3}$$

where $\Gamma_c^\varepsilon := \partial\Omega_\varepsilon \backslash \Gamma_b^\varepsilon$, and n^ε denotes the unit outward normal vector to Ω_ε, while the *body loads* $\tilde{b}^\varepsilon$ and the *contact loads* $\tilde{c}^\varepsilon$ are simply given by

$$\tilde{b}^\varepsilon := -\mathrm{div}\tilde{F}^\varepsilon \quad \text{in } \Omega_\varepsilon, \qquad \tilde{c}^\varepsilon := \tilde{F}^\varepsilon n^\varepsilon \quad \text{in } \Gamma_c^\varepsilon. \tag{4}$$

Note that given $\tilde{b}^\varepsilon \in L^2(\Omega_\varepsilon;\mathbb{R}^3)$ and $\tilde{c}^\varepsilon \in H^{-1/2}(\partial\Omega_\varepsilon;\mathbb{R}^3)$ it is always possible to find an $\tilde{F}^\varepsilon \in H(\mathrm{div},\Omega_\varepsilon)$ which satisfies (4).

3 The rescaled problem

It is convenient to work with the domain Ω instead of the domain Ω_ε. We therefore rescale the problem by means of the scaling map $s_\varepsilon : \Omega \to \Omega_\varepsilon$,

$$s_\varepsilon(x_1,x_2,x_3) = (\varepsilon^2 x_1, \varepsilon x_2, x_3).$$

Let $\tilde{u}^\varepsilon$ be the solution of (2); then we define the "rescaled solution" u^ε by

$$u_1^\varepsilon := \varepsilon^2 \tilde{u}_1^\varepsilon \circ s_\varepsilon, \quad u_2^\varepsilon := \varepsilon \tilde{u}_2^\varepsilon \circ s_\varepsilon, \quad u_3^\varepsilon := \tilde{u}_3^\varepsilon \circ s_\varepsilon.$$

Let E^ε be the "rescaled strain" defined by

$$E^\varepsilon(\varphi) := \begin{pmatrix} \frac{1}{\varepsilon^4}E_{11}(\varphi) & \frac{1}{\varepsilon^3}E_{12}(\varphi) & \frac{1}{\varepsilon^2}E_{13}(\varphi) \\ \frac{1}{\varepsilon^3}E_{21}(\varphi) & \frac{1}{\varepsilon^2}E_{22}(\varphi) & \frac{1}{\varepsilon}E_{23}(\varphi) \\ \frac{1}{\varepsilon^2}E_{31}(\varphi) & \frac{1}{\varepsilon}E_{32}(\varphi) & E_{33}(\varphi) \end{pmatrix}. \tag{5}$$

It follows that $E^\varepsilon(u^\varepsilon) = E(\tilde{u}^\varepsilon) \circ s_\varepsilon$.

We further assume that there exists a $\mathbb{C} \in L^\infty(\Omega; \mathbb{R}^{3\times3\times3\times3})$ such that

$$\mathbb{C}^\varepsilon = \mathbb{C} \circ s_\varepsilon,$$

and we denote with $F^\varepsilon = \tilde{F}^\varepsilon \circ s_\varepsilon^{-1} \in L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}})$. With this notation u^ε turns out to be the unique solution of

$$\begin{cases} u^\varepsilon \in H^1_{dn}(\Omega; \mathbb{R}^3), \\ \displaystyle\int_\Omega \mathbb{C}E^\varepsilon(u^\varepsilon) \cdot E^\varepsilon(\varphi)\, dx = \int_\Omega F^\varepsilon \cdot E^\varepsilon(\varphi)\, dx \quad \forall \varphi \in H^1_{dn}(\Omega; \mathbb{R}^3), \end{cases} \tag{6}$$

where $\Gamma_b := \partial\Omega \cap \{x_3 = 0\}$ and

$$H^1_{dn}(\Omega; \mathbb{R}^3) := \Big\{\varphi \in H^1(\Omega; \mathbb{R}^3) : \varphi = 0 \text{ on } \Gamma_b\Big\}.$$

By taking $\varphi = u^\varepsilon$ and using (1), we find

$$c\|E^\varepsilon(u^\varepsilon)\|_{L^2(\Omega)} \le \|F^\varepsilon\|_{L^2(\Omega)}. \tag{7}$$

Thus a uniform bound on $\|F^\varepsilon\|_{L^2(\Omega)}$ would lead to rescaled strains uniformly bounded in ε. We augment this requirement by assuming

$$F^\varepsilon \to F \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}}). \tag{8}$$

Remark **3.1** Let $S^\varepsilon := \mathrm{diag}(1/\varepsilon^2, 1/\varepsilon, 1)$; then $E^\varepsilon(\varphi) = S^\varepsilon E(\varphi) S^\varepsilon$. If we assume $F^\varepsilon \in H(\mathrm{div}, \Omega)$, then we can write

$$\begin{aligned} \int_\Omega F^\varepsilon \cdot E^\varepsilon(\varphi)\, dx &= \int_\Omega S^\varepsilon F^\varepsilon S^\varepsilon \cdot E(\varphi)\, dx \\ &= -\int_\Omega \mathrm{div}(S^\varepsilon F^\varepsilon S^\varepsilon) \cdot \varphi\, dx + \langle S^\varepsilon F^\varepsilon S^\varepsilon n, \varphi\rangle_{\partial\Omega}, \end{aligned}$$

for all $\varphi \in H^1_{dn}(\Omega; \mathbb{R}^3)$, and conclude that instead of considering F^ε we could have been using the following body and contact forces

$$b^\varepsilon := -\mathrm{div}(S^\varepsilon F^\varepsilon S^\varepsilon) \quad \text{in } \Omega \quad \text{and} \quad c^\varepsilon := S^\varepsilon F^\varepsilon S^\varepsilon n \quad \text{in } \Gamma_c.$$

Note that if $F^\varepsilon = S^{\varepsilon^{-1}} F^{(0)} S^{\varepsilon^{-1}}$ for some $F^{(0)} \in H(\mathrm{div}, \Omega)$ the body and the contact forces would be independent of ε. Since $S^{\varepsilon^{-1}} = \mathrm{diag}(\varepsilon^2, \varepsilon, 1)$, the sequence $\{S^{\varepsilon^{-1}} F^{(0)} S^{\varepsilon^{-1}}\}$, with $F^{(0)} \in H(\mathrm{div}, \Omega)$, strongly converges in

$L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})$ and hence satisfies assumption (8). Assumption (8) though allows us to consider "stronger" forces than $F^\varepsilon = S^{\varepsilon^{-1}}F^{(0)}S^{\varepsilon^{-1}}$, like $F^\varepsilon = F^{(0)}$ or, more generally,

$$F^\varepsilon = F^{(0)} + \varepsilon F^{(1)} + \varepsilon^2 F^{(2)} + \varepsilon^3 F^{(3)} + \varepsilon^4 F^{(4)},$$

with $F^{(i)} \in H(\mathrm{div},\Omega)$ for $i = 0,1,\dots,4$, where, for instance, the term $\varepsilon^4 F^{(4)}_{11}$ would lead to the definition of body and contact forces independent of ε.

4 Partial Korn's inequalities

In this section we state and prove several Korn's inequalities. The proofs of Theorems 4.2 and 4.3 follow some of the lines of that of Theorem 4.4, which is due to Monneau, Murat, and Sili [7].

Theorem 4.1 *There exists a constant C such that*

$$\Big\| u_1 - \fint_{\Omega_1} u_1\,dx_1 \Big\|_{L^2(\Omega)} \le C \Big\| \frac{\partial u_1}{\partial x_1} \Big\|_{L^2(\Omega)}$$

for every $u_1 \in H^1(\Omega_1; L^2(\Omega_{23}))$.

Proof. By density we may restrict ourselves to considering $u_1 \in C^1(\bar\Omega)$. For every x_2 and x_3 there exists a $\xi = \xi(x_2,x_3)$ such that $u_1(\xi,x_2,x_3) = \fint_{\Omega_1} u_1(s,x_2,x_3)\,ds$ and

$$u_1(x_1,\cdot,\cdot) - u_1(\xi,\cdot,\cdot) = \int_\xi^{x_1} \frac{\partial u_1}{\partial x_1}(s,\cdot,\cdot)\,ds.$$

Taking squares and applying Jensen's inequality we conclude the proof. □

Theorem 4.2 *There exists a constant C such that*

$$\Big\| u_2 - \Big(\fint_{\Omega_1} u_2\,dx_1 - x_1 \frac{\partial}{\partial x_2} \fint_{\Omega_1} u_1\,dx_1 \Big) \Big\|_{H^{-1}(\Omega_2; L^2(\Omega_{13}))} \le C \left(\|E_{11}(u)\|_{L^2(\Omega)} + \|E_{12}(u)\|_{L^2(\Omega)} \right)$$

for every $u \in H^1(\Omega;\mathbb{R}^2)$.

Proof. Let

$$\bar u_1 := u_1 - \fint_{\Omega_1} u_1\,dx_1, \quad \bar u_2 := u_2 - \fint_{\Omega_1} u_2\,dx_1 + x_1 \frac{\partial}{\partial x_2} \fint_{\Omega_1} u_1\,dx_1,$$

and note that $\fint_{\Omega_1} \bar{u}_2 \, dx_1 = 0$ and $\bar{u}_2 \in H^1(\Omega_1; L^2(\Omega_{23}))$. Let $\psi \in H^1_0(\Omega_2)$; then

$$\begin{aligned} \frac{\partial}{\partial x_1} \int_{\Omega_2} \psi \bar{u}_2 \, dx_2 &= \int_{\Omega_2} \psi \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial}{\partial x_2} \fint_{\Omega_1} u_1 \, dx_1 \right) dx_2 \\ &= \int_{\Omega_2} \psi \left(2E_{12}(u) - \frac{\partial \bar{u}_1}{\partial x_2} \right) dx_2 \\ &= \int_{\Omega_2} 2\psi E_{12}(u) + \bar{u}_1 \frac{d\psi}{dx_2} \, dx_2. \end{aligned}$$

Since $\int_{\Omega_2} \psi \bar{u}_2 \, dx_2 \in H^1(\Omega_1; L^2(\Omega_{23}))$ and $\int_{\Omega_1} \int_{\Omega_2} \psi \bar{u}_2 \, dx_2 dx_1 = 0$, by Theorem 4.1 and the above equation we deduce

$$\begin{aligned} \left\| \int_{\Omega_2} \psi \bar{u}_2 \, dx_2 \right\|_{L^2(\Omega)} &\le C \left\| \frac{\partial}{\partial x_1} \int_{\Omega_2} \psi \bar{u}_2 \, dx_2 \right\|_{L^2(\Omega)} \\ &\le C \, \|\psi\|_{H^1(\Omega_2)} (\|E_{12}(u)\|_{L^2(\Omega)} + \|\bar{u}_1\|_{L^2(\Omega)}) \\ &\le C \, \|\psi\|_{H^1(\Omega_2)} (\|E_{12}(u)\|_{L^2(\Omega)} + \|E_{11}(u)\|_{L^2(\Omega)}). \end{aligned}$$

Let $\varphi \in L^2(\Omega_{13})$; then

$$\begin{aligned} \left| \int_{\Omega} \varphi \psi \bar{u}_2 \, dx \right| &\le \|\varphi\|_{L^2(\Omega_{13})} \left\| \int_{\Omega_2} \psi \bar{u}_2 \, dx_2 \right\|_{L^2(\Omega_{13})} \\ &\le C \, \|\varphi\|_{L^2(\Omega_{13})} \|\psi\|_{H^1(\Omega_2)} (\|E_{11}(u)\|_{L^2(\Omega_{13})} + \|E_{12}(u)\|_{L^2(\Omega)}). \end{aligned}$$

A density argument concludes the proof. Indeed, let $\{\varphi_n\}$ be an orthonormal basis of $L^2(\Omega_{13})$ and for any $v \in H^1_0(\Omega_2; L^2(\Omega_{13}))$, let

$$\psi_n(x_2) := \int_{\Omega_{13}} \varphi_n(x_1, x_3) v(x) \, dx_1 dx_3 \in H^1_0(\Omega_2).$$

Then for $v_N := \sum_{n=1}^N \psi_n \varphi_n$ we have

$$\left| \int_{\Omega} v_N \bar{u}_2 \, dx \right| \le \|v_N\|_{H^1_0(\Omega_2; L^2(\Omega_{13}))} (\|E_{11}(u)\|_{L^2(\Omega)} + \|E_{12}(u)\|_{L^2(\Omega)}),$$

and letting N go to infinity we conclude the proof. □

Remark **4.1** In spite of the fact that the left-hand side belongs to $L^2(\Omega)$, the inequality of Theorem 4.2 does not hold true if one replaces the norm H^{-1} with the norm of L^2, because of the following counterexample, which is inspired by an example contained in [7] in a quite similar framework.

Consider two scalar smooth functions $\varphi_\alpha \in C^\infty(\bar{\Omega}_\alpha)$ with φ_1 satisfying

$$\int_{\Omega_1} \varphi_1 \, dx_1 = \int_{\Omega_1} \frac{\partial \varphi_1}{\partial x_1} \, dx_1 = 0.$$

Define

$$u_1 := -\varphi_2 \frac{\partial \varphi_1}{\partial x_1}, \quad u_2 := \varphi_1 \frac{\partial \varphi_2}{\partial x_2}, \quad u_3 := 0.$$

Then $u \in H^1(\Omega; \mathbb{R}^3)$, and the inequality of Theorem 4.2 reduces to

$$\left\| \varphi_1 \frac{\partial \varphi_2}{\partial x_2} \right\|_{H^{-1}(\Omega_2; L^2(\Omega_1))} \leq C \left\| \varphi_2 \frac{\partial^2 \varphi_1}{\partial x_1^2} \right\|_{L^2(\Omega)},$$

which cannot be true if we replace H^{-1} by L^2 because in such a case, taking $\|\varphi_1\|_{L^2(\Omega_1)} = \|\partial^2 \varphi_1 / \partial x_1^2\|_{L^2(\Omega_1)}$ would imply that

$$\left\| \frac{\partial \varphi_2}{\partial x_2} \right\|_{L^2(\Omega)} \leq C \|\varphi_2\|_{L^2(\Omega)}$$

for any $\varphi_2 \in C^\infty(\bar{\Omega}_2)$, which is clearly impossible.

Define (using the summation convention)

$$rd_2 = \{r \in L^2(\Omega_{12}; \mathbb{R}^2) : \exists\, c \in \mathbb{R},\, d \in \mathbb{R}^2 \text{ such that } r_\alpha(y) = \mathcal{E}_{\beta\alpha}\, x_\beta\, d + c_\alpha\},$$

where $\mathcal{E}$ denotes the Ricci's symbol. The elements of rd_2 are the infinitesimal rigid displacements on Ω_{12}. It is easy to see that $rd_2 \subset H^1(\Omega_{12}; \mathbb{R}^2)$; moreover, being finite-dimensional, it is closed in $L^2(\Omega_{12}; \mathbb{R}^2)$. Thus, the orthogonal projection operator of $L^2(\Omega_{12}; \mathbb{R}^2)$ on rd_2, which will be denoted by $\wp$, is well defined. Given a vector function $v \in L^2(\Omega_{12}, \mathbb{R}^m)$ with $m \geq 2$, we define

$$\vartheta(v) := \frac{1}{I_O} \int_{\Omega_{12}} (x_1 v_2 - x_2 v_1)\, dx', \text{ where } I_O = \int_{\Omega_{12}} (x_1^2 + x_2^2)\, dx'. \tag{9}$$

If $v \in L^2(\Omega_{12}; \mathbb{R}^2)$, then the components of $\wp$ turn out to be

$$\wp_\alpha(v) = \mathcal{E}_{\beta\alpha}\, x_\beta \vartheta(v) + \fint_{\Omega_{12}} v_\alpha\, dx'. \tag{10}$$

Furthermore, the two-dimensional Korn's inequality can be written as

$$\|v - \wp(v)\|_{H^1(\Omega_{12}; \mathbb{R}^2)} \leq C \sum_{\alpha,\beta} \|E_{\alpha\beta}(v)\|_{L^2(\Omega_{12}; \mathbb{R}^{2\times 2})} \tag{11}$$

for all $v \in H^1(\Omega_{12}; \mathbb{R}^2)$ with a constant C which is independent of v.

Similarly, given a vector function $u \in L^2(\Omega_3; L^2(\Omega_{12}, \mathbb{R}^m))$ with $m \geq 2$, we analogously define $\vartheta(u) \in L^2(\Omega_3)$ and $\wp_\alpha(u)$. The operator $\wp$ associates to any $u \in L^2(\Omega_3; L^2(\Omega_{12}, \mathbb{R}^m))$ a function $\wp(u) \in L^2(\Omega_3; L^2(\Omega_{12}, \mathbb{R}^2))$ which is an infinitesimal rigid displacement on Ω_{12} for almost every $x_3 \in \Omega_3$.

Let us observe that the orthogonal complement, with respect to the $L^2(\Omega_{12}; \mathbb{R}^2)$ inner product, of rd_2 in $H^1(\Omega_{12}; \mathbb{R}^2)$ can be then characterized as

$$rd_2^\perp = \{v \in H^1(\Omega_{12}; \mathbb{R}^2) \ : \ \wp(v) = 0\} = \{v \in H^1_m(\Omega_{12}; \mathbb{R}^2) \ : \ \vartheta(v) = 0\}.$$

Moreover, we denote by

$$RD_2^{\perp}(\Omega) = \{v \in L^2(\Omega_3; H^1_m(\Omega_{12}; \mathbb{R}^2)) \;:\; \vartheta(v) = 0 \text{ a.e. } x_3 \in \Omega_3\}. \tag{12}$$

Hereafter, for any $u \in L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^m))$, $m \geq 2$, we set

$$\tilde{u}_\alpha := u_\alpha - \wp_\alpha(u). \tag{13}$$

Of course $\tilde{u} \in L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2))$ and $\fint_{\Omega_{12}} \tilde{u}\, dx_1 dx_2 = 0$ and $\vartheta(\tilde{u}) = 0$ a.e. in Ω_3, where the latter follows from the linearity of ϑ and the fact that $\vartheta(u) = \vartheta(\wp(u))$. Thus $\tilde{u} \in RD_2^{\perp}(\Omega)$.

Lemma 4.1 *There exists a constant C such that*

$$\|\tilde{u}\|_{L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2))} \leq C \sum_{\alpha,\beta} \|E_{\alpha\beta}(u)\|_{L^2(\Omega_3; L^2(\Omega_{12}))}$$

for every $u \in L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^m))$, $m \geq 2$.

Proof. Since $u_\alpha(\cdot, \cdot, x_3) \in L^2(\Omega_{12})$ for almost every $x_3 \in \Omega_3$, from (10) and (11) we have that the relations

$$\wp_\alpha(u) = \mathcal{E}_{\beta\alpha}\, x_\beta\, \vartheta(u) + \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} u_\alpha \, dx', \tag{14}$$

$$\|\tilde{u}\|_{H^1(\Omega_{12}; \mathbb{R}^2)} \leq C \sum_{\alpha,\beta} \|E_{\alpha\beta}(u)\|_{L^2(\Omega_{12})} \tag{15}$$

hold for almost every x_3 in Ω_3, and the claimed inequality follows by integration. □

A different proof of the lemma above can be found in Le Dret [6].

Proposition 4.1 $RD_2^{\perp}(\Omega)$ *is a Hilbert space with the norm*

$$\|v\|_{RD_2^{\perp}(\Omega)} := \left(\sum_{\alpha,\beta} \|E_{\alpha\beta}(v)\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Proof. We have only to prove that $\|v\|_{RD_2^{\perp}(\Omega)}$ is equivalent to the norm induced on $RD_2^{\perp}(\Omega)$ by $L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2))$ since the former space is a closed subspace of the latter. For any $v \in RD_2^{\perp}(\Omega)$, recalling that $\wp(v) = 0$, so that $v = \tilde{v}$, and using Lemma 4.1, we then have

$$\|v\|_{L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2))} = \|\tilde{v}\|_{L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2))} \leq C \|v\|_{RD_2^{\perp}(\Omega)}$$

while the opposite inequality is trivially satisfied. □

Theorem 4.3 *There exists a constant C such that*

$$\left\| u_3 - \left(x_1 x_2 \frac{d\vartheta(u)}{dx_3} - x_1 \frac{d}{dx_3} \fint_{\Omega_{12}} u_1 \, dx' + \fint_{\Omega_1} u_3 \, dx_1 \right) \right\|_{H^{-1}(\Omega_3; L^2(\Omega_{12}))} \leq C \left(\sum_{\alpha,\beta} \|E_{\alpha\beta}(u)\|_{L^2(\Omega)} + \|E_{13}(u)\|_{L^2(\Omega)} \right)$$

for every $u \in H^1(\Omega; \mathbb{R}^3)$.

Proof. Let $\tilde{u}_\alpha := u_\alpha - \wp_\alpha(u)$ as in (13) and

$$\tilde{u}_3 := u_3 - \left(x_1 x_2 \frac{d\vartheta(u)}{dx_3} - x_1 \frac{d}{dx_3} \fint_{\Omega_{12}} u_1 \, dx' + \fint_{\Omega_1} u_3 \, dx_1 \right).$$

Since $E_{13}(u) = E_{13}(\tilde{u})$,

$$\frac{\partial \tilde{u}_3}{\partial x_1} = 2E_{13}(u) - \frac{\partial \tilde{u}_1}{\partial x_3}.$$

Let $\psi \in H^1_0(\Omega_3)$; then

$$\begin{aligned} \frac{\partial}{\partial x_1} \int_{\Omega_3} \psi \tilde{u}_3 \, dx_3 &= \int_{\Omega_3} \psi \Big(2E_{13}(u) - \frac{\partial \tilde{u}_1}{\partial x_3} \Big) \, dx_3 \\ &= \int_{\Omega_3} 2 \Big(\psi E_{13}(u) + \tilde{u}_1 \frac{d\psi}{dx_3} \Big) \, dx_3. \end{aligned}$$

Since $\int_{\Omega_3} \psi \tilde{u}_3 \, dx_3 \in H^1(\Omega)$ and $\int_{\Omega_1} \int_{\Omega_3} \psi \tilde{u}_3 \, dx_3 dx_1 = 0$, by Theorem 4.1, Lemma 4.1, and the previous equality we deduce

$$\begin{aligned} \left\| \int_{\Omega_3} \psi \tilde{u}_3 \, dx_3 \right\|_{L^2(\Omega)} &\leq C \left\| \frac{\partial}{\partial x_1} \int_0^\ell \psi \tilde{u}_3 \, dx_3 \right\|_{L^2(\Omega)} \\ &\leq C \|\psi\|_{H^1(\Omega_3)} (\|E_{13}(u)\|_{L^2(\Omega)} + \|\tilde{u}_1\|_{L^2(\Omega)}) \\ &\leq C \|\psi\|_{H^1(\Omega_3)} (\|E_{13}(u)\|_{L^2(\Omega)} + \textstyle\sum_{\alpha,\beta} \|E_{\alpha\beta}(u)\|_{L^2(\Omega)}). \end{aligned}$$

Let $\varphi \in L^2(\Omega_{12})$; then

$$\begin{aligned} \left| \int_\Omega \varphi \psi \tilde{u}_3 \, dx \right| &= \left| \int_{\Omega_{12}} \varphi \int_{\Omega_3} \psi \tilde{u}_3 \, dx_3 dx' \right| \leq \|\varphi\|_{L^2(\Omega_{12})} \left\| \int_{\Omega_3} \psi \tilde{u}_3 \, dx_3 \right\|_{L^2(\Omega_{12})} \\ &\leq C \|\varphi\|_{L^2(\Omega_{12})} \|\psi\|_{H^1(\Omega_3)} \left(\|E_{13}(u)\|_{L^2(\Omega)} + \sum_{\alpha,\beta} \|E_{\alpha\beta}(u)\|_{L^2(\Omega)} \right). \end{aligned}$$

Arguing as in the proof of Theorem 4.2, we conclude the proof. □

Remark **4.2** As in Remark 4.1, in spite of the fact that the left-hand side belongs to $L^2(\Omega)$, the inequality of Theorem 4.3 does not hold true if one replaces the norm H^{-1} with the norm of L^2, because of the following counterexample.

Consider three scalar smooth functions $\varphi_i \in C^\infty(\bar{\Omega}_i)$ with φ_α satisfying

$$\int_{\Omega_\alpha} \varphi_\alpha \, dx_\alpha = \int_{\Omega_\alpha} \frac{\partial \varphi_\alpha}{\partial x_\alpha} \, dx_\alpha = 0.$$

Define

$$u_1 := -\varphi_2 \frac{\partial \varphi_1}{\partial x_1} \varphi_3, \quad u_2 := -\varphi_1 \frac{\partial \varphi_2}{\partial x_2} \varphi_3, \quad u_3 := +\varphi_1 \varphi_2 \frac{\partial \varphi_3}{\partial x_3}.$$

Then $u \in H^1(\Omega; \mathbb{R}^3)$, and the inequality of Theorem 4.3 reduces to

$$\left\| \varphi_1 \varphi_2 \frac{\partial \varphi_3}{\partial x_3} \right\|_{H^{-1}(\Omega_2; L^2(\Omega_1))} \leq C \left\| \varphi_3 \left(\frac{\partial^2 \varphi_1}{\partial x_1^2} \varphi_2 + \varphi_1 \frac{\partial^2 \varphi_2}{\partial x_2^2} \right) \right\|_{L^2(\Omega)},$$

which cannot be true if we replace H^{-1} by L^2, because in such a case taking $\|\varphi_\alpha\|_{L^2(\Omega_\alpha)} = \|\partial^2 \varphi_\alpha / \partial x_1^\alpha\|_{L^2(\Omega_\alpha)}$ would imply that

$$\left\| \frac{\partial \varphi_3}{\partial x_3} \right\|_{L^2(\Omega)} \leq C \|\varphi_3\|_{L^2(\Omega)}$$

for any $\varphi_3 \in C^\infty(\bar{\Omega}_3)$, which is clearly impossible.

The next partial Korn's inequality is proved in Monneau, Murat, and Sili [7].

Theorem 4.4 *There exists a constant C such that*

$$\left\| u_3 - \left(\fint_{\Omega_{12}} u_3 \, dx' - x_\alpha \frac{d}{dx_3} \fint_{\Omega_{12}} u_\alpha \, dx' \right) \right\|_{H^{-1}(\Omega_3; L^2(\Omega_{12}))}$$
$$\leq C \left(\sum_{\alpha\beta} \|E_{\alpha\beta}(u)\|_{L^2(\Omega)} + \sum_\alpha \|E_{\alpha 3}(u)\|_{L^2(\Omega)} \right)$$

for every $u \in H^1_{dn}(\Omega; \mathbb{R}^3)$.

5 Limit strain characterization

Let

$$\begin{aligned} H^1_m(\Omega_{12}) &:= \{z \in H^1(\Omega_{12}) : \textstyle\int_{\Omega_{12}} z \, dx' = 0\}, \\ H^1_{m_1}(\Omega_1; L^2(\Omega_{23})) &:= \{z \in H^1(\Omega_1; L^2(\Omega_{23})) : \textstyle\int_{\Omega_1} z \, dx_1 = 0 \text{ a.e. in } \Omega_{23}\}, \\ H^{-1}_{m_1}(\Omega_3; L^2(\Omega_{12})) &:= \{v \in H^{-1}(\Omega_3; L^2(\Omega_{12})) : \langle z_3, \varphi \rangle = 0 \\ &\qquad \forall\, \varphi \in H^1_0(\Omega_3; L^2(\Omega_2))\}, \end{aligned}$$

where the bracket in the last definition has to be understood in the sense of the duality $H^{-1}(\Omega_3; L^2(\Omega_{12})) \times H_0^1(\Omega_3; L^2(\Omega_{12}))$.

Let (u^ε) be the sequence of solutions to problems (6). From (7) and assumption (8) it follows that

$$\sup_\varepsilon \|E^\varepsilon(u^\varepsilon)\|_{L^2(\Omega)} < +\infty. \tag{16}$$

Hence, possibly passing to a subsequence, we have that

$$E^\varepsilon(u^\varepsilon) \rightharpoonup E \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}})$$

for some $E \in L^2(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}})$.

In this section we characterize the limit strain E. For clarity we state several lemmas.

Lemma 5.1 (component 33) *There exists a function $\bar{u}$ in the set of the so-called Bernoulli–Navier displacements*

$$\mathscr{U} := \{u \in H^1_{dn}(\Omega; \mathbb{R}^3) : E_{\alpha i}(u) = 0\}$$

such that

$$E_{33} = \frac{\partial \bar{u}_3}{\partial x_3} = E_{33}(\bar{u}).$$

Proof. From (16), the structure of E^ε, and Korn's inequality, we have that

$$C \geq \|E^\varepsilon(u^\varepsilon)\|_{L^2(\Omega)} \geq \|E(u^\varepsilon)\|_{L^2(\Omega)} \geq C_K \|u^\varepsilon\|_{H^1(\Omega)},$$

where C is a constant independent of ε and C_K is Korn's constant. Hence, up to a subsequence $u^\varepsilon \rightharpoonup \bar{u}$ in $H^1(\Omega; \mathbb{R}^3)$, for some $\bar{u} \in H^1_{dn}(\Omega; \mathbb{R}^3)$. The claim follows by noticing that $\|E_{\alpha i}(u^\varepsilon)\|_{L^2(\Omega)} \leq C\varepsilon$, and $E^\varepsilon_{33}(u^\varepsilon) = \partial u^\varepsilon_3 / \partial x_3$. In fact it follows that $\bar{u} \in \{u \in H^1_{dn}(\Omega; \mathbb{R}^3) : E_{\alpha i}(u) = 0\}$. □

Remark **5.1 (representation of the space $\mathscr{U}$)** It is well known (see, for instance, Le Dret [5]) that the space of Bernoulli–Navier displacements admits the following representation:

$$\mathscr{U} := \Big\{ u \in H^2_{dn}(\Omega)^2 \times H^1_{dn}(\Omega) : \text{exists } \zeta \in H^2_{dn}(\Omega_3)^2 \times H^1_{dn}(\Omega_3) \text{ such that}$$
$$u_1 = \zeta_1,\ u_2 = \zeta_2,\ u_3 = \zeta_3 - x_1 \frac{d\zeta_1}{dx_3} - x_2 \frac{d\zeta_2}{dx_3} \Big\}.$$

Moreover, $\mathscr{U}$ is a Hilbert space with the norm

$$\|u\|_{\mathscr{U}} := \|E_{33}(u)\|_{L^2(\Omega)},$$

which is equivalent to that induced by $H^1_{dn}(\Omega; \mathbb{R}^3)$ (see [8]).

Lemma 5.2 (component 23) *There exists a function $\bar{v}$ in the space*

$$\mathscr{V} := \Big\{ v \in H^1_{dn}(\Omega)^2 \times L^2(\Omega_3; H^1_m(\Omega_{12})) : \textit{exist } \vartheta \in H^1(\Omega_3) \textit{ such that } \vartheta(0) = 0$$
$$\textit{and } \varrho \in L^2(\Omega_3; H^1(\Omega_2)) \textit{ such that } v_1(x) = -x_2\vartheta(x_3),\ v_2(x) = x_1\vartheta(x_3),$$
$$v_3(x) = x_1 x_2 \frac{d\vartheta}{dx_3}(x_3) + \varrho(x_2, x_3) \Big\}$$

such that

$$E_{23} = E_{23}(\bar{v}).$$

Proof. Let $v_i^\varepsilon := \frac{1}{\varepsilon} u_i^\varepsilon$ and $\vartheta^\varepsilon := \vartheta(v^\varepsilon)$ (see (9)).

First of all, by adapting an argument of [4] and Lemmas 4.4 and 4.5, we prove that there exists $\vartheta \in H^1(\Omega_3)$ with $\vartheta(0) = 0$, such that, up to subsequences,

$$\vartheta^\varepsilon \to \vartheta \text{ in } L^2(\Omega_3). \tag{17}$$

Applying Lemma 4.1, there exists a constant C such that

$$\|\tilde{v}^\varepsilon\|_{L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2))} \le C \sum_{\alpha,\beta} \|E_{\alpha\beta}(v^\varepsilon)\|_{L^2(\Omega_3; L^2(\Omega_{12}))}. \tag{18}$$

Since, furthermore,

$$E_{11}(v^\varepsilon)_{11} = \varepsilon^3 E^\varepsilon_{11}(u^\varepsilon),\ E_{12}(v^\varepsilon) = \varepsilon^2 E^\varepsilon_{12}(u^\varepsilon),\ E_{22}(v^\varepsilon) = \varepsilon E^\varepsilon_{22}(u^\varepsilon),$$

and using (16) we have

$$\|E_{\alpha\beta}(v^\varepsilon)\|_{L^2(\Omega)} \le \varepsilon \|E^\varepsilon_{\alpha\beta}(u^\varepsilon)\|_{L^2(\Omega)} \le C\,\varepsilon. \tag{19}$$

From (18) we get

$$\|\tilde{v}^\varepsilon\|_{L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2))} \le C\varepsilon; \tag{20}$$

hence

$$\tilde{v}^\varepsilon \to 0 \text{ in } L^2(\Omega_3; H^1(\Omega_{12}; \mathbb{R}^2)). \tag{21}$$

Now let $\eta \in C^\infty_c(\Omega_{12})$ be such that

$$\int_{\Omega_{12}} \eta\, dx' = -\frac{I_O}{2}.$$

Then, taking into account (14), we have

$$
\begin{aligned}
I_O\vartheta^\varepsilon &= -2\vartheta^\varepsilon\int_{\Omega_{12}}\eta\,dx' = -\vartheta^\varepsilon\int_{\Omega_{12}}\eta D_\alpha x_\alpha\,dx'\\
&= \vartheta^\varepsilon\int_{\Omega_{12}}D_\alpha\eta\,x_\alpha\,dx' = \vartheta^\varepsilon\int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}\mathcal{E}_{\beta\gamma}D_\alpha\eta\,x_\beta\,dx'\\
&= \int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\mathcal{E}_{\beta\gamma}x_\beta\vartheta^\varepsilon\,dx'\\
&= \int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\Big(\wp_\gamma(v^\varepsilon)-\frac{1}{|\Omega_{12}|}\int_{\Omega_{12}}v^\varepsilon_\gamma\,dx'\Big)\,dx'\\
&= \int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\,\wp_\gamma(v^\varepsilon)\,dx'\\
&= \int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\,v^\varepsilon_\gamma\,dx' - \int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\big(v^\varepsilon-\wp_\gamma(v^\varepsilon)\big)\,dx'.
\end{aligned}
$$

Hence, denoting by

$$\tilde\vartheta^\varepsilon = \frac{1}{I_O}\int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\,w^\varepsilon_\gamma\,dx'$$

and recalling (21), we find

$$\vartheta^\varepsilon - \tilde\vartheta^\varepsilon \to 0 \ \text{ in } L^2(\Omega_3). \tag{22}$$

We now show that $D_3\tilde\vartheta^\varepsilon$ is bounded in L^2. Since $\mathcal{E}_{\alpha\gamma}D_\alpha D_\gamma\eta = 0$ everywhere in Ω_3 and $D_\alpha\eta = 0$ on $\partial\Omega_3$, we have

$$I_O D_3\tilde\vartheta^\varepsilon = \int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\,D_3 v^\varepsilon_\gamma\,dx' = 2\int_{\Omega_{12}}\mathcal{E}_{\alpha\gamma}D_\alpha\eta\,E_{\gamma 3}(v^\varepsilon)\,dx',$$

but $E_{13}(v^\varepsilon) = \varepsilon E^\varepsilon_{13}(u^\varepsilon)$ and $E_{23}(v^\varepsilon) = E^\varepsilon_{23}(u^\varepsilon)$, and therefore $D_3\tilde\vartheta^\varepsilon$ is bounded in $L^2(\Omega_3)$. Since $\tilde\vartheta^\varepsilon(0) = 0$, $\tilde\vartheta^\varepsilon$ is then bounded in $H^1(\Omega_3)$ so that there exists $\vartheta \in H^1(\Omega_3)$ with $\vartheta(0) = 0$, such that, up to subsequences,

$$\tilde\vartheta^\varepsilon \rightharpoonup \vartheta \text{ in } H^1(\Omega_3).$$

Thus, from (22) we obtain (17).

Let us now set

$$\bar v^\varepsilon_1 := v^\varepsilon_1 - \fint_{\Omega_{12}} v^\varepsilon_1\,dx', \quad \bar v^\varepsilon_2 := v^\varepsilon_2 - \fint_{\Omega_{12}} v^\varepsilon_2\,dx',$$

and

$$\bar v^\varepsilon_3 := v^\varepsilon_3 - \Big(\fint_{\Omega_{12}} v^\varepsilon_3\,dx' - x_\alpha\frac{d}{dx_3}\fint_{\Omega_{12}} v^\varepsilon_\alpha\,dx'\Big).$$

Observing that, by the definitions,

$$\bar{v}_1^\varepsilon = \tilde{v}_1^\varepsilon - x_2\vartheta^\varepsilon, \quad \bar{v}_2^\varepsilon = \tilde{v}_2^\varepsilon + x_1\vartheta^\varepsilon,$$

from (17) and (21) we have that

$$\bar{v}_1^\varepsilon \to -x_2\vartheta, \qquad \bar{v}_2^\varepsilon \to +x_1\vartheta \qquad \text{in } L^2(\Omega_3; H^1(\Omega_{12})).$$

By Theorem 4.4, we have that

$$\|\bar{v}_3^\varepsilon\|_{H^{-1}(\Omega_3;L^2(\Omega_{12}))} \leq C,$$

and hence there exists $\bar{v}_3 \in H^{-1}(\Omega_3; L^2(\Omega_{12}))$ such that, up to subsequences,

$$\bar{v}_3^\varepsilon \rightharpoonup: \bar{v}_3 \text{ in } H^{-1}(\Omega_3; L^2(\Omega_{12})). \tag{23}$$

Moreover, let us set

$$\bar{v}_1 = -x_2\vartheta, \quad \bar{v}_2 = +x_1\vartheta,$$

and check that the vector field $\bar{v}$ so defined satisfies the properties claimed in the statement of Lemma 4.1. A simple computation shows that

$$E_{13}(\bar{v}^\varepsilon) = E_{13}(v^\varepsilon) \quad \text{and} \quad E_{23}(\bar{v}^\varepsilon) = E_{23}(v^\varepsilon).$$

Noticing that $E_{13}(v^\varepsilon) = \varepsilon E_{13}^\varepsilon(u^\varepsilon) \to 0$ in $L^2(\Omega)$ and $E_{13}(\bar{v}^\varepsilon) \to E_{13}(\bar{v})$ in $\mathscr{D}'(\Omega)$, we obtain that $E_{13}(\bar{v}) = 0$. Hence,

$$\frac{\partial \bar{v}_3}{\partial x_1} = -\frac{\partial \bar{v}_1}{\partial x_3} = x_2\frac{d\vartheta}{dx_3}$$

and, integrating with respect to x_1,

$$\bar{v}_3 = x_1x_2\frac{d\vartheta}{dx_3} + \varrho(x_2, x_3)$$

for some function $\varrho \in L^2(\Omega_3; H^1(\Omega_2))$. Moreover, $\bar{v}_3 \in L^2(\Omega_3; H^1(\Omega_{12}))$ and, from (23) and the fact that $\bar{v}_3$, $\bar{v}_3^\varepsilon \in L^2(\Omega)$, we then obtain easily that $⨍_{\Omega_{12}} \bar{v}_3 dx' = 0$, which concludes the proof. □

Lemma 5.3 (characterization of the space $\mathscr{V}$) *The space $\mathscr{V}$ admits the following characterization:*

$$\begin{aligned}\mathscr{V} = \{v \in H^1_{dn}(\Omega)^2 \times L^2(\Omega_3; H^1_m(\Omega_{12})) : E_{\alpha\beta}(v) = 0,\ E_{13}(v) = 0,\\ E_{23}(v) \in L^2(\Omega) \ \textit{and} \ \textstyle\int_{\Omega_{12}} v_\alpha\, dx' = 0 \ a.e.\}.\end{aligned} \tag{24}$$

Moreover, it is a Hilbert space with the norm

$$\|v\|_{\mathscr{V}} := \|W_{13}(v)\|_{L^2(\Omega)} + \|E_{23}(v)\|_{L^2(\Omega)},$$

where

$$W_{13}(v) = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right).$$

PROOF. Let us call V the space at the right-hand side of equality (24) and let $\mathscr{V}$ be as in the statement of Lemma 5.2. It is trivial to check that $\mathscr{V} \subseteq V$. Let us prove the opposite inclusion. Let $v \in V$. Since $v_\alpha \in H^1_{dn}(\Omega)$ and $E_{\alpha\beta}(v) = 0$, by integration there exists $\vartheta \in L^2(\Omega_3)$ such that

$$v_1 = -x_2\vartheta(x_3) + a_1(x_3), \quad v_2 = x_1\vartheta(x_3) + a_2(x_3),$$

and since $\int_{\Omega_{12}} v_\alpha \, dx' = 0$, we have $a_1 = a_2 = 0$ a.e.. From the resulting expression of v_α it follows that $\vartheta \in H^1_{dn}(\Omega)$ and $\vartheta(0) = 0$.

Since $E_{13}(v) = 0$ we obtain that $\partial v_3/\partial x_1 = x_2\vartheta'(x_3)$. Then there exists $\varrho \in L^2(\Omega_{23})$ such that

$$v_3 = x_1x_2\vartheta'(x_3) + \varrho(x_2, x_3),$$

from which it follows also that $\varrho \in L^2(\Omega_3; H^1_m(\Omega_2))$ and that $E_{23}(v) \in L^2(\Omega)$.

The last part of the claim follows from the fact that $L^2(\Omega_3; H^1_m(\Omega_{12})$ is a Hilbert space with the scalar product

$$\langle u_3, v_3\rangle_{L^2(\Omega_3;H^1_m(\Omega_{12}))} = \int_{\Omega_3} \langle u_3(x_3), v_3(x_3)\rangle_{H^1_m(\Omega_{12})} \, dx_3,$$

and $\mathscr{V}$ is a closed subspace of $H^1_{dn}(\Omega)^2 \times L^2(\Omega_3; H^1_m(\Omega_{12}))$ which is Hilbert with the product norm

$$\|v_1\|_{H^1_{dn}(\Omega)} + \|v_2\|_{H^1_{dn}(\Omega)} + \|v_3\|_{L^2(\Omega_3;H^1_m(\Omega_{12}))}$$

induced by $H^1_{dn}(\Omega)^2 \times L^2(\Omega_3; H^1_m(\Omega_{12}))$. The proof that this norm is equivalent to $\|v\|_{\mathscr{V}}$ is an easy consequence of the Poincaré inequality, the representation lemma, Lemma 5.2, and the characterization of the space $\mathscr{V}$ proved above. □

Lemma 5.4 (components 22 and 13) *There exists a function $\bar{w}$ in the space*

$$\mathscr{W} = \{w \in RD^\perp_2(\Omega) \times H^{-1}_{m_1}(\Omega_3; L^2(\Omega_{12})) \ : \ E_{11}(w) = E_{12}(w) = 0, \quad E_{13}(w) \in L^2(\Omega)\}$$

such that

$$E_{22} = \frac{\partial \bar{w}_2}{\partial x_2} = E_{22}(\bar{w}), \quad E_{13} = \frac{1}{2}\left(\frac{\partial \bar{w}_1}{\partial x_3} + \frac{\partial \bar{w}_3}{\partial x_1}\right) = E_{13}(\bar{w}).$$

Moreover, $\mathscr{W}$ is a Hilbert space with the norm

$$\|w\|_{\mathscr{W}} := \|E_{22}(w)\|_{L^2(\Omega)} + \|E_{13}(w)\|_{L^2(\Omega)} + \|w_3\|_{H^{-1}_{m_1}(\Omega_3;L^2(\Omega_{12}))}.$$

PROOF. Let

$$w^\varepsilon := \frac{1}{\varepsilon^2}u^\varepsilon;$$

then

$$\begin{aligned} &\|E_{11}(w^\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^2, \qquad \|E_{12}(w^\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon, \\ &\|E_{22}(w^\varepsilon)\|_{L^2(\Omega)} \le C, \qquad \|E_{13}(w^\varepsilon)\|_{L^2(\Omega)} \le C. \end{aligned} \tag{25}$$

Let us recall that $\tilde{w}^\varepsilon_\alpha := w^\varepsilon_\alpha - \wp_\alpha(w^\varepsilon)$. By Lemma 4.1 and using (25), we have

$$\|\tilde{w}^\varepsilon\|_{L^2(\Omega_3;H^1(\Omega_{12};\mathbb{R}^2))} \le C \sum_{\alpha,\beta} \|E_{\alpha\beta}(w^\varepsilon)\|_{L^2(\Omega)} \le C,$$

$⨍_{\Omega_{12}} \tilde{w}^\varepsilon \, dx' = 0$, and $\vartheta(\tilde{w}^\varepsilon) = 0$. Thus, up to a subsequence,

$$\tilde{w}^\varepsilon_\alpha \rightharpoonup \bar{w}_\alpha \quad \text{ in } L^2(\Omega_3; H^1(\Omega_{12})) \text{ for } \alpha = 1, 2; \tag{26}$$

moreover, a.e.,

$$⨍_{\Omega_{12}} \bar{w}_\alpha dx' = 0, \ \vartheta(\bar{w}) = 0. \tag{27}$$

Hence $(\bar{w}_1, \bar{w}_2) \in RD^{\perp}_{2}(\Omega)$ (see (12)). Using (25) we also obtain

$$E_{11}(\bar{w}) = E_{12}(\bar{w}) = 0.$$

Moreover,

$$E^\varepsilon_{22}(u^\varepsilon) = E_{22}(w^\varepsilon) = E_{22}(\tilde{w}^\varepsilon) \rightharpoonup E_{22}(\bar{w}) \text{ in } L^2(\Omega).$$

Let

$$\tilde{w}^\varepsilon_3 := w^\varepsilon_3 - \left(x_1 x_2 \frac{d\vartheta(w^\varepsilon)}{dx_3} - x_1 \frac{d}{dx_3} ⨍_{\Omega_{12}} w^\varepsilon_1 \, dx' + ⨍_{\Omega_1} w^\varepsilon_3 \, dx_1 \right).$$

By Theorem 4.3 we have that

$$\|\tilde{w}^\varepsilon_3\|_{H^{-1}(\Omega_3;L^2(\Omega_{12}))} \le C \left(\sum_{\alpha,\beta} \|E_{\alpha\beta}(w^\varepsilon)\|_{L^2(\Omega)} + \|E_{13}(w^\varepsilon)\|_{L^2(\Omega)} \right) \le C.$$

Hence, up to a subsequence,

$$\tilde{w}^\varepsilon_3 \rightharpoonup w_3 \quad \text{ in } H^{-1}(\Omega_3; L^2(\Omega_{12})).$$

Taking into account that (see (16))

$$\wp_1(w^\varepsilon) = ⨍_{\Omega_{12}} w^\varepsilon_1 \, dx' - x_2 \vartheta(w^\varepsilon), \quad \wp_2(w^\varepsilon) = ⨍_{\Omega_{12}} w^\varepsilon_2 \, dx' + x_1 \vartheta(w^\varepsilon),$$

we easily deduce that $E^\varepsilon_{13}(u^\varepsilon) = E_{13}(w^\varepsilon) = E_{13}(\tilde{w}^\varepsilon)$ and hence that

$$E_{13} = \frac{1}{2} \left(\frac{\partial \bar{w}_1}{\partial x_3} + \frac{\partial \bar{w}_3}{\partial x_1} \right).$$

Finally, since $⨍_{\Omega_1} \tilde{w}^\varepsilon_3 \, dx_1 = 0$, we have that

$$\langle \bar{w}_3, \varphi \rangle_{H^{-1}(\Omega_3;L^2(\Omega_{12})) \times H^1_0(\Omega_3;L^2(\Omega_{12}))} = 0$$

for all $\varphi \in H^1_0(\Omega_3; L^2(\Omega_2))$.

The last part of the claim follows from the fact that $\mathscr{W}$ is a closed subspace of

$$\{z \in RD_2^\perp(\Omega) \times H^{-1}_{m_1}(\Omega_3; L^2(\Omega_{12})) \ : \ E_{13}(z) \in L^2(\Omega)\}$$

which, in turn, is a Hilbert space under the scalar product

$$\langle z, \zeta \rangle := \langle z, \zeta \rangle_{RD_2^\perp(\Omega)\times H^{-1}_{m_1}(\Omega_3;L^2(\Omega_{12}))} + \int_\Omega E_{13}(z) E_{13}(\zeta)\, dx$$

and from an application of Proposition 4.1. □

Lemma 5.5 (representation of the space $\mathscr{W}$) *The space $\mathscr{W}$ admits the following representation:*

$$\mathscr{W} = \Big\{ w \in L^2(\Omega_3; H^1(\Omega_{12}))^2 \times H^{-1}_{m_1}(\Omega_3; L^2(\Omega_{12})) \ : \ E_{13}(w) \in L^2(\Omega),$$
$$\text{there exists } \eta_1 \in L^2(\Omega_3; H^2(\Omega_2)) \text{ and } \eta_2 \in L^2(\Omega_3; H^1(\Omega_2)) \text{ such that}$$
$$w_1(x) = \eta_1(x_2, x_3), \quad w_2(x) = -x_1 \frac{\partial \eta_1}{\partial x_2}(x_2, x_3) + \eta_2(x_2, x_3),$$
$$\int_{\Omega_2} \eta_\alpha \, dx_2 = 0, \quad \int_{\Omega_2} \left(\frac{a_1^2}{12} \frac{\partial \eta_1}{\partial x_2} + x_2 \eta_1 \right) dx_2 = 0 \Big\}. \tag{28}$$

Proof. Let us call W the set on the right-hand side of equality (28), and let $\mathscr{W}$ be as in the statement of Lemma 5.4. Then it is trivial to check that $W \subseteq \mathscr{W}$. Let us prove the converse inequality. Let $w \in \mathscr{W}$ as in Lemma 5.4. Since $E_{11}(w) = E_{12}(w) = 0$, by integration we deduce that there exist $\eta_1 \in L^2(\Omega_3; H^2(\Omega_2))$ and $\eta_2 \in L^2(\Omega_3; H^1(\Omega_2))$ such that

$$w_1(x) = \eta_1(x_2, x_3), \quad w_2(x) = -x_1 \frac{\partial \eta_1}{\partial x_2}(x_2, x_3) + \eta_2(x_2, x_3).$$

Since

$$\fint_{\Omega_{12}} w_\alpha dx' = 0, \qquad \vartheta(w) = 0, \tag{29}$$

we have that

$$\int_{\Omega_2} \eta_\alpha \, dx_2 = 0 \qquad \text{a.e. for } \alpha = 1, 2,$$

and

$$\int_{\Omega_2} \left(\frac{a_1^2}{12} \frac{\partial \eta_1}{\partial x_2} + x_2 \eta_1 \right) dx_2 = 0,$$

a.e.. □

Lemma 5.6 (component 12) *There exists a vector function $\bar{p}$ in the set*

$$\mathscr{P} = \{0\} \times H^1_{m_1}(\Omega_1; L^2(\Omega_{23})) \times \{0\}$$

such that

$$E_{12} = \frac{1}{2}\frac{\partial \bar{p}_2}{\partial x_1} = E_{12}(\bar{p}). \tag{30}$$

Moreover, $\mathscr{P}$ is a Hilbert space with the norm

$$\|p\|_{\mathscr{P}} := \|E_{12}(p)\|.$$

Proof. Let

$$p^\varepsilon_\alpha := \frac{1}{\varepsilon^3} u^\varepsilon_\alpha \quad \text{for } \alpha = 1, 2,$$

and

$$\bar{p}^\varepsilon_1 := p^\varepsilon_1 - \fint_{\Omega_1} p^\varepsilon_1 \, dx_1, \quad \bar{p}^\varepsilon_2 := p^\varepsilon_2 - \fint_{\Omega_1} p^\varepsilon_2 \, dx_1 + x_1 \frac{\partial}{\partial x_2} \fint_{\Omega_1} p^\varepsilon_1 \, dx_1.$$

Then $E_{11}(\bar{p}^\varepsilon) = E_{11}(p^\varepsilon) = \varepsilon E^\varepsilon_{11}(u^\varepsilon)$; hence, by Theorem 4.1 we have

$$\bar{p}^\varepsilon_1 \to 0, \quad \frac{\partial \bar{p}^\varepsilon_1}{\partial x_1} \to 0 \quad \text{in } L^2(\Omega).$$

Since $E_{12}(\bar{p}^\varepsilon) = E_{12}(p^\varepsilon) = E^\varepsilon_{12}(u^\varepsilon)$, by Theorem 4.2 we also have that, up to subsequences,

$$\bar{p}^\varepsilon_2 \rightharpoonup \bar{p}_2 \text{ in } H^{-1}(\Omega_2; L^2(\Omega_{13})).$$

Setting $\bar{p} := (0, \bar{p}_2, 0)$, we have

$$E_{12} = E_{12}(\bar{p}) = \frac{1}{2}\frac{\partial \bar{p}_2}{\partial x_1},$$

that is, (30). It remains then to prove that $\bar{p}_2 \in H^1(\Omega_1; L^2(\Omega_{23}))$ and that $\fint_{\Omega_1} \bar{p}_2 \, dx_1 = 0$.

Since $\bar{p}_2 \in H^{-1}(\Omega_2; L^2(\Omega_{13}))$, for any $\varphi \in H^1_0(\Omega_2)$ and any $\psi \in L^2(\Omega_{13})$ the product $\varphi \otimes \psi$ belongs to $H^1_0(\Omega_2; L^2(\Omega_{13}))$, and the linear map

$$P_\varphi : L^2(\Omega_{13}) \to \mathbb{R}, \quad \langle P_\varphi, \psi \rangle := \langle \bar{p}_2, \varphi \otimes \psi \rangle$$

satisfies the estimate

$$|\langle P_\varphi, \psi \rangle| \leq \|\bar{p}_2\|_{H^{-1}} \|\varphi\|_{H^1_0} \|\psi\|_{L^2}.$$

Thus $P_\varphi \in L^2(\Omega_{13})$. Moreover, from the definition of P_φ and the fact that $\frac{\partial \bar{p}_2}{\partial x_1} \in L^2(\Omega)$, we obtain that $\frac{\partial P_\varphi}{\partial x_1} \in L^2(\Omega_{13})$ and also that

$$\frac{\partial P_\varphi}{\partial x_1} = \int_{\Omega_2} \frac{\partial \bar{p}_2}{\partial x_1} \varphi \, dx_2. \tag{31}$$

Since $\fint_{\Omega_1} \bar{p}_2^\varepsilon \, dx_1 = 0$ it follows from the definitions that

$$\fint_{\Omega_1} P_\varphi(s, x_3) \, ds = \langle \bar{p}_2, \varphi \rangle = 0 \tag{32}$$

for almost every $x_3 \in \Omega_3$; using this fact, the following Poincaré inequality holds:

$$\|P_\varphi(\cdot, x_3)\|_{L^2(\Omega_1)} \leq C \left\| \frac{\partial P_\varphi}{\partial x_1}(\cdot, x_3) \right\|_{L^2(\Omega_1)},$$

where the constant C depends only on the domain Ω_1 and is therefore independent of x_3. By substituting (31) inside the Poincaré inequality, we obtain that

$$\|P_\varphi\|_{L^2(\Omega_{13})} \leq C \left\| \frac{\partial \bar{p}_2}{\partial x_1} \right\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega_2)}. \tag{33}$$

Using the density of $C_c^\infty(\Omega_2) \otimes C_c^\infty(\Omega_{13})$ in $L^2(\Omega)$ (see, for instance, Treves [10, Theorem 39.2 and subsequent Corollary 3]), the fact that $P_\varphi \in L^2(\Omega_{13})$, and inequality (33), we have that

$$\begin{aligned}
\|\bar{p}_2\|_{L^2(\Omega)} &= \sup_{\varphi \in C_c^\infty(\Omega_2),\ \psi \in C_c^\infty(\Omega_{13})} \frac{|\langle \bar{p}_2, \varphi\psi \rangle|}{\|\varphi\|_{L^2(\Omega_2)} \|\psi\|_{L^2(\Omega_{13})}} \\
&= \sup_{\varphi \in C_c^\infty(\Omega_2),\ \psi \in C_c^\infty(\Omega_{13})} \frac{|\langle P_\varphi, \psi \rangle|}{\|\varphi\|_{L^2(\Omega_2)} \|\psi\|_{L^2(\Omega_{13})}} \\
&\leq \sup_{\varphi \in C_c^\infty(\Omega_2)} \frac{\|P_\varphi\|_{L^2(\Omega_{13})}}{\|\varphi\|_{L^2(\Omega_2)}} \leq C \left\| \frac{\partial \bar{p}_2}{\partial x_1} \right\|_{L^2(\Omega)};
\end{aligned}$$

hence $\bar{p}_2 \in L^2(\Omega)$. Thus $\bar{p}_2 \in H^1(\Omega_1; L^2(\Omega_{23}))$, and (32) implies $\fint_{\Omega_1} \bar{p}_2 \, dx_1 = 0$.

The last part of the claim follows from the fact that $H^1_{m_1}(\Omega_1; L^2(\Omega_{23}))$ is a Hilbert space with the norm

$$\|p\|_{H^1_{m_1}(\Omega_1; L^2(\Omega_{23}))} := \left\| \frac{\partial p_2}{\partial x_1} \right\|_{L^2(\Omega)},$$

which, by Theorem 4.1, turns out to be equivalent to the canonical one. □

Lemma 5.7 (component 11) *There exists a function $\bar{q}$ in the space*

$$\mathscr{Q} := H^1_{m_1}(\Omega_1; L^2(\Omega_{23})) \times \{0\}^2$$

such that

$$E_{11} = \frac{\partial \bar{q}_1}{\partial x_1} = E_{11}(\bar{q}).$$

Moreover, $\mathscr{Q}$ is a Hilbert space with the norm

$$\|q\|_{\mathscr{Q}} := \|E_{11}(q)\|.$$

PROOF. Let

$$q_1^\varepsilon := \frac{1}{\varepsilon^4}\left(u_1^\varepsilon - \fint_{\Omega_1} u_1^\varepsilon dx_1\right);$$

then

$$\sup_\varepsilon \left\| \frac{\partial q_1^\varepsilon}{\partial x_1} \right\|_{L^2(\Omega)} = \sup_\varepsilon \|E_{11}^\varepsilon(u^\varepsilon)\|_{L^2(\Omega)} \leq C,$$

and by Theorem 4.1 we have $\sup_\varepsilon \|q_1^\varepsilon\|_{L^2(\Omega)} \leq C$. Then, up to a subsequence, $q_1^\varepsilon \rightharpoonup \bar{q}_1$ in $L^2(\Omega)$, and $E_{11}^\varepsilon(u^\varepsilon) = \partial \bar{q}_1^\varepsilon/\partial x_1 \rightharpoonup \partial \bar{q}_1/\partial x_1$ in $L^2(\Omega)$ for some $\bar{q}_1 \in L^2(\Omega)$.

The last part of the claim follows from the fact that $H^1_{m_1}(\Omega_1; L^2(\Omega_{23}))$ is a Hilbert space with the norm

$$\|q\|_{H^1_{m_1}(\Omega_1;L^2(\Omega_{23}))} := \|q_{1,1}\|_{L^2(\Omega)},$$

which, by Theorem 4.1, turns out to be equivalent to the canonical one. □

6 The limit problem

Let us consider the space $\mathscr{A} := \mathscr{U} \times \mathscr{V} \times \mathscr{W} \times \mathscr{P} \times \mathscr{Q}$. According to the notation and the results proved in the previous section, $\mathscr{A}$ is a Hilbert space when endowed with the product norm

$$\|(u, v, w, p, q)\|_{\mathscr{A}} := \|u\|_{\mathscr{U}} + \|v\|_{\mathscr{V}} + \|w\|_{\mathscr{W}} + \|p\|_{\mathscr{P}} + \|q\|_{\mathscr{Q}}.$$

Given a 5-tuple of vector valued distributions $(u, v, w, p, q) \in \mathcal{D}'(\Omega; \mathbb{R}^3)^5$, let us define

$$E(u, v, w, p, q) := \begin{pmatrix} E_{11}(q) & E_{12}(p) & E_{13}(w) \\ & E_{22}(w) & E_{23}(v) \\ \text{Sym.} & & E_{33}(u) \end{pmatrix}. \tag{34}$$

We are now in a position to state the main result of this paper.

Theorem 6.1 *Let $\mathbb{C}$ be a positive definite fourth order tensor field on Ω with the minor symmetries, i.e., $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}$. Let F^ε be a second order symmetric tensor field which belongs to $L^2(\Omega; \mathbb{R}^{3\times3})$. Then problem (6), that is,*

$$\begin{cases} u^\varepsilon \in H^1_{dn}(\Omega; \mathbb{R}^3), \\ \displaystyle\int_\Omega \mathbb{C}E^\varepsilon(u^\varepsilon) \cdot E^\varepsilon(\varphi)\, dx = \int_\Omega F^\varepsilon \cdot E^\varepsilon(\varphi)\, dx \quad \forall \varphi \in H^1_{dn}(\Omega; \mathbb{R}^3), \end{cases} \tag{35}$$

admits a unique solution u^ε. Moreover, if $F^\varepsilon \to F$ in $L^2(\Omega; \mathbb{R}^{3\times3})$, then we have the following:

1. *the problem*

$$\int_\Omega \mathbb{C}E(\bar{u},\bar{v},\bar{w},\bar{p},\bar{q})\cdot E(u,v,w,p,q)\,dx = \int_\Omega F\cdot E(u,v,w,p,q)\,dx \qquad \forall\,(u,v,w,p,q)\in\mathscr{A} \tag{36}$$

admits a unique solution $(\bar{u},\bar{v},\bar{w},\bar{p},\bar{q})\in\mathscr{A}$*;*

2. $u^\varepsilon \rightharpoonup \bar{u}$ *in* $H^1(\Omega;\mathbb{R}^3)$*;*
3. $E^\varepsilon(u^\varepsilon)\to E(\bar{u},\bar{v},\bar{w},\bar{p},\bar{q})$ *in* $L^2(\Omega;\mathbb{R}^{3\times3})$*.*

The following corollary can be seen as a corrector result.

Corollary 6.1 *If the solution* $(\bar{u},\bar{v},\bar{w},\bar{p},\bar{q})$ *of problem* (36) *is such that*

$$\frac{\partial\bar{v}_3}{\partial x_3},\ E_{23}(\bar{w}),\ \frac{\partial\bar{w}_3}{\partial x_3},\ \frac{\partial\bar{q}_1}{\partial x_2},\ \frac{\partial\bar{q}_1}{\partial x_3},\ \frac{\partial\bar{p}_2}{\partial x_2},\ \frac{\partial\bar{p}_2}{\partial x_3}\ \in\ L^2(\Omega), \tag{37}$$

then

$$\|E^\varepsilon(u^\varepsilon)-E^\varepsilon(\bar{u}^\varepsilon)\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\to 0,$$

where

$$\bar{u}^\varepsilon := \bar{u}+\varepsilon\bar{v}+\varepsilon^2\bar{w}+\varepsilon^3\bar{p}+\varepsilon^4\bar{q}.$$

Proof. Since

$$E^\varepsilon(\bar{u}^\varepsilon)=E(\bar{u},\bar{v},\bar{w},\bar{p},\bar{q})+\varepsilon\begin{pmatrix}0 & E_{12}(\bar{q}) & 0\\ & E_{22}(\bar{p}) & E_{23}(\bar{w})\\ \text{Sym.} & & E_{33}(\bar{v})\end{pmatrix}+\varepsilon^2\begin{pmatrix}0 & 0 & E_{13}(\bar{q})\\ & 0 & E_{23}(\bar{p})\\ \text{Sym.} & & E_{33}(\bar{w})\end{pmatrix}$$

the additional regularity assumptions imply that $E^\varepsilon(\bar{u}^\varepsilon)\in L^2(\Omega;\mathbb{R}^{3\times3})$ and

$$\|E^\varepsilon(\bar{u}^\varepsilon)-E(\bar{u},\bar{v},\bar{w},\bar{p},\bar{q})\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\to 0.$$

Then the claim follows from step 3 of Theorem 6.1. □

In order to prove Theorem 6.1, we introduce the subspaces of $H^1_{dn}(\Omega;\mathbb{R}^3)$,

$$\hat{\mathscr{U}} := \{u\in H^1_{dn}(\Omega;\mathbb{R}^3)\ :\ E_{\alpha\beta}(u)=E_{\alpha3}(u)=0\},$$
$$\hat{\mathscr{V}} := \{v\in H^1_{dn}(\Omega;\mathbb{R}^3)\ :\ E_{\alpha\beta}(v)=E_{13}(v)=0\},$$
$$\hat{\mathscr{W}} := \{w\in H^1_{dn}(\Omega;\mathbb{R}^3)\ :\ E_{1\beta}(w)=0\},$$
$$\hat{\mathscr{P}} := \{p\in H^1_{dn}(\Omega;\mathbb{R}^3)\ :\ E_{11}(p)=0\},$$
$$\hat{\mathscr{Q}} := H^1_{dn}(\Omega;\mathbb{R}^3),$$

and define $\hat{\mathscr{A}} := \hat{\mathscr{U}}\times\hat{\mathscr{V}}\times\hat{\mathscr{W}}\times\hat{\mathscr{P}}\times\hat{\mathscr{Q}}$.

Let us note that $\mathscr{U}=\hat{\mathscr{U}}$, but similar equalities are not true for the spaces $\mathscr{V}$, $\mathscr{W}$, $\mathscr{P}$, and $\mathscr{Q}$. Nevertheless, for such spaces we can prove the following approximation lemma.

Lemma 6.1 *For every $v \in \mathscr{V}$, $w \in \mathscr{W}$, $p \in \mathscr{P}$, and $q \in \mathscr{Q}$ there exist sequences $(\hat{v}^n)$ in $\hat{\mathscr{V}}$, $(\hat{w}^n)$ in $\hat{\mathscr{W}}$, $(\hat{p}^n)$ in $\hat{\mathscr{P}}$, and $(\hat{q}^n)$ in $\hat{\mathscr{Q}}$ such that the following convergences hold in the norm of $L^2(\Omega)$:*

(i) $E_{23}(\hat{v}^n) \to E_{23}(v)$,

(ii) $E_{13}(\hat{w}^n) \to E_{13}(w)$ *and* $E_{22}(\hat{w}^n) \to E_{22}(w)$,

(iii) $E_{12}(\hat{p}^n) \to E_{12}(p)$,

(iv) $E_{11}(\hat{q}^n) \to E_{11}(q)$.

Proof. Let us prove (i). Since $v \in \mathscr{V}$ (see Lemma 5.2), there exist $\vartheta \in H^1(\Omega_3)$ with $\vartheta(0) = 0$ and $\varrho \in L^2(\Omega_3; H^1(\Omega_2))$ such that

$$v_1(x) = -x_2\vartheta(x_3), \quad v_2(x) = x_1\vartheta(x_3),$$
$$v_3(x) = x_1 x_2 \frac{d\vartheta}{dx_3}(x_3) + \varrho(x_2, x_3).$$

Then there exist sequences $\vartheta_n \in H^2_{dn}(\Omega_3)$ and $\varrho_n \in H^1_{dn}(\Omega_{23})$ such that

$$\vartheta_n \to \vartheta \text{ in } H^1(\Omega_3), \quad \varrho_n \to \varrho \text{ in } L^2(\Omega_3; H^1(\Omega_2)).$$

Setting

$$\hat{v}^n_1(x) = -x_2\vartheta_n(x_3), \quad \hat{v}^n_2(x) = x_1\vartheta_n(x_3),$$
$$\hat{v}^n_3(x) = x_1 x_2 \frac{d\vartheta_n}{dx_3}(x_3) + \varrho_n(x_2, x_3),$$

we obtain the claim.

Let us prove (ii). As $w \in \mathscr{W}$ (see Lemma 5.4), there exist $\eta_1 \in L^2(\Omega_3; H^2(\Omega_2))$ and $\eta_2 \in L^2(\Omega_3; H^1(\Omega_2))$ such that

$$w_1(x) = \eta_1(x_2, x_3), \quad w_2(x) = -x_1 \frac{\partial \eta_1}{\partial x_2}(x_2, x_3) + \eta_2(x_2, x_3).$$

Then, there exist a sequence $\eta^n_1 \in H^2_{dn}(\Omega)$ with $\partial\eta^n_1/\partial x_1 = 0$ such that

$$\eta^n_1 \to \eta_1 \quad \text{in } L^2(\Omega_3; H^2(\Omega_2))$$

and a sequence $\eta^n_2 \in H^1_{dn}(\Omega)$ with $\partial\eta^n_2/\partial x_1 = 0$ such that

$$\eta^n_2 \to \eta_2 \quad \text{in } L^2(\Omega_3; H^1(\Omega_2)).$$

Since $\partial w_3/\partial x_1 = 2E_{13}(w) - \partial\eta_1/\partial x_3$ and $E_{13}(w) \in L^2(\Omega)$, by integration, there exists $G_{13} \in H^1(\Omega_1; L^2(\Omega_{23}))$ such that

$$\frac{\partial G_{13}}{\partial x_1} = E_{13}(w) \quad \text{and} \quad w_3 = 2G_{13} - x_1 \frac{\partial \eta_1}{\partial x_3},$$

where we have also used the fact that η_1 does not depend on x_1.

We may also find a sequence $G_{13}^n \in H^1_{dn}(\Omega)$ such that

$$G_{13}^n \to G_{13} \quad \text{in } H^1(\Omega_1; L^2(\Omega_{23}))$$

and define

$$\hat{w}_1^n := \eta_1^n, \quad \hat{w}_2^n := -x_1 \frac{\partial \eta_1^n}{\partial x_2} + \eta_2^n, \quad \hat{w}_3^n := G_{13}^n - x_1 \frac{\partial \eta_1^n}{\partial x_3}.$$

Then $\hat{w}^n \in H^1_{dn}(\Omega)$, $E_{11}(\hat{w}^n) = \partial \eta_1^n / \partial x_1 = 0$, and $E_{12}(\hat{w}^n) = 0$ so that $\hat{w}^n \in \hat{\mathscr{W}}$. Moreover,

$$E_{13}(\hat{w}^n) = \frac{\partial G_{13}^n}{\partial x_1} \to \frac{\partial G_{13}}{\partial x_1} = E_{13}(w) \quad \text{in } L^2(\Omega),$$

and

$$E_{22}(\hat{w}^n) = \frac{\partial \hat{w}_2^n}{\partial x_2} = -x_1 \frac{\partial^2 \eta_1^n}{\partial x_2^2} + \frac{\partial \eta_2^n}{\partial x_2} \to \frac{\partial \hat{w}_2}{\partial x_2} = E_{22}(\hat{w}) \quad \text{in } L^2(\Omega).$$

To prove (iii) it is enough to consider, for a given $p = (0, p_2, 0) \in \mathscr{P}$, a sequence $\hat{p}_2^n \in H^1_{dn}(\Omega)$, which converges to p_2 in the norm of $H^1(\Omega_1; L^2(\Omega_{23}))$, and set $\hat{p}^n = (0, \hat{p}_2^n, 0)$. Finally, claim (iv) simply follows from the density of $H^1_{dn}(\Omega)$ in $H^1_{m_1}(\Omega_1; L^2(\Omega_{23}))$. □

Proof of Theorem 6.1. The existence and uniqueness of the solution of problem (36) follows from an application of the Lax–Milgram lemma to the symmetric bilinear form defined on $\mathscr{A}$ by

$$a[(u, v, w, p, q), (\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q})] := \int_\Omega \mathbb{C} E(u, v, w, p, q) \cdot E(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q})\, dx,$$

which is continuous and coercive with respect to the Hilbertian norm on $\mathscr{A}$ defined at the beginning of this section.

Part 2 of the statement of Theorem 6.1 is actually a consequence of step 3. Let us now prove part 3. According to the results proved in the previous section, we have

$$E^\varepsilon(u^\varepsilon) \rightharpoonup E \text{ in } L^2(\Omega; \mathbb{R}^{3\times 3}), \tag{38}$$

and there exists a $(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q}) \in \mathscr{A}$ such that

$$E = \begin{pmatrix} E_{11}(\bar{q}) & E_{12}(\bar{p}) & E_{13}(\bar{w}) \\ & E_{22}(\bar{w}) & E_{23}(\bar{v}) \\ \text{Sym.} & & E_{33}(\bar{u}) \end{pmatrix} = E(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q}).$$

The result will be achieved in two steps: (i) we prove that $(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q})$ satisfies equality (36) and therefore coincides with the unique solution of the variational problem, and (ii) we show that the convergence in (38) is indeed strong.

Let $(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q}) \in \hat{\mathscr{A}}$ and set

$$\hat{\varphi}^\varepsilon := \hat{u} + \varepsilon \hat{v} + \varepsilon^2 \hat{w} + \varepsilon^3 \hat{p} + \varepsilon^4 \hat{q};$$

then $\hat{\varphi}^\varepsilon \in H^1_{dn}(\Omega; \mathbb{R}^3)$ and an easy computation shows that, as $\varepsilon \to 0$,

$$E^\varepsilon(\hat{\varphi}^\varepsilon) \to \begin{pmatrix} E_{11}(\hat{q}) & E_{12}(\hat{p}) & E_{13}(\hat{w}) \\ & E_{22}(\hat{w}) & E_{23}(\hat{v}) \\ \text{Sym.} & & E_{33}(\hat{u}) \end{pmatrix} = E(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q}) \tag{39}$$

in the norm convergence of $L^2(\Omega; \mathbb{R}^{3\times 3})$. Taking $\varphi = \hat{\varphi}^\varepsilon$ in (35) and passing to the limit we find

$$\int_\Omega \mathbb{C}E(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q}) \cdot E(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q})\, dx = \int_\Omega F \cdot E(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q})\, dx,$$

for every $(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q}) \in \hat{\mathscr{A}}$. This equality holds in fact for any $(u, v, w, p, q) \in \mathscr{A}$ in place of $(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q}) \in \hat{\mathscr{A}}$ because of the approximation Lemma 6.1 which ensures that there exists a sequence $(u, \hat{v}^n, \hat{w}^n, \hat{p}^n, \hat{q}^n) \in \hat{\mathscr{A}}$ such that

$$\|(u, \hat{v}^n, \hat{w}^n, \hat{p}^n, \hat{q}^n) - (u, v, w, p, q)\|_{\mathscr{A}} \to 0.$$

To show that the convergence in (38) is indeed strong, it suffices to prove that $\lim_{\varepsilon\to 0} \|E^\varepsilon(u^\varepsilon)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} = \|E(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q})\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$ or, equivalently, that

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\Omega \mathbb{C}E^\varepsilon(u^\varepsilon) \cdot E^\varepsilon(u^\varepsilon)\, dx &= \lim_{\varepsilon\to 0} \int_\Omega F^\varepsilon \cdot E^\varepsilon(u^\varepsilon)\, dx \\ &= \int_\Omega F \cdot E(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q})\, dx \\ &= \int_\Omega \mathbb{C}E(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q}) \cdot E(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{q})\, dx, \end{aligned}$$

where we passed to the limit thanks to the strong convergence of F^ε. □

7 Equilibrium differential equations

In this last section we derive the differential formulation of the limit problem. For simplicity we assume here that the elasticity tensor also satisfies the major symmetries; that is, $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$ for any i, j, k, l. Nevertheless, the same computation can be performed also in the general case.

To make (36) more explicit and to keep the notation compact, in writing the elasticity tensor components $\mathbb{C}_{ijkl}$ we associate to a pair of components ij a single component s following the rule $11 \mapsto 1, 22 \mapsto 2, 33 \mapsto 3, 23 \mapsto 4, 13 \mapsto 5, 12 \mapsto 6$, and we write, for instance, c_{14} for $\mathbb{C}_{1123}$; see Auld [1] for more details on the notation used. Clearly $c_{ij} = c_{ji}$. Still, for brevity, define $\bar{e}_1 = E_{11}(\bar{q})$, $\bar{e}_2 = E_{22}(\bar{w})$, $\bar{e}_3 = E_{33}(\bar{u})$, $\bar{e}_4 = 2E_{23}(\bar{v})$, $\bar{e}_5 = 2E_{13}(\bar{w})$, $\bar{e}_6 = 2E_{12}(\bar{p})$. Letting

$$\mathscr{L}(u,v,w,p,q) := \int_\Omega F \cdot E(u,v,w,p,q)\,dx,$$

we can then rewrite (36) as

$$\int_\Omega \sum_{j=1}^{6} \big[c_{1j}\bar{e}_j E_{11}(q) + c_{2j}\bar{e}_j E_{22}(w) + c_{3j}\bar{e}_j E_{33}(u) + 2c_{4j}\bar{e}_j E_{23}(v) \\ + 2c_{5j}\bar{e}_j E_{13}(w) + 2c_{6j}\bar{e}_j E_{12}(p)\big]\,dx = \mathscr{L}(u,v,w,p,q),$$

for every $(u,v,w,p,q) \in \mathscr{A}$. Thus

$$\int_\Omega \sum_{j=1}^{6} c_{1j}\bar{e}_j E_{11}(q)\,dx = \mathscr{L}(0,0,0,0,q), \tag{40}$$

$$\int_\Omega \sum_{j=1}^{6} 2c_{6j}\bar{e}_j E_{12}(p)\,dx = \mathscr{L}(0,0,0,p,0), \tag{41}$$

$$\int_\Omega \sum_{j=1}^{6} \big[c_{2j}\bar{e}_j E_{22}(w) + 2c_{5j}\bar{e}_j E_{13}(w)\big]\,dx = \mathscr{L}(0,0,w,0,0), \tag{42}$$

$$\int_\Omega \sum_{j=1}^{6} 2c_{4j}\bar{e}_j E_{23}(v)\,dx = \mathscr{L}(0,v,0,0,0), \tag{43}$$

$$\int_\Omega \sum_{j=1}^{6} c_{3j}\bar{e}_j E_{33}(u)\,dx = \mathscr{L}(u,0,0,0,0). \tag{44}$$

In this section, for simplicity, we assume

$$\begin{gathered}\mathscr{L}(0,0,\cdot,\cdot,\cdot) = 0, \quad \mathscr{L}(0,v,0,0,0) = \int_0^\ell m(x_3)\vartheta(x_3)\,dx_3, \\ \mathscr{L}(u,0,0,0,0) = \int_\Omega b \cdot u\,dx + \int_{\Gamma_\ell} s \cdot u\,d\mathscr{H}^2,\end{gathered} \tag{45}$$

where $\Gamma_\ell = \partial\Omega \cap \{x_3 = \ell\}$; $u \in \mathscr{U}$; $b \in L^2(\Omega)$; $s \in L^2(\Gamma_\ell)$; $v \in \mathscr{V}$; $\vartheta \in H^1(\Omega_3)$, $\vartheta(0) = 0$, is related to v as in Lemma 5.2; and $m \in L^2(\Omega_3)$. Such assumptions are quite often satisfied in engineering applications.

We now derive the equilibrium equations in differential form. Let $\psi \in L^2(\Omega)$ and define

$$q_1 := \int_{-a_1/2}^{x_1} \psi(s,\cdot,\cdot)\,ds - \fint_{\Omega_1} \int_{-a_1/2}^{x_1} \psi(s,\cdot,\cdot)\,ds\,dx_1.$$

Then $q := (q_1, 0, 0) \in \mathscr{Q}$ and $E_{11}(q) = \psi$; hence, from (40) and (45) we deduce

$$\sum_{j=1}^{6} c_{1j}\bar{e}_j = 0 \quad \text{a.e..} \tag{46}$$

With the same argument it follows from (41) that

$$\sum_{j=1}^{6} c_{6j}\bar{e}_j = 0 \quad \text{a.e..} \tag{47}$$

From (46) and (47) we deduce, since $c_{11}c_{66} - c_{16}^2 > 0$, that

$$\bar{e}_1 = E_{11}(\bar{q}) = -\sum_{j=2}^{5} \frac{c_{66}c_{1j} - c_{16}c_{6j}}{c_{11}c_{66} - c_{16}^2}\,\bar{e}_j \quad \text{a.e.,} \tag{48}$$

$$\bar{e}_6 = 2E_{12}(\bar{p}) = -\sum_{j=2}^{5} \frac{c_{11}c_{6j} - c_{16}c_{1j}}{c_{11}c_{66} - c_{16}^2}\,\bar{e}_j \quad \text{a.e..} \tag{49}$$

Using (45), (48), and (49) we can rewrite (42), after setting

$$\tilde{c}_{ij} = c_{ij} - c_{i1}\frac{c_{66}c_{1j} - c_{16}c_{6j}}{c_{11}c_{66} - c_{16}^2} - c_{i6}\frac{c_{11}c_{6j} - c_{16}c_{1j}}{c_{11}c_{66} - c_{16}^2},$$

for $i, j = 2, \ldots, 5$, as

$$\int_\Omega \sum_{j=2}^{5} \left[\tilde{c}_{2j}\bar{e}_j E_{22}(w) + 2\tilde{c}_{5j}\bar{e}_j E_{13}(w)\right] dx = \mathscr{L}(0, 0, w, 0, 0) = 0. \tag{50}$$

Since $w \in \mathscr{W}$, it then admits the representation given in Lemma 5.5 in terms of functions η_1 and η_2. Choosing $\eta_1 = \eta_2 = 0$, so that $E_{22}(w) = 0$, and w_3 like it has been chosen q_1 previously, we find from (50) that

$$\sum_{j=2}^{5} \tilde{c}_{5j}\bar{e}_j = 0 \quad \text{a.e..} \tag{51}$$

Let $\psi \in L^2(\Omega_{23})$. Taking $\eta_1 = w_3 = 0$ and

$$\eta_2 := \int_{-a_2/2}^{x_2} \psi(s, \cdot)\, ds - ⨍_{\Omega_2} \int_{-a_2/2}^{x_2} \psi(s, \cdot)\, ds\, dx_2$$

so that $E_{22}(w) = \psi$, we find from (50) that

$$\sum_{j=2}^{5} \int_{\Omega_1} \tilde{c}_{2j}\bar{e}_j\, dx_1 = 0 \quad \text{a.e..} \tag{52}$$

Taking instead $\eta_2 = w_3 = 0$ and

$$\eta_1 := \int_{-a_2/2}^{x_2} \int_{-a_2/2}^{t} \psi(s,\cdot)\,ds\,dt - K_1 x_2 - K_2,$$

where the constants K_1 and K_2 are chosen in order to satisfy the mean integral conditions required on η_1 by (28), we have $E_{22}(w) = -x_1\psi$, and hence, from (50) we deduce

$$\sum_{j=2}^{5} \int_{\Omega_1} x_1 \tilde{c}_{2j} \bar{e}_j \, dx_1 = 0 \quad \text{a.e..} \tag{53}$$

From (51), and observing that the positive definiteness of the elastic tensor implies $\tilde{c}_{55} > 0$, we find

$$\bar{e}_5 = 2E_{13}(\bar{w}) = -\sum_{j=2}^{4} \frac{\tilde{c}_{5j}}{\tilde{c}_{55}}\, \bar{e}_j \quad \text{a.e..} \tag{54}$$

To solve (52) and (53) we need to write explicitly $\bar{e}_2 = E_{22}(\bar{w})$. Since $\bar{w} \in \mathscr{W}$, by Lemma 5.5, we can write

$$\bar{w}_1(x) = \bar{\eta}_1(x_2,x_3), \quad \bar{w}_2(x) = -x_1 \frac{\partial \bar{\eta}_1}{\partial x_2}(x_2,x_3) + \bar{\eta}_2(x_2,x_3),$$

where $\bar{\eta}_1$ and $\bar{\eta}_2$ belong to the appropriate spaces. Since

$$\bar{e}_2 = -x_1 \frac{\partial^2 \bar{\eta}_1}{\partial x_2^2} + \frac{\partial \bar{\eta}_2}{\partial x_2} \tag{55}$$

and using (54), we can rewrite (52) and (53) as

$$\begin{aligned} s_{1;22} \frac{\partial^2 \bar{\eta}_1}{\partial x_2^2} - s_{0;22} \frac{\partial \bar{\eta}_2}{\partial x_2} &= \sum_{j=3}^{4} \fint_{\Omega_1} \hat{c}_{2j} \bar{e}_j \, dx_1, \\ s_{2;22} \frac{\partial^2 \bar{\eta}_1}{\partial x_2^2} - s_{1;22} \frac{\partial \bar{\eta}_2}{\partial x_2} &= \sum_{j=3}^{4} \fint_{\Omega_1} x_1 \hat{c}_{2j} \bar{e}_j \, dx_1, \end{aligned}$$

where we have set

$$\hat{c}_{ij} := \frac{\tilde{c}_{55}\tilde{c}_{ij} - \tilde{c}_{i5}\tilde{c}_{j5}}{\tilde{c}_{55}}, \qquad s_{k;ij} := \fint_{\Omega_1} x_1^k \, \hat{c}_{ij} \, dx_1, \tag{56}$$

for $i, j = 2, 3, 4$ and $k = 0, 1, 2$. From these equations we find

$$\frac{\partial^2 \bar{\eta}_1}{\partial x_2^2} = \frac{1}{s_{0;22}s_{2;22} - s_{1;22}^2} \left(s_{0;22} \sum_{j=3}^{4} \fint_{\Omega_1} x_1 \hat{c}_{2j} \bar{e}_j \, dx_1 - s_{1;22} \sum_{j=3}^{4} \fint_{\Omega_1} \hat{c}_{2j} \bar{e}_j \, dx_1 \right),$$

$$\frac{\partial \bar{\eta}_2}{\partial x_2} = \frac{1}{s_{0;22}s_{2;22} - s_{1;22}^2} \left(s_{1;22} \sum_{j=3}^{4} \fint_{\Omega_1} x_1 \hat{c}_{2j} \bar{e}_j \, dx_1 - s_{2;22} \sum_{j=3}^{4} \fint_{\Omega_1} \hat{c}_{2j} \bar{e}_j \, dx_1 \right),$$

and then by integration $\bar{\eta}_1$ and $\bar{\eta}_2$ (the fact that $s_{0;22}s_{2;22} - s_{1;22}^2 > 0$ can be checked, for instance, by using Hölder's inequality; see Wheeden and Zygmund [13, Chapter 8, Exercise 4]).

According to Remark 5.1 and Lemma 5.2, we let

$$\bar{e}_3 = \bar{\zeta}_3' - x_1\bar{\zeta}_1'' - x_2\bar{\zeta}_2'', \qquad \bar{e}_4 = 2x_1\bar{\vartheta}' + \frac{\partial\bar{\varrho}}{\partial x_2}. \tag{57}$$

Setting

$$S_{mpqr}^{ijkl} := \frac{s_{i;2j}s_{k;2l} - s_{m;2p}s_{q;2r}}{s_{0;22}s_{2;22} - s_{1;22}^2},$$

we then have

$$\begin{aligned} \frac{\partial^2\bar{\eta}_1}{\partial x_2^2} &= S_{0312}^{0213}\big(\bar{\zeta}_3' - x_2\bar{\zeta}_2''\big) - S_{1213}^{0223}\bar{\zeta}_1'' + 2S_{1214}^{0224}\bar{\vartheta}' + S_{0412}^{0214}\frac{\partial\bar{\varrho}}{\partial x_2}, \\ \frac{\partial\bar{\eta}_2}{\partial x_2} &= S_{0322}^{1213}\big(\bar{\zeta}_3' - x_2\bar{\zeta}_2''\big) - S_{2213}^{1223}\bar{\zeta}_1'' + 2S_{1422}^{1224}\bar{\vartheta}' + S_{0422}^{1214}\frac{\partial\bar{\varrho}}{\partial x_2}, \end{aligned}$$

and taking into account the relations (48), (49), (54), (55), and (57), we find

$$\begin{aligned} \sum_{j=1}^{6} c_{ij}\bar{e}_j = {} & \big(\hat{c}_{i3} - x_1\hat{c}_{i2}S_{0312}^{0213} + \hat{c}_{i2}S_{0322}^{1213}\big)\big(\bar{\zeta}_3' - x_2\bar{\zeta}_2''\big) \\ & -\big(x_1\hat{c}_{i3} - x_1\hat{c}_{i2}S_{1213}^{0223} + \hat{c}_{i2}S_{2213}^{1223}\big)\bar{\zeta}_1'' \\ & +2\big(x_1\hat{c}_{i4} - x_1\hat{c}_{i2}S_{1214}^{0224} + \hat{c}_{i2}S_{1422}^{1224}\big)\bar{\vartheta}' \\ & +\big(\hat{c}_{i4} - x_1\hat{c}_{i2}S_{0412}^{0214} + \hat{c}_{i2}S_{0422}^{1214}\big)\frac{\partial\bar{\varrho}}{\partial x_2}, \end{aligned} \tag{58}$$

for $i = 3, 4$. Now let $v \in \mathscr{V}$ and ϑ and ϱ be as in Lemma 5.2. With $\psi \in L^2(\Omega_{23})$ and $\vartheta = 0$ and

$$\varrho := \int_{-a_2/2}^{x_2} \psi(s,\cdot)\,ds - \fint_{\Omega_2}\int_{-a_2/2}^{x_2} \psi(s,\cdot)\,ds\,dx_2,$$

we find from (43) that

$$\fint_{\Omega_1}\sum_{j=1}^{6} c_{4j}\bar{e}_j\,dx_1 = 0.$$

It then follows that

$$\begin{aligned} 0 = {} & \big(s_{0;43} - s_{1;42}S_{0312}^{0213} + s_{0;42}S_{0322}^{1213}\big)\big(\bar{\zeta}_3' - x_2\bar{\zeta}_2''\big) \\ & -\big(s_{1;43} - s_{1;42}S_{1213}^{0223} + s_{0;42}S_{2213}^{1223}\big)\bar{\zeta}_1'' \\ & +2\big(s_{1;44} - s_{1;42}S_{1214}^{0224} + s_{0;42}S_{1422}^{1224}\big)\bar{\vartheta}' \\ & +\big(s_{0;44} - s_{1;42}S_{0412}^{0214} + s_{0;42}S_{0422}^{1214}\big)\frac{\partial\bar{\varrho}}{\partial x_2}, \end{aligned}$$

and provided that the last coefficient $s_{0;44} - s_{1;42}S^{0214}_{0412} + s_{0;42}S^{1214}_{0422} \neq 0$, one finds

$$\frac{\partial \bar{\varrho}}{\partial x_2} = \frac{-1}{s_{0;44} - s_{1;42}S^{0214}_{0412} + s_{0;42}S^{1214}_{0422}} \cdot \Big[\big(s_{0;43} - s_{1;42}S^{0213}_{0312} + s_{0;42}S^{1213}_{0322}\big)\big(\bar{\zeta}_3' - x_2\bar{\zeta}_2''\big) - \big(s_{1;43} - s_{1;42}S^{0223}_{1213} + s_{0;42}S^{1223}_{2213}\big)\bar{\zeta}_1'' + \big(s_{1;44} - s_{1;42}S^{0224}_{1214} + s_{0;42}S^{1224}_{1422}\big)2\bar{\vartheta}'\Big].$$

We may rewrite (58) as

$$\sum_{j=1}^{6} c_{ij}\bar{e}_j = F_{i3}\bar{\zeta}_3' - F_{i2}\bar{\zeta}_2'' - F_{i1}\bar{\zeta}_1'' + F_{i4}\bar{\vartheta}' \tag{59}$$

for $i = 3, 4$, where

$$\begin{aligned}
F_{i1} =\ & (x_1\hat{c}_{i3} - x_1\hat{c}_{i2}S^{0223}_{1213} + \hat{c}_{i2}S^{1223}_{2213}) \\
& - \frac{(\hat{c}_{i4} - x_1\hat{c}_{i2}S^{0214}_{0412} + \hat{c}_{i2}S^{1214}_{0422})\big(s_{1,43} - s_{1,42}S^{0223}_{1213} + s_{0,42}S^{1223}_{2213}\big)}{s_{0,44} - s_{1,42}S^{0214}_{0412} + s_{0,42}S^{1214}_{0422}}, \\
F_{i2} =\ & x_2F_{i3}, \\
F_{i3} =\ & (\hat{c}_{i3} - x_1\hat{c}_{i2}S^{0213}_{0312} + \hat{c}_{i2}S^{1213}_{0322}) \\
& - \frac{(\hat{c}_{i4} - x_1\hat{c}_{i2}S^{0214}_{0412} + \hat{c}_{i2}S^{1214}_{0422})(s_{0,43} - s_{1,42}S^{0213}_{0312} + s_{0,42}S^{1213}_{0322})}{s_{0,44} - s_{1,42}S^{0214}_{0412} + s_{0,42}S^{1214}_{0422}}, \\
F_{i4} =\ & 2(x_1\hat{c}_{i4} - x_1\hat{c}_{i2}S^{0224}_{1214} + \hat{c}_{i2}S^{1224}_{1422}) \\
& -2\frac{(\hat{c}_{i4} - x_1\hat{c}_{i2}S^{0214}_{0412} + \hat{c}_{i2}S^{1214}_{0422})(s_{1,44} - s_{1,42}S^{0224}_{1214} + s_{0,42}S^{1224}_{1422})}{s_{0,44} - s_{1,42}S^{0214}_{0412} + s_{0,42}S^{1214}_{0422}}.
\end{aligned}$$

Let

$$A_{ij}(x_3) := \int_{\Omega_{12}} F_{ij}(\cdot,\cdot,x_3)\,dx_1dx_2$$

and

$$K_{ij}(x_3) := \int_{\Omega_{12}} x_1F_{ij}(\cdot,\cdot,x_3)\,dx_1dx_2, \quad L_{ij}(x_3) := \int_{\Omega_{12}} x_2F_{ij}(\cdot,\cdot,x_3)\,dx_1dx_2.$$

Then, from (43), (44), and (59) we finally deduce the following system of equilibrium differential equations:

$$\begin{cases}
(A_{33}\bar{\zeta}_3' - A_{31}\bar{\zeta}_1'' - A_{32}\bar{\zeta}_2'' + A_{34}\bar{\vartheta}')' = -p_3, \\
(K_{33}\bar{\zeta}_3' - K_{31}\bar{\zeta}_1'' - K_{32}\bar{\zeta}_2'' + K_{34}\bar{\vartheta}')'' = -p_1, \\
(L_{33}\bar{\zeta}_3' - L_{31}\bar{\zeta}_1'' - L_{32}\bar{\zeta}_2'' + L_{34}\bar{\vartheta}')'' = -p_2, \\
2(K_{43}\bar{\zeta}_3' - K_{41}\bar{\zeta}_1'' - K_{42}\bar{\zeta}_2'' + K_{44}\bar{\vartheta}')' = -m,
\end{cases} \tag{60}$$

where

$$p_1 = \left(\int_{\Omega_{12}} x_1 b_3 \, dx_1 dx_2 \right)' + \int_{\Omega_{12}} b_1 \, dx_1 dx_2,$$
$$p_2 = \left(\int_{\Omega_{12}} x_2 b_3 \, dx_1 dx_2 \right)' + \int_{\Omega_{12}} b_2 \, dx_1 dx_2,$$
$$p_3 = \int_{\Omega_{12}} b_3 \, dx_1 dx_2.$$

The system (60) should then be completed with the suitable boundary conditions.

7.1 The homogeneous beam

In the general inhomogeneous and anisotropic case, the torsional, flexional, and extensional problems are all coupled together in the equilibrium differential system (60). A partial decoupling occurs already in the homogeneous fully anisotropic case. Indeed, in this case, from (56), we have

$$s_{0;ij} = \hat{c}_{ij}, \quad s_{1;ij} = 0, \quad s_{2;ij} = \frac{a_1^3}{12} \hat{c}_{ij},$$

and therefore

$$S^{0213}_{0312} = S^{1223}_{2213} = S^{0214}_{0412} = S^{1224}_{1422} = 0, \quad S^{0223}_{1213} = \frac{\hat{c}_{23}}{\hat{c}_{22}}, \quad S^{1213}_{0322} = -\frac{\hat{c}_{23}}{\hat{c}_{22}}, \quad S^{1214}_{1422} = \frac{\hat{c}_{24}}{\hat{c}_{22}},$$

which causes many of the coefficients of the system A_{ij}, K_{ij}, and L_{ij} to be zero. In this case, the system (60) simply rewrites as

$$\begin{cases} A_{33} \bar{\zeta}_3'' = -p_3, \\ (-K_{31} \bar{\zeta}_1'' + K_{34} \bar{\vartheta}')'' = -p_1, \\ -L_{32} \bar{\zeta}_2^{(iv)} = -p_2, \\ 2(-K_{41} \bar{\zeta}_1'' + K_{44} \bar{\vartheta}')' = -m, \end{cases} \tag{61}$$

where

$$A_{33} = a_1 a_2 \left[\frac{\hat{c}_{22}\hat{c}_{33} - \hat{c}_{23}^2}{\hat{c}_{22}} + \frac{(\hat{c}_{22}\hat{c}_{34} - \hat{c}_{23}\hat{c}_{24})^2}{\hat{c}_{22}(\hat{c}_{22}\hat{c}_{44} - \hat{c}_{24}^2)} \right],$$

$$K_{31} = \frac{a_1^3 a_2}{12} \frac{\hat{c}_{22}\hat{c}_{33} - \hat{c}_{23}^2}{\hat{c}_{22}}, \; K_{34} = \frac{a_1^3 a_2}{6} \frac{\hat{c}_{22}\hat{c}_{34} - \hat{c}_{23}\hat{c}_{24}}{\hat{c}_{22}},$$

$$K_{41} = \frac{a_1^3 a_2}{12} \frac{\hat{c}_{22}\hat{c}_{34} - \hat{c}_{23}\hat{c}_{24}}{\hat{c}_{22}}, \; K_{44} = \frac{a_1^3 a_2}{6} \frac{\hat{c}_{22}\hat{c}_{44} - \hat{c}_{24}^2}{\hat{c}_{22}},$$

$$L_{32} = \frac{a_1 a_2^3}{12} \left[\frac{\hat{c}_{22}\hat{c}_{33} - \hat{c}_{23}^2}{\hat{c}_{22}} + \frac{(\hat{c}_{22}\hat{c}_{34} - \hat{c}_{23}\hat{c}_{24})^2}{\hat{c}_{22}(\hat{c}_{22}\hat{c}_{44} - \hat{c}_{24}^2)} \right].$$

Thus for a fully anisotropic but homogeneous beam there is only coupling between twisting and bending in direction 1.

7.2 The homogeneous orthotropic/isotropic beam

When the material is orthotropic a complete decoupling occurs. Indeed, for orthotropic material we have that $c_{ki} = 0$ for $k = 1, 2, 3$ and $i = 4, 5, 6$, and $c_{45} = c_{46} = c_{56} = 0$. It then follows that $K_{34} = K_{41} = 0$ and the system (61) reduces to

$$\begin{cases} A_{33}\bar{\zeta}_3'' = -p_3, \\ -K_{31}\bar{\zeta}_1^{(iv)} = -p_1, \\ -L_{32}\bar{\zeta}_2^{(iv)} = -p_2, \\ 2K_{44}\bar{\vartheta}'' = -m. \end{cases} \tag{62}$$

Finally, if the material is isotropic, that is, if it is orthotropic and $c_{12} = c_{23} = c_{13} =: \lambda$, $c_{44} = c_{55} = c_{66} =: \mu$, and $c_{11} = c_{22} = c_{33} = \lambda + 2\mu$, where λ and μ are the Lamé moduli, then we have that

$$A_{33} = a_1 a_2 E, \quad K_{31} = \frac{a_1^3 a_2}{12} E, \quad 2K_{44} = \frac{a_1^3 a_2}{3} \mu, \quad L_{32} = \frac{a_1 a_2^3}{12} E,$$

where $E := \mu \frac{3\lambda+2\mu}{\lambda+\mu}$ is the Young modulus of the material. Hence, in the isotropic and homogeneous case we recover the usual form of the differential system of equilibrium equations.

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