ENTIRE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^3$ AND A CONJECTURE OF DE GIORGI

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Abstract. In 1978 De Giorgi formulated the following conjecture. *Let u be a solution of $\Delta u = u^3 - u$ in all of $\mathbb{R}^n$ such that $|u| \le 1$ and $\partial_n u > 0$ in $\mathbb{R}^n$. Is it true that all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if $n \le 8$?* Equivalently, does u depend only on one variable? When $n = 2$, this conjecture was proved in 1997 by N. Ghoussoub and C. Gui. In the present paper we prove it for $n = 3$. The question, however, remains open for $n \ge 4$. The results for $n = 2$ and 3 apply also to the equation $\Delta u = F'(u)$ for a large class of nonlinearities F.

1. Introduction

This paper is concerned with the study of bounded solutions of semilinear elliptic equations $\Delta u - F'(u) = 0$ in the whole space $\mathbb{R}^n$, under the assumption that u is monotone in one direction, say, $\partial_n u > 0$ in $\mathbb{R}^n$. The goal is to establish the one-dimensional character or symmetry of u, namely, that u only depends on one variable or, equivalently, that the level sets of u are hyperplanes. This type of symmetry question was raised by De Giorgi in 1978, who made the following conjecture – we quote (3), page 175 of [DG]:

Conjecture ([DG]). *Let us consider a solution $u \in C^2(\mathbb{R}^n)$ of*

$$\Delta u = u^3 - u$$

such that

$$|u| \le 1 \ , \quad \partial_n u > 0$$

in the whole $\mathbb{R}^n$. Is it true that all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if $n \le 8$?

When $n = 2$, this conjecture was recently proved by Ghoussoub and Gui [GG]. In the present paper we prove it for $n = 3$. The conjecture, however, remains open

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in all dimensions $n \geq 4$. The proofs for $n = 2$ and 3 use some techniques in the linear theory developed by Berestycki, Caffarelli and Nirenberg [BCN] in one of their papers on qualitative properties of solutions of semilinear elliptic equations.

The question of De Giorgi is also connected with the theories of minimal hypersurfaces and phase transitions. As we explain later in the introduction, the conjecture is sometimes referred to as "the ε-version of Bernstein problem for minimal graphs". This relation with Bernstein problem is probably the reason why De Giorgi states "at least if $n \leq 8$" in the above quotation.

Most articles dealing with the question of De Giorgi have also considered the conjecture in a slightly simpler version. It consists of assuming that, in addition,

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \qquad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{1.1}$$

Here, the limits are *not* assumed to be uniform in $x' \in \mathbb{R}^{n-1}$. Even in this simpler form, the conjecture was first proved in [GG] for $n = 2$, in the present article for $n = 3$, and it remains open for $n \geq 4$.

The positive answers to the conjecture for $n = 2$ and 3 apply to more general nonlinearities than the scalar Ginzburg-Landau equation $\Delta u + u - u^3 = 0$. Throughout the paper, we assume that $F \in C^2(\mathbb{R})$ and that u is a bounded solution of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$ satisfying $\partial_n u > 0$ in $\mathbb{R}^n$. Under these assumptions, Ghoussoub and Gui [GG] have established that, when $n = 2$, u is a function of one variable only (see section 2 for the proof). Here, the only requirement on the nonlinearity is that $F \in C^2(\mathbb{R})$.

The following are our results for $n = 3$. We start with the simpler case when the solution satisfies (1.1).

Theorem 1.1. *Let u be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad \text{in } \mathbb{R}^3 \tag{1.2}$$

satisfying

$$\partial_3 u > 0 \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad \lim_{x_3 \to \pm\infty} u(x', x_3) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^2. \tag{1.3}$$

Assume that $F \in C^2(\mathbb{R})$ and that

$$F \geq \min\{F(-1), F(1)\} \quad \text{in } (-1, 1). \tag{1.4}$$

Then the level sets of u are planes, i.e., there exist $a \in \mathbb{R}^3$ and $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(a \cdot x) \quad \text{for all } x \in \mathbb{R}^3.$$

Note that the direction a of the variable on which u depends is not known apriori. Indeed, if u is a one-dimensional solution satisfying (1.3), we can "slightly" rotate coordinates to obtain a new solution still satisfying (1.3). Instead, if we further assume that the limits in (1.1) are uniform in $x' \in \mathbb{R}^{n-1}$, then we are imposing an apriori choice of the direction a, namely, $a \cdot x = x_n$. In this respect, it has been established in [GG] for $n = 3$, and more recently in [BBG], [BHM] and [F2] for every dimension n, that if the limits in (1.1) are assumed to be uniform in $x' \in \mathbb{R}^{n-1}$ then u only depends on the variable x_n, that is, $u = u(x_n)$. This result applies to equation (1.2) for various classes of nonlinearities F which always include the Ginzburg-Landau model.

Theorem 1.1 applies to $F'(u) = u^3 - u$ since $F(u) = (1-u^2)^2/4$ is a double-well potential with absolute minima at $u = \pm 1$. For this nonlinearity, the explicit one-dimensional solution (which is unique up to a translation of the independent variable) is given by $\tanh(s/\sqrt{2})$. Hence, in this case the conclusion of Theorem 1.1 is that

$$u(x) = \tanh\left(\frac{a \cdot x - c}{\sqrt{2}}\right) \quad \text{in } \mathbb{R}^3,$$

for some $c \in \mathbb{R}$ and $a \in \mathbb{R}^3$ with $|a| = 1$ and $a_3 > 0$.

The hypothesis (1.4) made on F in Theorem 1.1 is a necessary condition for the existence of a one-dimensional solution as in the theorem; see Lemma 3.2(i). At the same time, most of the equations considered in Theorem 1.1 admit a one-dimensional solution. More precisely, if $F \in C^2(\mathbb{R})$ satisfies $F > F(-1) = F(1)$ in $(-1,1)$ and $F'(-1) = F'(1) = 0$, then $h'' - F'(h) = 0$ has an increasing solution $h(s)$ (which is unique up to a translation in s) such that $\lim_{s\to\pm\infty} h(s) = \pm 1$; see Lemma 3.2(ii).

The following result establishes for $n = 3$ the conjecture of De Giorgi in the form stated in [DG]. Namely, we do not assume that $u \to \pm 1$ as $x_3 \to \pm\infty$. The result applies to a class of nonlinearities which includes the model case $F'(u) = u^3 - u$ and also $F'(u) = \sin u$, for instance.

Theorem 1.2. *Let u be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad \textit{in } \mathbb{R}^3$$

satisfying

$$\partial_3 u > 0 \quad \textit{in } \mathbb{R}^3.$$

Assume that $F \in C^2(\mathbb{R})$ and that

$$F \geq \min\{F(m), F(M)\} \quad \textit{in } (m, M) \tag{1.5}$$

for each pair of real numbers $m < M$ satisfying $F'(m) = F'(M) = 0$, $F''(m) \geq 0$ and $F''(M) \geq 0$. Then the level sets of u are planes, i.e., there exist $a \in \mathbb{R}^3$ and $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(a \cdot x) \quad \textit{for all } x \in \mathbb{R}^3.$$

Our proof of Theorem 1.1 will only require $F \in C^{1,1}(\mathbb{R})$, i.e., F' Lipschitz. However, in Theorem 1.2 we need F' of class C^1.

Question. Do Theorems 1.1 and 1.2 hold for every nonlinearity $F \in C^2$? That is, can one remove hypotheses (1.4) and (1.5) in these results?

The first partial result on the question of De Giorgi was found in 1980 by Modica and Mortola [MM2]. They gave a positive answer to the conjecture for $n = 2$ under the additional assumption that the level sets of u are the graphs of an equi-Lipschitzian family of functions. Note that, since $\partial_n u > 0$, each level set of u is the graph of a function of x'.

In 1985, Modica [M1] proved that if $F \geq 0$ in $\mathbb{R}$ then every bounded solution u of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$ satisfies the gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq F(u) \qquad \text{in } \mathbb{R}^n. \tag{1.6}$$

In 1994, Caffarelli, Garofalo and Segala [CGS] generalized this bound to more general equations. They also showed that, if equality occurs in (1.6) at some point of $\mathbb{R}^n$ then the conclusion of the conjecture of De Giorgi is true. More recently, Ghoussoub and Gui [GG] have proved the conjecture in full generality when $n = 2$ (see also [F3], where weaker assumptions than $\partial_2 u > 0$ and more general elliptic operators are considered).

Under the additional assumption that $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$ uniformly in $x' \in \mathbb{R}^{n-1}$, it is known that u only depends on the variable x_n; here, the hypothesis $\partial_n u > 0$ is not needed. This result was first proved in [GG] for $n = 3$, and more recently in any dimension n by Barlow, Bass and Gui [BBG], Berestycki, Hamel and Monneau [BHM], and Farina [F2]. Their results apply to various classes of nonlinearities F, which always include the Ginzburg-Landau model. These papers also contain related results where the assumption on the uniformity of the limits $u \to \pm 1$ is replaced by various hypotheses on the level sets of u. The paper [BBG] uses probabilistic methods, [BHM] uses the sliding method, and [GG] and [F2] are based on the moving planes method.

Using a one-dimensional arrangement argument, Farina [F1] proved the conclusion $u = u(x_n)$ provided that u minimizes the energy functional in an infinite cylinder $\omega \times \mathbb{R}$ (with ω bounded) among the functions satisfying $v(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$ uniformly in $x' \in \omega$.

Our proof of the conjecture of De Giorgi in dimension 3 proceeds as the proof given in [BCN] and [GG] for $n = 2$. That is, for every coordinate x_i, we consider the function $\sigma_i = \partial_i u / \partial_n u$. The goal is to show that σ_i is constant (then the conjecture follows immediately) and this will be achieved using a Liouville type result (Proposition 2.1 below) for a non-uniformly elliptic equation satisfied by σ_i. The following energy estimate is the key result that will allow us to apply such Liouville type theorem when $n = 3$. This energy estimate holds, however, in all dimensions and for arbitrary $C^2(\mathbb{R})$ nonlinearities.

Theorem 1.3. *Let u be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad \text{in } \mathbb{R}^n,$$

where F is an arbitrary $C^2(\mathbb{R})$ function. Assume that

$$\partial_n u > 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \lim_{x_n \to +\infty} u(x', x_n) = 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}.$$

For every $R > 1$, let $B_R = \{|x| < R\}$. Then,

$$\int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + F(u) - F(1) \right\} dx \leq C R^{n-1}$$

for some constant C independent of R.

The energy functional in B_R,

$$E_R(u) = \int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + F(u) - F(1) \right\} dx,$$

has $\Delta u - F'(u) = 0$ as Euler-Lagrange equation. In 1989, Modica [M2] proved a monotonicity formula for the energy. It states that if

$$F \geq F(1) \quad \text{in } \mathbb{R}$$

and u is a bounded solution of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$, then the quantity

$$\frac{E_R(u)}{R^{n-1}}$$

is a nondecreasing function of R. Theorem 1.3 establishes that this quotient is, in addition, bounded from above. Moreover, the monotonicity formula shows that the upper bound in Theorem 1.3 is optimal: indeed, if $E_R(u)/R^{n-1} \to 0$ as $R \to \infty$ then we would obtain that $E_R(u) = 0$ for any $R > 0$, and hence that u is constant in $\mathbb{R}^n$.

Note that the estimate of Theorem 1.3 is clearly true assuming that u is a one-dimensional solution; see (3.7) in Lemma 3.2(i). The estimate is also easy to prove for u as in Theorem 1.3 under the additional assumption that u is a local minimizer of the energy; see Remark 2.3. In this case, the estimate already appears as a lemma in the work of Caffarelli and Córdoba [CC] on the convergence of intermediate level surfaces in phase transitions. The proof of the estimate for u as in Theorem 1.3 involves a new idea. It originated from the proof for local minimizers and from a relation between the key hypothesis $\partial_n u > 0$ and the second variation of energy; see section 2.

Finally, we recall the heuristic argument that connects the conjecture of De Giorgi with Bernstein problem for minimal graphs. For simplicity let us suppose that $F(u) = (1 - u^2)^2/4$. With u as in the conjecture, consider the blown-down sequence

$$u_\varepsilon(y) = u(y/\varepsilon) \qquad \text{for } y \in B_1 \subset \mathbb{R}^n,$$

and the penalized energy of u_ε in B_1:

$$H_\varepsilon(u_\varepsilon) = \int_{B_1} \left\{ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right\} dy.$$

Note that $H_\varepsilon(u_\varepsilon)$ is a bounded sequence, by Theorem 1.3. As $\varepsilon \to 0$, the functionals H_ε Γ-converge to a functional which is finite only for characteristic functions with values in $\{-1, 1\}$ and equal (up to the multiplicative constant $2\sqrt{2}/3$) to the area of the hypersurface of discontinuity; see [MM1] and [LM]. Heuristically, the sequence u_ε is expected to converge to a characteristic function whose hypersurface of discontinuity S has minimal area or is at least stationary. The set S describes the behavior at infinity of the level sets of u, and S is expected to be the graph of a function defined on $\mathbb{R}^{n-1}$ (since the level sets of u are graphs due to hypothesis $\partial_n u > 0$). The conjecture of De Giorgi states that the level sets are hyperplanes. The connection with the Bernstein problem (see chapter 17 of [G] for a complete survey on this topic) is due to the fact that every minimal graph of a function defined on $\mathbb{R}^m = \mathbb{R}^{n-1}$ is known to be a hyperplane whenever $m \le 7$, i.e., $n \le 8$. On the other hand, Bombieri, De Giorgi and Giusti [BDG] established the existence of a smooth and entire minimal graph of a function of 8 variables different than a hyperplane.

In a forthcoming work [AAC] with Alberti, we will use new variational methods to study the conjecture of De Giorgi in higher dimensions.

In section 2 we prove Theorems 1.1 and 1.3. Section 3 is devoted to establish Theorem 1.2.

2. Proof of Theorem 1.1

To prove the conjecture of De Giorgi in dimension 3, we will use the energy estimate of Theorem 1.3. It is this estimate that will allow us to apply, when $n = 3$, the following Liouville type result for the equation $\nabla \cdot (\varphi^2 \nabla \sigma) = 0$, where $\varphi = \partial_n u$, $\sigma = \partial_i u / \partial_n u$, and $\nabla \cdot$ denotes the divergence operator.

Proposition 2.1. *Let $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ be a positive function. Suppose that $\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies*

$$\sigma \, \nabla \cdot (\varphi^2 \nabla \sigma) \ge 0 \quad \text{in } \mathbb{R}^n \tag{2.1}$$

in the distributional sense. For every $R > 1$, let $B_R = \{|x| < R\}$ and assume that

$$\int_{B_R} (\varphi\sigma)^2 \le C R^2, \tag{2.2}$$

for some constant C independent of R. Then σ is constant.

The study of this type of Liouville property, its connections with the spectrum of linear Schrödinger operators, as well as its applications to symmetry properties of solutions of nonlinear elliptic equations, were developed by Berestycki, Caffarelli and Nirenberg [BCN]. In the papers [BCN] and [GG], this Liouville property was shown to hold under various decay assumptions on $\varphi\sigma$. These hypotheses, which were more restrictive than (2.2), could not be verified when trying to establish the conjecture of De Giorgi for $n \geq 3$. We then realized that hypothesis (2.2) could be verified when (and only when) $n \leq 3$ and that, at the same time, (2.2) was sufficient to carry out the proof of the Liouville property given in [BCN]. For convenience, we include below their proof of Proposition 2.1. See Remark 2.2 for another question regarding this Liouville property.

Before proving Theorem 1.3 and Proposition 2.1, we use these results to give the detailed proof of Theorem 1.1. First, we establish some simple bounds and regularity results for the solution u. We assume that u is a bounded solution of $\Delta u - F'(u) = 0$ in the distributional sense in $\mathbb{R}^n$. It follows that u is of class C^1, and that ∇u is bounded in the whole $\mathbb{R}^n$, i.e.,

$$|\nabla u| \in L^\infty(\mathbb{R}^n). \tag{2.3}$$

Indeed, applying interior $W^{2,p}$ estimates, with $p > n$, to the equation $\Delta u = F'(u) \in L^\infty$ in every ball $B_2(y)$ of radius 2 in $\mathbb{R}^n$, we find that

$$\|u\|_{W^{2,p}(B_1(y))} \leq C\left\{\|u\|_{L^\infty(B_2(y))} + \|F'(u)\|_{L^p(B_2(y))}\right\} \leq C$$

with C independent of y. Using the Sobolev embedding $W^{2,p}(B_1(y)) \subset C^1(\overline{B}_1(y))$ for $p > n$, we conclude (2.3) and that $u \in C^1$.

Next, we verify that

$$u \in W^{3,p}_{\mathrm{loc}}(\mathbb{R}^n) \qquad \text{for all } 1 \leq p < \infty;$$

in particular, we have that

$$u \in C^{2,\alpha}(\mathbb{R}^n) \qquad \text{for all } 0 < \alpha < 1.$$

Indeed, since F' is C^1, and u and ∇u are bounded, we have that $F'(u) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, $\nabla F'(u) = F''(u)\nabla u$, and

$$\Delta\, \partial_j u - F''(u)\, \partial_j u = 0 \tag{2.4}$$

in the weak sense, for every index j. Since $F''(u)\partial_j u \in L^\infty(\mathbb{R}^n) \subset L^p_{\mathrm{loc}}(\mathbb{R}^n)$, we obtain $\partial_j u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof of Theorem 1.1. For each $i \in \{1,2\}$, we consider the functions

$$\varphi = \partial_3 u \qquad \text{and} \qquad \sigma_i = \frac{\partial_i u}{\partial_3 u}.$$

Note that σ_i is well defined since $\partial_3 u > 0$. We also have that σ_i is $C^{1,\alpha}$ (see the remarks made above about the regularity of u) and that

$$\varphi^2 \nabla \sigma_i = \partial_3 u \, \nabla \partial_i u - \partial_i u \, \nabla \partial_3 u.$$

Since the right hand side of the last equality belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^3)$, we can use that $\partial_i u$ and $\partial_3 u$ satisfy the same linearized equation $\Delta w - F''(u)w = 0$ to conclude that

$$\nabla \cdot (\varphi^2 \nabla \sigma_i) = 0$$

in the weak sense in $\mathbb{R}^3$.

Our goal is to apply to this equation the Liouville property of Proposition 2.1. Since

$$\varphi \sigma_i = \partial_i u,$$

condition (2.2) will be established if we show that, for each $R > 1$,

$$\int_{B_R} |\nabla u|^2 \leq C R^2 \tag{2.5}$$

for some constant C independent of R.

Recall that, by assumption, $F \geq \min\{F(-1), F(1)\}$ in $(-1, 1)$. Suppose first that $\min\{F(-1), F(1)\} = F(1)$. In this case we have $F(u) - F(1) \geq 0$ in $\mathbb{R}^3$. Hence, applying Theorem 1.3 with $n = 3$ (it is here and only here that we use $n = 3$), we conclude that

$$\frac{1}{2} \int_{B_R} |\nabla u|^2 \leq \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + F(u) - F(1) \right\} \leq C R^2.$$

This proves (2.5). In case that $\min\{F(-1), F(1)\} = F(-1)$, we obtain the same conclusion by applying the previous argument with $u(x', x_3)$ replaced by $-u(x', -x_3)$ and with $F(v)$ replaced by $F(-v)$.

By Proposition 2.1, we have that σ_i is constant, that is

$$\partial_i u = c_i \, \partial_3 u$$

for some constant c_i. Hence, u is constant along the directions $(1, 0, -c_1)$ and $(0, 1, -c_2)$. We conclude that u is a function of the variable $a \cdot x$ alone, where $a = (c_1, c_2, 1)$. □

When carried out in dimension 2, the previous proof is essentially the one given in [GG] to establish their extended version of the conjecture of De Giorgi for $n = 2$. The proof above shows that every bounded solution u of $\Delta u - F'(u) = 0$ in $\mathbb{R}^2$, with $\partial_2 u > 0$ and $F \in C^2(\mathbb{R})$, is a function of one variable only. Here, no other assumption on F is required, since there is no need to apply Theorem 1.3. Indeed, when $n = 2$, (2.5) is obviously satisfied since ∇u is bounded.

Remark 2.2. In [BCN], the authors raised the following question: does Proposition 2.1 hold for $n \geq 3$ under the assumption $\varphi\sigma \in L^\infty(\mathbb{R}^n)$ – instead of (2.2)? If the answer were yes, then the previous proof would establish the conjecture of De Giorgi in dimension n, since we have that $\varphi\sigma_i = \partial_i u$ is bounded in $\mathbb{R}^n$. However, it has been established by Ghoussoub and Gui [GG] for $n \geq 7$, and later by Barlow [B] for $n \geq 3$, that the answer to the above question is negative.

We turn now to the

Proof of Theorem 1.3. We consider the functions

$$u^t(x) = u(x', x_n + t),$$

defined for $x = (x', x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For each t, we have

$$\Delta u^t - F'(u^t) = 0 \qquad \text{in } \mathbb{R}^n$$

and

$$|u^t| + |\nabla u^t| \leq C \qquad \text{in } \mathbb{R}^n,$$

by (2.3); throughout the proof, C will denote different positive constants independent of R and t. Note also that

$$\lim_{t\to+\infty} u^t(x) = 1 \qquad \text{for all } x \in \mathbb{R}^n.$$

Denoting the derivative of $u^t(x)$ with respect to t by $\partial_t u^t(x)$, we have

$$\partial_t u^t(x) = \partial_n u(x', x_n + t) > 0 \qquad \text{for all } x \in \mathbb{R}^n.$$

We consider the energy of u^t in the ball $B_R = B_R(0)$ defined by

$$E_R(u^t) = \int_{B_R} \left\{ \frac{1}{2}|\nabla u^t|^2 + F(u^t) - F(1) \right\} dx.$$

Note that

$$\lim_{t\to+\infty} E_R(u^t) = 0. \tag{2.6}$$

Indeed, the term $\int_{B_R}\{F(u^t) - F(1)\}$ tends to zero as $t \to +\infty$ by the Lebesgue dominated convergence theorem. To see that the term $\int_{B_R}(1/2)|\nabla u^t|^2$ also tends to zero, we multiply $\Delta u^t - F'(u^t) = 0$ by $u^t - 1$ and we integrate by parts in B_R. We obtain

$$\int_{B_R} |\nabla u^t|^2 = \int_{\partial B_R} \frac{\partial u^t}{\partial \nu}(u^t - 1) - \int_{B_R} F'(u^t)(u^t - 1).$$

Clearly, the last two integrals converge to zero, again by the dominated convergence theorem.

Next, we compute and bound the derivative of $E_R(u^t)$ with respect to t. We use the equation $\Delta u^t - F'(u^t) = 0$, the L^∞ bounds for u^t and ∇u^t, and the crucial fact $\partial_t u^t > 0$. We find that

$$\begin{aligned} \partial_t E_R(u^t) &= \int_{B_R} \nabla u^t \nabla(\partial_t u^t) + \int_{B_R} F'(u^t)\partial_t u^t \\ &= \int_{\partial B_R} \frac{\partial u^t}{\partial \nu} \partial_t u^t \\ &\geq -C \int_{\partial B_R} \partial_t u^t. \end{aligned} \tag{2.7}$$

Hence, for each $T > 0$, we have

$$\begin{aligned} E_R(u) &= E_R(u^T) - \int_0^T dt\ \partial_t E_R(u^t) \\ &\leq E_R(u^T) + C \int_0^T dt \int_{\partial B_R} d\sigma(x)\, \partial_t u^t(x) \\ &= E_R(u^T) + C \int_{\partial B_R} d\sigma(x) \int_0^T dt\, \partial_t[u^t(x)] \\ &= E_R(u^T) + C \int_{\partial B_R} d\sigma(x)\, (u^T - u)(x) \\ &\leq E_R(u^T) + C|\partial B_R| = E_R(u^T) + CR^{n-1}. \end{aligned} \tag{2.8}$$

Letting $T \to +\infty$ and using (2.6), we obtain the desired estimate. □

Now that Theorems 1.3 and 1.1 are proved, we can verify that these results only require F' Lipschitz – instead of $F \in C^2$. The only delicate point to be checked is the linearized equation (2.4), which is then used to derive the equation satisfied by σ_i in the weak sense. To verify (2.4), we use that $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and that F' is Lipschitz. It follows (see Theorem 2.1.11 of [Z]) that $F'(u) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ and $\nabla F'(u) = F''(u)\nabla u$ almost everywhere. Using this, we derive (2.4).

Remark 2.3. Recall that u is said to be a local minimizer if, for every bounded domain Ω, u is an absolute minimizer of the energy in Ω on the class of functions agreeing with u on $\partial\Omega$. It is easy to prove estimate $E_R(u) \leq CR^{n-1}$, for $R > 1$, whenever $u \in L^\infty(\mathbb{R}^n)$ is a local minimizer. We just compare the energy of u with the energy of a function satisfying $v \equiv 1$ in B_{R-1} and $v = u$ on ∂B_R. Take, for instance, $v = \eta + (1-\eta)u$, where $0 \leq \eta \leq 1$ has compact support in B_R and $\eta \equiv 1$ in B_{R-1}. Then

$$\begin{aligned} E_R(u) \leq E_R(v) &= \int_{B_R \setminus B_{R-1}} \left\{ \frac{1}{2}|\nabla v|^2 + F(v) - F(1) \right\} dx \\ &\leq C|B_R \setminus B_{R-1}| \leq CR^{n-1}, \end{aligned}$$

with C independent of R.

This proof suggested to look for an appropriate path connecting u with the constant function 1, in the general case of Theorem 1.3. We have seen that this is given by the solution u itself. Indeed, *sliding* u in the direction x_n, we obtain the path $u^t(x) = u(x', x_n + t)$ connecting u for $t = 0$ and the function 1 for $t = +\infty$ in the ball $B_R = B_R(0)$. Moreover, this path is made by functions which are all solutions of the same Euler-Lagrange equation.

At the same time, it is interesting to observe that the condition $\partial_n u > 0$ forces the second variation of energy in B_R at u (and hence, also at each function u^t in the path) to be nonnegative under perturbations vanishing on ∂B_R. Indeed, $\partial_n u$ is a positive solution of the linearized equation $\Delta\, \partial_n u - F''(u)\, \partial_n u = 0$. By a well known result in the theory of the maximum principle, this implies that the first eigenvalue of the operator $-\Delta + F''(u)$ in every ball B_R is nonnegative (this result will be needed and established in the proof of Lemma 3.1). Therefore,

$$\int_{B_R} |\nabla \xi|^2 + F''(u)\xi^2 \geq 0 \qquad \text{for all } \xi \in C_c^\infty(B_R),$$

which means that the second variation of energy is nonnegative under perturbations vanishing on ∂B_R.

Finally, we present the proof of the Liouville property exactly as given in [BCN].

Proof of Proposition 2.1. Let ζ be a C^∞ function on $\mathbb{R}^+$ such that $0 \leq \zeta \leq 1$ and

$$\zeta = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t \geq 2. \end{cases}$$

For $R > 1$, let

$$\zeta_R(x) = \zeta\left(\frac{|x|}{R}\right) \qquad \text{for } x \in \mathbb{R}^n.$$

Multiplying (2.1) by ζ_R^2 and integrating by parts in $\mathbb{R}^n$, we obtain

$$\begin{aligned} \int \zeta_R^2\, \varphi^2 |\nabla \sigma|^2 &\leq -2 \int \zeta_R\, \varphi^2 \sigma\, \nabla \zeta_R \cdot \nabla \sigma \\ &\leq 2 \left[\int_{\{R<|x|<2R\}} \zeta_R^2\, \varphi^2 |\nabla \sigma|^2 \right]^{1/2} \left[\int \varphi^2 \sigma^2 |\nabla \zeta_R|^2 \right]^{1/2} \\ &\leq C \left[\int_{\{R<|x|<2R\}} \zeta_R^2\, \varphi^2 |\nabla \sigma|^2 \right]^{1/2} \left[\frac{1}{R^2} \int_{B_{2R}} (\varphi\sigma)^2 \right]^{1/2}, \end{aligned}$$

for some constant C independent of R. Using hypothesis (2.2), we infer that

$$\int \zeta_R^2\, \varphi^2 |\nabla \sigma|^2 \leq C \left[\int_{\{R<|x|<2R\}} \zeta_R^2\, \varphi^2 |\nabla \sigma|^2 \right]^{1/2}, \tag{2.9}$$

again with C independent of R. This implies that $\int \zeta_R^2 \varphi^2 |\nabla\sigma|^2 \le C$ and, letting $R \to \infty$, we obtain

$$\int_{\mathbb{R}^n} \varphi^2 |\nabla\sigma|^2 \le C.$$

It follows that the right hand side of (2.9) tends to zero as $R \to \infty$, and hence

$$\int_{\mathbb{R}^n} \varphi^2 |\nabla\sigma|^2 = 0.$$

We conclude that σ is constant. $\square$

3. Proof of Theorem 1.2

To prove Theorem 1.2, we proceed as in the previous section. We need to establish the energy estimate $E_R(u) \le CR^2$. In the definition of $E_R(u)$, we now replace the term $F(1)$ of the previous section by $F(\sup u)$. Looking at the proof of Theorem 1.3, we see that the difficulty arises when trying to show (2.6), i.e., $\lim_{t\to+\infty} E_R(u^t) = 0$ – since we no longer assume $\lim_{x_3\to+\infty} u(x', x_3) = \sup u$ for all x'. Hence, we consider the function

$$\overline{u}(x') = \lim_{x_3\to+\infty} u(x', x_3),$$

which is a solution of the same semilinear equation, but now in $\mathbb{R}^2$. Using a method developed by Berestycki, Caffarelli and Nirenberg [BCN] to study symmetry of solutions in half spaces, we establish a stability property for $\overline{u}$ which will imply that $\overline{u}$ is actually a solution depending on one variable only. As a consequence, we will obtain that the energy of $\overline{u}$ in a two-dimensional ball of radius R is bounded by CR and, hence, that

$$\limsup_{t\to+\infty} E_R(u^t) \le CR^2.$$

Proceeding exactly as in the proof of Theorem 1.3, this estimate will suffice to establish $E_R(u) \le CR^2$ and, under the assumptions made on F, the conjecture. The rest of this section is devoted to give the precise proof of Theorem 1.2.

We start with a lemma that states the stability property of $\overline{u}$ and its consequences.

Lemma 3.1. *Let $F \in C^2(\mathbb{R})$ and let u be a bounded solution of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$ satisfying $\partial_n u > 0$ in $\mathbb{R}^n$. Then, the function*

$$\overline{u}(x') = \lim_{x_n\to+\infty} u(x', x_n) \tag{3.1}$$

is a bounded solution of

$$\Delta\overline{u} - F'(\overline{u}) = 0 \quad \text{in } \mathbb{R}^{n-1} \tag{3.2}$$

and, in addition, there exists a positive function $\varphi \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ *for every* $p < \infty$*, such that*

$$\Delta\varphi - F''(\overline{u})\varphi \leq 0 \quad \text{in } \mathbb{R}^{n-1}. \tag{3.3}$$

As a consequence, if $n = 3$ *then* $\overline{u}$ *is a function of one variable only. More precisely, either*
(a) $\overline{u}$ *is equal to a constant* M *satisfying* $F''(M) \geq 0$*, or*
(b) there exist $b \in \mathbb{R}^2$*, with* $|b| = 1$*, and a function* $h \in C^2(\mathbb{R})$ *such that* $h' > 0$ *in* $\mathbb{R}$ *and*

$$\overline{u}(x') = h(b \cdot x') \quad \text{for all } x' \in \mathbb{R}^2.$$

The following lemma, which is elementary, is concerned with one-dimensional solutions. We will use its first part.

Lemma 3.2. *Let* F *be a* $C^2(\mathbb{R})$ *function.*
(i) Suppose that there exists a bounded function $h \in C^2(\mathbb{R})$ *satisfying*

$$h'' - F'(h) = 0 \quad \text{and} \quad h' > 0 \quad \text{in } \mathbb{R}. \tag{3.4}$$

Let $m_1 = \inf_{\mathbb{R}} h$ *and* $m_2 = \sup_{\mathbb{R}} h$*. Then, we have*

$$F'(m_1) = F'(m_2) = 0, \tag{3.5}$$

$$F > F(m_1) = F(m_2) \quad \text{in } (m_1, m_2), \tag{3.6}$$

and

$$\int_{-\infty}^{+\infty} \left\{ \frac{1}{2} h'(s)^2 + F(h(s)) - F(m_2) \right\} ds < +\infty. \tag{3.7}$$

(ii) Conversely, assume that $m_1 < m_2$ *are two real numbers such that* F *satisfies* (3.5) *and* (3.6)*. Then there exists an increasing solution* h *of* $h'' - F'(h) = 0$ *in* $\mathbb{R}$*, with* $\lim_{s\to-\infty} h(s) = m_1$ *and* $\lim_{s\to+\infty} h(s) = m_2$*. Such solution is unique up to a translation of the independent variable* s*.*

We start with the proof of Lemma 3.1. Here, we employ several ideas taken from section 3 of [BCN].

Proof of Lemma 3.1. The fact that $\overline{u}$ is solution of $\Delta\overline{u} - F'(\overline{u}) = 0$ in $\mathbb{R}^{n-1}$ is easily verified viewing $\overline{u}$ as a function of n variables, limit as $t \to +\infty$ of the functions

$u^t(x', x_n) = u(x', x_n + t)$. By standard elliptic theory, $u^t \to \overline{u}$ uniformly in the C^1 sense on compact sets of $\mathbb{R}^n$.

To check the existence of $\varphi > 0$ satisfying (3.3), we use that

$$\partial_n u > 0 \qquad \text{and} \qquad \Delta\,\partial_n u - F''(u)\,\partial_n u = 0 \quad \text{in } \mathbb{R}^n. \tag{3.8}$$

It is well known in the theory of the maximum principle that (3.8) leads to

$$\int_{\mathbb{R}^n} |\nabla \xi|^2 + F''(u)\xi^2 \geq 0 \qquad \text{for all } \xi \in C_c^\infty(\mathbb{R}^n); \tag{3.9}$$

that is, the first eigenvalue (with Dirichlet boundary conditions) of $-\Delta + F''(u)$ in every bounded domain is nonnegative. Indeed, (3.9) can be easily proved multiplying the equation in (3.8) by $\xi^2/\partial_n u$ – recall that $\partial_n u \in C^{1,\alpha}$ – and integrating by parts to obtain

$$\int \frac{2\xi}{\partial_n u} \nabla \partial_n u \cdot \nabla \xi + F''(u)\xi^2 = \int \frac{\xi^2}{(\partial_n u)^2} |\nabla \partial_n u|^2.$$

Then, (3.9) follows by the Cauchy-Schwarz inequality.

Next, we claim that

$$\int_{\mathbb{R}^{n-1}} |\nabla \eta|^2 + F''(\overline{u})\eta^2 \geq 0 \qquad \text{for all } \eta \in C_c^\infty(\mathbb{R}^{n-1}). \tag{3.10}$$

To show this, we take $\rho > 0$ and $\psi_\rho \in C^\infty(\mathbb{R})$ with $0 \leq \psi_\rho \leq 1$, $0 \leq \psi_\rho' \leq 2$, $\psi_\rho = 0$ in $(-\infty, \rho) \cup (2\rho + 2, +\infty)$, and $\psi_\rho = 1$ in $(\rho + 1, 2\rho + 1)$, and we apply (3.9) with $\xi(x) = \eta(x')\psi_\rho(x_n)$. We obtain, after dividing the expression by $\alpha_\rho = \int \psi_\rho^2$, that

$$\int_{\mathbb{R}^{n-1}} |\nabla \eta(x')|^2 + \int_{\mathbb{R}^{n-1}} \eta^2(x') \int_{\mathbb{R}} \frac{(\psi_\rho')^2(x_n)}{\alpha_\rho} + \int_{\mathbb{R}^{n-1}} \eta^2(x') \int_{\mathbb{R}} F''(u(x', x_n)) \frac{\psi_\rho^2(x_n)}{\alpha_\rho}$$

is nonnegative. Passing to the limit as $\rho \to +\infty$, and using $F \in C^2$ and that $u(x', x_n)$ converges to $\overline{u}(x')$ as $x_n \to +\infty$ uniformly in compact sets of $\mathbb{R}^{n-1}$, we conclude (3.10). This is the crucial point where we need $F \in C^2$, and not only $F \in C^{1,1}$.

Now, (3.10) implies that the first eigenvalue $\lambda_{1,R}$ of $-\Delta + F''(\overline{u})$ in the ball $B_R' = \{x' \in \mathbb{R}^{n-1} : |x'| < R\}$ is nonnegative for every $R > 1$. Let $\varphi_R > 0$ be the corresponding first eigenfunction in B_R':

$$\begin{cases} \Delta\varphi_R - F''(\overline{u})\varphi_R = -\lambda_{1,R}\,\varphi_R & \text{in } B_R' \\ \varphi_R = 0 & \text{on } \partial B_R', \end{cases}$$

normalized such that $\varphi_R(0) = 1$. Note that $\lambda_{1,R} \geq 0$ is decreasing in R and, hence, bounded. Therefore, Harnack inequality gives that φ_R are bounded, uniformly in R, on every compact set of $\mathbb{R}^{n-1}$. By $W^{2,p}$ estimates, it follows that a subsequence of φ_R converges in $W^{2,p}_{\mathrm{loc}}$ to a positive function $\varphi \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^{n-1})$, for every $p < \infty$, satisfying $\Delta\varphi - F''(\overline{u})\varphi \leq 0$ in $\mathbb{R}^{n-1}$ (since $\lambda_{1,R} \geq 0$ for every R).

Finally, assume that $n = 3$. For each $i \in \{1, 2\}$, we consider the function

$$\sigma_i = \frac{\partial_i \overline{u}}{\varphi} \qquad \text{in } \mathbb{R}^2.$$

Note that σ_i is well defined and we have enough regularity to compute:

$$\nabla \cdot (\varphi^2 \nabla \sigma_i) = \varphi\, \Delta \partial_i \overline{u} - \partial_i \overline{u}\, \Delta\varphi,$$

and hence

$$\begin{aligned} \sigma_i \nabla \cdot (\varphi^2 \nabla \sigma_i) &= \partial_i \overline{u}\, \Delta \partial_i \overline{u} - (\partial_i \overline{u})^2 (\Delta\varphi/\varphi) \\ &= (\partial_i \overline{u})^2 F''(\overline{u}) - (\partial_i \overline{u})^2 (\Delta\varphi/\varphi) \\ &\geq 0, \end{aligned}$$

by (3.3).

Next, we apply the Liouville property of Proposition 2.1 to this inequality in $\mathbb{R}^2$. Since $\varphi\sigma_i = \partial_i \overline{u}$ is bounded and the dimension is two, condition (2.2) holds. We obtain that σ_i is constant, that is,

$$\partial_i \overline{u} = c_i \varphi \tag{3.11}$$

for some constant c_i. If $c_1 = c_2 = 0$ then $\overline{u}$ is equal to a constant M. In this case, (3.10) obviously implies that $F''(M) \geq 0$.

If at least one c_i is not zero, then u is constant along the direction $(c_2, -c_1)$, by (3.11). Hence, taking $b = |(c_1, c_2)|^{-1}(c_1, c_2)$, we find that $\overline{u}(x') = h(b \cdot x')$ in $\mathbb{R}^2$ for some function h. Using this relation and (3.11), we see that $c_i\varphi = c_i|(c_1, c_2)|^{-1} h'(b \cdot x')$, and hence $h' > 0$ in $\mathbb{R}$. $\square$

Next, we sketch the proof of Lemma 3.2, which is elementary.

Proof of Lemma 3.2. (i) Multiplying the equation by h' and integrating, we find that $2F(h) - (h')^2 = c$ is constant in $\mathbb{R}$. Since h has finite limits as $s \to \pm\infty$ we obtain

$$\liminf_{s \to \pm\infty} h'(s) = 0, \tag{3.12}$$

whence c is equal to both $2F(m_1)$ and $2F(m_2)$. Since $2F(h) = c + (h')^2 > c$ and the image of h is (m_1, m_2), we infer (3.6). The equalities $F'(m_1) = F'(m_2) = 0$ follow from the equation $h'' - F'(h) = 0$ and from (3.12), using the mean value theorem. Finally, the integral in (3.7) is equal to $\int_{-\infty}^{+\infty} (h')^2\, ds$, which can be estimated by

$$(m_2 - m_1) \sup h' \leq (m_2 - m_1)\sqrt{2D} < +\infty,$$

where $D = \sup_{t\in(m_1,m_2)} F(t) - F(m_1)$.
(ii) Let $m \in (m_1, m_2)$ and let $\phi : (m_1, m_2) \to \mathbb{R}$ be the function

$$\phi(t) = \int_m^t \frac{1}{\sqrt{2F(z) - 2F(m_1)}}\, dz,$$

well defined thanks to (3.6). By (3.5) and $F(m_1) = F(m_2)$, we infer that the image of ϕ is the whole real line, and it is easy to check by integration that the unique increasing solution of $h'' - F'(h) = 0$ in $\mathbb{R}$ satisfying $h(0) = m$ is the inverse function of ϕ. □

Finally, we give the

Proof of Theorem 1.2. Since $\partial_3 u > 0$, the proof of Theorem 1.1 shows that Theorem 1.2 will be established if we prove (2.5) for every $R > 1$, i.e.,

$$\int_{B_R} |\nabla u|^2 \le CR^2$$

for some constant C independent of R.

Let

$$m = \inf_{\mathbb{R}^3} u \qquad \text{and} \qquad M = \sup_{\mathbb{R}^3} u,$$

and consider the functions

$$\underline{u}(x') = \lim_{x_3\to-\infty} u(x', x_3) \quad \text{and} \quad \overline{u}(x') = \lim_{x_3\to+\infty} u(x', x_3).$$

Note that $\underline{u} < \overline{u}$ in $\mathbb{R}^2$, $m = \inf_{\mathbb{R}^2} \underline{u}$, and $M = \sup_{\mathbb{R}^2} \overline{u}$. We apply Lemma 3.1. If $\overline{u}$ is constant then necessarily $\overline{u} \equiv M$, $F'(M) = 0$ by (3.2), and $F''(M) \ge 0$ as stated in Lemma 3.1. In case (b) of Lemma 3.1, we see that the function h satisfies (3.4). Hence, we can apply Lemma 3.2(i) with $m_1 = \inf \overline{u} < m_2 = M = \sup \overline{u}$, and we obtain again that $F'(M) = 0$ and, using (3.6), that $F''(M) \ge 0$. Hence, we have proved that we always have

$$F'(M) = 0 \qquad \text{and} \qquad F''(M) \ge 0.$$

In an analogous way, arguing with $\underline{u}$ (or simply replacing $u(x', x_3)$ by $-u(x', -x_3)$, and $F(v)$ by $F(-v)$), we see that

$$F'(m) = 0 \qquad \text{and} \qquad F''(m) \ge 0.$$

By the hypothesis made on F, it follows that $F \ge \min\{F(m), F(M)\}$ in (m, M). Suppose first that $\min\{F(m), F(M)\} = F(M)$ (the other case reduces to this one, again by the same change of u and F as before). Then, $F(u) - F(M) \geq 0$ in $\mathbb{R}^3$. Hence, the theorem will be proved if we show that

$$\int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + F(u) - F(M) \right\} dx \leq CR^2$$

for each $R > 1$.

To establish this, we proceed as in the proof of Theorem 1.3. That is, we consider the functions $u^t(x) = u(x', x_n + t)$ defined for $x = (x', x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$, and the energy of u^t in the ball $B_R = B_R(0)$, defined now by

$$E_R(u^t) = \int_{B_R} \left\{ \frac{1}{2} |\nabla u^t|^2 + F(u^t) - F(M) \right\} dx.$$

We need to show that $E_R(u) = E_R(u^0) \leq CR^2$. The computations leading to inequalities (2.7) and (2.8) are still valid here – since the extra hypothesis of Theorem 1.1, $\lim_{x_3 \to \pm\infty} u(x', x_3) = \pm 1$, was only used in the proof of Theorem 1.3 to establish (2.6), i.e., $\lim_{t \to +\infty} E_R(u^t) = 0$. Using (2.8) we see that $E_R(u) \leq CR^2$ will hold if we verify

$$\limsup_{t \to +\infty} E_R(u^t) \leq CR^2.$$

This inequality is an easy consequence of Lemmas 3.1 and 3.2(i). Indeed, using standard elliptic estimates and that $u^t(x)$ increases in B_R to $\overline{u}(x')$ as $t \to +\infty$, we have

$$\begin{aligned}\lim_{t \to +\infty} E_R(u^t) &= \int_{B_R} \left\{ \frac{1}{2} |\nabla \overline{u}(x')|^2 + F(\overline{u}(x')) - F(M) \right\} dx \\ &\leq CR \int_{B'_R} \left\{ \frac{1}{2} |\nabla \overline{u}(x')|^2 + F(\overline{u}(x')) - F(M) \right\} dx',\end{aligned}$$

where $B'_R = \{|x'| < R\} \subset \mathbb{R}^2$. But the last integral $\int_{B'_R} \{(1/2)|\nabla \overline{u}(x')|^2 + F(\overline{u}(x')) - F(M)\} \, dx'$, which is computed in a two-dimensional ball, is bounded by CR, since $\overline{u}$ is a function of one variable only (by Lemma 3.1), and in this variable the energy is integrable on all the real line, by (3.7). The proof is now complete. $\square$

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