Convergence analysis for a smeared crack approach in brittle fracture

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Abstract. Our analysis focuses on the mechanical energies involved in the propagation of fractures: the elastic energy, stored in the bulk, and the fracture energy, dissipated in the crack. We consider a finite element model based on a smeared crack approach: the fracture is approximated geometrically by a stripe of elements and mechanically by a softening constitutive law. We define in this way a discrete free energy G_h (being h the element size) which accounts both for elastic displacements and fractures. Our main interest is the behaviour of G_h as $h \to 0$. We prove that, only for a suitable choice of the (mesh dependent) constitutive law, G_h converges to a limit functional G_ϕ with a positive (anisotropic) term concentrated on the crack. We discuss the mesh bias and compute it explicitly in the case of a structured triangulation.

1 Introduction

Smeared cracking is a well known finite element model for simulating strain localization phenomena, among which the propagation of fractures. Starting from the ideas of Rashid [23] this technique has attracted much attention in the field of computational mechanics, being continuously studied and improved till now (see for instance [4], [24] and the references therein). Besides academic research, smeared cracking has been used also for real life applications and has been implemented in some commercial softwares for structural mechanics.

In its "classical" form, the idea consists basically in replacing the fracture with a band of finite elements (see Figure 4). This geometrical approximation is then accompanied with a softening constitutive law of damage type. These two aspects, geometrical and constitutive, characterize the different versions of this approach and have been the main source of investigation and discussion. Different authors has noted that, for some choices of the softening law, when the size of the triangulation becomes too small the model gives a wrong mechanical response where no dissipation occurs. Moreover, often it has been pointed out that the geometry of the mesh introduces an artificial bias which may affect strongly the direction of propagation. These points have been treated and often circumvented in different ways, for instance by means of constitutive laws depending on the mesh size [21], [22] and/or by a non-local approach [17].

Apart from physical considerations, for which we remand to the specific engineering literature, to our knowledge a mathematical treatment of the smeared cracking model was still missing. The point of view adopted here is purely based on the energies related to fracture propagation and mathematically relies strongly on the theories of SBD functions [2] and Γ-convergence [14]. Without entering into technical details, let us explain briefly the main result, starting from the continuum model. We consider a reference two dimensional domain Ω with in-plane, possibly discontinuous, deformations. Denoting by u the displacement field, the fracture will be always identified with the set $J(u)$ where u is discontinuous. We will consider a brittle material and assume cracking to be governed by Griffith's model. Thus, the energy concentrated on the fracture will be of the form $\gamma\,|J(u)|$ (where $|\cdot|$ denotes the length measure and γ is a material parameter) while the energy stored in the bulk will be the linearized elastic energy $W^e(\boldsymbol{\varepsilon}(u))$ (where $\boldsymbol{\varepsilon}(u)$ is the symmetrized gradient of u). Hence, in the continuum setting, the free energy will be of the form

$$G(u) = \int_\Omega W^e\big(\boldsymbol{\varepsilon}(u)\big)\,dx + \gamma\,|J(u)|\,.$$

Fracture propagation can be modeled as an evolution associated with such an energy. Recently, much attention has been payed to this subject, in particular as far as the quasi-static propagation based on minimizing movements, i.e. on sequences of minimizers of G. We refer the interested reader to [15], [11], [16] and [8]. This aspect is out of the purposes of this paper. At the present stage we limit our analysis to the problem of giving a rigorous approximation of G, in terms of a smeared cracking model, which, however, is a prior question to time dependent models.

Our finite element method is defined for the prototype case of piecewise linear elements on a structured mesh $\boldsymbol{T}_h$ and the energy is simply of the form

$$G_h(u_h) = \int_\Omega W_h\big(\boldsymbol{\varepsilon}(u_h)\big)\,.$$

The density W_h is defined by means of a damage constitutive law of the following kind

$$\frac{\partial W_h}{\partial \boldsymbol{\varepsilon}} = \Big[1 - d\big(hW^e(\boldsymbol{\varepsilon})\big)\Big]\boldsymbol{\sigma} = \Big[1 - d\big(hW^e(\boldsymbol{\varepsilon})\big)\Big]\frac{\partial W^e}{\partial \boldsymbol{\varepsilon}}\,,$$

for a suitable choice of the damage function d which governs the transition between the elastic and fracture regimes. We remark that d depends on the elastic energy density through a rescaling factor h which is crucial for our convergence result. As we will see this factor plays the role of the characteristic length in [21]. Denoting by f a primitive of $(1-d)$ clearly we can write

$$W_h(\boldsymbol{\varepsilon}) = \frac{1}{h} f\big(hW^e(\boldsymbol{\varepsilon})\big)\,.$$

Our main interest is the convergence of G_h as the element size h tends to zero, or, in other terms, to understand when G_h is an approximation of G. More precisely, we prove that G_h Γ-converges (as $h \to 0$ tends to zero and with respect to the L^1-topology) to an energy of the form

$$G_\phi(u) = \int_\Omega W^e\big(\boldsymbol{\varepsilon}(u)\big)\,dx + \gamma\int_{J(u)} \phi(\nu)\,ds\,,$$

which reminds a similar formula obtained in [21]. The density ϕ represents the mesh bias due to the geometry of the triangulation $\boldsymbol{T}_h$. Note that ϕ depends on the normal vector ν to the set $J(u)$ and thus it depends on the orientation of the fracture. This term can be explicitly computed in the case of structured triangulations. Unfortunately, for a general mesh it seems not possible to characterize it in a precise way. We recall that a convergence result of this kind, with a slightly different fracture energy, has been obtained in [1] using an "atomistic" approach by finite differences. Finally, in the case of a scale independent law we can shows that the limit energy G would be identically zero. This fact explains, in mathematical terms, why in some cases it has been observed that the dissipation was almost vanishing for small mesh sizes.

From the technical point of view the main difficulty lies in the Γ-liminf inequality. Its proof requires first a suitable application of the slicing technique, to take into account the geometry of the triangulation, and then a "strange" one-dimensional estimate, depending at the same time on two different sections. Unfortunately, these technical aspects, mainly due to symmetrized gradients, makes it difficult to extend our convergence result to the case of non-local and cohesive models. Hopefully these extensions will be the subject for future investigations.

2 Anisotropic limit for structured triangulations

Let Ω be an open bounded Lipschitz set in $\boldsymbol{R}^2$ and let $\{\boldsymbol{T}_h\}$ (for $h > 0$) be a regular family of structured triangulations. For the sake of simplicity we will consider the prototype mesh represented in Figure 1 which is defined on the grid $(h/\sqrt{2})\boldsymbol{Z}^2$ (in such a way that the diameter of the elements is h). We denote by $V_h(\Omega, \boldsymbol{R}^2)$ the finite element set of piecewise affine functions on $\boldsymbol{T}_h$ taking values in $\boldsymbol{R}^2$ and uniformly bounded in L^∞ by some positive constant k. Note that, as k can be arbitrarily large, this constraint does not play any role in the applications; it is indeed a purely technical hypothesis which avoids truncations of the displacement field. In the sequel we will always write V_h instead of $V_h(\Omega, \boldsymbol{R}^2)$ and similarly for other functional spaces.

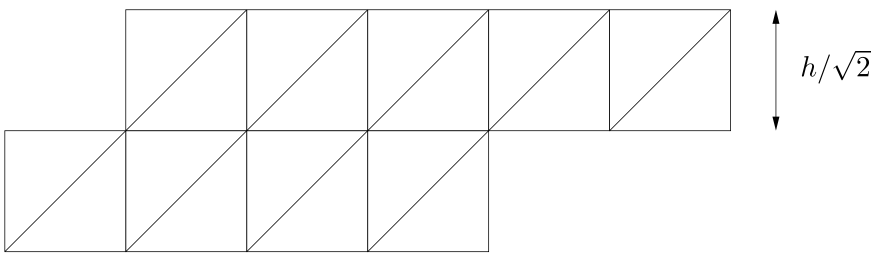

Figure 1. The structure of the triangulation $\boldsymbol{T}_h$.

Let $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(u)$ be the symmetrized gradient and consider the linearized elastic energy density

$$W^e\big(\boldsymbol{\varepsilon}(u)\big) = \mu|\boldsymbol{\varepsilon}(u)|^2 + \tfrac{\lambda}{2}|\mathrm{tr}\,\boldsymbol{\varepsilon}(u)|^2\,,$$

where μ and λ are material parameters (the Lamè constants). Let $f : [0, +\infty) \to \boldsymbol{R}$ be an increasing, continuous function such that

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1 \qquad \lim_{t\to+\infty} f(t) = \gamma\,. \tag{1}$$

Let $f_h(t) = f(ht)/h$ be a rescaling of f and define

$$W_h\big(\boldsymbol{\varepsilon}(u)\big) = f_h\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big)\,.$$

The discrete energy is then given by

$$G_h(u_h) = \begin{cases} \displaystyle\int_\Omega W_h\big(\boldsymbol{\varepsilon}(u_h)\big)\,dx & u_h \in V_h \\ +\infty & u_h \in L^1 \setminus V_h\,. \end{cases} \tag{2}$$

The convergence result for G_h is summarized in the following Theorem.

Theorem 2.1 *The functionals G_h Γ-converge, with respect to the L^1-topology, to the functional*

$$G(u) = \begin{cases} \displaystyle\int_\Omega W^e\big(\boldsymbol{\varepsilon}(u)\big)\,dx + \gamma \int_{J(u)} \phi(\nu)\,ds & u \in SBD^2 \text{ with } \|u\|_{L^\infty} < k \\ +\infty & \text{otherwise in } L^1. \end{cases} \tag{3}$$

The density ϕ, appearing in the length term and depending on the normal ν to $J(u)$, represents the mesh bias and depends only on the discrete geometry of the triangulations (for more details we refer to Section 3). For simplicity we denote by ds the one-dimensional Hausdorff measure, i.e. the length measure.

Finally, if u_h is a family in V_h which is equibounded in energy, namely with $G_h(u_h) \le c$, then u_h is strongly precompact in L^p for every $1 \le p < +\infty$.

Remark 2.2 *Note that the function f, and its rescaling f_h, introduce in the discrete energy a local softening which depends on the size h of the elements. Scaling the constitutive law in this way is necessary to recover in the limit the correct fracture energy. Indeed, consider the scale independent energy*

$$E_h(u) = \begin{cases} \displaystyle\int_\Omega f\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big)\,dx & \text{if } u \in V_h \\ +\infty & \text{if } u \in L^1 \setminus V_h. \end{cases}$$

Then E_h converges to a functional E which is identically equal to zero. This fact has been observed numerically by many authors and usually known as non-objectivity of the triangulation: for $h \to 0$ the deformation localizes in narrow bands while the energy decreases to zero. From the theoretical point of view this effect can be explained in terms of relaxation. More precisely: for $u \in H^1$ let $u_h \in V_h$ converging to u with respect to the H^1-norm. As $f(W^e(M))$ is a Lipschitz function from $\boldsymbol{R}^{2\times 2}$ (the space of 2×2 matrices) into $\boldsymbol{R}$ we get

$$\left| \int_\Omega f\Big(W^e\big(\boldsymbol{\varepsilon}(u_h)\big)\Big)\,dx - f\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big)\,dx \right| \le L \int_\Omega |\boldsymbol{\varepsilon}(u_h) - \boldsymbol{\varepsilon}(u)|^2\,dx\,.$$

Thus

$$\limsup_{h\to 0} E_h(u_h) \le \int_\Omega f\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big)\,dx\,. \tag{4}$$

Let us denote by $\tilde{E}$ the functional

$$\tilde{E}(u) = \begin{cases} \displaystyle\int_\Omega f\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big)\,dx & \text{if } u \in H^1 \\ +\infty & \text{if } u \in L^1 \setminus H^1. \end{cases}$$

Note that, being f bounded from above, $\tilde{E}$ is not lower semicontinuous, with respect to the L^1-topology, and its lower semicontinuous envelope is identically equal to zero. Since the Γ-limit E is lower semicontinuous, by (4) it follows that E is identically zero as well.

The main consequence of Γ-convergence is the convergence of minimizers. Let us consider a free energy of the form

$$F(u) = G_\phi(u) - \int_\Omega lu\,dx = \int_\Omega W_h\big(\boldsymbol{\varepsilon}(u)\big)\,dx + \gamma \int_{J(u)} \phi(\nu)\,ds - \int_\Omega lu\,dx\,,$$

where l is a (dead) load belonging to L^2. It's well known that the equilibrium (Euler-Lagrange) equations reads

$$\begin{cases} -div\big(\boldsymbol{\sigma}(u)\big) = l & \text{in } \Omega \setminus J(u) \\ \boldsymbol{\sigma}(u)\cdot\nu = 0 & \text{in } \partial\Omega \\ \boldsymbol{\sigma}^{\pm}(u)\cdot\nu = 0 & \text{in } J(u), \end{cases}$$

where $\boldsymbol{\sigma}(u) = \partial W^e/\partial\boldsymbol{\varepsilon}(u)$. First, let us look for global minimizers of F in the space SBD^2 with the constraint $\|u\|_\infty \le k$. If u_i is a minimizing sequence for F then $F(u_i) \le c$ and thus we get easily

$$G_\phi(u_i) \le c + \int_\Omega lu_i\,dx \le c + k\|l\|_{L^1}\,.$$

Since $G(u_i)$ is bounded from above, Theorem 2.1 tells that there exists a subsequence u_j such that $u_j \to u$ in L^1. By the lower semicontinuity of G_ϕ it follows that

$$G_\phi(u) - \int_\Omega lu\,dx \le \liminf_{j\to+\infty} G_\phi(u_j) - \int_\Omega lu_j\,dx\,.$$

Thus u is a minimizer of the limit energy G. (Note that in general for these kind of non-convex problems the minimizer is not unique). Now, let $l_h \in V_h$ such that $l_h \to l$ in L^2 and let the discrete energy be defined by

$$F_h(u_h) = G_h(u_h) - \int_\Omega l_h u_h\,dx = \int_\Omega W_h\big(\boldsymbol{\varepsilon}(u_h)\big)\,dx - \int_\Omega l_h u_h\,dx\,,$$

for u_h in the finite element space V_h. Clearly F_h Γ-converges to F with respect to the L^1-topology. Moreover, if u_h is a family of minimizers for F_h then $F_h(u_h) \le c$ and thus $G_h(u_h)$ is bounded. It follows by Theorem 2.1 that u_h is precompact in L^1. Hence there exists a subsequence u_{h_n} of u_h converging to some function u in L^1. By a standard result on Γ-convergence, we know that u is a minimizer of G and that $G_h(u_h) \to G_\phi(u)$. In other terms, the minimizers u_{h_n} in V_{h_n} approximates the minimizer u with respect to the L^1-topology while the energies $G_{h_n}(u_{h_n})$ converges to $G_\phi(u)$.

Finally, we remark that a similar result holds true also in the case of Dirichlet boundary conditions on a subset $\partial_D\Omega$ of the boundary $\partial\Omega$. On the contrary, for technical reason, it is more difficult to understand the behaviour in the case of Neumann conditions.

If f is concave, as it is in most of the applications, by (1) f' is non-increasing and $0 \le f'(t) \le 1$. Therefore, the damage variable $d = (1 - f')$ will be non-decreasing and such that

$$\lim_{t\to0^+} d(t) = 0 \qquad \lim_{t\to+\infty} d(t) = 1\,.$$

Thus we can write

$$\frac{\partial W_h}{\partial\boldsymbol{\varepsilon}} = f'\big(hW^e(\boldsymbol{\varepsilon})\big)\frac{\partial W^e}{\partial\boldsymbol{\varepsilon}} = \Big(1 - d\big(hW^e(\boldsymbol{\varepsilon})\big)\Big)\boldsymbol{\sigma}\,,$$

which is a standard form for a damage constitutive law. Usual choices for f are

$$f(t) = 1 - e^{-t} \qquad f(t) = t \wedge 1 = \begin{cases} t & \text{for } t < 1 \\ 1 & \text{otherwise.} \end{cases}$$

The first, being smooth, is more suitable for numerical computations and resembles the behaviour of quasi-brittle materials, such as concrete. On the contrary the second shows better the underling idea of this approach and introduces a sharp elasto-fracture transition which is typical of brittle materials. In this case the energy density W_h can be written as

$$W_h\big(\varepsilon(u_h)\big) = \begin{cases} W^e\big(\varepsilon(u_h)\big) & \text{if } W^e\big(\varepsilon(u_h)\big) < 1/h \\ \\ 1/h & \text{if } W^e\big(\varepsilon(u_h)\big) \geq 1/h. \end{cases}$$

The first regime presents clearly the elastic behaviour. The second accounts for fractures, indeed in this case the local energy is simply $|T|/h = h/4$. This quantity represents the (anisotropic) length of a crack "embedded" in the element T and resembles a similar argument of [21]. Note that this is the only energy "stored" in T and thus, once a fracture is created, the element is considered traction free, in agreement with the equilibrium (Euler-Lagrange) equations. Finally we remark that the fracture criterion depends on the element size h and scales like $1/h$. This order of convergence is in accordance with the behaviour of the strain in the vicinity of the crack tip. Indeed, it is well known that in the continuous model the strain field u has a singularity of the form $\sqrt{r}$ (in the standard system (r, θ) of polar coordinates centred in the crack tip). In the discrete setting, considering the Lagrange interpolation u_h of u, the elastic energy density W^e will be just of the form $1/h$.

Finally let us remark that the isotropic fracture energy appearing in Griffith's model can be written as

$$\gamma |J(u)| = \gamma \int_{J(u)} |\nu| \, ds = \gamma \int_{J(u)} ds \,,$$

where $|\nu|$ denotes the usual Euclidean norm. It is clear that ϕ measures the distortion from the isotropic case due to the triangulation $\boldsymbol{T}_h$.

3 Definition and properties of the mesh bias

3.1 Anisotropy for the structured triangulation

The anisotropy function ϕ, appearing in (3), can be defined in several equivalent ways. Let $\widehat{T}$ be a triangle of $\boldsymbol{T}_h$ for $h = 1$, i.e. with diameter equal to one, and let v be a vector in $\boldsymbol{R}^2$. For every $\xi \in \boldsymbol{S}^1$ (the unit circle) let τ_ξ be the height of $\widehat{T}$ in direction ξ (see Figure 3). Then we can define

$$\phi(v) = \sup\left\{\tau_\xi \langle v, \xi \rangle \text{ for } \xi \in \boldsymbol{S}^1 \right\}. \tag{5}$$

As a matter of fact $\phi(v)$ can be written more easily considering only a finite number of vectors ξ. Indeed $\xi \mapsto \tau_\xi \xi$ is a one to one mapping of $\boldsymbol{S}^1$ in ∂H, the boundary of the hexagon H obtained by the union of the triangles $\widehat{T}$ with a common vertex in the origin. Then we can write

$$\phi(v) = \sup\left\{\langle v, y \rangle \text{ for } y \in \partial H \right\} = \sup\left\{|\langle v, y \rangle| \text{ for } y \in \partial H \right\}, \tag{6}$$

which is clearly equivalent to (5). As H is a convex set, we get $\phi(v) = \sup\{\langle v, y_i\rangle\}$ where y_i are the vertices of H. Writing $y_i = \tau_i \xi_i$ we get

$$\phi(v) = \sup\{\tau_i |\langle v, \xi_i\rangle| \text{ for } i = 1, \ldots, 3\}, \tag{7}$$

where

$$\begin{array}{lll} \xi_1 = (1,0) & \xi_2 = (\sqrt{2}/2, \sqrt{2}/2) & \xi_3 = (0,1) \\ \tau_1 = \sqrt{2}/2 & \tau_2 = 1 & \tau_3 = \sqrt{2}/2\,. \end{array}$$

It follows that ϕ is a norm in $\boldsymbol{R}^2$. Its unit ball $\{\phi(v) = 1\}$ is represented in Figure 2 and shows clearly a strong dependence on the orientation. Finally, from (6) we can regard $\phi(\nu)$ (for $\nu \in \boldsymbol{S}^1$) as the one dimensional measure of $\widehat{T}_\nu$, the projection of $\widehat{T}$ on the subspace $\langle \nu \rangle$.

Remark 3.1 *Whenever the topology of the mesh is similar to that of Figure 1 it is possible to compute the associated anisotropy function ϕ. More precisely, assume that the triangulation has the following periodicity: the union of the elements with a common vertex is an hexagon H which, up to translations, is independent of the vertex itself. Under this hypothesis ϕ can be computed as in (6) or (7). For instance, considering a mesh of equilateral triangles, the set H will be regular hexagon and then the unit ball of the anisotropy ϕ will be a regular hexagon as well (see [18]).*

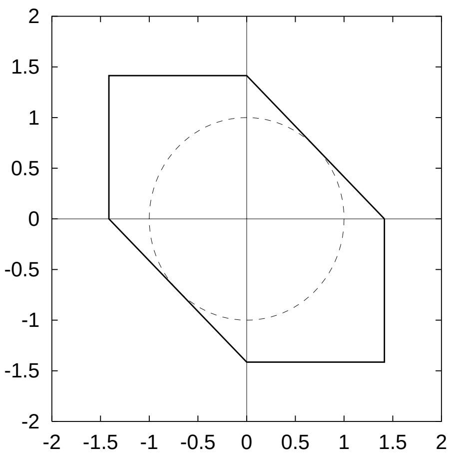

Figure 2. The anisotropic unit ball $\{\phi(v) = 1\}$ compared with the unit circle $|v| = 1$.

The definition of ϕ given in (5) is designed mainly for the proof of the Γ-liminf inequality. For the Γ-limsup inequality we will use instead the following property.

Lemma 3.2 *Let J be a segment in $\boldsymbol{R}^2$ with normal ν. Let h_n be an infinitesimal sequence. Assume that for every h_n the set J does not contain any vertex of the triangulation $\boldsymbol{T}_{h_n}$ and define the covering J_{h_n} as the union of the triangles T of $\boldsymbol{T}_{h_n}$ which intersect J (see Figure 3). Then*

$$\limsup_{n\to+\infty} \frac{|J_{h_n}|}{h_n} = \phi(\nu)\mathcal{H}^1(J)\,. \tag{8}$$

Proof. Let $\zeta \in \boldsymbol{S}^1$ such that $\phi(\nu) = \tau_\zeta \langle \zeta, \nu \rangle$ (see Figure 3). For $z \in J$ let J_h^z be the one dimensional section $\{y \in J_h \,:\, y = z + s\zeta \text{ for } s \in \boldsymbol{R}\}$. Since ζ is aligned with the lattice $(h/\sqrt{2})\boldsymbol{Z}^2$ it is not difficult to check that the measure of J_h^z does not depend on z and is equal to $\tau_\zeta h$. Thus we have $|J_{h_n}| = \mathcal{H}^1(J)\tau_\zeta h_n \langle \zeta, \nu \rangle + O(h_n^2)$, where $O(h_n^2)$ takes into account the behaviour close to the endpoints of J. As $h_n \to 0$ we get (8). ∎

An alternative proof can be found also in [18] Lemma 3.3.

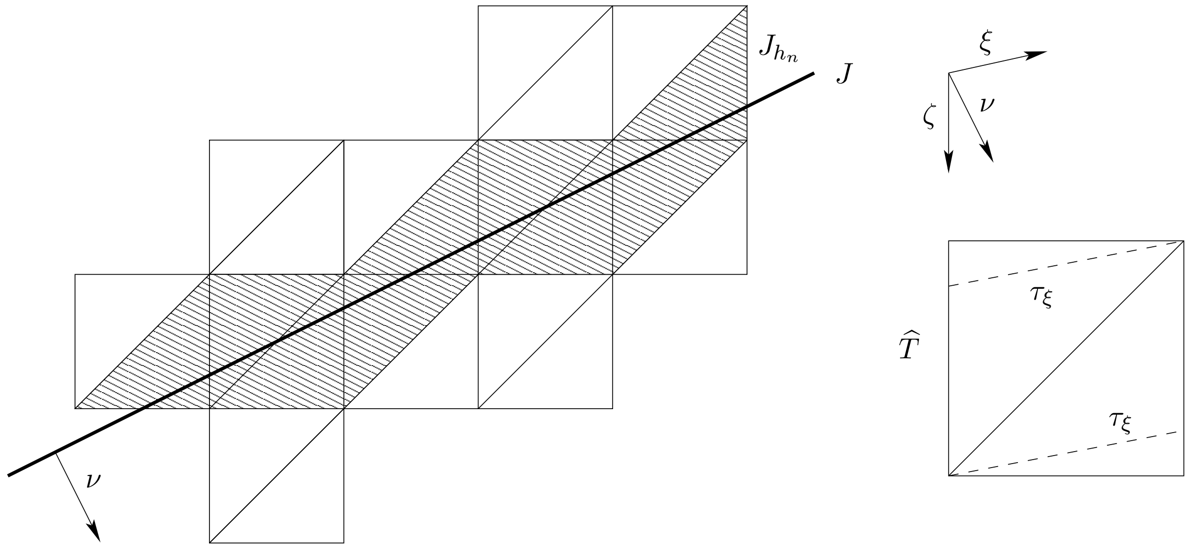

Figure 3. The covering J_{h_n} of a segment J and the height τ_ξ.

Now, let us consider the example represented in Figure 4. A fracture set $J(u)$ is plotted with two triangulations $\boldsymbol{T}_h$ having different sizes. The elements covering the fracture give the smeared representation of $J(u)$. We can clearly see that such coverings form two neighbourhood of $J(u)$ which are not uniform and depend strongly on the orientation of the curve (i.e. on its normal ν). The difference is really evident comparing the behaviour close to the endpoints of $J(u)$. As shown in the right picture, this phenomenon does not change when the size becomes smaller. In particular it will not disappear as $h \to 0$ and in the limit it will generate the anisotropic density ϕ.

Finally, we remark that in the case of a general family of regular triangulations, it is not possible to find an anisotropy function. Indeed, the regularity property does not give any information on the orientation of the elements, which by contrast is fundamental for the definition of the covering and then in the computation of (8). A reduction of the mesh bias can be obtained when the coverings J_h are close to a uniform (tubular) neighbourhood of J. This is the case when we employ a non-local functional like

$$G_h(u_h) = \int_\Omega f_h\Big(W^e\big(\varepsilon(u_h)\big) * \rho_h\Big)\,.$$

The non-local operator, with weight ρ_h and radius r_h, is denoted by $W^e\big(\varepsilon(u_h)\big) * \rho_h$ (a convolution with kernel ρ_h) while the mesh size h is assumed to be sufficiently smaller than r_h. Often this radius is called "internal length" and is consider a physical parameter depending on the properties of the material at the meso scale. It seems interesting to understand the behaviour when the internal length is very small, namely when $r_h \to 0$: from the mechanical

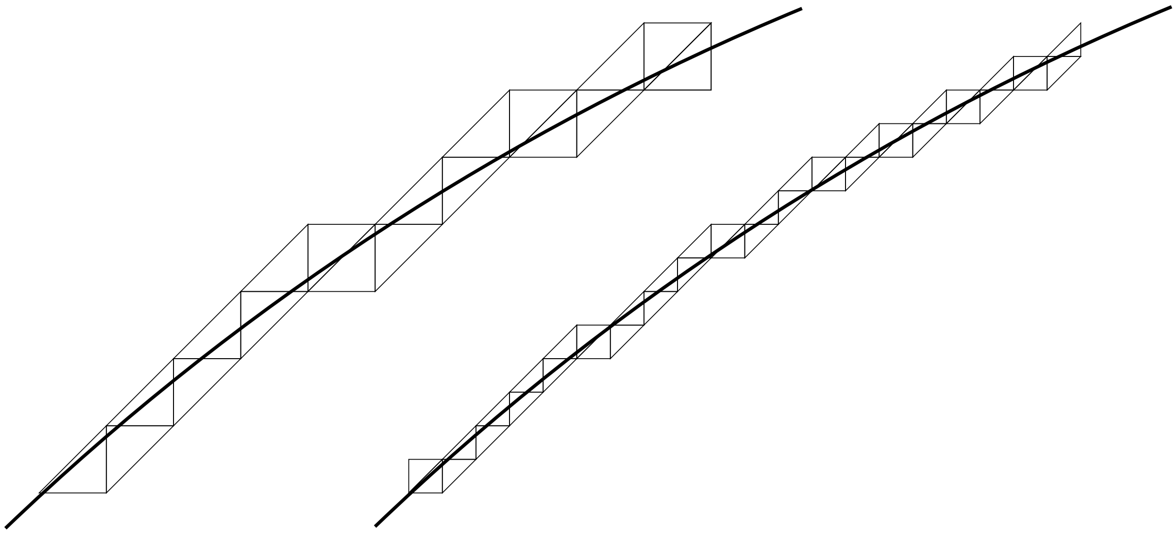

Figure 4. A fracture set and its smeared representation for two mesh sizes.

point of view we will expect a transition between a quasi-brittle and a brittle material, while from the mathematical point of view we will be able to estimate the mesh bias. More precisely, let us assume that ρ_h is obtained by $\rho_h(z) = \rho(z/h)/h$. Now, let $f_h(t) = f(r_h t)/r_h$. Note that here the scaling factor is r_h. It is reasonable to expect that for $r_h \to 0$ the energy G_h will converge again to a functional G_ϕ like (2) where the anisotropy ϕ will be closer to the Euclidean norm.

4 The Γ-limsup inequality

For our type of functional the Γ-limsup inequality is almost straightforward and based on standard arguments. For completeness we report in this section a short sketch, which touches the main points of the proof, while we refer to [10] or [18] for the details in a similar case. Let us consider a sequence $h_n \to 0$ and the discrete functionals

$$G_n(u) = \begin{cases} \displaystyle\int_\Omega W_n\big(\boldsymbol{\varepsilon}(u)\big)\, dx & u_n \in V_n \\ +\infty & u_n \in L^1 \setminus V_n\,. \end{cases}$$

(for simplicity in the notation the subscript h_n is replaced by n).

Thanks to a recent density result [12] (see also Proposition A.3) it is sufficient to take into account a class of very regular functions of bounded deformation, namely those u belonging to the Sobolev space $W^{k,\infty}(\Omega \setminus J(u))$ (for k arbitrarily large) and such that $J(u)$ is the union of the disjoint segments J^i (for $i = 1, \ldots, m$). By a simple translation argument, it is not restrictive to assume that $J(u)$ does not contain any vertex of the triangulations $\boldsymbol{T}_n$. In this way we can define u_n as the Lagrange interpolation of u in the space V_n. We denote by J_n^i the coverings of J^i as defined in the previous section and let J_n be their union. Clearly we can write that

$$\limsup_{n\to+\infty} G_n(u_n) \le \limsup_{n\to+\infty} \int_{\Omega\setminus J_n} W_n\big(\boldsymbol{\varepsilon}(u_n)\big)\, dx + \limsup_{n\to+\infty} \int_{J_n} W_n\big(\boldsymbol{\varepsilon}(u_n)\big)\, dx\,.$$

As $|J_n| \to 0$, by the regularity of u and classical results on finite element interpolation, the first limit gives the bulk energy, i.e.

$$\limsup_{n\to+\infty} \int_{\Omega\setminus J_n} W_n\big(\boldsymbol{\varepsilon}(u_n)\big)\, dx \le \int_{\Omega\setminus J(u)} W^e\big(\boldsymbol{\varepsilon}(u_n)\big)\, dx\,.$$

The second limit will give the fracture energy. Indeed, as $W_n\big(\boldsymbol{\varepsilon}(u_n)\big) \le \gamma$, by Lemma 3.2 we can write

$$\begin{aligned}\limsup_{n\to+\infty} \int_{J_n} W_n\big(\boldsymbol{\varepsilon}(u_n)\big)\, dx &\le \sum_{i=1}^m \limsup_{n\to+\infty} \int_{J_n^i} W_n\big(\boldsymbol{\varepsilon}(u_n)\big)\, dx \\ &\le \gamma \sum_{i=1}^m \limsup_{n\to+\infty} \frac{|J_n^i|}{h_n} \\ &\le \gamma \sum_{i=1}^m \phi(\nu)\mathcal{H}^1(J^i) = \gamma \int_{J(u)} \phi(\nu)\, ds\,.\end{aligned}$$

It follows easily that $\limsup_{n\to+\infty} G_n(u_n) \le G_\phi(u)\,.$

5 The Γ-liminf inequality

The proof of the Γ-liminf is based on a measure theoretic argument (Lemma A.4) which allows to consider separately the estimate for the bulk and surface energy. As it is not restrictive we will consider again the sequence of functionals G_n defined in the previous section. Moreover, for every open subset A of Ω, the localized functionals will be defined as

$$G_n(u,A) = \begin{cases} \displaystyle\int_A W_n\big(\boldsymbol{\varepsilon}(u)\big)\, dx & u \in V_n \\ +\infty & u \in L^1 \setminus V_n\,, \end{cases} \tag{9}$$

$$G(u,A) = \begin{cases} \displaystyle\int_A W^e\big(\boldsymbol{\varepsilon}(u)\big)\, dx + \int_{J(u)\cap A} \phi(\nu)\, ds & u \in SBD^2 \\ +\infty & u \in L^1 \setminus SBD^2\,. \end{cases} \tag{10}$$

5.1 Estimate for the bulk energy and compactness

For $0<\delta<1$ let $\delta' > 0$ such that $f(t) \ge (1-\delta)t \wedge \delta'$. For $A \subset \Omega$ let $A_\delta = \{x \in A : d(x, \partial A) > \delta\}$. First we will show that (if n is sufficiently large) for every $u \in V_n$ there exists $v \in SBD$ (depending on δ) such that $\|v-u\|_{L^1} \le \|u\|_{L^\infty} h_n G_n(u,A)/\delta'$ and

$$(1-\delta)\int_{A_\delta} W^e\big(\boldsymbol{\varepsilon}(v)\big)\, dx + c\,\delta'\mathcal{H}^1(J(v)\cap A_\delta) \le G_n(u_n, A)\,, \tag{11}$$

where the parameter $c > 0$ depends only on the geometry of the mesh. From this estimate we will prove both compactness and the lower inequality for the bulk energy.

The proof of (11) follows closely [10] and thus we will write a short sketch. Let $f^\delta(t) = (1-\delta)t \wedge \delta'$ and $f_n^\delta(t) = f^\delta(h_n t)/h_n = (1-\delta)t \wedge (\delta'/h_n)$. Then

$$W_n\big(\boldsymbol{\varepsilon}(u)\big) = f_n\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big) \ge f_n^\delta\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big)\,.$$

Using the function f_n^δ the elastic and fracture regimes are clearly distinguished. Indeed

$$f_n^\delta\Big(W^e\big(\varepsilon(u)\big)\Big) = \begin{cases} (1-\delta)W^e\big(\varepsilon(u)\big) & \text{if } \ h_n(1-\delta)W^e\big(\varepsilon(u)\big) < \delta' \\ \\ \delta'/h_n & \text{if } \ h_n(1-\delta)W^e\big(\varepsilon(u)\big) \geq \delta'. \end{cases}$$

According to this behaviour we introduce the (relatively close) set $A_n^\sharp$ which represents the smeared fracture and is obtained by the intersection of A with all the triangles T where $h_n(1-\delta)W^e\big(\varepsilon(u)\big) \geq \delta'$. Finally we define $A_n^\flat = A \setminus A_n^\sharp$ and using this notation we write the lower bound

$$G_n(u,A) \geq (1-\delta)\int_{A_n^\flat} W^e\big(\varepsilon(u)\big)\,dx + \delta'|A_n^\sharp|/h_n\,. \tag{12}$$

Let the function v be defined as

$$v = \begin{cases} u \text{ in } A_n^\flat \\ 0 \text{ otherwise.} \end{cases}$$

Obviously $v \in SBD^2$ and $J(v) \subset \partial A_n^\sharp$. Since $v = u$ in $A_n^\flat$, from (12) we get easily $\|v-u\|_{L^1} \leq \|u\|_{L^\infty} h_n G_n(u,A)/\delta'$ while inequality (11) will be obtained estimating the $\mathcal{H}^1$-measure of $J(v)$ in terms of $|A_n^\sharp|/h_n$. Clearly $J(v)$ is covered by the boundaries of the elements T which form $A_n^\sharp$. For a suitable constant c (independent of h_n) we can write $|T| = ch_n\mathcal{H}^1(\partial T)$ for every triangle T. Therefore we could get (11) from (12) if $|T\cap A| = ch_n\mathcal{H}^1(\partial T\cap A)$. Unfortunately this bound is not always true, indeed it may happen that an element T crosses the boundary of A in such a way that $|T\cap A|$ is less than $ch_n\mathcal{H}^1(\partial T\cap A)$. This small difficulty can be easily overcome considering the estimate only in the set A_δ and choosing n sufficiently large, in such a way that $h_n < \delta$.

Now we can prove the liminf inequality for the bulk energy. Let $u_n \in V_n$ converging to u in $L^1(\Omega)$. For an open subset A of Ω, after passing to a subsequence (which is not relabelled) we can assume without loss of generality that

$$\liminf_{n\to+\infty} G_n(u_n,A) = \lim_{n\to+\infty} G_n(u_n,A) < +\infty\,. \tag{13}$$

Now, for $\delta > 0$ let $v_n \in SBD$ satisfying (11). Since $\|v_n - u_n\|_{L^1} \leq \|u_n\|_{L^\infty} h_n G_n(u_n,A)/\delta'$ we get $v_n \to u$ in $L^1(\Omega)$. Then for every open set $A \subset \Omega$ by lower semicontinuity we obtain

$$(1-\delta)\int_{A_\delta} W^e\big(\varepsilon(u)\big)\,dx \leq \liminf_{n\to+\infty}\,(1-\delta)\int_{A_\delta} W^e\big(\varepsilon(v_n)\big)\,dx \leq \liminf_{n\to+\infty} G_n(u_n,A)\,.$$

Taking the supremum as $\delta \searrow 0$ we get the localized lower inequality for the bulk energy:

$$\int_A W^e\big(\varepsilon(u)\big)\,dx \leq \liminf_{n\to+\infty} G_n(u_n,A)\,. \tag{14}$$

Finally we can prove compactness. Let $u_n \in V_n$ be a sequence with bounded energy. For $\delta > 0$ let $v_n \in SBD$ satisfying (11) for $A = \Omega$, namely

$$(1-\delta)\int_{\Omega_\delta} W^e\big(\varepsilon(v_n)\big)\,dx + c\,\delta'\mathcal{H}^1(J(v_n)\cap\Omega_\delta) \leq G_n(u_n) \leq C\,.$$

By [5] it follows that v_n is precompact in $L^1(A_\delta)$. Since $\|v_n - u_n\| \leq h_n C/\delta'$ the sequence u_n is precompact in $L^1(\Omega_\delta)$. As δ is arbitrarily small and u_n is uniformly bounded in $L^\infty(\Omega)$ it follows easily that u_n is precompact in $L^1(\Omega)$.

5.2 Slicing estimate for the fracture energy

Let us consider a sequence $u_n \in V_n$ converging to u in $L^1(\Omega)$ and satisfying (13). In this section we have to prove that

$$\liminf_{n\to+\infty} G_n(u_n, A) \geq \gamma \int_{J(u)\cap A} \phi(\nu)\, ds\,.$$

The first step will be the following: for every vector of the form $\xi = \zeta/|\zeta|$, where $\zeta \in \boldsymbol{Z}^2$, we have

$$\liminf_{n\to+\infty} G_n(u_n, A) \geq \gamma \int_{J^\xi(u)\cap A} \eta_\xi |\langle \nu, \xi\rangle|\, ds\,, \tag{15}$$

for a suitable choice of $\eta_\xi \leq \tau_\xi$ to be chosen later. As a matter of fact it will be sufficient to show that

$$\liminf_{n\to+\infty} \int_A f_n\big(\mu|D_\xi u_n^\xi|^2\big)\, dx \geq \gamma \int_{J^\xi(u)\cap A} \eta_\xi |\langle \nu, \xi\rangle|\, ds\,.$$

Indeed by Proposition A.2 we get

$$W^e\big(\boldsymbol{\varepsilon}(u)\big) \geq \mu|\boldsymbol{\varepsilon}(u)|^2 \geq \mu\,\|\boldsymbol{\varepsilon}(u)\|^2 \geq \mu|\langle \xi, \boldsymbol{\varepsilon}(u)\,\xi\rangle|^2 = \mu|D_\xi u^\xi|^2$$

and then, being f_n non-decreasing, we can write

$$W_n\big(\boldsymbol{\varepsilon}(u)\big) = f_n\Big(W^e\big(\boldsymbol{\varepsilon}(u)\big)\Big) \geq f_n\big(\mu|D_\xi u^\xi|^2\big)\,.$$

As anticipated, in this section we will use the slicing technique and thus we will need some more notation (see Figure 6 and 7). For $\xi = \zeta/|\zeta|$ and $\zeta \in \boldsymbol{Z}^2$ let $\xi' \in \{\xi_i\}$ such that $|\langle \xi, \xi'\rangle| = \min_i |\langle \xi, \xi_i\rangle|$. (The vector ξ' will replace $\xi^\perp$ in the slicing estimates). Let us denote by $Y = \langle \xi\rangle$ and by $Z = \langle \xi'\rangle$ the subspaces generated by ξ and ξ'. Now, considering a square Q_n obtained by the union of two triangles $T \in \boldsymbol{T}_n$ (see Figure 5), we define h_n^ξ and b_n^ξ as the lengths of the section of Q_n in direction ξ and ξ' respectively. Then $2|T| = h_n^\xi b_n^\xi c_\xi$ where $c_\xi = |\langle \xi', \xi^\perp\rangle|$. Note that the intersection between the set of knots $(h_n/\sqrt{2})\boldsymbol{Z}^2$ and the subspace Z will be a uniform grid of the form $b_n^\xi \boldsymbol{Z}$.

At this point, for the sake of clarity, it is better to consider separately the easier case $\xi = \xi_i$ (for $i = 1, ..., 3$) and the general case $\xi \neq \xi_i$.

5.2.1 Slicing estimate for $\boldsymbol{\xi = \xi_i}$

Let A be a open subset of Ω. For $z \in Z$ let $A^{\xi,z}$ be the section $\{x \in A \,:\, x = z + t\xi \text{ for } t \in \boldsymbol{R}\}$. For $z \in b_n^\xi \boldsymbol{Z}_n$ we define $A_n^{\xi,z}$ as the stripe obtained by the union of the squares $Q_n^{\xi,z}$ which are contained in A and which covers the section $A^{\xi,z}$ as represented in Figure 6. As the sets $A_n^{\xi,z}$ are pairwise disjoint we can write

$$\int_A f_n\big(\mu|D_\xi u_n^\xi|^2\big)\, dx \geq \sum_{z\in b_n^\xi \boldsymbol{Z}} \int_{A_n^{\xi,z}} f_n\big(\mu|D_\xi u_n^\xi|^2\big)\, dx\,.$$

For every set $A_n^{\xi,z}$ let us denote by $l_n^{\xi,z}$ and $r_n^{\xi,z}$ the restrictions of u_n^ξ respectively on $A^{\xi,z} \cap A_n^{\xi,z}$ and $A^{\xi,z+b_n^\xi} \cap A_n^{\xi,z}$. By abuse of notation, we will consider $l_n^{\xi,z}$ and $r_n^{\xi,z}$ as defined on a lattice of the form ih_n^ξ for $i = 0, ..., k_n$ (where k_n may depend on z).

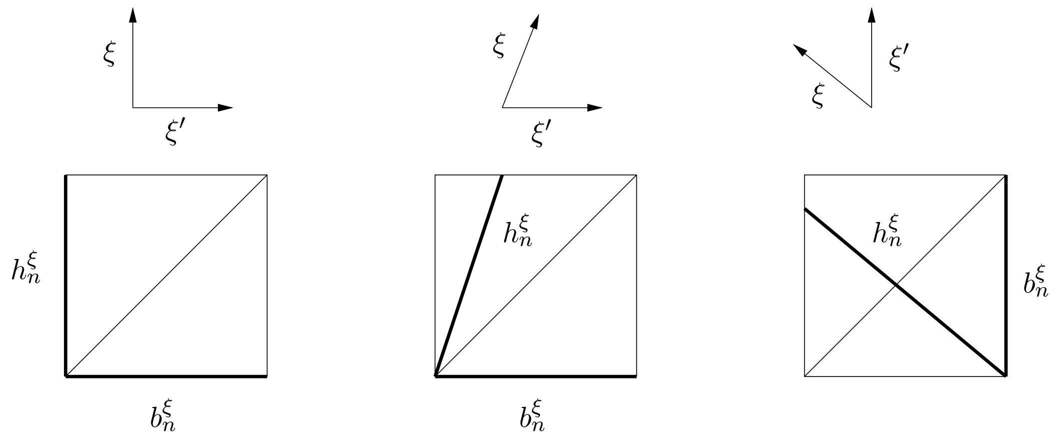

Figure 5. Some examples for the definition of h_n^ξ and b_n^ξ.

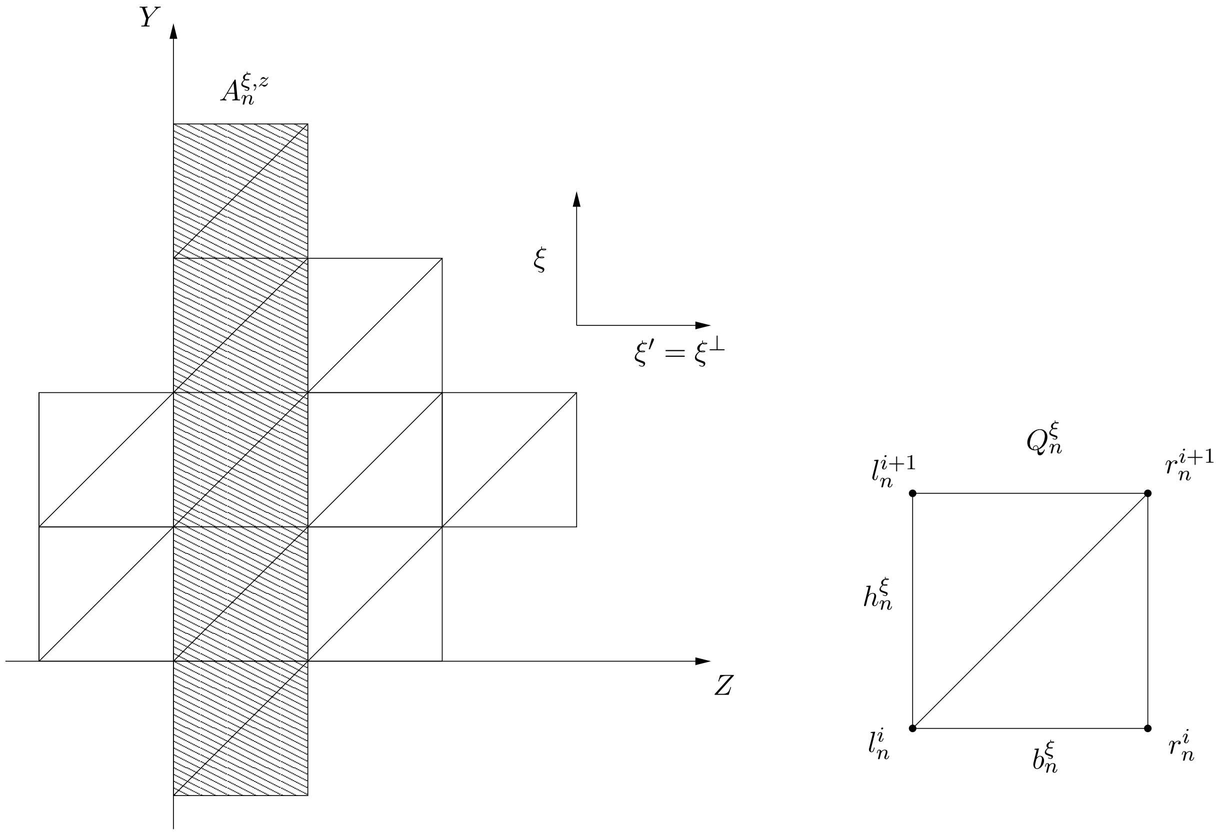

Figure 6. Representation of the slicing for $\xi = \xi_1$; l_n^i denotes $l_n^{\xi,z}(ih_n^\xi)$ etc.

Clearly we can write $D_\xi u_n^\xi$ in terms of $l_n^{\xi,z}$ and $r_n^{\xi,z}$. For simplicity, whenever possible we will drop the dependence on ξ and z and we will denote by l_n^i the value $l_n^{\xi,z}(ih_n^\xi)$. For every triangle T in $A_n^{\xi,z}$ either

$$D_\xi u_n^\xi = D^i l_n := \frac{l_n^{i+1} - l_n^i}{h_n^\xi} \qquad \text{or} \qquad D_\xi u_n^\xi = D^i r_n := \frac{r_n^{i+1} - r_n^i}{h_n^\xi}.$$

Thus

$$\int_{A_n^{\xi,z}} f_n\big(\mu|D_\xi u_n^\xi|^2\big)\,dx = \sum_{i=0}^{k_n-1} |T| f_n\big(\mu|D^i l_n|^2\big) + |T| f_n\big(\mu|D^i r_n|^2\big)$$
$$= \sum_{i=0}^{k_n-1} \big(\tfrac{1}{2} b_n^\xi c_\xi\big)\, h_n^\xi f_n\big(\mu|D^i l_n|^2\big) + \sum_{i=0}^{k_n-1} \big(\tfrac{1}{2} b_n^\xi c_\xi\big)\, h_n^\xi f_n\big(\mu|D^i r_n|^2\big)\,.$$

For $\eta_\xi = h_n^\xi/h_n$ and $\mu_\xi = \mu/\eta_\xi$, we define the one dimensional functional

$$F_n(l_n) = \sum_{i=0}^{k_n-1} h_n^\xi f_n\big(\mu|D^i l_n|^2\big) = \eta_\xi h_n^\xi \sum_{i=0}^{k_n-1} \frac{1}{h_n^\xi} f\left(h_n^\xi\,\mu_\xi |D^i l_n|^2\right)\,. \tag{16}$$

Then we can write

$$\int_{A_n^{\xi,z}} f_n\big(\mu|D_\xi u_n^\xi|^2\big)\,dx \geq \big(\tfrac{1}{2} b_n^\xi c_\xi\big) F_n(l_n) + \big(\tfrac{1}{2} b_n^\xi c_\xi\big) F_n(r_n)\,.$$

Let l_n^ξ and r_n^ξ be the extensions of $l_n^{\xi,z}$ and $r_n^{\xi,z}$ to $A_n^{\xi,z}$ (for every $z \in b_n^\xi \boldsymbol{Z}$) defined by $l_n^\xi(x + t\xi') = l_n^{\xi,z}(x)$ for $x \in A^{\xi,z} \cap A_n^{\xi,z}$ and $r_n^\xi(x - t\xi') = r_n^{\xi,z}(x)$ for $x \in A^{\xi,z+b_n^\xi} \cap A_n^{\xi,z}$. For $x \notin A_n^{\xi,z}$ we set $l_n^\xi = r_n^\xi = 0$. Note that by definition l_n^ξ and r_n^ξ are piecewise affine on the squares $Q_n^{\xi,z}$ and are possibly discontinuous along $\partial A_n^{\xi,z}$. Moreover we have the following property.

Lemma 5.1 *The sequences l_n^ξ and r_n^ξ converge to u^ξ in $L^1(A)$.*

Proof. Let m_n and M_n be piecewise constant function on the squares Q_n^ξ defined by $m_n = \min u_n^\xi$ and $M_n = \max u_n^\xi$, for every Q_n^ξ (note that u_n^ξ takes scalar values). Let $k \in N$, for $i, j \in \boldsymbol{N}$ with $|i| \leq k$ and $|j| \leq k$ let us introduce the function $w_n^{i,j}(x) = u_n^\xi(x + y_i\xi + z_j\xi')$ where $y_i = (i/k)h_n^\xi$ and $z_j = (i/k)b_n^\xi$. Finally, we consider the functions M_n^k and m_n^k given by

$$M_n^k(x) = \max_{i,j}\{w_n^{i,j}(x)\} \qquad\qquad m_n^k(x) = \min_{i,j}\{w_n^{i,j}(x)\}\,.$$

Clearly $w_n^{i,j} \to u^\xi$ for every i, j and then M_n^k and m_n^k converges to u^ξ as well. For $\eta > 0$ let $A_\eta = \{x \in A : d(x, \partial A) > \eta\}$. As u_n^ξ is piecewise affine and uniformly bounded in L^∞ it follows that for every $\delta > 0$ there exists k, sufficiently large, in such a way that for every $x \in A_\eta$ we have

$$M_n^k(x) \geq M_n(x) - \delta \qquad\qquad m_n^k(x) \leq m_n(x) + \delta\,.$$

Then for n sufficiently large we can write that $\|M_n^k - M_n\|_{L^1(A_\eta)} \leq \delta|A_\eta|$ and similarly for m_n^k. Since δ is arbitrary by a diagonal argument it follows that M_n and m_n converge to u^ξ as well. In the set A_η we have $m_n \leq l_n^\xi \leq M_n$; then we get $l_n^\xi \to u^\xi$ and the same for r_n^ξ. For $\eta \searrow 0$ we recover the convergence in $L^1(A)$. ■

For $w \in Z$ let us denote by $l_n^{\xi,w}$ the function l_n^ξ restricted to the section $A^{\xi,w}$. Then, by the definition of l_n^ξ, we can write

$$b_n^\xi F_n(l_n^{\xi,z}) = \int_z^{z+b_n^\xi} F_n(l_n^{\xi,w})\,dw$$

and similarly for r_n^ξ. Then

$$\int_A f_n\big(\mu|D_\xi u_n^\xi|^2\big)\,dx \geq (\tfrac{1}{2}c_\xi)\int_Z F_n(l_n^{\xi,w})\,dw + (\tfrac{1}{2}c_\xi)\int_Z F_n(r_n^{\xi,w})\,dw\,.$$

Since $l_n^{\xi,w}$ converges to $u^{\xi,w}$ for $\mathcal{H}^1$-a.e. $w \in Z$, we get (see for instance [9])

$$\liminf_{n\to+\infty} F_n(l_n^{\xi,w}) \geq \gamma\eta_\xi \#(J(u^{\xi,w}) \cap A^{\xi,w})\,.$$

Clearly the same holds for $r_n^{\xi,w}$. From this inequality we conclude easily, indeed by Fatou's lemma and Proposition A.2 we get

$$\begin{aligned}\liminf_{n\to+\infty}\int_A f_n\big(\mu|D_\xi u_n^\xi|^2\big)\,dx &\geq \gamma c_\xi \int_Z \eta_\xi \#(J(u^{\xi,w}) \cap A^{\xi,w})\,dw \\ &= \gamma \int_{\langle\xi^\perp\rangle} \eta_\xi \#(J(u^{\xi,w}) \cap A^{\xi,w})\,dw = \gamma\int_{J^\xi(u)\cap A} \eta_\xi |\langle\nu,\xi\rangle|\,ds\,.\end{aligned}$$

Finally, we remark that for $\xi \in \{\xi_i\}$ we have $\eta_\xi = \tau_\xi$.

Remark 5.2 *Note that it is not possible to use simply to one dimensional restriction of the functional G_n as given by Fubini's theorem. Such a choice would result in a wrong estimate of the surface energy due to the fact that the corresponding one-dimensional functional would be defined on a non-uniform lattice. In some sense the triangulation introduces a sort of non local effect at the element scale h_n which makes not possible to consider directly the one dimensional sections. Indeed G_n has been restricted first to the "smeared" section $A_n^{\xi,z}$ for $z \in b_n^\xi \boldsymbol{Z}$ and then, through the functions l_n^ξ and r_n^ξ, to the "real" section $A^{\xi,w}$ for $w \in Z$.*

5.2.2 Slicing estimate for $\boldsymbol{\xi \neq \xi_i}$.

For $\xi \neq \xi_i$, in a similar way as for $\xi = \xi_i$, we define for $z \in b_n^\xi \boldsymbol{Z}$ the stripe $A_n^{\xi,z}$ obtained this time by the union of the quadrilaterals $Q_n^{\xi,z}$ which are contained in A and which covers the section $A^{\xi,z}$, as represented in Figure 7. The sets $Q_n^{\xi,z}$ are always the union of two neighbouring triangles of $\boldsymbol{T}_h$ which can form either a square or a parallelogram. Note that if $z, w \in b_n^\xi \boldsymbol{Z}$ and $z \neq w$ then $|A_n^{\xi,z} \cap A_n^{\xi,w}| = 0$.

The definition of a suitable section in direction ξ is more delicate as the lattice $(h_n/\sqrt{2})\boldsymbol{Z}^2$ is not aligned with ξ. As a matter of fact we will transform $(h_n/\sqrt{2})\boldsymbol{Z}^2$ into $(\xi' b_n^\xi \boldsymbol{Z}) \times (\xi h_n^\xi \boldsymbol{Z})$. More precisely, each stripe $A_n^{\xi,z}$ is mapped in a stripe $B_n^{\xi,z}$ obtained as in Figure 7, namely transforming the quadrilaterals $Q_n^{\xi,z}$ in the parallelograms $P_n^{\xi,z}$. Note that this map does not coincide with the linear map of $(h_n/\sqrt{2})\boldsymbol{Z}^2$ in $(\xi' b_n^\xi \boldsymbol{Z}) \times (\xi h_n^\xi \boldsymbol{Z})$. Such a choice would not be right for our energy estimate. We remark that the new mesh $\boldsymbol{T}_n'$, obtained by this mapping, is no longer periodic with respect to $h_n^\xi \xi$; indeed the orientation of the elements inside the parallelograms $P_n^{\xi,z}$ is no longer uniform (see again Figure 7).

Finally, u_n^ξ, defined in $\boldsymbol{T}_n$, are transformed in v_n^ξ, defined on the deformed triangulation $\boldsymbol{T}_n'$. Now, let $l_n^{\xi,z}$ and $r_n^{\xi,z}$ be the restrictions of v_n^ξ respectively on $A^{\xi,z} \cap B_n^{\xi,z}$ and $A^{\xi,z+b_n^\xi} \cap B_n^{\xi,z}$. Moreover, as in the previous section, we denote by l_n^ξ and r_n^ξ the right and left extensions of $l_n^{\xi,z}$ and $r_n^{\xi,z}$. Using Lemma 5.1 it is not difficult to check that v_n^ξ converges to u^ξ in $L^1(\Omega)$ and thus the same holds for l_n^ξ and r_n^ξ. (We remark that L^1-convergence is not preserved by the linear mapping of $(h_n/\sqrt{2})\boldsymbol{Z}^2$ in $(\xi' b_n^\xi \boldsymbol{Z}) \times (\xi h_n^\xi \boldsymbol{Z})$).

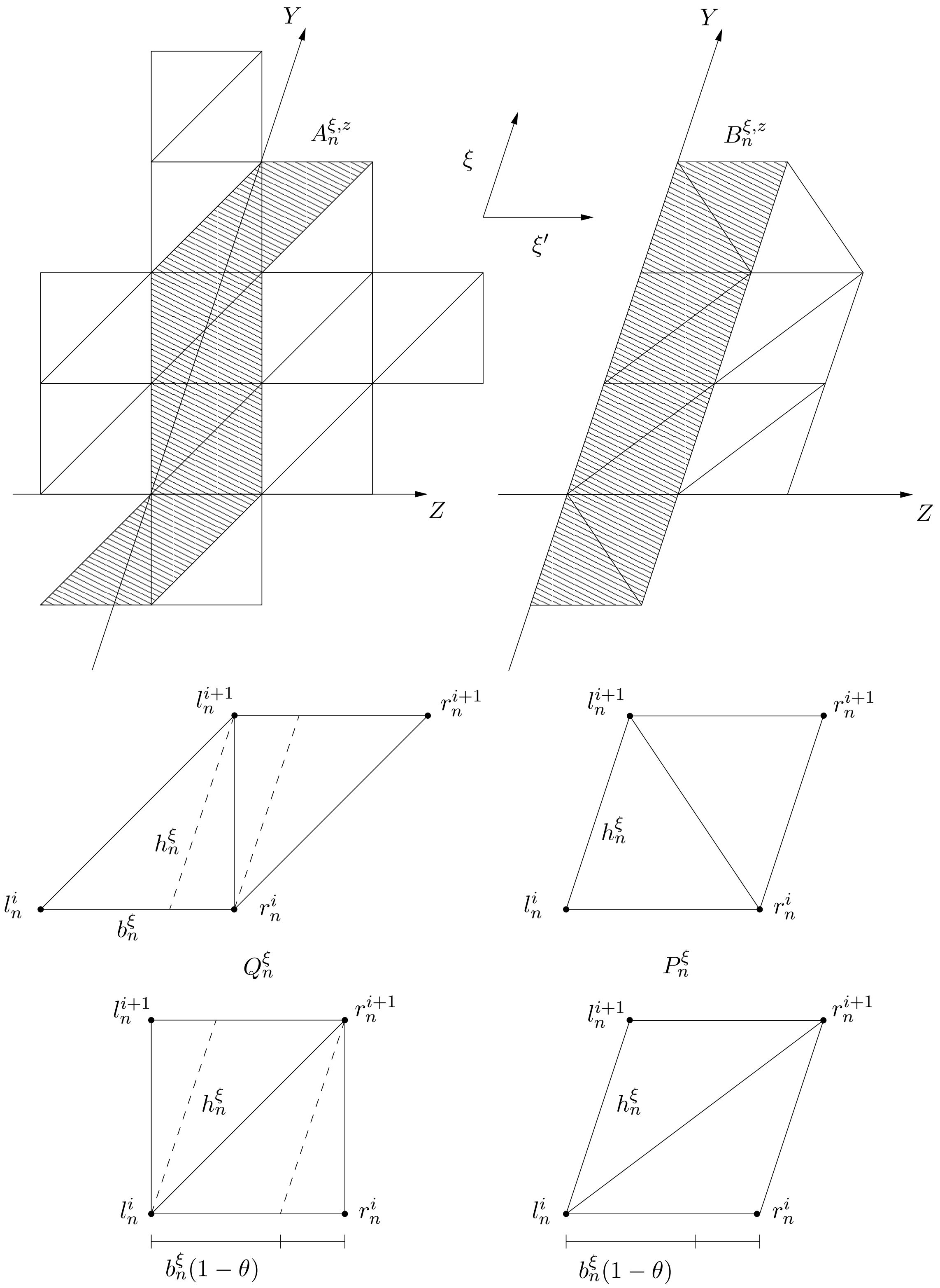

Figure 7. Representation of the slicing for $\xi \neq \xi_i$.

As in the previous section, the next step is to write the directional derivative $D_\xi u_n^\xi$ in terms of l_n^ξ and r_n^ξ. It is not restrictive to consider $B_n^{\xi,z}$ of the form $(-k_n h_n^\xi, k_n h_n^\xi)\xi \times (z, z+b_n^\xi)\xi'$ with k_n depending on z. Accordingly, we consider $l_n^{\xi,z}$ and $r_n^{\xi,z}$ to be defined on the lattice ih_n^ξ for $i=-k_n, ..., k_n$. Moreover, whenever it is convenient, we will write l_n instead of $l_n^{\xi,z}$ and denote by l_n^i the value $l_n^{\xi,z}(ih_n)$ etc.

Depending whether the quadrilateral Q_n^ξ is a square or not, for $\theta = |\langle \xi, \xi' \rangle|$ (see also Figure 7) we have

$$
\begin{aligned}
h_n^\xi D_\xi u_n^\xi = h_n^\xi D_L^i(l_n, r_n) &:= l_n^{i+1} - \Big((1-\theta)r_n^i + \theta l_n^i\Big) = (l_n^{i+1} - l_n^i) - (1-\theta)(r_n^i - l_n^i) \\
h_n^\xi D_\xi u_n^\xi = h_n^\xi D_R^i(l_n, r_n) &:= \Big((1-\theta)l_n^{i+1} + \theta r_n^{i+1}\Big) - r_n^i = (r_n^{i+1} - r_n^i) - (1-\theta)(r_n^{i+1} - l_n^{i+1}),
\end{aligned}
$$

or

$$
\begin{aligned}
h_n^\xi D_\xi u_n^\xi = h_n^\xi D_L^i(l_n, r_n) &:= \Big((1-\theta)l_n^{i+1} + \theta r_n^{i+1}\Big) - l_n^i = (l_n^{i+1} - l_n^i) + \theta(r_n^{i+1} - l_n^{i+1}) \\
h_n^\xi D_\xi u_n^\xi = h_n^\xi D_R^i(l_n, r_n) &:= r_n^{i+1} - \Big((1-\theta)r_n^i + \theta l_n^i\Big) = (r_n^{i+1} - r_n^i) + \theta(r_n^i - l_n^i).
\end{aligned}
$$

(Note that here the directional derivative $D_\xi u_n^\xi$ depends both on l_n^ξ and r_n^ξ). Therefore

$$
\begin{aligned}
\int_{B_n^{\xi,z}} f_n\big(\mu|D_\xi u_n^\xi|^2\big)\,dx &= \sum_{i=-k_n}^{k_n-1} |T| f_n\big(\mu|D_L^i(l_n,r_n)|^2\big) + |T| f_n\big(\mu|D_R^i(l_n,r_n)|^2\big) \\
&= (b_n^\xi c_\xi) \sum_{i=-k_n}^{k_n-1} \tfrac{1}{2} h_n^\xi f_n\big(\mu|D_L^i(l_n,r_n)|^2\big) + \tfrac{1}{2} h_n^\xi f_n\big(\mu|D_R^i(l_n,r_n)|^2\big).
\end{aligned}
$$

Now we can define the functional

$$
F_n^{\xi,z}(l_n^{\xi,z}, r_n^{\xi,z}, B_n^{\xi,z}) := \sum_{i=-k_n}^{k_n-1} \tfrac{1}{2} h_n^\xi f_n\big(\mu|D_L^i(l_n,r_n)|^2\big) + \tfrac{1}{2} h_n^\xi f_n\big(\mu|D_R^i(l_n,r_n)|^2\big).
$$

Then we can write

$$
\int_{B_n^{\xi,z}} f_n\big(\mu|D_\xi u_n^\xi|^2\big)\,dx = c_\xi \int_z^{z+b_n^\xi} F_n^{\xi,w}(l_n^{\xi,w}, r_n^{\xi,w}, B_n^{\xi,w})\,dw
$$

and thus

$$
\int_A f_n\big(\mu|D_\xi u_n^\xi|^2\big)\,dx \geq c_\xi \int_Z F_n^{\xi,w}(l_n^{\xi,w}, r_n^{\xi,w}, B_n^{\xi,w})\,dw.
$$

Arguing as in the previous section it will be sufficient to show that

$$
\liminf_{n\to+\infty} F_n^{\xi,w}(l_n^{\xi,w}, r_n^{\xi,w}, B_n^{\xi,w}) \geq \gamma \eta_\xi \#(J(u^{\xi,w}) \cap A^{\xi,w}).
$$

For the proof of this inequality we can consider, by a localization argument, that $A^{\xi,z}$ is an interval I of the form $(-k,k)$ and that $J(u^{\xi,z}) = \{0\}$. Moreover, by an approximation argument, it is not restrictive to assume that $f(t) = t \wedge 1$. Thus we will show that

$$
\liminf_{n\to+\infty} F_n^{\xi,w}(l_n^{\xi,w}, r_n^{\xi,w}, B_n^{\xi,w}) \geq \eta_\xi. \tag{17}
$$

We remark that in this case it is not possible to consider separately the limit for $l_n^{\xi,z}$ and $r_n^{\xi,z}$, as we did in the previous section, because the derivatives D_R^i and D_L^i depend on both of them.

Let e_n be defined in the knots ih_n by $e_n^i = r_n^i - l_n^i$ (for $-k_n \leq i \leq k_n$). Clearly $e_n \to 0$ in L^1. Depending again whether $Q_n^{\xi,z}$ is a square we have

$$h_n^\xi D_R^i(l_n, r_n) - h_n^\xi D_L^i(l_n, r_n) = (1-\theta)(r_n^{i+1} - l_n^{i+1}) + (\theta - 1)(r_n^i - l_n^i)\,,$$

or

$$h_n^\xi D_R^i(l_n, r_n) - h_n^\xi D_L^i(l_n, r_n) = \theta(r_n^{i+1} - l_n^{i+1}) - \theta(r_n^i - l_n^i)\,,$$

which give respectively

$$(1-\theta)(e_n^{i+1} - e_n^i) = h_n^\xi \big(D_R^i(l_n, r_n) - D_L^i(l_n, r_n)\big) \tag{18}$$

$$\theta(e_n^{i+1} - e_n^i) = h_n^\xi \big(D_R^i(l_n, r_n) - D_L^i(l_n, r_n)\big)\,. \tag{19}$$

For convenience we introduce another interval $I' = (-k', k')$ for $0 < k' < k$ and define $\bar{e}_n = \max\{|e_n^i| \,:\, ih_n^\xi \in I'\} = e_n(y_n)$ for $y_n = j_n h_n^\xi$. We will consider two different cases, according to $\liminf_{n\to+\infty} \bar{e}_n > 0$ or $\liminf_{n\to+\infty} \bar{e}_n = 0$.

Let us consider first that $\liminf_{n\to\infty} \bar{e}_n = e' > 0$. Possibly passing to a subsequence, we can assume that $y_n \to y$ for $y \in [-k', k']$ and that for $\delta > 0$ and n sufficiently large there exists an index i_n such that $y - \delta < i_n h_n^\xi < y$ and $|e_n^{i_n}| < \bar{e}_n/2$. Thus from (18) and (19) we can write

$$\begin{aligned}\sum_{i=i_n}^{j_n-1} h_n^\xi |D_L^i(l_n, r_n) - D_R^i(l_n, r_n)| &\geq \big(\theta \wedge (1-\theta)\big) \sum_{i=i_n}^{j_n-1} |e_n^{i+1} - e_n^i| \\ &\geq \big(\theta \wedge (1-\theta)\big)|e_n^{i_n} - e_n^{j_n}| \geq 2c\,\bar{e}_n\,,\end{aligned}$$

where the constant c depends only on θ. Hence

$$2c\,\bar{e}_n \leq \sum_{i=i_n}^{j_n-1} h_n^\xi |D_L^i(l_n, r_n) - D_R^i(l_n, r_n)| \leq \sum_{i=i_n}^{j_n-1} h_n^\xi |D_L^i(l_n, r_n)| + \sum_{i=i_n}^{j_n-1} h_n^\xi |D_R^i(l_n, r_n)|\,.$$

Then one of the two terms in the previous sum is greater than or equal to $c\,\bar{e}_n$. Let us assume it is the first and define a piecewise affine function $L_n : I \to \boldsymbol{R}$ in the following way:

$$L_n^i = \begin{cases} 0 & \text{for } i \leq i_n \\ L_n^{i-1} + h_n^\xi |D_L^i(l_n, r_n)| & \text{for } i_n + 1 \leq i \leq j_n \\ L_n^{j_n} & \text{for } i > j_n. \end{cases}$$

Since $\sum_{i=i_n}^{j_n-1} h_n^\xi |D_L^i(l_n, r_n)| \geq c\,\bar{e}_n$ we have $L_n^{j_n} \geq c\,\bar{e}_n$. Now we show that for n sufficiently large we have

$$\sum_{i=i_n}^{j_n-1} \tfrac{1}{2} h_n^\xi f_n\big(\mu |D_L^i(l_n, r_n)|^2\big) \geq \tfrac{1}{2}\eta_\xi\,.$$

This inequality is obvious if, for some index i, we have $h_n\mu|D_L^i(l_n,r_n)|^2 > 1$. Otherwise we can write

$$\begin{aligned}\sum_{i=i_n}^{j_n-1} \tfrac{1}{2}h_n^\xi f_n\big(\mu|D_L^i(l_n,r_n)|^2\big) &\geq \tfrac{1}{2}\mu\sum_{i=j_n}^{j_n-1} h_n^\xi|D_L^i(l_n,r_n)|^2 \\ &\geq \tfrac{1}{2}\mu\int_I |L_n'|^2\,dy \geq c'\,\bar{e}_n^{\,2}/\delta\,.\end{aligned}$$

As $\bar{e}_n \to e' > 0$, for δ sufficiently small we get the required inequality.

Clearly the same argument holds for another index j_n such that $y < j_n h_n^\xi < y+\delta$ and thus (17) is proved.

Now, let us consider the case $\liminf_{n\to+\infty}\bar{e}_n = 0$. For $\delta > 0$ we define

$$g_h^\delta(t) = \begin{cases} |t|/\delta & |t| < (\delta/h) \\ 1/h & \text{otherwise.}\end{cases}$$

It is easy to see that $g_h^\delta(t) \leq f_h(t^2) + (1/\delta^2)$. We recall that $J(u^{\xi,z}) = \{0\}$ and we denote by $[u^{\xi,z}]$ its jump. For $\delta > 0$ let j_n^+ and j_n^- such that $j_n^\pm h_n^\xi \to \pm\delta^3$. Then, for $n \gg 1$ and $\mu' = \sqrt{\mu}$ we get

$$\begin{aligned}F_n^{\xi,z}(l_n^{\xi,z},r_n^{\xi,z},B_n^{\xi,z}) &\geq \sum_{i=j_n^-}^{j_n^+-1} \tfrac{1}{2}h_n^\xi f_n\big(\mu|D_L^i(l_n,r_n)|^2\big) + \tfrac{1}{2}h_n^\xi f_n\big(\mu|D_R^i(l_n,r_n)|^2\big) \\ &\geq \sum_{i=j_n^-}^{j_n^+-1} \tfrac{1}{2}h_n^\xi g_n^\delta\big(\mu' D_L^i(l_n,r_n)\big) + \tfrac{1}{2}h_n^\xi g_n^\delta\big(\mu' D_R^i(l_n,r_n)\big) - 3\delta\,.\end{aligned}$$

Now we distinguish between two cases. First we consider that for every index $i = j_n^-,...,j_n^+-1$ we have $\big|\mu' D_L^i(l_n,r_n)\big| \wedge \big|\mu' D_R^i(l_n,r_n)\big| \leq (\delta/h)$. Then

$$F_n^{\xi,z}(l_n^{\xi,z},r_n^{\xi,z},B_n^{\xi,z}) \geq \frac{c}{\delta}\sum_{i=j_n^-}^{j_n^+-1} h_n^\xi\big|D_L^i(l_n,r_n)\big| + h_n^\xi\big|D_R^i(l_n,r_n)\big| - 3\delta\,.$$

Remembering that $D_L^i(l_n,r_n) = D_\xi u_n^{\xi,z}$, and similarly for $D_R^i(l_n,r_n)$, we can write

$$F_n^{\xi,z}(l_n^{\xi,z},r_n^{\xi,z},B_n^{\xi,z}) \geq \frac{c}{\delta}\sum_{i=j_n^-}^{j_n^+-1} h_n^\xi\big|D_\xi u_n^{\xi,z}\big| - 3\delta \geq \frac{c}{\delta}\int_{y_n^-}^{y_n^+}\big|(u_n^{\xi,z})'\big|\,dy - 3\delta\,.$$

Hence, by the lower semicontinuity of the total variation (with respect to the L^1-convergence), we get

$$\liminf_{n\to+\infty} F_n^{\xi,z}(l_n^{\xi,z},r_n^{\xi,z},B_n^{\xi,z}) \geq \frac{c}{\delta}|[u^{\xi,z}]| - 3\delta\,,$$

which gives (17) for δ sufficiently small.

Now we consider that for some index i we have $\big|\mu' D_L^i(l_n,r_n)\big| \wedge \big|\mu' D_R^i(l_n,r_n)\big| > (\delta/h)$. Then, being g_n^δ Lipschitz continuous with constant $1/\delta$, by (18) and (19) we can write

$$\begin{aligned}\Big|g_n^\delta\big(\mu' D_L^i(l_n,r_n)\big) - g_n^\delta\big(\mu' D_R^i(l_n,r_n)\big)\Big| &\leq \frac{1}{\delta}\,\big|\mu' D_L^i(l_n,r_n) - \mu' D_R^i(l_n,r_n)\big| \\ &\leq c\,(\bar{e}_n/h_n^\xi)\,,\end{aligned}$$

where c depends on δ, θ and μ. Then, assuming for instance that $|\mu' D^i_R(l_n, r_n)| \geq (\delta/h)$ we get

$$g^\delta_n\big(\mu' D^i_L(l_n, r_n)\big) + g^\delta_n\big(\mu' D^i_R(l_n, r_n)\big) \geq 2\,(1/h_n) - c\,(\bar{e}_n/h^\xi_n)\,.$$

It follows that

$$\begin{aligned} &\tfrac{1}{2} h^\xi_n g^\delta_n\big(\mu |D^i_L(l_n, r_n)|^2\big) + \tfrac{1}{2} h^\xi_n g^\delta_n\big(\mu |D^i_L(l_n, r_n)|^2\big) \geq \\ &\qquad\qquad \geq \tfrac{1}{2} h^\xi_n\big((2/h_n) - c\,(\bar{e}_n/h^\xi_n)\big) = \eta_\xi - c'\bar{e}_n\,. \end{aligned}$$

As $\bar{e}_n \to 0$ we get the right inequality.

Now we can complete the proof of the Γ-liminf inequality for the length energy. In the previous two subsections we have shown that (15) holds true for every sequence $u_n \in V_n$ converging to u in $L^1(\Omega)$. Thus, for every $\xi = \zeta/|\zeta|$, with $\zeta \in \boldsymbol{Z}^2$, we have

$$\Gamma\text{-}\liminf_{n\to+\infty} G_n(u, A) \geq \gamma \int_{J^\xi(u)\cap A} \eta_\xi |\langle \nu, \xi\rangle|\, ds = \gamma \int_{J(u)\cap A} \psi_\xi\, ds\,, \tag{20}$$

where

$$\psi_\xi(x) = \begin{cases} \eta_\xi |\langle \nu, \xi\rangle| & \text{if } x \in J^\xi(u) \\ 0 & \text{otherwise.} \end{cases}$$

For $d\lambda = ds$ and $\mu(A) = \Gamma\text{-liminf}_{n\to+\infty} G_n(u, A)$, by Proposition A.4 we are allowed to take in (20) the pointwise supremum of the family ψ_ξ. Hence it will be sufficient to show that $\sup_\xi \psi_\xi(x) \geq \phi(\nu)$. For $x \in J(u)$ let $\nu = \nu(x)$ be the normal to $J(u)$ and let $\phi(\nu) = \tau_i |\langle \nu, \xi_i\rangle|$ for an index $i = 1, .., 3$. Let $u^+ = u^+(x)$ and $u^- = u^-(x)$ be the traces of u on $J(u)$. If $\langle u^+ - u^-, \xi_i\rangle \neq 0$ then $x \in J^{\xi_i}(u)$ and $\psi_{\xi_i}(x) = \eta_{\xi_i} |\langle \nu, \xi_i\rangle| = \tau_i |\langle \nu, \xi_i\rangle|$. Obviously $\sup_\xi \psi_\xi(x) \geq \tau_i |\langle \nu, \xi_i\rangle| = \phi(\nu)$. On the contrary, if $\langle u^+ - u^-, \xi_i\rangle = 0$ let us take a sequence ξ^k which converges to ξ_i and such that $\xi^k \neq \xi_i$ for $k \in \boldsymbol{N}$. As $u^+ - u^- \neq 0$ and the vectors ξ^k and ξ_i are independent, it follows that $\langle u^+ - u^-, \xi^k\rangle \neq 0$. Therefore $x \in J^{\xi^k}(u)$ and $\psi_{\xi^k}(x) = \eta_{\xi^k} |\langle \nu, \xi^k\rangle|$. Since $\eta_\xi = h^\xi_n/h_n$, it is easy to check that η_{ξ^k} converges to $\eta_{\xi_i} = \tau_i$ (see Figure 5). Then $\sup_\xi \psi_\xi(x) \geq \sup_k \psi_{\xi^k}(x) \geq \tau_i |\langle \nu, \xi_i\rangle| = \phi(\nu)$.

A Some technical results

A.1 Basic facts about the space SBD

In this appendix we recall briefly some results about the space SBD of special functions with bounded deformation. In the sequel we will always assume that Ω is an open, bounded, Lipschitz set in $\boldsymbol{R}^2$. A vector field $u \in L^1(\Omega, \boldsymbol{R}^2)$ belongs to $SBD(\Omega)$ if the symmetrized distributional derivative $Eu = \frac{1}{2}(Du + Du^T)$ is a finite measure which can be written as

$$Eu = \boldsymbol{\varepsilon}(u)\, \mathcal{L}^n + (u^+ - u^-) \odot \nu_u\, \mathcal{H}^1 \llcorner J(u)\,.$$

For simplicity in the sequel we will denote by dx the Lebesgue measure and by ds the one-dimensional Hausdorff measure. When $\boldsymbol{\varepsilon}(u) \in L^p(\Omega, \boldsymbol{R}^{2\times 2})$ for $p > 1$ and $\mathcal{H}^1(J(u)) < +\infty$ we will say that $u \in SBD^p(\Omega)$.

The following compactness and lower semicontinuity result is proved in [5], the extension to anisotropic energies can be found in [19].

Proposition A.1 *Let ϕ be a norm in $\boldsymbol{R}^2$ and u_k be a sequence in $SBD^2(\Omega)$ such that*

$$\int_\Omega W^e\big(\varepsilon(u_k)\big)\,dx + \mathcal{H}^1\big(J(u_k)\big) + \|u_k\|_\infty \le C\,.$$

Then there exist a subsequence u_n of u_k and a function $u \in SBD^2(\Omega)$ such that $u_n \to u$ in $L^1(\Omega, \boldsymbol{R}^2)$, $\varepsilon(u_k) \rightharpoonup \varepsilon(u)$ in $L^1(\Omega, \boldsymbol{R}^{2\times 2})$ and

$$\int_{J(u)} \phi(\nu)\,ds \le \liminf_{n\to+\infty} \int_{J(u_n)} \phi(\nu)\,ds\,.$$

For $\xi \in \boldsymbol{S}^1$ we denote by Z the subspace $\langle \xi^\perp \rangle$ generated by $\xi^\perp$ and by $J^\xi(u)$ the subset of $J(u)$ where $\langle u^+ - u^-, \xi\rangle \neq 0$. For $z \in Z$ let $\Omega^{\xi,z}$ be the section $\{x \in \Omega\,:\, x = z + t\xi \text{ for } t \in \boldsymbol{R}\}$ and let $u^{\xi,z}$ be the (scalar) function

$$u^{\xi,z}(t) = \langle u(z+t\xi), \xi\rangle\,.$$

The functions in $SBD(\Omega)$ have the following slicing properties [2].

Proposition A.2 *Let $u \in SBD(\Omega)$. For a.e. $\xi \in \boldsymbol{S}^1$ and for a.e. $z \in Z$ the function $u^{\xi,z}$ belongs to $SBV(\Omega^{\xi,z})$. Moreover, for every open set $B \subset \Omega$ we have*

$$\int_Z \mathcal{H}^0\Big(J(u^{\xi,z}) \cap B^{\xi,z}\Big)\,dz = \int_{J^\xi(u)\cap B} |\langle \nu, \xi\rangle|\,ds\,.$$

The Γ-limsup inequality is based on the following density result proved in [12], the extension to anisotropic energies is due to Dal Maso and is unpublished.

Proposition A.3 *Let ϕ be a norm in $\boldsymbol{R}^2$. Then for every $u \in SBD^2(\Omega) \cap L^\infty(\Omega, \boldsymbol{R}^2)$ there exists a sequence u_n in $SBD^2(\Omega)$ such that $J(u_n)$ is a finite union of disjoint segments, $u_n \in W^{k,\infty}(\Omega, \boldsymbol{R}^2)$ (for k arbitrarily large), $\|u_n\|_\infty \le \|u\|_\infty$ and*

$$u_n \to u \text{ in } L^1(\Omega, \boldsymbol{R}^2) \qquad \varepsilon(u_n) \to \varepsilon(u) \text{ in } L^2(\Omega, \boldsymbol{R}^{2\times 2})$$

$$\lim_{n\to+\infty} \int_{J(u_n)} \phi(\nu)\,ds \le \int_{J(u)} \phi(\nu)\,ds\,.$$

A.2 The supremum of a family of measures

This measure theoretic result, proved e.g. in [6], will be used for the Γ-liminf inequality.

Proposition A.4 *We denote by $\mathcal{A}(\Omega)$ the collections of open sets contained in Ω. Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be super-additive on open sets with disjoint compact closures and let λ be a positive Borel measure in Ω. Let ψ_i be a family of positive Borel functions such that*

$$\int_A \psi_i\,d\lambda \le \mu(A) \quad \textit{for all } A \in \mathcal{A}(\Omega)\,.$$

Then

$$\int_A \sup_i \psi_i(x)\,d\lambda \le \mu(A) \quad \textit{for all } A \in \mathcal{A}(\Omega)\,.$$

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